Higher-Order Small Time Asymptotic Expansion of Itô Semimartingale Characteristic Function with Application to Estimation of Leverage from Options*

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Abstract

In this paper, we derive a higher-order asymptotic expansion of characteristic functions of an Itô semimartingale over asymptotically shrinking time intervals. The leading term in the expansion is determined by the value of the diffusive coefficient at the beginning of the interval. The higher-order terms are determined by the jump compensator as well as the coefficients appearing in the diffusion dynamics. The result is applied to develop a nearly rate-efficient estimator of the leverage coefficient of an asset price, i.e., the coefficient in its volatility dynamics that appears in front of the Brownian motion that drives also the asset price.

Keywords: characteristic function, higher-order asymptotic expansion, Itô semimartingale, leverage effect, nonparametric inference, options.

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1 Introduction

In this paper we are interested in the asymptotic behavior over small time scales of the following Itô semimartingale

$$x_t = x_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s + \sum_{s \le t} \Delta x_s, \tag{1.1}$$

defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{Q}\right)$, and where α and σ are processes with càdlàg paths and W is a Brownian motion. The diffusion coefficient σ_t, which we henceforth refer to as stochastic volatility, is another Itô semimartingale given by

$$\sigma_t = \sigma_0 + \int_0^t b_s ds + \int_0^t \eta_s dW_s + \int_0^t \widetilde{\eta}_s d\widetilde{W}_s + \sum_{s\le t} \Delta\sigma_s, \tag{1.2}$$

where b, η and $\widetilde{\eta}$ are processes with càdlàg paths, and $\widetilde{W}$ is a Brownian motion independent of W. The general Itô semimartingale model given by (1.1)-(1.2) is used in many applications, and in particular in finance for modeling financial asset prices. Indeed, most asset pricing models are nested in the above general setup, e.g., models in the popular exponential-affine class of Duffie et al. (2003) (a notable exception are models driven by fractional Brownian motion that have been recently studied by Gatheral et al. (2018)). It is well known that over short time intervals, the diffusion component of x determines its behavior. More specifically, under fairly weak conditions for the processes in (1.1)-(1.2), we have the following convergence result (see e.g., Jacod and Protter (2012))

$$\frac{x_T - x_0}{\sqrt{T}\sigma_0} \xrightarrow{\mathcal{L}} Z, \quad \text{as } T \downarrow 0, \tag{1.3}$$

where Z is a standard normal variable and provided $\sigma_0 \neq 0$. This result has been used by Todorov and Tauchen (2011) to study whether a discretely-observed Itô semimartingale contains a diffusion coefficient. Related to this and denoting with $\mathbb{E}_0^{\mathbb{Q}}(\cdot) = \mathbb{E}^{\mathbb{Q}}(\cdot|\mathcal{F}_0)$, Jacod and Todorov (2014) and Todorov (2019) show that

$$\mathbb{E}_0^{\mathbb{Q}}(e^{iux_T/\sqrt{T}}) = e^{-u^2\sigma_0^2/2} + O_p(\sqrt{T}), \text{ locally uniformly in } u, \tag{1.4}$$

for $T \downarrow 0$ and where the residual term on the right hand side of (1.4) is due to the jumps in x and the variation in the characteristics of x over the time interval. Todorov and Tauchen (2012a,b), Bull (2014), Kong et al. (2015), Jacod and Todorov (2014, 2018) and Todorov (2019), among others, use the empirical characteristic function of high-frequency increments of a semimartingale process in various applications. In particular, Jacod and Todorov (2014, 2018) show that the result in (1.4) can be used to separate in an efficient way stochastic volatility from jumps from discrete observations of x and Todorov (2019) shows that this result can be used to do this separation from short-dated options.

For many applications of interest, however, the variation in σ can be rather nontrivial and therefore the expansion in (1.4), which ignores this variation, can provide an inaccurate approximation to $\mathbb{E}_0^{\mathbb{Q}}(e^{iux_T/\sqrt{T}})$ when applied with small but finite T. The goal of this paper, therefore, is to generalize the result in (1.4) by expanding $\mathbb{E}_0^{\mathbb{Q}}(e^{iux_T/\sqrt{T}})$, for $T \downarrow 0$, by explicitly accounting for the variation in σ, in particular. This result is derived under the assumption of finite variation jumps driving the dynamics of x (such an assumption is required also for deriving many of the existing results that deal with separation of diffusions from jumps in high-frequency settings). More specifically, we derive a higher-order expansion result with a residual term of asymptotic order of only $O_p(T^{3/2-r/2} \vee T^{3/2}|\log(T)|)$, where $r \in [0,1]$ captures the so-called degree of activity of the jumps in x, see e.g., Section 3.2 of Jacod and Protter (2012) and our assumptions A1-r and A2-r below (in particular, $r = 0$ corresponds to jumps of finite activity often used in applications). The additional terms in this higher-order expansion are due to the jumps in x as well as the time-variation in the characteristics of x, i.e., the time variation in the drift, volatility and jump compensator of x.

The higher-order asymptotic expansion of the conditional characteristic function implies for $T \downarrow 0$ and locally uniformly in u:

$$\mathbb{E}_0^{\mathbb{Q}}(e^{iux_T/\sqrt{T}}) = e^{iu\sqrt{T}\alpha_0 - u^2\sigma_0^2/2 + T\phi_0(u/\sqrt{T}) + i\sqrt{T}\eta_0 f(u,\sigma_0)} + O_p(T), \tag{1.5}$$

for some known smooth function f of u and σ_0, and where $\phi_0(u)$ is a quantity associated with the jump compensator at time 0. The order of magnitude of $T\phi_0(u/\sqrt{T})$ is $O_p(T^{1-r/2})$, where recall r denotes the index of jump activity.

We apply the above asymptotic expansion result to propose a (nearly) rate-efficient estimator of the coefficient η, entering the dynamics of σ, from short-dated options written on an asset price x. This coefficient, together with σ, determines the continuous part of the quadratic covariation between x and σ, which is referred to as leverage effect, following the influential work of Black (1976) who provided economic rationale for it. Ait-Sahalia et al. (2013), Wang and Mykland (2014), Aït-Sahalia et al. (2017) and Curato (2019) propose estimation of leverage from high-frequency record of x, see also Vetter (2015) for the related work on estimation of diffusive volatility of volatility, but the rate of convergence is rather slow. This is to be expected as the signal about η in a discrete record of x is rather weak. Kalnina and Xiu (2017), see also Andersen et al. (2015), propose instead to use proxies for volatility from high-frequency option data, such as the volatility VIX index, when estimating leverage. If volatility is treated as observable, then the rates of convergence improve but the mapping between the latent spot volatility and the option-based VIX index is in general not one-to-one, and hence with this approach we do not recover η.

In this article, we develop an alternative method for making inference about η which uses the option data alone for the inference. Our estimation strategy is based on the higher-order expansion of the conditional characteristic function in (1.5). In particular, this expansion result suggests that an easy estimator of η_0 can be formed from the principal argument of $\mathbb{E}_0^{\mathbb{Q}}(e^{iux_T/\sqrt{T}})$ together with an estimator of the spot volatility σ_0^2 by ignoring the presence of $T\phi_0(u/\sqrt{T})$ in the above asymptotic expansion. For the estimator of σ_0^2, we can follow Todorov (2019) and use the absolute value of $\mathbb{E}_0^{\mathbb{Q}}(e^{iux_T/\sqrt{T}})$. The resulting estimate of η_0, however, will have a bias of order $O_p(T^{1/2-r/2})$, which is rather nontrivial when r approaches 1. The major source of this bias is due to the asymmetry in the compensator of the jumps in x.

To improve on the above estimator, we can use $\mathbb{E}_0^{\mathbb{Q}}(e^{iux_{T_1}/\sqrt{T_1}})$ as well as $\mathbb{E}_0^{\mathbb{Q}}(e^{iux_{T_2}/\sqrt{T_2}})$, for two different T_1 and T_2, which are both approaching asymptotically zero. An appropriate combination of the two characteristic functions (for different values of u) can annihilate the leading terms in them that is due to the jumps in x, i.e., the terms $T_1\phi_0(u/\sqrt{T_1})$ and $T_2\phi_0(u/\sqrt{T_2})$, and lead to an estimate of η_0 which has an asymptotic bias of order only $O_p(\sqrt{T_1})$. This bias is of even smaller asymptotic order in cases when x and its volatility σ do not jump together or if the jump compensator is smooth.

To make the above estimation of η_0 feasible, we need an estimator for $\mathbb{E}_0^{\mathbb{Q}}(e^{iux_T/\sqrt{T}})$. For this, we follow Qin and Todorov (2019) and Todorov (2019) and combine in an appropriate way available noisy observations of options with different strikes written on the asset at time $t = 0$ and which expire at time $t = T$. For related work regarding rate-efficient estimators of the Lévy density from options with fixed time to maturity in exponential Lévy models, see e.g., Belomestny and Reiß (2006, 2015), Cont and Tankov (2004), Söhl (2014), Söhl and Trabs (2014) and Trabs (2014, 2015). The error in recovering $\mathbb{E}_0^{\mathbb{Q}}(e^{iux_T/\sqrt{T}})$ from the options is $O_p(\sqrt{\Delta}/T^{1/4})$, for Δ denoting the mesh of the log-strike grid and which is shrinking to zero simultaneously with $T \downarrow 0$. We derive an associated Central Limit Theorem (CLT) which in turn allows for quantifying the precision in estimating η_0 from the available options. We further show that our estimator of η_0 is nearly efficient in a minimax sense, i.e., the best possible rate of convergence for an estimator of η from noisy option data at a fixed point in time is at most $O_p\left(\frac{T^{3/4}(\log(1/T))^{7/4}}{\sqrt{\Delta}}\right)$ while that of our estimator is $O_p\left(\frac{T^{3/4}}{\sqrt{\Delta}}\right)$. Finally, if high-frequency option data is available, then we can improve the efficiency of our option-based estimator of η_0 by combining estimates of it from options in a local window around $t = 0$.

The rest of the paper is organized as follows. We start with our assumptions for the asset price dynamics in Section 2. The higher-order asymptotic expansion of the characteristic function of increments of the process over short time is given in Section 3. This expansion is used in Section 4 to develop an option-based nonparametric estimator of the leverage coefficient. Section 4 contains also a feasible CLT for the estimator. In Section 5 we derive a bound for the optimal rate of convergence for recovering the leverage coefficient from noisy option data. Section 6 contains a Monte Carlo study. Proofs are given in Section 7.

2 Assumptions for the Asset Price Dynamics

We start with stating the assumptions for the dynamics of x (the log-price) that we need for deriving our higher-order expansion of the characteristic function of the increments of x. Without loss of generality, throughout the analysis we will set $x_0 = 0$. The dynamics of x is given in (1.1)-(1.2) and the jumps in x and σ are specified by

$$\sum_{s\leq t} \Delta x_s = \int_0^t \int_{\mathbb{R}} \delta_x(s,z)\mu(ds,dz), \quad \sum_{s\leq t} \Delta \sigma_s = \int_0^t \int_{\mathbb{R}} \delta_\sigma(s,z)\mu(ds,dz), \tag{2.1}$$

where μ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\lambda(ds,dz) = ds \otimes dz$, and $\delta_x : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\delta_y : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are two predictable functions. We note that we do not impose any restriction on the connection between δ_x and δ_σ, so that arbitrary dependence between price and volatility jumps is allowed.

For the higher-order expansion of the conditional characteristic function of the increments of x, we will need to impose further some structure on the dynamics of α, η and $\widetilde{\eta}$ as this dynamics will play a role in the asymptotic expansion as we will see in the next section. Just like x and σ, the processes α, η and $\widetilde{\eta}$ will be general Itô semimartingales with the following dynamics

$$\begin{aligned} \alpha_t = \alpha_0 &+ \int_0^t b_s^\alpha ds + \int_0^t \sigma_s^\alpha dW_s + \int_0^t \widetilde{\sigma}_s^\alpha d\widetilde{W}_s + \int_0^t \overline{\sigma}_s^\alpha d\overline{W}_s \\ &+ \int_0^t \int_{\mathbb{R}} \delta_\alpha(s,z)\mu(ds,dz), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \eta_t = \eta_0 &+ \int_0^t b_s^\eta ds + \int_0^t \sigma_s^\eta dW_s + \int_0^t \widetilde{\sigma}_s^\eta d\widetilde{W}_s + \int_0^t \overline{\sigma}_s^\eta d\overline{W}_s \\ &+ \int_0^t \int_{\mathbb{R}} \delta_\eta(s,z)\mu(ds,dz), \end{aligned} \tag{2.3}$$

$$\begin{aligned} \widetilde{\eta}_t = \widetilde{\eta}_0 &+ \int_0^t b_s^{\widetilde{\eta}} ds + \int_0^t \sigma_s^{\widetilde{\eta}} dW_s + \int_0^t \widetilde{\sigma}_s^{\widetilde{\eta}} d\widetilde{W}_s + \int_0^t \overline{\sigma}_s^{\widetilde{\eta}} d\overline{W}_s \\ &+ \int_0^t \int_{\mathbb{R}} \delta_{\widetilde{\eta}}(s,z)\mu(ds,dz), \end{aligned} \tag{2.4}$$

where b^α, b^η, $b^{\widetilde{\eta}}$, σ^α, $\widetilde{\sigma}^\alpha$, $\overline{\sigma}^\alpha$, σ^η, $\widetilde{\sigma}^\eta$, $\overline{\sigma}^\eta$, $\sigma^{\widetilde{\eta}}$, $\widetilde{\sigma}^{\widetilde{\eta}}$ and $\overline{\sigma}^{\widetilde{\eta}}$ are processes with càdlàg paths, $\overline{\sigma}_t^\alpha$, $\overline{\sigma}_t^\eta$ and $\overline{\sigma}_t^{\widetilde{\eta}}$ are 1×3 vectors and the rest of the above processes are $\mathbb{R}$-valued, $\overline{W}$ is a 3-dimensional Brownian motion orthogonal to W and $\widetilde{W}$, and $\delta_\alpha : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\delta_\eta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\delta_{\widetilde{\eta}} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are predictable functions.

The dimension of $\overline{W}$ is sufficiently large to accommodate arbitrary continuous quadratic covariation between x, σ, α, η and $\widetilde{\eta}$. Similarly, since the functions δ_x, δ_σ, δ_η and $\delta_{\widetilde{\eta}}$ are left unrestricted, the relationship between the jumps in x, σ, α, η and $\widetilde{\eta}$ can be arbitrary. Most asset pricing models used to date satisfy the above dynamics in (1.1)-(1.2) and (2.2)-(2.4). We now state our assumptions for the dynamics of x which we need for the asymptotic analysis. Since later we will use different probability measures, to avoid confusion in the statements that follow, we will denote with $\mathbb{E}^{\mathbb{Q}}$ the expectation under $\mathbb{Q}$ and similarly $\mathbb{E}_t^{\mathbb{Q}}$ will be the $\mathcal{F}_t$-conditional expectation under $\mathbb{Q}$.

A1-r. *There exist $\mathcal{F}_0$-adapted random variables $C_0 > 0$ and $t > 0$ such that for $s \in [0, t]$:*

$$\mathbb{E}_0^{\mathbb{Q}} |z_s|^2 < C_0, \tag{2.5}$$

for z being each one of the processes b, b^α, b^η, $b^{\widetilde{\eta}}$, σ^α, $\widetilde{\sigma}^\alpha$, $\overline{\sigma}^\alpha$, σ^η, $\widetilde{\sigma}^\eta$, $\overline{\sigma}^\eta$, $\sigma^{\widetilde{\eta}}$, $\widetilde{\sigma}^{\widetilde{\eta}}$ and $\overline{\sigma}^{\widetilde{\eta}}$. In addition, for some $r \in [0, 1]$, we have

$$\mathbb{E}_0^{\mathbb{Q}} \left(\int_{\mathbb{R}} |\delta_x(s,z)|^r dz \right) + \mathbb{E}_0^{\mathbb{Q}} \left(\int_{\mathbb{R}} (|\delta_\sigma(s,z)| \vee |\delta_\sigma(s,z)|^4) dz \right)^2 < C_0, \tag{2.6}$$

and further for some $\iota > 0$

$$\begin{aligned} &\mathbb{E}_0^{\mathbb{Q}} \left(\int_{\mathbb{R}} (|\delta_\eta(s,z)| \vee |\delta_\eta(s,z)|^{1+\iota}) dz \right)^2 \\ &\quad + \mathbb{E}_0^{\mathbb{Q}} \left(\int_{\mathbb{R}} (|\delta_{\widetilde{\eta}}(s,z)| \vee |\delta_{\widetilde{\eta}}(s,z)|^{1+\iota}) dz \right)^2 < C_0. \end{aligned} \tag{2.7}$$

A2-r. *There exist $\mathcal{F}_0$-adapted random variables $C_0 > 0$ and $t > 0$ such that for $u, s \in [0, t]$:*

$$\mathbb{E}_0^{\mathbb{Q}} |z_s - z_u|^2 \leq C_0 |s - u|, \tag{2.8}$$

for z being each one of the processes: b, σ^α, $\widetilde{\sigma}^\alpha$, $\overline{\sigma}^\alpha$, σ^η, $\widetilde{\sigma}^\eta$, $\overline{\sigma}^\eta$, $\sigma^{\widetilde{\eta}}$, $\widetilde{\sigma}^{\widetilde{\eta}}$ and $\overline{\sigma}^{\widetilde{\eta}}$. Furthermore, for some $r \in [0, 1]$ and $\iota > 0$, we have

$$\mathbb{E}_0^{\mathbb{Q}} \left(\int_{\mathbb{R}} |\delta_x(s,z) - \delta_x(u,z)|^r ds dz \right) \leq C_0 \sqrt{|s-u|}, \tag{2.9}$$

$$\mathbb{E}_0^{\mathbb{Q}} \left(\int_{\mathbb{R}} \left(|\delta_\sigma(s,z) - \delta_\sigma(u,z)| \vee |\delta_\sigma(s,z) - \delta_\sigma(u,z)|^{1+\iota} \right) ds dz \right)^2 \leq C_0 |s - u|, \tag{2.10}$$

Assumption A1-r is a moment condition for the various processes appearing in the dynamics of x. Assumption A2-r imposes smoothness in expectation. In particular, we note that the condition in (2.8) will be satisfied as soon as the processes it applies to are Itô semimartingales themselves, which is the case in most applications. Similarly, the conditions in (2.9) and (2.10) for the jumps will be satisfied if they are generated by Lévy-driven SDE-s or if the jumps are time-changed Lévy processes with absolutely continuous time-change process, whose time derivative is an Itô semimartingale. Finally, the constant r in A1-r and A2-r captures the so-called degree of activity of the jumps in x, with $r = 0$ corresponding to finitely active jumps (which are of finite number on any finite time interval) and $r = 1$ to the most active jumps which are only summable, see e.g., Section 3.2 of Jacod and Protter (2012).

3 Higher-Order Asymptotic Expansion of Characteristic Functions of Itô Semimartingale Increments

We continue with the asymptotic expansion of the characteristic function of x_T, for small T. For stating our result, we will need notation for the characteristic exponent of the jumps in x. Towards this end, we denote with $\nu_t(x, y)$ the image of the Lebesgue measure λ, restricted to the set $\{z : ((\delta_x(t, z), \delta_\sigma(t, z)) \neq 0\}$, under the map $z \to (\delta_x(t, z), \delta_\sigma(t, z))$. The measure $\nu_t(x, y)$ is the compensator of the jumps in x and σ. We then set

$$\phi_t(u) = \int_{\mathbb{R}^2} (e^{iux} - 1)\nu_t(dx, dy). \tag{3.1}$$

Further, we denote the conditional characteristic function of the scaled increments of x with

$$\mathcal{L}_T(u) = \mathbb{E}_0^{\mathbb{Q}} \left(e^{iux_T/\sqrt{T}} \right), \quad u \in \mathbb{R}. \tag{3.2}$$

The reason for the scaling of x_T by $\sqrt{T}$ is easiest to see in the case when $x_t = W_t$. In this case $x_T/\sqrt{T}$ is $O_p(1)$ when $T \downarrow 0$. This last result continuous to hold in the case of the general Itô semimartingale model in (1.1), provided of course $\sigma_0 \neq 0$.

In what follows, for a generic process $\xi_T(u)$ indexed by $u \in \mathbb{R}$ and a deterministic sequence α_T, $\xi_T(u) = O_p^{\mathrm{lu}}(\alpha_T)$ means $\sup_{u \in \mathcal{U}} |\xi_T(u)| = O_p(\alpha_T)$ as $T \downarrow 0$, for any compact set $\mathcal{U} \in \mathbb{R}$. The short time asymptotic expansion of $\mathcal{L}_T(u)$ is given in the following theorem.

Theorem 1 *Assume A1-r and A2-r hold for the process x with dynamics given in (1.1)-(1.2) and (2.2)-(2.4) with $\sigma_0 \neq 0$. For $T \downarrow 0$, we then have:*

$$\mathcal{L}_T(u) = \Psi_T(u) + O_p^{lu}(T^{3/2-r/2} \vee T^{3/2} |\log(T)|), \tag{3.3}$$

where

$$
\begin{aligned}
\Psi_T(u) = \exp\Bigg(& iu\sqrt{T}\alpha_0 + T\phi_0(u/\sqrt{T}) - \frac{u^2\sigma_0^2}{2} - i\frac{u^3\sqrt{T}}{2}\frac{\sigma_0^2\eta_0}{1+u^2T\eta_0^2}\Bigg) \\
& - \frac{1}{2}e^{-\frac{u^2\sigma_0^2}{2}}u^2T\left(\sigma_0\sigma_0^\alpha + \sigma_0 b_0 + \frac{1}{2}\eta_0^2 + \frac{1}{2}\widetilde{\eta}_0^2\right) \\
& + \frac{1}{2}e^{-\frac{u^2\sigma_0^2}{2}}u^4T\left(\sigma_0^2\eta_0^2 + \frac{1}{3}\sigma_0^2\widetilde{\eta}_0^2 + \frac{1}{3}\sigma_0^\eta\sigma_0^3\right) \\
& + Te^{-\frac{u^2\sigma_0^2}{2}}\int_0^1\int_{\mathbb{R}^2} e^{i\frac{ux}{\sqrt{T}}}\left(e^{-u^2y\sigma_0 s - \frac{u^2}{2}y^2 s} - 1\right)\nu_0(dx,dy)ds.
\end{aligned} \tag{3.4}
$$

The result of this theorem generalizes in a nontrivial way the analysis of Jacod and Todorov (2014) and Todorov (2019), where a bound for the difference $\mathcal{L}_T(u) - e^{-\frac{u^2\sigma_0^2}{2}}$ is only provided (for the purposes of volatility estimation). The result in (3.3) is a higher-order expansion of the conditional characteristic function of the price increment which allows us to study the effect of the variation in the characteristics of the semimartingale x on $\mathcal{L}_T(u)$. In particular, for the goal of this paper, Theorem 1 allows us to estimate η_0. We note, however, that there are other applications of the above result, e.g., for less biased volatility estimation than what is derived in Todorov (2019), which we leave for future work.

We note that $\Psi_T(u)$ is an asymptotic expansion of $\mathbb{E}_0^{\mathbb{Q}}\left(e^{iu\overline{x}_T/\sqrt{T}}\right)$, for $\overline{x}_T$ being an approximation of x_T given by:

$$
\overline{x}_t = \int_0^t (\alpha_0 + \sigma_0^\alpha W_s + \widetilde{\sigma}_0^\alpha \widetilde{W}_s)ds + \int_0^t \overline{\sigma}_s dW_s + \int_0^t\int_{\mathbb{R}} \delta_x(0,z)\mu(ds,dz), \tag{3.5}
$$

where

$$
\overline{\sigma}_t = \sigma_0 + b_0 t + \eta_0 W_t + \frac{1}{2}\sigma_0^\eta(W_t^2 - t) + \widetilde{\eta}_0\widetilde{W}_t + \int_0^t\int_{\mathbb{R}} \delta_\sigma(0,z)\mu(ds,dz). \tag{3.6}
$$

The approximation $\overline{x}_T$ of x_T is one in which we take only the leading terms in the dynamics of α and σ and "freeze" the jump size function $\delta_x(t,z)$ to its value at the beginning of the interval. Since $\Psi_T(u)$ is the leading term in an asymptotic expansion of $\mathbb{E}_0^{\mathbb{Q}}\left(e^{iu\overline{x}_T/\sqrt{T}}\right)$, it does not necessarily correspond to a characteristic function of a random variable itself, but this does not matter for developing consistent estimators of various quantities (such as σ_0 and η_0) on the basis of the result in Theorem 1.

The first component of $\Psi_T(u)$, given by

$$
\exp\left(iu\sqrt{T}\alpha_0 + T\phi_0(u/\sqrt{T}) - \frac{u^2\sigma_0^2}{2}\right),
$$

corresponds to the conditional characteristic function of the price increment, when the characteristics of the process x are frozen at their value at the beginning of the interval. All other terms in

$\Psi_T(u)$ are due to the time-variation in the characteristics of x. In particular, the last term in the exponential on the first line in (3.4) is due to the presence of leverage, i.e., due to the dependence of the dynamics of σ on the Brownian motion W that drives x. This term affects the imaginary part of $\mathcal{L}_T(u)$ and is of order $O_p(\sqrt{T})$. The rest of the terms on the right-hand side of (3.4), appearing in the second, third and fourth lines of (3.4), are all of order $O_p(T)$ and they all affect the real part of $\mathcal{L}_T(u)$. Some of these terms reflect the first-order effect from different components of the dynamics of the process σ: its drift, diffusion coefficients as well as the jumps in it. That is, these terms reflect the effect on $\mathcal{L}_T(u)$ from a process σ whose characteristics are frozen at their values at the beginning of the interval. It is interesting to note, however, that terms that appear in the dynamics of the drift α and the leverage coefficient η also appear in this expansion and are of the same asymptotic order as those that are due to the first-order effect of the dynamics of σ described above. For the terms that reflect the dynamics of α and η, however, only the first-order effect from their dependence on W matters.

We further note that the jumps in x do not play a leading role neither in the real nor in the imaginary part of $\mathcal{L}_T(u)$, provided $\sigma_0 \neq 0$ and $\eta_0 \neq 0$. In particular, the asymptotic order of $T\phi_0(u/\sqrt{T})$ is $O_p(T^{1-r/2})$. The reason for this negligible role of the jumps is that the characteristic function of x_T is evaluated at $u/\sqrt{T}$ and $T \downarrow 0$, i.e., we are evaluating the conditional characteristic function at asymptotically increasing values of the characteristic exponent. If we were to consider, instead, the behavior of the conditional characteristic function $\mathbb{E}_0^{\mathbb{Q}}\left(e^{iux_T}\right)$, for $u \in \mathbb{R}$, then it is easy to show that the leading term in its behavior will also depend on the jumps in x (in this case the effect of $\eta_0 \neq 0$ will be only of higher asymptotic order).

The size of the error $\mathcal{L}_T(u) - \Psi_T(u)$ depends on the time variation in α, b, η, $\widetilde{\eta}$ and ν. In the case when the jumps in x are of Lévy type, then this error becomes only $O_p(T^{3/2}|\log(T)|)$.

Example 1 *As an example, we compute $\Phi_T(u)$ for the special case when x_t satisfies the following Lévy-driven SDE:*

$$dx_t = f(x_{t-})(\alpha dt + \sigma dW_t + \int_{\mathbb{R}} z\mu(dt, dz)), \tag{3.7}$$

where f is some twice continuously differentiable function, α and σ are some constants, W is a Brownian motion and μ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $dt \otimes \nu(dx)$, for some

Lévy measure ν. In this special case $\Phi_T(u)$ equals

$$\begin{aligned}
&\exp\Bigg(iu\sqrt{T}\alpha f_0 + T\int_{\mathbb{R}}(e^{iuf_0/\sqrt{T}}-1)\nu(dz) - \frac{u^2}{2}\sigma^2 f_0^2 - \frac{iu^3\sqrt{T}}{2}\frac{\sigma^4 f_0^3 f_0^{'}}{1+u^2T\sigma^4 f_0^2(f_0^{'})^2}\Bigg)\\
&-\frac{1}{2}e^{-\frac{u^2}{2}\sigma^2 f_0^2}u^2Tf_0^2\left(2\alpha\sigma^2 f_0^{'} + \frac{\sigma^4}{2}f_0^{''}f_0 + \frac{\sigma^4}{2}(f_0^{'})^2\right)\\
&+\frac{1}{2}e^{-\frac{u^2}{2}\sigma^2 f_0^2}u^4Tf_0^4\sigma^6\left(\frac{4}{3}(f_0^{'})^2 + \frac{1}{3}f_0^{''}f_0\right)\\
&+Te^{-\frac{u^2}{2}\sigma^2 f_0^2}\int_0^1\int_{\mathbb{R}} e^{\frac{iuf_0z}{\sqrt{T}}}\left(e^{-\frac{u^2}{2}\sigma^2 sf^2(x_0+f_0z)+\frac{u^2}{2}\sigma^2 sf_0^2}-1\right)\nu(dz)ds,
\end{aligned} \tag{3.8}$$

where in the above we use the shorthand notation $f_0 = f(x_0)$, $f_0^{'} = f^{'}(x_0)$ and $f_0^{''} = f^{''}(x_0)$.

We continue with showing how Theorem 1 can be used to make inference for η_0. For the results that follow, it is convenient to denote separately the imaginary and real parts of $\mathcal{L}_T(u)$:

$$\mathcal{I}_T(u) = \Im(\mathcal{L}_T(u)),\quad \mathcal{R}_T(u) = \Re(\mathcal{L}_T(u)),\ u\in\mathbb{R}. \tag{3.9}$$

Not surprisingly, given the fact that leverage generates asymmetry in the return distribution, η_0 plays a leading role in the imaginary part of the characteristic function, $\mathcal{I}_T(u)$, or equivalently in $\text{Arg}(\mathcal{L}_T(u))$, where $\text{Arg}(z)$ is the principal argument of the complex number z (i.e., the unique real number $\theta\in(-\pi,\pi]$ that satisfies $z = |z|(\cos(\theta)+i\sin(\theta))$).

There are two additional terms in an asymptotic expansion of $\text{Arg}(\mathcal{L}_T(u))$ which complicate, however, identification of η_0. These terms are due to the drift term in the price as well as the jumps. In addition, the unknown spot volatility, σ_0, also shows up in the leading term of $\text{Arg}(\mathcal{L}_T(u))$. To remove the term in $\text{Arg}(\mathcal{L}_T(u))$ due to the drift, we can difference (in u) appropriately $\text{Arg}(\mathcal{L}_T(u))$. Second, the term due to the jumps in $\text{Arg}(\mathcal{L}_T(u))$ is of order $O_p(T^{1-\frac{r}{2}})$ and hence it is dominated by the terms involving η_0 in $\mathcal{I}_T(u)$, provided r in A1-r and A2-r is strictly less than 1. Finally, we can estimate σ_0 from $|\mathcal{L}_T(u)|$ and use it to recover η_0 from $\mathcal{I}_T(u)$. We now provide the details. First, our estimator of volatility is given by

$$\sigma_{0,T}^2(u) = -\frac{2}{u^2}\log(|\mathcal{L}_T(u)|\vee T),\ u\in\mathbb{R}_+. \tag{3.10}$$

Using the result of Theorem 1 (which implies $\mathcal{I}_T(u) = O_p(\sqrt{T})$), we can write

$$\text{Arg}(\mathcal{L}_T(u)) = u\sqrt{T}\alpha_0 + T\phi_0(u/\sqrt{T}) - \frac{u^3\sqrt{T}}{2}\frac{\sigma_0^2\eta_0}{1+u^2T\eta_0^2} + O_p(T),\ \text{for some fixed } u\in\mathbb{R}_+. \tag{3.11}$$

As a result, an estimator for η_0 is given by

$$\eta_{0,T}(u,v) = \frac{8}{3\sqrt{T}}\frac{2\,\text{Arg}(\mathcal{L}_T(u/2)) - \text{Arg}(\mathcal{L}_T(u))}{u^3\sigma_{0,T}^2(v)},\quad u,v\in\mathbb{R}_+. \tag{3.12}$$

Applying Theorem 1, we have

$$\sigma_{0,T}^2(v) - \sigma_0^2 = O_p\left(T^{1-\frac{r}{2}}\right) \text{ and } \eta_{0,T}(u,v) - \eta_0 = O_p\left(T^{\frac{1-r}{2}}\right), \quad \text{for some fixed } u, v \in \mathbb{R}_+. \quad (3.13)$$

The error in recovering η_0 via $\eta_{0,T}(u,v)$ is of larger asymptotic order than the one in recovering σ_0^2 using $\sigma_{0,T}^2(u)$. The error in $\eta_{0,T}(u,v)$ is rather nontrivial when r is close to 1, i.e., when the jumps have rather high activity. In fact, $\eta_{0,T}(u,v)$ does not converge to η_0 for $r = 1$. The reason for this relatively poor performance of $\eta_{0,T}(u,v)$ is the term $T\Im(\phi_0(u/\sqrt{T}))$ in $\mathrm{Arg}(\mathcal{L}_T(u))$. There is a relatively easy way, however, to remove that term in the estimation of η_0 if one can use conditional characteristic functions over two different (short) intervals $[0, T_1]$ and $[0, T_2]$, with $0 < T_1 < T_2$. We denote $\tau = T_2/T_1$ and consider asymptotics with $T_1 \downarrow 0$ and $\tau > 1$ being fixed. Then, we can form an estimator of η_0 from the two characteristic functions via

$$\eta_{0,T_1,T_2}(u,v) = \frac{2}{\sqrt{T_1}} \frac{\mathrm{Arg}(\mathcal{L}_{T_1}(u)) - \mathrm{Arg}(\mathcal{L}_{T_2}(\sqrt{\tau}u))/\tau}{(\tau-1)u^3\sigma_{0,T_1}^2(v)}, \quad u, v \in \mathbb{R}_+. \quad (3.14)$$

Note that now, we no longer need to take a difference of $\mathrm{Arg}(\mathcal{L}_T(u))$ for two values of u to remove the effect of α_0, as the term due to it automatically cancels out in $\mathrm{Arg}(\mathcal{L}_{T_1}(u)) - \mathrm{Arg}(\mathcal{L}_{T_2}(\sqrt{\tau}u))/\tau$. Using Theorem 1, which implies the expansion in (3.11) above, it is easy to show the following

$$\eta_{0,T_1,T_2}(u,v) - \eta_0 = O_p\left(\sqrt{T_1}\right), \quad \text{for some fixed } u, v \in \mathbb{R}_+. \quad (3.15)$$

Comparing (3.13) and (3.15), we see that the error in recovering η_0 is reduced nontrivially when using the two characteristic functions in the estimation. Furthermore, if there are no price-volatility co-jumps, (i.e., if ν_0 is concentrated on the x and y axes in $\mathbb{R}^2$) or under some mild smoothness conditions for ν_0 (recall that smoothness of a function translates into decay of its Fourier transform, see e.g., Theorem 3.2.9 in Grafakos (2008)), one can show that the last term in $\Psi_T(u)$ is equal to $Te^{-\frac{u^2\sigma_0^2}{2}} \int_0^1 \int_{\mathbb{R}^2} 1_{\{x=0\}} \left(e^{-u^2 y\sigma_0 s - \frac{u^2}{2} y^2 s} - 1\right) \nu_0(dx,dy)ds + O_p^{\mathrm{lu}}(T^{3/2-r/2})$. In this situation, it is easy to show that the following slightly stronger result holds

$$\eta_{0,T_1,T_2}(u,v) - \eta_0 = O_p\left(T_1^{1-\frac{r}{2}}\right), \quad \text{for some fixed } u, v \in \mathbb{R}_+. \quad (3.16)$$

Note that the error term in estimating η_0 in (3.16) is determined by $\mathcal{L}_T(u) - \Psi_T(u)$ and $\mathcal{L}_T(v) - \Psi_T(v)$. Therefore, given the result in Theorem 1, further reductions in the order of magnitude of the error in recovering η_0 seem to be impossible in general settings if one is to use $\mathcal{L}_{T_1}(u)$ and $\mathcal{L}_{T_2}(u)$ for two different and small T_1 and T_2. We note that a higher-order extension of the result in Theorem 1, combined with using $\mathcal{L}_T(u)$ for many distinct small T, might lead to further improvements of $\eta_{0,T_1,T_2}(u,v)$ but such an extension seems highly nontrivial theoretically.

4 Estimation of Leverage Effect from Options

Conditional characteristic functions of price increments are of course not directly observable. However, we can estimate them from observations of options written on the asset, following Qin and Todorov (2019) and Todorov (2019). This will render the estimates of η_0 given in the previous section feasible.

4.1 Option Observation Scheme and Assumptions

We start with introducing the option observation scheme and stating the necessary assumptions for the feasible estimation of η_0. Our inference is based on European style call and put options recorded at the fixed time $t = 0$. A call option gives the owner the right to buy the underlying asset at a pre-specified strike price on the future maturity date of the option, while the put option, similarly, gives the owner the right to sell at the strike price at maturity. For each strike and maturity, we rely on the so-called out-of-the-money (OTM) option price–the cheaper of the call and the put for the given strike–which would be worth zero, if the option were to expire today. We denote OTM option price observed at time $t = 0$ by $O_T(k)$, where T is the time to expiration and k is the log-strike of the option. The OTM option price, $O_T(k)$, is a call if $k > \log(F_T)$, and a put, if $k \leq \log(F_T)$, where F_T is the time-0 futures price of the asset with expiration date T. As known from finance theory, see e.g., Duffie (2001), the option prices are conditional expectations of options' terminal payoff under the so-called risk-neutral probability measure and discounted at the risk-free interest rate back to time 0. We denote the risk-neutral probability measure with $\mathbb{Q}$. The dynamics of x in (1.1)-(1.2) and (2.2)-(2.4) are under this probability measure. The underlying asset price and true option prices are all defined on $\Omega^{(0)}$, with the associated σ-algebra $\mathcal{F}^{(0)}$ and filtration $(\mathcal{F}_t^{(0)})_{t\geq 0}$. The statistical (true) probability measure is denoted with $\mathbb{P}^{(0)}$ and finance theory implies the local equivalence of $\mathbb{P}^{(0)}$ and $\mathbb{Q}$ (so that, in particular, all diffusion coefficients in the dynamics in (1.1)-(1.2) and (2.2)-(2.4) are the same under the two probability measures).

Our data consists of options observed at time $t = 0$ and expiring at two different maturity dates T_1 and T_2. For each maturity date, T_l, where $l = 1, 2$, we observe N_l options with log-strikes given by,

$$\underline{k}_l \;\equiv\; k_{l,1} \;<\; k_{l,2} \;<\; \cdots k_{l,N_l} \;\equiv\; \overline{k}_l, \quad l = 1, 2. \tag{4.1}$$

We denote

$$\underline{k} = \underline{k}_1 \vee \underline{k}_2 \text{ and } \overline{k} = \overline{k}_1 \wedge \overline{k}_2, \tag{4.2}$$

and we further use the notation $\underline{K} = e^{\underline{k}}$ and $\overline{K} = e^{\overline{k}}$. The gap between the log-strikes is denoted $\Delta_{l,i} = k_{l,i} - k_{l,i-1}$, for $i = 2, ..., N_l$ and $l = 1, 2$. The log-strike grids need not be equidistant, i.e.,

$\Delta_{l,i}$ may differ across i's. The asymptotic theory developed below is of infill type, i.e., the mesh of the log-strike grid, $\sup_{i=2,\ldots,N_l} \Delta_{l,i}$, shrinks towards zero.

Finally, we allow for observation error, i.e., instead of observing $O_{T_l}(k_{l,j})$ directly, we observe,

$$\widehat{O}_{T_l}(k_{l,j}) = O_{T_l}(k_{l,j}) + \epsilon_{T_l}(k_{l,j}), \quad j = 1, ..., N_l, \ l = 1, 2. \tag{4.3}$$

where the errors $\epsilon_{T_l}(k_{l,j})$ are defined on a space $\Omega^{(1)} = \mathbb{R}^{\mathbb{R}} \times \mathbb{R}^{\mathbb{R}}$ which is equipped with the product Borel σ-field $\mathcal{F}^{(1)}$, and transition probability $\mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)})$ from the probability space $\Omega^{(0)}$, on which X is defined, to $\Omega^{(1)}$. We further define,

$$\Omega \,=\, \Omega^{(0)} \times \Omega^{(1)}, \quad \mathcal{F} \,=\, \mathcal{F}^{(0)} \times \mathcal{F}^{(1)},$$

and

$$\mathbb{P}(d\omega^{(0)}, d\omega^{(1)}) \,=\, \mathbb{P}^{(0)}(d\omega^{(0)})\, \mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)})\,.$$

For simplicity, when expectations are under the true probability measure $\mathbb{P}$, we will simply denote them with $\mathbb{E}$, i.e., without using superscript $\mathbb{P}$ in their notation. For quantifying the error in measuring the characteristic functions from the options, we will need the following assumptions:

A3. *We have $\sigma_0 > 0$ and there exist $\mathcal{F}_0^{(0)}$-adapted random variables C_0 and $\bar{t} > 0$ such that for $s \in [0, \bar{t}]$:*

$$\mathbb{E}_0^{\mathbb{Q}}|\alpha_s|^4 + \mathbb{E}_0^{\mathbb{Q}}|\sigma_s|^6 + \mathbb{E}_0^{\mathbb{Q}}(e^{4|x_s|}) + \mathbb{E}_0^{\mathbb{Q}}\left(\int_{\mathbb{R}^2}(e^{3|x|} - 1)\nu_s(dx, dy)\right)^4 < C_0. \tag{4.4}$$

A4. *The log-strike grids $\{k_{l,j}\}_{j=1}^{N_l}$, for $l = 1, 2$, are $\mathcal{F}^{(0)}$-adapted and we have*

$$c_0\Delta \le k_{l,j} - k_{l,j-1} \le C_0\Delta, \quad l = 1, 2, \quad \text{as } \Delta \downarrow 0, \tag{4.5}$$

where Δ is a deterministic sequence, and $c_0 > 0$ and $C_0 < \infty$ are $\mathcal{F}^{(0)}$-adapted random variables. In addition, for some arbitrary small $\zeta > 0$:

$$\sup_{j:|k_j|<\zeta}\left|\frac{k_{l,j} - k_{l,j-1}}{\Delta} - \psi_l(k_{l,j-1})\right| \overset{\mathbb{P}}{\to} 0, \quad l = 1, 2, \quad \text{as } \Delta \downarrow 0, \tag{4.6}$$

where $\psi_l(k)$ are $\mathcal{F}^{(0)}$-adapted functions which are continuous in k at 0.

A5. *We have $\epsilon_{T_l}(k_{l,j}) = \zeta_l(k_{l,j})\bar{\epsilon}_{l,j}O_{T_l}(k_{l,j})$ for $l = 1, 2$, where for k in a neighborhood of zero, we have $|\zeta_l(k) - \xi_l(0)| \le C_0|k|^\iota$, for some $\iota > 0$ and $C_0 < \infty$ being an $\mathcal{F}^{(0)}$-adapted random variable. The two sequences $\{\bar{\epsilon}_{1,j}\}_{j=1}^{N_1}$ and $\{\bar{\epsilon}_{2,j}\}_{j=1}^{N_2}$ are defined on $\mathcal{F}^{(1)}$, are i.i.d and independent of each* *other and of $\mathcal{F}^{(0)}$. We further have $\mathbb{E}(\overline{\epsilon}_{l,j}|\mathcal{F}^{(0)}) = 0$, $\mathbb{E}((\overline{\epsilon}_{l,j})^2|\mathcal{F}^{(0)}) = 1$ and $\mathbb{E}(|\overline{\epsilon}_{l,j}|^\kappa|\mathcal{F}^{(0)}) < \infty$, for some $\kappa \geq 4$ and $l = 1, 2$.*

The above assumptions are adapted from Todorov (2019). Assumption A3 is a moment condition on the increments of x and processes that show in its dynamics. It is stronger than A1-r as we need stronger conditions for the option prices to be finite. Assumption A4 is about the strike grid and A5 is about the option observation error. We note, in particular, that we allow for heteroskedasticity in the observation error across strikes. In fact, since the option prices are of different order of magnitude, depending on the distance of their log-strike from zero, this difference in asymptotic order carries over to the corresponding option observation errors. Finally, we note that in A4 and A5, we impose slightly more structure on $\psi_l(k)$ and $\zeta_l(k)$ for k around zero. The reason for that is that the asymptotic behavior of our statistics will be driven by the options with strikes in vicinity of zero.

4.2 Option-Based Estimators of the Leverage Coefficient

As implied by the results in Carr and Madan (2001), the conditional characteristic function of the price increment can be "spanned" by a portfolio of options in the following way:

$$\mathcal{L}_T(u) = 1 - \left(\frac{u^2}{T} + i\frac{u}{\sqrt{T}}\right) \int_{\mathbb{R}} e^{(iu/\sqrt{T}-1)k} O_T(k) dk, \; u \in \mathbb{R}, \tag{4.7}$$

provided the dividend yield of the underlying asset and the risk-free interest rate are both zero, an assumption that we will maintain here for simplicity given the fact that our asymptotics is for $T \downarrow 0$. Using this result, our estimate for the conditional characteristic function of the price increment from the available options is given by

$$\widehat{\mathcal{L}}_{T_l}(u) = 1 - \left(\frac{u^2}{T} + i\frac{u}{\sqrt{T}}\right) \sum_{j=2}^{N_l} e^{(iu/\sqrt{T_l}-1)k_{l,j-1}} \widehat{O}_{T_l}(k_{l,j-1}) \Delta_{l,j}, \; u \in \mathbb{R}, \tag{4.8}$$

and we further denote

$$\widehat{\mathcal{I}}_{T_l}(u) = \Im(\widehat{\mathcal{L}}_{T_l}(u)), \; \widehat{\mathcal{R}}_{T_l}(u) = \Re(\widehat{\mathcal{L}}_{T_l}(u)). \tag{4.9}$$

Under assumptions A3-A5, we have $\widehat{\mathcal{L}}_{T_l}(u) \overset{\mathbb{P}}{\to} \mathcal{L}_{T_l}(u)$ locally uniformly in u, for $l = 1, 2$. This means that the feasible counterpart of $\eta_{0,T_1,T_2}(u, v_1, v_2)$ based on $\widehat{\mathcal{L}}_{T_1}(u)$ and $\widehat{\mathcal{L}}_{T_2}(u)$ will consistently estimate η_0. To derive a CLT for such an estimator, we will first need to derive a CLT for the estimator of the conditional characteristic function, $\widehat{\mathcal{L}}_{T_l}(u)$. This is what we do next.

For stating the result, we introduce some notation needed for defining the estimate of the $\mathcal{F}^{(0)}$-conditional asymptotic variance of the limiting distribution in the CLT. In particular, using the smoothness of $O_T(k)$ in k, we form the following estimates of the observation errors for $j = 2, ..., N_l - 1$ and $l = 1, 2$:

$$\widehat{\epsilon}_{T_l}(k_{l,j}) = \sqrt{\frac{2}{3}} \left(\widehat{O}_{T_l}(k_{l,j}) - \frac{1}{2} \left(\widehat{O}_{T_l}(k_{l,j-1}) + \widehat{O}_{T_l}(k_{l,j+1}) \right) \right), \tag{4.10}$$

and we further set $\widehat{\epsilon}_{T_l}(k_{l,1}) = \widehat{\epsilon}_{T_l}(k_{l,2})$ as well as $\widehat{\epsilon}_{T_l}(k_{l,N_l}) = \widehat{\epsilon}_{T_l}(k_{l,N_l-1})$. We denote next with J_l^*, the smallest element of the set of integers $(1, 2, ..., N_l)$ such that $|k_{J_l^*}| \leq |k_{l,j}|$ for $j = 1, .., N_l$ and $l = 1, 2$. That is, $k_{J_l^*}$ is the available log-strike that is closest to the current log-price (recall that $x_0 = 0$). We then modify the estimate of the error corresponding to $k_{J_l^*}$ by replacing it with

$$\widehat{\epsilon}_{T_l}(k_{l,J_l^*}) = \frac{1}{2} \left(|\widehat{\epsilon}_{T_l}(k_{l,J_l^*-1})| + |\widehat{\epsilon}_{T_l}(k_{l,J_l^*+1})| \right), \quad l = 1, 2. \tag{4.11}$$

Using these estimates, we define

$$\begin{aligned} \overline{\zeta}_{T_l}(j, u) &= -\left(\frac{u^2}{T_l} + i \frac{u}{\sqrt{T_l}} \right) e^{(iu/\sqrt{T_l}-1)k_{l,j-1}} \widehat{\epsilon}_{T_l}(k_{l,j-1}) \Delta_{l,j}, \\ \zeta_{T_l}(j, u) &= \left(\Im(\overline{\zeta}_{T_l}(j, u)) \, \Re(\overline{\zeta}_{T_l}(j, u)) \right)^\top, \end{aligned} \tag{4.12}$$

for $j = 2, ..., N_l$ and $l = 1, 2$. The estimate of the $\mathcal{F}^{(0)}$-conditional asymptotic variance of $\widehat{\mathcal{L}}_{T_l}(u)$ is then given by

$$\widehat{\Sigma}_{T_l}(u, v) = \sum_{j=2}^{N_l} \zeta_{T_l}(j, u) \zeta_{T_l}^\top(j, v), \quad u, v \in \mathbb{R}, \ l = 1, 2. \tag{4.13}$$

Finally, the limiting distribution will depend on the following function

$$\widetilde{\Phi}(k) = f(k) + |k|\Phi(-|k|), \quad k \in \mathbb{R}, \tag{4.14}$$

where f and Φ denote the pdf and cdf of a standard normal random variable.

We are now ready to state our CLT result. The convergence in distribution of the centered $\widehat{\mathcal{L}}_{T_l}(u)$ holds $\mathcal{F}^{(0)}$-conditionally. This is denoted by $\xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}}$ and formally means convergence in probability of the conditional probability laws when the latter are considered as random variables taking values in the space of probability measures equipped with the weak topology, see e.g., VIII.5.26 of Jacod and Shiryaev (2003).

Theorem 2 *Suppose assumptions A3-A5 hold. Let $T_1 \downarrow 0$, $T_2 \asymp T_1$, $\Delta \asymp T_1^\alpha$, $\underline{K} \asymp T_1^\beta$, $\overline{K} \asymp T_1^{-\gamma}$, for $\beta, \gamma > 0$ and $\alpha > \frac{1}{2}$. Then, for any compact $\mathcal{U} \in \mathbb{R}$, we have*

$$\frac{T_l^{1/4}}{\sqrt{\Delta}} \begin{pmatrix} \widehat{\mathcal{I}}_{T_l}(u) - \mathcal{I}_{T_l}(u) \\ \widehat{\mathcal{R}}_{T_l}(u) - \mathcal{R}_{T_l}(u) \end{pmatrix} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \begin{pmatrix} Z_{l,I}(u) \\ Z_{l,R}(u) \end{pmatrix}, \ l = 1, 2, \tag{4.15}$$

uniformly in $u \in \mathcal{U}$, and where $\mathcal{F}^{(0)}$-conditionally $(Z_{1,I}(u)\ \ Z_{1,R}(u))$ and $(Z_{2,l}(u)\ \ Z_{2,R}(u))$ are independent of each other and are centered Gaussian processes with

$$\begin{aligned} &\begin{pmatrix} \mathbb{E}\left(Z_{l,I}(u)Z_{l,I}(v)\middle|\mathcal{F}^{(0)}\right) & \mathbb{E}\left(Z_{l,I}(u)Z_{l,R}(v)\middle|\mathcal{F}^{(0)}\right) \\ \mathbb{E}\left(Z_{l,R}(u)Z_{l,I}(v)\middle|\mathcal{F}^{(0)}\right) & \mathbb{E}\left(Z_{l,R}(u)Z_{l,R}(v)\middle|\mathcal{F}^{(0)}\right) \end{pmatrix} \\ &= \Sigma_l(u,v) := \begin{pmatrix} \Sigma_{l,I}(u,v) & 0 \\ 0 & \Sigma_{l,R}(u,v) \end{pmatrix}, \end{aligned} \tag{4.16}$$

and

$$\Sigma_{l,I}(u,v) = \sigma_0^3\psi_l(0)\zeta_l^2(0)\int_{\mathbb{R}} \sin(\sigma_0^2 uk)\sin(\sigma_0^2 vk)\widetilde{\Phi}^2(k)dk, \tag{4.17}$$

$$\Sigma_{l,R}(u,v) = \sigma_0^3\psi_l(0)\zeta_l^2(0)\int_{\mathbb{R}} \cos(\sigma_0^2 uk)\cos(\sigma_0^2 vk)\widetilde{\Phi}^2(k)dk, \tag{4.18}$$

for $l = 1, 2$. Moreover, we have

$$\frac{\sqrt{T_l}}{\Delta}\widehat{\Sigma}_{T_l}(u,v) \overset{\mathbb{P}}{\to} \Sigma_l(u,v), \quad \textit{uniformly in } u,v \in \mathcal{U},\ l = 1,2. \tag{4.19}$$

We make several observations regarding the above result. First, we note that the convergence in the above theorem holds in a joint asymptotic setting in which the time horizons T_1 and T_2 shrink, the mesh of the log-strike grids shrinks and the log-strike ranges increase. The asymptotic requirements on the log-strike grid are needed for feasible implementation of the result of Carr and Madan (2001) (as $\widehat{\mathcal{L}}_{T_l}(u)$ involves Riemann sum approximation of the integral on the right-hand side of (4.7)) and are commonly used in other applications involving option portfolios with different strikes (e.g., for establishing the consistency of the popular VIX volatility index that is widely used in academic work and in practice) as well as for consistent risk-neutral density estimation, following Breeden and Litzenberger (1978). Second, the CLT holds locally uniformly in u. This is very convenient for applications as one can choose u adaptively based on some estimates from the available data. This is what we will do in our implementation. Third, since we look at the characteristic function for asymptotically increasing values of its argument, the limiting distribution in Theorem 2 is governed by the measurement error of the options with log-strikes in vicinity of zero. Therefore, since the observation error is proportional to the option it is attached to, both the rate of convergence and the limiting asymptotic variance are governed by the asymptotic behavior of these near-the-money options. Their values, in turn, are dominated asymptotically by the diffusive component of the underlying asset.

We note that in the CLT result in (4.15), the "reference" mesh of the log-strike grid, Δ, appears (recall assumption A4). This "reference" Δ is not necessarily an observable quantity and is used to capture the order of magnitude of the mesh of the log-strike grid (see (4.5)). However, given the convergence result in (4.19), for feasible implementation of the CLT, e.g., for constructing confidence intervals for $\mathcal{I}_{T_l}(u)$ and $\mathcal{R}_{T_l}(u)$, one does not need to know Δ.

We proceed with feasible estimation of the leverage coefficient. Using the estimate of the conditional characteristic function from the data, we can define the feasible counterparts of $\sigma_{0,T_l}(u)$, $\eta_{0,T_l}(u,v)$ and $\eta_{0,T_1,T_2}(u,v)$ as follows:

$$\widehat{\sigma}^2_{0,T_l}(u) = -\frac{2}{u^2}\log\left(|\widehat{\mathcal{L}}_{T_l}(u)| \vee N_l^{-1}\right), \tag{4.20}$$

$$\widehat{\eta}_{0,T_l}(u,v) = \frac{8}{3\sqrt{T_l}}\frac{2\,\mathrm{Arg}(\widehat{\mathcal{L}}_{T_l}(u/2)) - \mathrm{Arg}(\widehat{\mathcal{L}}_{T_l}(u))}{u^3\widehat{\sigma}^2_{0,T_l}(v)}, \tag{4.21}$$

$$\widehat{\eta}_{0,T_1,T_2}(u,v) = \frac{2}{\sqrt{T_1}}\frac{\mathrm{Arg}(\widehat{\mathcal{L}}_{T_1}(u)) - \mathrm{Arg}(\widehat{\mathcal{L}}_{T_2}(\sqrt{\tau}u))/\tau}{(\tau-1)u^3\widehat{\sigma}^2_{0,T_1}(v)}, \tag{4.22}$$

for $u,v\in\mathbb{R}_+$ and $l=1,2$. Given the fact that $\widehat{\eta}_{0,T_1,T_2}(u,v)$ contains significantly less bias due to the jumps in x, we will present a feasible CLT only for this estimator of η. To state this result, we introduce some additional notation. In particular, for $l=1,2$, we set

$$\begin{aligned}\widehat{G}_1(u,v) = \frac{2}{\widehat{\sigma}^2_{0,T_1}(v)}\Bigg(&\frac{\widehat{\mathcal{R}}_{T_1}(u)}{(\tau-1)\sqrt{T_1}u^3|\widehat{\mathcal{L}}_{T_1}(u)|^2}, -\frac{\widehat{\mathcal{I}}_{T_1}(u)}{(\tau-1)\sqrt{T_1}u^3|\widehat{\mathcal{L}}_{T_1}(u)|^2},\\ &\frac{\widehat{\eta}_{0,T_1,T_2}(u,v)\widehat{\mathcal{I}}_{T_1}(v)}{v^2|\widehat{\mathcal{L}}_{T_1}(v)|^2}, \frac{\widehat{\eta}_{0,T_1,T_2}(u,v)\widehat{\mathcal{R}}_{T_1}(v)}{v^2|\widehat{\mathcal{L}}_{T_1}(v)|^2}\Bigg),\end{aligned} \tag{4.23}$$

$$\widehat{G}_2(u,v) = -\frac{2}{\tau\widehat{\sigma}^2_{0,T_1}(v)|\widehat{\mathcal{L}}_{T_2}(\sqrt{\tau}u)|^2}\left(\frac{\widehat{\mathcal{R}}_{T_2}(\sqrt{\tau}u)}{(\tau-1)\sqrt{T_1}u^3}, -\frac{\widehat{\mathcal{I}}_{T_2}(\sqrt{\tau}u)}{(\tau-1)\sqrt{T_1}u^3}\right), \tag{4.24}$$

$$\widehat{H}_1(u,v) = \begin{pmatrix} \widehat{\Sigma}_1(u,u) & \widehat{\Sigma}_1(u,v) \\ \widehat{\Sigma}_1(u,v) & \widehat{\Sigma}_1(v,v)\end{pmatrix}. \tag{4.25}$$

We then denote

$$\widehat{\mathrm{Avar}(\eta_0)} = \widehat{G}_1(u,v)\widehat{H}_1(u,v)\widehat{G}_1(u,v)^\top + \widehat{G}_2(u,v)\widehat{\Sigma}_2(u\sqrt{\tau},u\sqrt{\tau})\widehat{G}_2(u,v)^\top. \tag{4.26}$$

The feasible CLT for $\widehat{\eta}_{0,T_1,T_2}(u,v)$ is given in the following corollary.

Corollary 1 *Suppose assumptions A1-r, A2-r and A3-A5 hold and further* $\sigma_0 \neq 0$. *Let* $T_1 \downarrow 0$, $T_2 = \tau T_1$ *for some* $\tau > 1$, $\Delta \asymp T_1^\alpha$, $\underline{K} \asymp T_1^\beta$, $\overline{K} \asymp T_1^{-\gamma}$, *for* $\beta,\gamma > 0$ *and* $\frac{1}{2} < \alpha < \frac{5}{2}$. *Then, we have*

$$\frac{\widehat{\eta}_{0,T_1,T_2}(u,v) - \eta_0}{\sqrt{\widehat{Avar(\eta_0)}}} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} N(0,1). \tag{4.27}$$

Corollary 1 follows by an application of the Delta method and Theorems 1 and 2. If $\alpha > \frac{3}{2}$, the estimator is consistent and $\frac{T_1^{3/4}}{\sqrt{\Delta}}(\widehat{\eta}_{0,T_1,T_2}(u,v) - \eta_0)$ is asymptotically mixed Gaussian. This implies slower rate of convergence for recovering η_0 than for estimating the conditional characteristic function. The reason for this is that η_0 appears in $\text{Arg}(\mathcal{L}_T(u))$ multiplied by $\sqrt{T}$. As we will show in the next section, $\widehat{\eta}_{0,T_1,T_2}(u,v)$ achieves nearly the optimal rate for estimating η_0 from noisy short-dated options in a minimax sense. We also note, that the error in estimating σ_0 plays an asymptotically negligible role in $\widehat{\eta}_{0,T_1,T_2}(u,v) - \eta_0$.

In the case $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$, $\widehat{\text{Avar}(\eta_0)}$ is exploding asymptotically, and therefore $\widehat{\eta}_{0,T_1,T_2}(u,v)$ is not consistent for η_0. In practice, this will result in confidence intervals for η_0 which are too big for meaningful inference. That is, in this case, the measurement error in the observed options is big relative to the signal contained in them regarding the leverage coefficient. In such a situation, however, one can achieve consistency of the leverage coefficient estimator by simply adding more data. This can be done by considering an asymptotically increasing number of option cross-sections observed in a local window around time $t = 0$, i.e., in a high-frequency option panel setup (or alternatively option cross-sections observed at disjoint times). We leave the formal analysis of such an extension of the current setup for future work.

5 Minimax Risk for Estimation of Leverage from Options

We will now derive a lower bound for the minimax risk for estimating the leverage coefficient from noisy short-dated option data. This result will show that our nonparametric estimator, developed in the previous section, is nearly rate-efficient. For establishing this result, it will suffice to work in the special setting where the underlying log-price dynamics under $\mathbb{Q}$ is given by

$$x_t = at + \int_0^t (\sigma + \eta W_s) dW_s + J_t^x, \quad t \geq 0, \tag{5.1}$$

where J_t^x is a compound Poisson jump process with unit intensity and deterministic jump size of $J < 0$. The risk-neutral law of x is uniquely identified by the three parameters (a, σ, η) (and of course the jump size J). Our interest is in the estimation of η. We will denote with $\mathcal{G}(R)$ the set of risk-neutral probability measures $\mathbb{Q}$ under which x has the dynamics in (5.1), with $a < R$, $\sigma \in (1/R, R)$ and $\eta \in (1/R, R)$, for some finite constant $R > 0$. We note that the risk-neutrality of $\mathbb{Q}$ implies $a = r_t - \frac{1}{2}(\sigma + \eta W_t)^2 - (e^J - 1)$, where r_t denotes the spot risk-free rate at time t. We could have alternatively considered the model in (5.1) in which a is replaced by $r - \frac{1}{2}(\sigma + \eta W_t)^2 - (e^J - 1)$, for some constant r, but for simplicity (although such an extension is certainly doable), we proceed with the model in (5.1).

For simplicity, as this has no impact on the derived result below, we will assume that we have short-dated options with only one common expiration date. The option observations are given by

$$\widehat{O}_T(k_i) = O_T(k_i) + (O_T(k_i) \vee T)\epsilon_i, \quad i = 1, ..., N, \tag{5.2}$$

for the strike grid $k_1 < k_2 < .... < k_n$, and where $\{\epsilon_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables, with standard normal distribution, defined on a product extension of $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t^{(0)})_{t\geq 0}, \mathbb{P}^{(0)})$ and independent of $\mathcal{F}^{(0)}$. We further have $c\Delta \leq k_i - k_{i-1} \leq \Delta$, for $i = 1, ..., N$ and some $\Delta > 0$ going down to zero as well as a constant $c \in (0, 1)$. Note that when x contains jumps, then in general the (true) option price for every strike is of order $O_p(T)$. Therefore, the truncation from below in the term multiplying ϵ_i in (5.2) does not change in general the order of magnitude of the observation error.

In what follows, we will denote with $\mathbb{E}_{\mathcal{T}}$ expectation under which the true option price $O_T(k)$ is computed according to the risk-neutral probability measure $\mathcal{T}$. We then have the following result.

Theorem 3 *In the setting of (5.1)-(5.2), let $\Delta \asymp T^{\alpha}$, $e^{k_1} \asymp T^{-\beta}$ and $e^{k_N} \asymp T^{-\gamma}$, for $\frac{3}{2} < \alpha < \frac{5}{2}$ and $\beta, \gamma \geq 0$ as $T \downarrow 0$. We then have*

$$\inf_{\widehat{\eta}} \sup_{\mathcal{T}\in\mathcal{G}(R)} \mathbb{E}_{\mathcal{T}} \left(\frac{T^{3/2}(\log(1/T))^{7/2}}{\Delta} |\widehat{\eta} - \eta|^2 \right) \geq c, \tag{5.3}$$

for some $c > 0$ and where $\widehat{\eta}$ is any estimator of σ based on the option data $\{\widehat{O}_T(k_i)\}_{i=1,...,N}$.

We note that the same result holds if one is to use options with two different times to maturity in the inference. The result of the theorem above, therefore, shows that the estimator of η_0 we constructed in the previous section, is nearly rate optimal (up to a log term).

6 Simulation Study

We now evaluate the performance of the leverage coefficient estimator $\widehat{\eta}_{0,T_1,T_2}(u, v)$ on simulated data. The dynamics of $X = e^x$ under the risk-neutral measure in the simulation is given by

$$\frac{dX_t}{X_{t-}} = \sqrt{V_t}dW_t + \int_{\mathbb{R}} (e^x - 1)\mu(dt, dx), \tag{6.1}$$

where W is a Brownian motion and μ is an integer-valued random measure with compensator

$$\nu(dt, dx) = V_t dt \times \left(c_- \frac{e^{-20|x|}}{|x|^{1+\beta}} 1_{\{x<0\}} + c_+ \frac{e^{-100|x|}}{|x|^{1+\beta}} 1_{\{x>0\}} \right) dx, \tag{6.2}$$

while the process V has the following dynamics

$$dV_t = 6(0.02 - V_t)dt + 2\eta\sqrt{V}_t dW_t + \sqrt{0.09 - 4\eta^2}\sqrt{V}_t d\widetilde{W}_t, \tag{6.3}$$

for $\widetilde{W}$ being a Brownian motion independent of W. The above model belongs to the exponentially-affine jump-diffusion class of models of Duffie et al. (2000). In particular, V is a square-root diffusion which is commonly used to model volatility in parametric option pricing. The jumps in X are with time-varying jump intensity that is proportional to the level of stochastic volatility. They can be represented as a time change (with the time change being the integrated variance $\int_0^t V_s ds$) of the flexible tempered stable Lévy process, see e.g., Carr et al. (2002). Part of the model parameters are fixed while we will vary the rest in the Monte Carlo in order to check the sensitivity of our estimator with respect to them.

We consider three cases for the parameter β capturing the jump activity (assumptions A1-r and A2-r hold for the model in the Monte Carlo with $\beta + \iota$, for any $\iota > 0$): case A with $\beta = -0.5$, case B with $\beta = 0$, and case C with $\beta = 0.5$. The first case corresponds to finite activity jumps and the last two to jumps of infinite activity. For each of the three cases for β, we consider two cases for the scale parameters $c_\pm$: case L corresponding to conditional spot jump volatility (i.e., $\int_\mathbb{R} x^2 \nu_t(dx)$) being half of V_t (diffusive volatility) and case H corresponding to conditional spot jump volatility that equals V_t. In all considered cases, the ratio of the conditional volatility of negative jumps to that of positive jumps is 9 to 1. This matches roughly earlier estimates of these quantities from observed index option data, see e.g., Andersen et al. (2017) and references therein. The jump parameters for the different considered cases are given in Table 1.

Table 1: Jump Parameter Settings used in the Monte Carlo

Model Case	β	c_-	c_+
A-L	-0.5	0.6177×10^3	3.4194×10^3
A-H	-0.5	1.2354×10^3	6.8388×10^3
B-L	0.0	1.8182×10^2	4.5454×10^2
B-H	0.0	3.6364×10^2	9.0908×10^2
C-L	0.5	0.4587×10^2	0.5129×10^2
C-H	0.5	0.9174×10^2	1.0258×10^2

Finally, η in the volatility dynamics above corresponds to the leverage coefficient, which in this model is a constant. This follows by an application of Itô's lemma by taking into account the fact that V remains strictly positive on any fixed time interval whenever started from a positive value, see e.g., equation IV.8.12 in Ikeda and Watanabe (1981). We consider two cases for it: $\eta = 0$ and $\eta = -0.135$. The first corresponds to zero covariation between X and V and the second one corresponds to strong negative covariation between X and V (the correlation $\langle x_t, V_t \rangle / \sqrt{\langle x_t, x_t \rangle \langle V_t, V_t \rangle}$ in this case equals -0.9).

We next describe the option observation setting. Options written on X are observed at time $t = 0$ with maturities of 3 and 5 business days and we set $X_0 = 2500$. The strike grid and range of the options are calibrated to match roughly available data on market index options, see e.g., Andersen et al. (2017). In particular, for each of the two maturities, the strike grid is equidistant with increments of 5. The strike range is determined by the requirement that the true option prices should be at least 0.05 in value. Finally, the option observation error is set to $\epsilon_{T_l}(k_{l,j}) = 0.05 \times O_{T_l}(k_{l,j})Z_{l,j}$, for $l = 1, 2$ and where $\{Z_{1,j}\}_{j\geq 1}$ and $\{Z_{2,j}\}_{j\geq 1}$ are two independent sequences of i.i.d. standard normal random variables.

For the implementation of the estimator $\widehat{\eta}_{0,T_1,T_2}(u, v)$, we need to set the values of u and v. We will do this in a data-driven way. We set the value of v, needed in the estimation of the volatility, to

$$\widehat{v} = \Big(\inf\{u \geq 0 : |\widehat{\mathcal{L}}_{T_1}(u)| \leq 0.2\}\Big) \wedge \mathrm{argmin}_{u\in[0,40]}|\widehat{\mathcal{L}}_{T_1}(u)|. \tag{6.4}$$

This choice of v aims at evaluating $\widehat{\mathcal{L}}_{T_1}(u)$ at a sufficiently high value of u in order to minimize the impact of the jumps in x on $\mathcal{L}_{T_1}(u)$ but such that estimation is still reliable. The second restriction on the right-hand side of (6.4) is asymptotically non-binding but is aimed to guard against potential finite sample distortions in the recovery of $\mathcal{L}_{T_1}(u)$. Our choice of u is given by

$$\widehat{u} = \left(\frac{4}{(\tau-1)\widehat{\sigma}^2_{0,T_1}(\widehat{v})}\right)^{1/3} \bigwedge \Big(\inf\{u \geq 0 : \mathrm{Arg}(\widehat{\mathcal{L}}_{T_2}(\sqrt{\tau}u)) \geq 0.5\}\Big). \tag{6.5}$$

First, note that the second restriction on the right-hand side of (6.5) is asymptotically non-binding because $\Im(\widehat{\mathcal{L}}_{T_2}(\sqrt{\tau}u)) \overset{\mathbb{P}}{\to} 0$. This restriction is used as a guard against finite-sample distortions and it ensures that we evaluate the principal argument of $\widehat{\mathcal{L}}_{T_2}(\sqrt{\tau}u)$ only when the latter is in a neighborhood of zero (which is needed for our asymptotic expansions). The rationale behind the leading term in $\widehat{u}$, i.e., the first term on the right-hand side of (6.5), is the following. Recall that

$$\frac{1}{\sqrt{T_1}}\Big(\mathrm{Arg}(\widehat{\mathcal{L}}_{T_1}(u)) - \mathrm{Arg}(\widehat{\mathcal{L}}_{T_2}(\sqrt{\tau}u))/\tau\Big) \overset{\mathbb{P}}{\to} \frac{\tau-1}{2}u^3\sigma_0^2\eta_0. \tag{6.6}$$

Therefore, by setting u equal to $\widehat{u}$, we aim for an estimator of η_0 with precision that is half that of $\frac{1}{\sqrt{T_1}}\Big(\mathrm{Arg}(\widehat{\mathcal{L}}_{T_1}(u)) - \mathrm{Arg}(\widehat{\mathcal{L}}_{T_2}(\sqrt{\tau}u))/\tau\Big)$. Higher choices of u will make even small errors in estimating the conditional characteristic function have rather big impact on the estimation of η_0. On the other hand, a smaller choice of u will make the role of jumps more prominent. Recall that our asymptotics is local uniform in u and the error bounds we provide here are not optimal for u approaching asymptotically zero, where the jumps will play the leading role.

The results from the Monte Carlo are reported in Table 2. In the case of no leverage effect, i.e., when $\eta = 0$, we see that our estimator has very small bias, which can be either positive or negative, depending on the particular case. The bias, however, is very small relative to the interquantile range of the estimator. Similar observations can be also made for the case with leverage effect, i.e., when $\eta = -0.135$. The bias of the estimator in this case is again very small relative to the sampling variation in the estimator. Since the observation errors are proportional to the option prices they are attached to and the latter are higher when the underlying process has bigger jump component, the error in estimating η is higher for all cases in which the spot jump volatility is equal to the diffusive volatility (cases H). Also, it is interesting to note that the performance of our leverage coefficient estimator is not very sensitive to the value of the jump activity parameter β, i.e., we do not see much difference in the performance of $\widehat{\eta}_{0,T_1,T_2}(u,v)$ across cases A, B and C. Finally, the performance of our estimator does not differ much across the different starting values for the diffusive volatility. Overall, the results reported in Table 2 suggest satisfactory performance of $\widehat{\eta}_{0,T_1,T_2}(u,v)$ in finite samples.

Table 2: Monte Carlo Results for $\widehat{\eta}_{0,T_1,T_2}(\widehat{u},\widehat{v})$

		$\eta = 0$		$\eta = -0.135$	
Model Case	**Volatility**	**Median**	**IQR**	**Median**	**IQR**
A-L	Low	0.0124	0.0389	−0.1255	0.0416
A-L	Median	0.0171	0.0464	−0.1153	0.0468
A-L	High	0.0198	0.0514	−0.1090	0.0558
A-H	Low	−0.0152	0.0513	−0.1543	0.0580
A-H	Median	−0.0014	0.0595	−0.1432	0.0665
A-H	High	0.0099	0.0705	−0.1260	0.0778
B-L	Low	0.0098	0.0380	−0.1206	0.0440
B-L	Median	0.0121	0.0421	−0.1147	0.0447
B-L	High	0.0067	0.0547	−0.1213	0.0570
B-H	Low	0.0110	0.0520	−0.1435	0.0661
B-H	Median	0.0201	0.0575	−0.1240	0.0670
B-H	High	0.0276	0.0741	−0.1124	0.0714
C-L	Low	−0.0061	0.0384	−0.1416	0.0426
C-L	Median	−0.0044	0.0412	−0.1384	0.0478
C-L	High	−0.0006	0.0556	−0.1311	0.0616
C-H	Low	0.0153	0.0521	−0.1235	0.0557
C-H	Median	0.0128	0.0586	−0.1207	0.0657
C-H	High	0.0138	0.0707	−0.1257	0.0696

IQR stands for inter-quantile range. Low, median and high volatility correspond to V_0 set to the 25th, 50th and 75th quantile, respectively, of its marginal distribution. Results are based on 1,000 replications.

7 Proofs

7.1 Proof of Theorem 1

In the proof of Theorem 1, we will drop the superscript $\mathbb{Q}$ when denoting expectations under $\mathbb{Q}$ as all expectations will be under this probability. The proof of Theorem 1 consists of the following two lemmas:

Lemma 1 *Assume A1-r and A2-r hold. For $T \downarrow 0$, we have:*

$$\mathbb{E}_0\left(e^{iux_T/\sqrt{T}}\right) - \mathbb{E}_0\left(e^{iu\overline{x}_T/\sqrt{T}}\right) = O_p^{lu}(T^{3/2-r/2} \vee T^{3/2}|\log(T)|). \tag{7.1}$$

Lemma 2 *For $T \downarrow 0$, we have:*

$$\mathbb{E}_0\left(e^{iu\overline{x}_T/\sqrt{T}}\right) = \Psi_T(u) + O_p^{lu}(T^{3/2}). \tag{7.2}$$

Throughout the proofs of the two lemmas, we will use the following shorthand notation

$$u_T = \frac{u}{\sqrt{T}}, \quad u \in \mathbb{R}. \tag{7.3}$$

Proof of Lemma 1. In the proof, we will denote with C_0 a finite-valued $\mathcal{F}_0$-adapted random variable that can change from one line to another.

Part I: Some Notation. Since we want to show order of magnitude locally uniformly in u, henceforth, we will assume that $|u| \le \mathcal{U}$ for some arbitrary positive number $\mathcal{U}$. We denote

$$\overline{\alpha}_t = \alpha_0 + \sigma_0^{\alpha} W_t + \widetilde{\sigma}_0^{\alpha}\widetilde{W}_t, \;\breve{\alpha}_t = \overline{\alpha}_t + \overline{\sigma}_0^{\alpha}\overline{W}_t. \tag{7.4}$$

We split $\sigma_t = \sigma_t^{(1)} + \sigma_t^{(2)}$ and $\overline{\sigma}_t = \overline{\sigma}_t^{(1)} + \overline{\sigma}_t^{(2)}$, where

$$\begin{aligned} \sigma_t^{(1)} &= \sigma_0 + \int_0^t b_s ds + \int_0^t \eta_s dW_s + \int_0^t \widetilde{\eta}_s d\widetilde{W}_s, \\ \overline{\sigma}_t^{(1)} &= \sigma_0 + b_0 t + \eta_0 W_t + \frac{\sigma_0^{\eta}}{2}(W_t^2 - t) + \widetilde{\eta}_0 \widetilde{W}_t, \end{aligned} \tag{7.5}$$

$$\sigma_t^{(2)} = \int_0^t \int_{\mathbb{R}} \delta_\sigma(s,z)\mu(ds,dz), \; \overline{\sigma}_t^{(2)} = \int_0^t \int_{\mathbb{R}} \delta_\sigma(0,z)\mu(ds,dz). \tag{7.6}$$

We further denote

$$\begin{aligned} \sigma_t^{(1,r)} = \;& \widetilde{\sigma}_0^{\eta} \int_0^t \widetilde{W}_s dW_s + \overline{\sigma}_0^{\eta}\int_0^t \overline{W}_s dW_s + \sigma_0^{\widetilde{\eta}} \int_0^t W_s d\widetilde{W}_s \\ &+ \widetilde{\sigma}_0^{\widetilde{\eta}} \int_0^t \widetilde{W}_s d\widetilde{W}_s + \overline{\sigma}_0^{\widetilde{\eta}} \int_0^t \overline{W}_s d\widetilde{W}_s. \end{aligned} \tag{7.7}$$

We also split $x_t = \sum_{i=0}^3 x_t^{(i)}$ and $\overline{x}_t = \sum_{i=0}^3 \overline{x}_t^{(i)}$, where

$$x_t^{(0)} = \int_0^t \alpha_s ds,\ \breve{x}_t^{(0)} = \int_0^t \breve{\alpha}_s ds,\ \overline{x}_t^{(0)} = \int_0^t \overline{\alpha}_s ds, \tag{7.8}$$

$$x_t^{(i)} = \int_0^t \sigma_s^{(i)} dW_s,\ \overline{x}_t^{(i)} = \int_0^t \overline{\sigma}_s^{(i)} dW_s,\ i = 1, 2, \tag{7.9}$$

$$x_t^{(3)} = \int_0^t \int_{\mathbb{R}} \delta_x(s,z)\mu(ds,dz),\ \overline{x}_t^{(3)} = \int_0^t \int_{\mathbb{R}} \delta_x(0,z)\mu(ds,dz). \tag{7.10}$$

Finally, we set

$$\breve{x}_t = \breve{x}_t^{(0)} + x_t^{(1)} + \overline{x}_t^{(2)} + \overline{x}_t^{(3)},\ \breve{\sigma}_t = \sqrt{(\sigma_t^{(1)})^2 + (\overline{\sigma}_t^{(2)})^2}. \tag{7.11}$$

Part II: Preliminary Estimates. We start with establishing several bounds that will be used in the proof. First, direct calculation shows

$$\mathbb{E}_0(\breve{\alpha}_t - \overline{\alpha}_t)^2 + \mathbb{E}_0(\overline{\sigma}_t^{(1)} - \sigma_0)^2 \le C_0 T, \quad t \in [0,T]. \tag{7.12}$$

Using Burkholder-Davis-Gundy inequality and the smoothness in expectation assumption for η and $\widetilde{\eta}$ in A2-r, we get

$$\mathbb{E}_0(\sigma_t^{(1)} - \overline{\sigma}_t^{(1)})^2 \le C_0 T^2,\ t \in [0,T]. \tag{7.13}$$

Inequality in means and the bound about $\breve{\alpha}_t - \overline{\alpha}_t$ in (7.12) implies

$$\mathbb{E}_0\left(\breve{x}_t^{(0)} - \overline{x}_t^{(0)}\right)^2 \le C_0 T \int_0^t \mathbb{E}_0\left(\breve{\alpha}_s - \overline{\alpha}_s\right)^2 ds \le C_0 T^3,\ t \in [0,T]. \tag{7.14}$$

By Itô isometry and the bound in (7.13), we have

$$\mathbb{E}_0(x_t^{(1)} - \overline{x}_t^{(1)})^2 = \mathbb{E}_0\left(\int_0^t (\sigma_s^{(1)} - \overline{\sigma}_s^{(1)})^2 ds\right) \le C_0 T^3,\ t \in [0,T]. \tag{7.15}$$

Combining the above two bounds, we get altogether

$$\mathbb{E}_0(\breve{x}_t - \overline{x}_t)^2 \le C_0 T^3,\ t \in [0,T]. \tag{7.16}$$

Next, using the definition of $\sigma_t^{(1)}$, $\overline{\sigma}_t^{(1)}$ and $\sigma_t^{(1,r)}$, we have

$$\begin{aligned}\sigma_t^{(1)} - \overline{\sigma}_t^{(1)} - \sigma_t^{(1,r)} = &\int_0^t (b_s - b_0) ds + \int_0^t (\eta_s - \eta_0 - \sigma_0^\eta W_s - \widetilde{\sigma}_0^\eta \widetilde{W}_s - \overline{\sigma}_0^\eta \overline{W}_s) dW_s \\ &+ \int_0^t (\widetilde{\eta}_s - \widetilde{\eta}_0 - \sigma_0^{\widetilde{\eta}} W_s - \widetilde{\sigma}_0^{\widetilde{\eta}} \widetilde{W}_s - \overline{\sigma}_0^{\widetilde{\eta}} \overline{W}_s) d\widetilde{W}_s.\end{aligned} \tag{7.17}$$

Applying Itô isometry twice and using the smoothness in expectation assumption for the processes σ^η, $\widetilde{\sigma}^\eta$, $\overline{\sigma}^\eta$, $\sigma^{\widetilde{\eta}}$, $\widetilde{\sigma}^{\widetilde{\eta}}$ and $\overline{\sigma}^{\widetilde{\eta}}$ in A2-r, we have

$$\mathbb{E}_0\left(\int_0^t (\eta_s - \eta_0 - \eta_s^{(j)} - \sigma_0^\eta W_s - \widetilde{\sigma}_0^\eta \widetilde{W}_s - \overline{\sigma}_0^\eta \overline{W}_s) dW_s\right)^2 \le C_0 T^3, \tag{7.18}$$

$$\mathbb{E}_0\left(\int_0^t (\widetilde{\eta}_s - \widetilde{\eta}_0 - \widetilde{\eta}_s^{(j)} - \sigma_0^{\widetilde{\eta}} W_s - \widetilde{\sigma}_0^{\widetilde{\eta}} \widetilde{W}_s - \overline{\sigma}_0^{\widetilde{\eta}} \overline{W}_s) d\widetilde{W}_s\right)^2 \le C_0 T^3, \tag{7.19}$$

where we denote $\eta_t^{(j)} = \sum_{s\le t} \Delta\eta_s^{(j)}$ and $\widetilde{\eta}_t^{(j)} = \sum_{s\le t} \Delta\widetilde{\eta}_s^{(j)}$. Next, using integration by parts, we can write

$$\int_0^t \eta_s^{(j)} dW_s = \eta_t^{(j)} W_t - \int_0^t W_s d\eta_s^{(j)}, \quad \int_0^t \widetilde{\eta}_s^{(j)} d\widetilde{W}_s = \widetilde{\eta}_t^{(j)} \widetilde{W}_t - \int_0^t \widetilde{W}_s d\widetilde{\eta}_s^{(j)}. \tag{7.20}$$

Using the integrability assumption for the jumps in A1-r, we have

$$\begin{aligned} &\mathbb{E}_0(|\eta_t^{(j)} W_t| 1_{\{|W_t|\le\sqrt{T}|\log(T)|\}}) + \mathbb{E}_0(|\widetilde{\eta}_t^{(j)} \widetilde{W}_t| 1_{\{|\widetilde{W}_t|\le\sqrt{T}|\log(T)|\}}) \\ &\qquad \le C_0 T^{3/2} |\log(T)|, \ t\in[0,T]. \end{aligned} \tag{7.21}$$

Using the fact that $\int_k^\infty x^p e^{-x^2/2} dx \sim k^{p-1} e^{-k^2/2}$ as $k\to\infty$ and for any $p>1$, we have (assuming without loss of generality that $T<1$):

$$\begin{aligned} &\mathbb{E}_0 ||W_t| 1_{\{|W_t|>\sqrt{T}|\log(T)|\}}|^p + \mathbb{E}_0 ||\widetilde{W}_t| 1_{\{|\widetilde{W}_t|>\sqrt{T}|\log(T)|\}}|^p \\ &\qquad \le C_p T^{p/2} |\log(T)|^{p-1} T^{-\frac{1}{2}\log(T)}, \ p>1. \end{aligned} \tag{7.22}$$

Using this bound, Hölder inequality and Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} &\mathbb{E}_0(|\eta_t^{(j)} W_t| 1_{\{|W_t|>\sqrt{T}|\log(T)|\}}) + \mathbb{E}_0(|\widetilde{\eta}_t^{(j)} \widetilde{W}_t| 1_{\{|\widetilde{W}_t|>\sqrt{T}|\log(T)|\}}) \\ &\qquad \le C_0 T^{3/2} |\log(T)|, \ t\in[0,T]. \end{aligned} \tag{7.23}$$

Next, we have

$$\begin{aligned} \mathbb{E}_0\left|\int_0^t W_s d\eta_s^{(j)}\right| &\le \mathbb{E}_0\left(\int_0^t |W_s| \int_{\mathbb{R}} |\delta_\eta(s,z)| dz ds\right), \\ \mathbb{E}_0\left|\int_0^t \widetilde{W}_s d\widetilde{\eta}_s^{(j)}\right| &\le \mathbb{E}_0\left(\int_0^t |\widetilde{W}_s| \int_{\mathbb{R}} |\delta_{\widetilde{\eta}}(s,z)| dz ds\right), \end{aligned} \tag{7.24}$$

and therefore by Cauchy-Schwarz inequality and the integrability assumptions for δ_η and $\delta_{\widetilde{\eta}}$ in A1-r, we have

$$\mathbb{E}_0\left|\int_0^t W_s d\eta_s^{(j)}\right| + \mathbb{E}_0\left|\int_0^t \widetilde{W}_s d\widetilde{\eta}_s^{(j)}\right| \le C_0 T^{3/2}. \tag{7.25}$$

Altogether, combining (7.21)-(7.25), we have

$$\mathbb{E}_0\left|\int_0^t \eta_s^{(j)} dW_s\right| + \mathbb{E}_0\left|\int_0^t \widetilde{\eta}_s^{(j)} d\widetilde{W}_s\right| \le C_0 T^{3/2} |\log(T)|, \ t\in[0,T]. \tag{7.26}$$

Part III: The difference $\mathbb{E}_0(e^{iu_T x_T} - e^{iu_T \check{x}_T})$. We have

$$\mathbb{E}_0|e^{iu_T x_T} - e^{iu_T \check{x}_T}| \le u_T \mathbb{E}_0|x_T^{(0)} - \check{x}_T^{(0)}| + u_T \mathbb{E}_0|x_T^{(2)} - \overline{x}_T^{(2)}| + u_T^r \mathbb{E}_0|x_T^{(3)} - \overline{x}_T^{(3)}|^r, \tag{7.27}$$

where r is the constant appearing in A1-r and A2-r. We analyze each of the terms on the right-hand-side of the above inequality. For the first term, we have

$$\begin{aligned}\alpha_t - \breve{\alpha}_t = & \int_0^t b_s^\alpha ds + \int_0^t (\sigma_s^\alpha - \sigma_0^\alpha) dW_s + \int_0^t (\widetilde{\sigma}_s^\alpha - \widetilde{\sigma}_0^\alpha) d\widetilde{W}_s \\ & + \int_0^t (\overline{\sigma}_s^\alpha - \overline{\sigma}_0^\alpha) d\overline{W}_s + \sum_{s\leq t} \Delta\alpha_s.\end{aligned} \tag{7.28}$$

From here, using the smoothness in expectation assumption for the processes σ^α, $\widetilde{\sigma}^\alpha$ and $\overline{\sigma}^\alpha$ in A2-r as well as the integrability assumption for the jumps in α in A1-r, we have

$$\mathbb{E}_0|\alpha_t - \breve{\alpha}_t| \leq C_0 T, \;\; t \in [0,T], \tag{7.29}$$

and therefore

$$\mathbb{E}_0|x_T^{(0)} - \breve{x}_T^{(0)}| \leq C_0 T^2. \tag{7.30}$$

We continue next with $x_T^{(2)} - \overline{x}_T^{(2)}$. Using integration by parts, we have

$$\begin{aligned}x_t^{(2)} - \overline{x}_t^{(2)} = & W_t \int_0^t \int_{\mathbb{R}} (\delta_\sigma(s,z) - \delta_\sigma(0,z)) \mu(ds,dz) \\ & - \int_0^t \int_{\mathbb{R}} [W_s(\delta_\sigma(s,z) - \delta_\sigma(0,z))] \mu(ds,dz).\end{aligned} \tag{7.31}$$

From here, using Hölder inequality, the smoothness in expectation assumption for the jumps in σ in A2-r as well as Burkholder-Davis-Gundy inequality and by considering separately the first summand on the sets $|W_t| \leq \sqrt{T}|\log(T)|$ and $|W_t| > \sqrt{T}|\log(T)|$ (exactly as in (7.21)-(7.23) above), we have

$$\mathbb{E}_0|x_T^{(2)} - \overline{x}_T^{(2)}| \leq C_0 T^2 |\log(T)|. \tag{7.32}$$

We are left with $x_T^{(3)} - \overline{x}_T^{(3)}$. Using the algebraic inequality $|\sum_{i\geq 1} a_i|^p \leq \sum_{i\geq 1} |a_i|^p$, for a sequence of real numbers $\{a_i\}_{i\geq 1}$ and $p \in (0,1]$, we have by application of A2-r

$$\begin{aligned}& \mathbb{E}_0 \left| \int_0^T \int_{\mathbb{R}} (\delta_x(s,z) - \delta_x(0,z)) \mu(ds,dz) \right|^r \\ & \leq \mathbb{E}_0 \left(\int_0^T \int_{\mathbb{R}} |\delta_x(s,z) - \delta_x(0,z)|^r \mu(ds,dz) \right) \leq C_0 T^{3/2}.\end{aligned} \tag{7.33}$$

Altogether, combining (7.30), (7.32) and (7.33), we have

$$\mathbb{E}_0|e^{iu_T x_T} - e^{iu_T \breve{x}_T}| \leq C_0 (u_T T^2 |\log(T)| + u_T^r T^{3/2}). \tag{7.34}$$

Part IV: The difference $\mathbb{E}_0(e^{iu_T\breve{x}_T} - e^{iu_T\overline{x}_T})$. Applying Itô's lemma, we have

$$\begin{aligned}\mathbb{E}_0\left(\cos(u_T\breve{x}_T)\right) - 1 = & -\mathbb{E}_0\left(\int_0^T (u_T \sin(u_T\breve{x}_s)\breve{\alpha}_s) ds + \frac{1}{2}\int_0^T (u_T^2 \cos(u_T\breve{x}_s)\breve{\sigma}_s^2) ds \right) \\ & + \mathbb{E}_0\left(\int_0^T \int_{\mathbb{R}^2} [\cos(u_T\breve{x}_s)(\cos(u_T x) - 1) - \sin(u_T\breve{x}_s)\sin(u_T x)] \nu_0(dx,dy) ds \right),\end{aligned} \tag{7.35}$$

$$
\begin{aligned}
\mathbb{E}_0\left(\sin(u_T\breve{x}_T)\right) &= \mathbb{E}_0\left(\int_0^T (u_T\cos(u_T\breve{x}_s)\breve{\alpha}_s)ds - \frac{1}{2}\int_0^T (u_T^2\sin(u_T\breve{x}_s)\breve{\sigma}_s^2)ds\right) \\
&+ \mathbb{E}_0\left(\int_0^T\int_{\mathbb{R}^2}\left[\sin(u_T\breve{x}_s)(\cos(u_Tx)-1)+\cos(u_T\breve{x}_s)\sin(u_Tx)\right]\nu_0(dx,dy)ds\right).
\end{aligned} \tag{7.36}
$$

We have similar result for $\mathbb{E}_0\left(\cos(u_T\overline{x}_T)\right)-1$ and $\mathbb{E}_0\left(\sin(u_T\overline{x}_T)\right)$ after replacing $\breve{x}_s$ with $\overline{x}_s$, $\breve{\alpha}_s$ with $\overline{\alpha}_s$ and $\breve{\sigma}_s^2$ with $\overline{\sigma}_s^2$ in the above two equalities. We denote for $t\in[0,T]$:

$$
\begin{aligned}
\alpha_{t,T}(u) &= \mathbb{E}_0\left(\cos(u_T\breve{x}_t)-\cos(u_T\overline{x}_t)\right), \\
\beta_{t,T}(u) &= \mathbb{E}_0\left(\sin(u_T\breve{x}_t)-\sin(u_T\overline{x}_t)\right).
\end{aligned} \tag{7.37}
$$

With this notation, we can write

$$
\begin{aligned}
|\alpha_{T,T}(u)| \le u_T &\int_0^T \left|\mathbb{E}_0\left(\sin(u_T\breve{x}_s)\breve{\alpha}_s-\sin(u_T\overline{x}_s)\overline{\alpha}_s\right)\right|ds \\
&+ \frac{u_T^2}{2}\int_0^T \left|\mathbb{E}_0\left(\cos(u_T\breve{x}_s)\breve{\sigma}_s^2-\cos(u_T\overline{x}_s)\overline{\sigma}_s^2\right)\right|ds \\
&+ C_0u_T^r\int_0^T (|\alpha_{s,T}(u)|+(|\beta_{s,T}(u)|)ds,
\end{aligned} \tag{7.38}
$$

$$
\begin{aligned}
|\beta_{T,T}(u)| \le u_T &\int_0^T \left|\mathbb{E}_0\left(\cos(u_T\breve{x}_s)\breve{\alpha}_s-\cos(u_T\overline{x}_s)\overline{\alpha}_s\right)\right|ds \\
&+ \frac{u_T^2}{2}\int_0^T \left|\mathbb{E}_0\left(\sin(u_T\breve{x}_s)\breve{\sigma}_s^2-\sin(u_T\overline{x}_s)\overline{\sigma}_s^2\right)\right|ds \\
&+ C_0u_T^r\int_0^T (|\alpha_{s,T}(u)|+(|\beta_{s,T}(u)|)ds.
\end{aligned} \tag{7.39}
$$

We can decompose

$$
\begin{aligned}
&\sin(u_T\breve{x}_s)\breve{\alpha}_s-\sin(u_T\overline{x}_s)\overline{\alpha}_s = (\sin(u_T\breve{x}_s)-\sin(u_T\overline{x}_s))(\breve{\alpha}_s-\overline{\alpha}_s) \\
&\ + (\sin(u_T\breve{x}_s)-\sin(u_T\overline{x}_s))(\overline{\alpha}_s-\alpha_0)+ \\
&\alpha_0(\sin(u_T\breve{x}_s)-\sin(u_T\overline{x}_s))+\sin(u_T\overline{x}_s)(\breve{\alpha}_s-\overline{\alpha}_s).
\end{aligned} \tag{7.40}
$$

We proceed with bounding the conditional expectations of each of the summands on the right-hand-side of the above equality. Using Cauchy-Schwarz inequality as well as the results in (7.12) and (7.16), we have

$$
\begin{aligned}
&\mathbb{E}_0\left|(\sin(u_T\breve{x}_s)-\sin(u_T\overline{x}_s))(\breve{\alpha}_s-\overline{\alpha}_s)\right| \\
&\quad + \mathbb{E}_0\left|(\sin(u_T\breve{x}_s)-\sin(u_T\overline{x}_s))(\overline{\alpha}_s-\alpha_0)\right| \le C_0u_TT^2,\ s\in[0,T].
\end{aligned} \tag{7.41}
$$

Further, since $\overline{x}_s$ and $\breve{\alpha}_s-\overline{\alpha}_s$ are $\mathcal{F}_0$-conditionally independent, we have

$$
\mathbb{E}_0[\sin(u_T\overline{x}_s)(\breve{\alpha}_s-\overline{\alpha}_s)] = 0. \tag{7.42}
$$

Altogether, we get

$$|\mathbb{E}_0[\sin(u_T\breve{x}_s)\breve{\alpha}_s - \sin(u_T\overline{x}_s)\overline{\alpha}_s]| \leq \alpha_0|\beta_{s,T}(u)| + C_0 u_T T^2,\ \ s \in [0,T]. \tag{7.43}$$

We turn next to $\cos(u_T\breve{x}_s)\breve{\sigma}_s^2 - \cos(u_T\overline{x}_s)\overline{\sigma}_s^2$. We have the following decomposition for $s \in [0,T]$:

$$\begin{aligned} &\cos(u_T\breve{x}_s)\breve{\sigma}_s^2 - \cos(u_T\overline{x}_s)\overline{\sigma}_s^2 = \cos(u_T\breve{x}_s)(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)})^2 \\ &\quad + 2\cos(u_T\breve{x}_s)(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)})(\overline{\sigma}_s^{(1)} - \sigma_0) \\ &\quad + 2\sigma_0(\cos(u_T\breve{x}_s) - \cos(u_T\overline{x}_s))(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)}) \\ &\quad + 2\sigma_0(\cos(u_T\overline{x}_s) - 1)(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)}) + 2\sigma_0(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)}) \\ &\quad + \sigma_0^2(\cos(u_T\breve{x}_s) - \cos(u_T\overline{x}_s)) + (\cos(u_T\breve{x}_s) - \cos(u_T\overline{x}_s))(\overline{\sigma}_s^2 - \sigma_0^2). \end{aligned} \tag{7.44}$$

We analyze the conditional expectations of each of the summands on the right-hand-side of the above equality. Using Cauchy-Schwarz inequality and the bounds in (7.12), (7.13) and (7.15), we have

$$|\mathbb{E}_0[\cos(u_T\breve{x}_s)(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)})^2]| + |\mathbb{E}_0[\cos(u_T\breve{x}_s)(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)})(\overline{\sigma}_s^{(1)} - \sigma_0)]| \leq C_0 T^{3/2}, \tag{7.45}$$

$$|\mathbb{E}_0[(\cos(u_T\breve{x}_s) - \cos(u_T\overline{x}_s))(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)})]| \leq C_0 u_T T^{5/2}, \tag{7.46}$$

$$|\mathbb{E}_0[(\cos(u_T\breve{x}_s) - \cos(u_T\overline{x}_s))(|\overline{\sigma}_s^{(1)} - \sigma_0| \vee |\overline{\sigma}_s^{(1)} - \sigma_0|^2)]| \leq C_0 u_T T^2, \tag{7.47}$$

$$|\mathbb{E}_0[(\cos(u_T\breve{x}_s) - \cos(u_T\overline{x}_s))(|\overline{\sigma}_s^{(2)}| \vee |\overline{\sigma}_s^{(2)}|^2)]| \leq C_0 u_T T^2. \tag{7.48}$$

Using the smoothness in expectation assumption for the process b in A2-r, we further have

$$|\mathbb{E}_0(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)})| \leq C_0 T^{3/2}. \tag{7.49}$$

Next, using the bounds in (7.18), (7.19), (7.21), (7.23) and (7.26), we have

$$|\mathbb{E}_0[(\cos(u_T\overline{x}_s) - 1)(\sigma_s^{(1)} - \overline{\sigma}_s^{(1)} - \sigma_s^{(1,r)})]| \leq C_0 T^{3/2}|\log(T)|, \tag{7.50}$$

and applying Cauchy-Schwarz inequality, we have

$$|\mathbb{E}_0[(\cos(u_T\overline{x}_s) - \cos(u_T(\sigma_0 W_s + \overline{x}_s^{(3)}))\sigma_s^{(1,r)})]| \leq C_0 u_T T^2. \tag{7.51}$$

Finally, using successive conditioning, we have

$$\mathbb{E}_0[\cos(u_T(\sigma_0 W_s + \overline{x}_s^{(3)}))\sigma_s^{(1,r)}] = 0. \tag{7.52}$$

Combining the above results, we get altogether,

$$\begin{aligned} &|\mathbb{E}_0(\cos(u_T\breve{x}_s)\breve{\sigma}_s^2 - \cos(u_T\overline{x}_s)\overline{\sigma}_s^2)| \\ &\quad \leq \sigma_0^2|\alpha_{s,T}(u)| + C_0(u_T T^2 + T^{3/2}|\log(T)|),\ \ s \in [0,T]. \end{aligned} \tag{7.53}$$

This result, together with (7.38) and (7.43), implies

$$|\alpha_{T,T}(u)| \leq C_0(u_T T^2 + T^{3/2}|\log(T)|) + C_0(u_T^2 \vee 1)\int_0^T (|\alpha_{s,T}(u)| + |\beta_{s,T}(u)|)ds. \tag{7.54}$$

Exactly the same calculations as the ones above (and using (7.39)) yield

$$|\beta_{T,T}(u)| \leq C_0(u_T T^2 + T^{3/2}|\log(T)|) + C_0(u_T^2 \vee 1)\int_0^T (|\alpha_{s,T}(u)| + |\beta_{s,T}(u)|)ds. \tag{7.55}$$

We can apply Grownwall's lemma, see e.g., Appendix 1 of Revuz and Yor (1999), for $|\alpha_{s,T}(u)| + |\beta_{s,T}(u)|$, when viewed as a function of s, to conclude

$$|\alpha_{T,T}(u)| + |\beta_{T,T}(u)| \leq C_0(\mathcal{U})(u_T T^2 + T^{3/2}|\log(T)|), \tag{7.56}$$

for some $\mathcal{F}_0$-adapted random variable $C_0(\mathcal{U})$ that depends on $\mathcal{U}$. □

Proof of Lemma 2. Using integration by parts and the independence of W and $\widetilde{W}$, we have

$$\int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t), \quad \int_0^t \widetilde{W}_s dW_s = \widetilde{W}_t W_t - \int_0^t W_s d\widetilde{W}_s, \quad \int_0^t \widetilde{W}_s ds = \int_0^t (t-s) d\widetilde{W}_s, \tag{7.57}$$

$$\int_0^t (W_s^2 - s) dW_s = \frac{W_t^3}{3} - tW_t, \quad \int_0^t W_s ds = \int_0^t (t-s) dW_s, \tag{7.58}$$

and

$$\int_0^t \int_0^s \int_{\mathbb{R}} \delta_\sigma(0,z)\mu(du,dz) dW_s = \int_0^t \int_{\mathbb{R}} (W_t - W_s)\delta_\sigma(0,z)\mu(du,dz). \tag{7.59}$$

Using Proposition 2.5 of Rajput and Rosinski (1989) and the independence of W and the Poisson measure μ, we have

$$\begin{aligned} &\mathbb{E}\left(e^{iu\int_0^t\int_{\mathbb{R}}((W_t - W_s)\delta_\sigma(0,z) + \delta_x(0,z))\mu(ds,dz)} \middle| \mathcal{F}^W \right) \\ &\quad = \exp\left(\int_0^t \int_{\mathbb{R}^2} (e^{iux + iu(W_t - W_s)y} - 1)\nu_0(dx,dy) \right), \end{aligned} \tag{7.60}$$

where $\mathcal{F}^W$ denotes the sigma algebra generated by the Brownian motion W. Combining these results and upon conditioning on $\mathcal{F}^W$, we get

$$\begin{aligned} \mathbb{E}_0\left(e^{iu_T \overline{x}_T}\right) = \mathbb{E}_0\Bigg[\exp\Bigg(& iu_T \int_0^T (\alpha_0 + \sigma_0^\alpha W_s) ds + iu_T \int_0^T (\sigma_0 + b_0 s) dW_s \\ & + \frac{iu_T}{2}\eta_0(W_T^2 - T) + \frac{iu_T}{2}\sigma_0^\eta \left(\frac{W_T^3}{3} - TW_T \right) - \frac{u_T^2}{2}\widetilde{\eta}_0^2 \int_0^T (W_T - W_s)^2 ds \\ & - \frac{u_T^2}{2}\int_0^T ((\widetilde{\sigma}_0^\alpha)^2 (T-s)^2 + 2\widetilde{\sigma}_0^\alpha \widetilde{\eta}_0 (T-s)(W_T - W_s)) ds \\ & + \int_0^T \int_{\mathbb{R}^2} (e^{iu_T x + iu_T(W_T - W_s)y} - 1)\nu_0(dx,dy) \Bigg) \Bigg]. \end{aligned} \tag{7.61}$$

Applying a second-order Taylor expansion and using the self-similarity of the Brownian motion and integrability of power moments of its increments, we can further write

$$
\begin{aligned}
&\mathbb{E}_0\left(e^{iu_T\overline{x}_T}\right) = e^{iu_T\alpha_0 T+T\phi_0(u_T)}\mathbb{E}_0\left(e^{iu_T\sigma_0^\alpha\int_0^T W_s ds+iu_T\int_0^T(\sigma_0+b_0 s)dW_s+\frac{iu_T}{2}\eta_0(W_T^2-T)}\right)\\
&+e^{iu_T\alpha_0 T+T\phi_0(u_T)}\mathbb{E}_0\Bigg[e^{iu_T\int_0^T(\sigma_0+b_0 s)dW_s+\frac{iu_T}{2}\eta_0(W_T^2-T)}\\
&\times\left(-\frac{u_T^2}{2}\widetilde{\eta}_0^2\int_0^T(W_T-W_s)^2ds+\frac{iu_T}{2}\sigma_0^\eta\left(\frac{W_T^3}{3}-TW_T\right)-u_T^2\widetilde{\sigma}_0^\alpha\widetilde{\eta}_0\int_0^T[(T-s)(W_T-W_s)]ds\right.\\
&\left.+\int_0^T\int_{\mathbb{R}^2}(e^{iu_Tx+iu_T(W_T-W_s)y}-e^{iu_Tx})\nu_0(dx,dy)\right)\Bigg]+O_p^{\text{lu}}\left(T^2\right).
\end{aligned}
\tag{7.62}
$$

By another Taylor expansion and using the fact that jumps in price and volatility are of finite variation, we can further simplify the above expression as follows:

$$
\begin{aligned}
&\mathbb{E}_0\left(e^{iu_T\overline{x}_T}\right) = e^{iu_T\alpha_0 T+T\phi_0(u_T)}\mathbb{E}_0\left(e^{iu_T\sigma_0^\alpha\int_0^T W_s ds+iu_T\int_0^T(\sigma_0+b_0 s)dW_s+\frac{iu_T}{2}\eta_0(W_T^2-T)}\right)\\
&-\mathbb{E}_0\left(e^{iu_T\sigma_0 W_T}\frac{u_T^2}{2}\widetilde{\eta}_0^2\int_0^T(W_T-W_s)^2ds\right)\\
&+\mathbb{E}_0\left(e^{iu_T\sigma_0 W_T}\int_0^T\int_{\mathbb{R}^2}(e^{iu_Tx+iu_T(W_T-W_s)y}-e^{iu_Tx})\nu_0(dx,dy)\right)\\
&+\mathbb{E}_0\left(e^{iu_T\sigma_0 W_T}\frac{iu_T}{2}\sigma_0^\eta\left(\frac{W_T^3}{3}-TW_T\right)\right)+O_p^{\text{lu}}\left(T^{3/2}\right):=\sum_{j=1}^4 A_T^{(j)}+O_p^{\text{lu}}\left(T^{3/2}\right).
\end{aligned}
\tag{7.63}
$$

We proceed with expanding the terms $A_T^{(1)}$, $A_T^{(2)}$, $A_T^{(3)}$ and $A_T^{(4)}$.

Part I: Expansion of $A_T^{(1)}$. In what follows, it is convenient to use the shorthand notation

$$
\widetilde{b}_t = b_t+\sigma_t^\alpha \text{ and } \overline{b}_t = \sigma_t^\alpha - b_t. \tag{7.64}
$$

We can decompose

$$
\begin{aligned}
&\sigma_0^\alpha\int_0^T W_s ds+\int_0^T(\sigma_0+b_0 s)dW_s\\
&=\left(\sigma_0+\frac{1}{2}\widetilde{b}_0 T\right)W_T+\overline{b}_0\int_0^T\left(\frac{T}{2}-s\right)dW_s,
\end{aligned}
\tag{7.65}
$$

and by direct calculation we have

$$
\mathbb{E}_0\left[W_T\int_0^T\left(\frac{T}{2}-s\right)dW_s\right]=\int_0^T\left(\frac{T}{2}-s\right)ds=0. \tag{7.66}
$$

This, combined with the fact that the pair $\left(W_T,\ \int_0^T\left(\frac{T}{2}-s\right)dW_s\right)$ is jointly normally distributed, implies that W_T is independent from $\int_0^T\left(\frac{T}{2}-s\right)dW_s$. Therefore, we can write

$$
\begin{aligned}
A_T^{(1)} &= e^{iu_T\alpha_0 T+T\phi_0(u_T)}\mathbb{E}_0\left(e^{iu_T(\sigma_0+\frac{1}{2}\widetilde{b}_0 T)W_T+\frac{iu_T}{2}\eta_0(W_T^2-T)}\right)\\
&\times\mathbb{E}_0\left(e^{iu_T\overline{b}_0\int_0^T(\frac{T}{2}-s)dW_s}\right).
\end{aligned}
\tag{7.67}
$$

By direct calculation:

$$\mathbb{E}_0\left(e^{iu_T\bar{b}_0\int_0^T\left(\frac{T}{2}-s\right)dW_s}\right)=e^{-\frac{u_T^2\bar{b}_0^2}{2}\int_0^T\left(\frac{T}{2}-s\right)^2ds}=1+O_p^{\mathrm{lu}}(T^2), \tag{7.68}$$

and therefore

$$A_T^{(1)}=e^{iu_T\alpha_0T+T\phi_0(u_T)}\mathbb{E}_0\left(e^{iu_T(\sigma_0+\frac{1}{2}\widetilde{b}_0T)W_T+\frac{iu_T}{2}\eta_0(W_T^2-T)}\right)+O_p^{\mathrm{lu}}(T^2). \tag{7.69}$$

To proceed further, we use the fact that if $Z\sim N(\mu,\sigma^2)$, then by using $\int_{\mathbb{R}}e^{-z^2/2}dz=\sqrt{2\pi}$, the characteristic function of Z^2 can be shown to be equal to

$$\mathbb{E}(e^{iuZ^2})=\exp\left(\frac{iu\mu^2}{1-2iu\sigma^2}\right)\frac{1}{\sqrt{1-2iu\sigma^2}}. \tag{7.70}$$

Using this result, we have

$$\begin{aligned}
&\mathbb{E}_0\left(e^{iu_T\left((\sigma_0+\frac{1}{2}\widetilde{b}_0T)W_T+\frac{\eta_0}{2}W_T^2\right)}\right)\\
&=\mathbb{E}_0\left(e^{iu_T\frac{\eta_0}{2}\left(W_T+\frac{\sigma_0+\widetilde{b}_0T/2}{\eta_0}\right)^2-iu_T\frac{\eta_0}{2}\frac{(\sigma_0+\widetilde{b}_0T/2)^2}{\eta_0^2}}\right)\\
&=\frac{1}{\sqrt{1-iu_T\eta_0T}}\exp\left(-iu_T\frac{(\sigma_0+\widetilde{b}_0T/2)^2}{2\eta_0}\left(1-\frac{1}{1-iu_T\eta_0T}\right)\right)\\
&=\frac{1}{\sqrt{1-iu_T\eta_0T}}\exp\left(-\frac{u_T^2T}{2}\frac{(\sigma_0+\widetilde{b}_0T/2)^2}{1-iu_T\eta_0T}\right).
\end{aligned} \tag{7.71}$$

By successive Taylor expansion, we can decompose

$$\begin{aligned}
&\exp\left(-\frac{u_T^2T}{2}\frac{(\sigma_0+\widetilde{b}_0T/2)^2}{1-iu_T\eta_0T}\right)=\exp\left(-\frac{u_T^2T}{2}\frac{\sigma_0^2+\widetilde{b}_0\sigma_0T}{1+u_T^2\eta_0^2T^2}(1+iu_T\eta_0T)\right)+O_p^{\mathrm{lu}}(T^2)\\
&=\exp\left(-\frac{u_T^2T}{2}\frac{\sigma_0^2+\widetilde{b}_0\sigma_0T}{1+u_T^2\eta_0^2T^2}-i\frac{u_T^3T^2}{2}\frac{\sigma_0^2\eta_0}{1+u_T^2\eta_0^2T^2}\right)+O_p^{\mathrm{lu}}(T^{3/2})\\
&=\exp\left(-\frac{u_T^2T\sigma_0^2}{2}-i\frac{u_T^3T^2}{2}\frac{\sigma_0^2\eta_0}{1+u_T^2\eta_0^2T^2}\right)\left(1-\frac{u_T^2T^2}{2}\widetilde{b}_0\sigma_0+\frac{u_T^4T^3}{2}\sigma_0^2\eta_0^2\right)+O_p^{\mathrm{lu}}(T^{3/2}).
\end{aligned} \tag{7.72}$$

To continue further, we note that since the function arcsin is continuously differentiable in a neighborhood of zero,

$$\arcsin\left(\frac{u_T\eta_0T}{\sqrt{1+u_T^2\eta_0^2T^2}}\right)-\arcsin\left(u_T\eta_0T\right)=O_p^{\mathrm{lu}}(T^{3/2}), \tag{7.73}$$

and further using the algebraic inequality $|\sin(x)-x|\leq|x|^3$ and the fact that the function arcsin has a positive bounded first derivative in a neighborhood of zero:

$$\arcsin\left(u_T\eta_0T\right)-u_T\eta_0T=O_p^{\mathrm{lu}}(T^{3/2}). \tag{7.74}$$

Using these inequalities and a Taylor expansion for the function $(1+x)^{-1/4}$, we have

$$\begin{aligned}\frac{1}{\sqrt{1-iu_T\eta_0 T}} &= \frac{1}{(1+u_T^2\eta_0^2T^2)^{1/4}}e^{\frac{iu_T T\eta_0}{2}} + O_p^{\mathrm{lu}}(T^{3/2}) \\ &= \left(1-\frac{u_T^2T^2}{4}\eta_0^2\right)e^{\frac{iu_T T\eta_0}{2}} + O_p^{\mathrm{lu}}(T^{3/2}).\end{aligned} \tag{7.75}$$

Combining the above results, we get altogether

$$\begin{aligned}A_T^{(1)} &= \exp\left(iu_T\alpha_0 T + T\phi_0(u_T) - \frac{u_T^2T\sigma_0^2}{2} - i\frac{u_T^3T^2}{2}\frac{\sigma_0^2\eta_0}{1+u_T^2\eta_0^2T^2}\right) \\ &\times\left(1-\frac{u_T^2T^2}{2}\widetilde{b}_0\sigma_0 + \frac{u_T^4T^3}{2}\sigma_0^2\eta_0^2 - \frac{u_T^2T^2}{4}\eta_0^2\right) + O_p^{\mathrm{lu}}(T^{3/2}).\end{aligned} \tag{7.76}$$

Part II: Expansion of $A_T^{(2)}$. Using successive conditioning and the fact that $\mathbb{E}(Z^2e^{iuZ}) = e^{-u^2\sigma^2/2}(\sigma^2 - u^2\sigma^4)$ for $Z\sim N(0,\sigma^2)$, we have

$$\begin{aligned}&\mathbb{E}_0\left(\int_0^T e^{iu_T\sigma_0(W_T-W_s)}(W_T-W_s)^2e^{iu_T\sigma_0W_s}ds\right) \\ &= \mathbb{E}_0\left(\int_0^T e^{-\frac{u_T^2}{2}\sigma_0^2(T-s)}(T-s)(1-u_T^2\sigma_0^2(T-s))e^{iu_T\sigma_0W_s}ds\right) \\ &= \int_0^T e^{-\frac{u_T^2}{2}\sigma_0^2(T-s)}(T-s)(1-u_T^2\sigma_0^2(T-s))e^{-\frac{u_T^2}{2}\sigma_0^2 s}ds.\end{aligned} \tag{7.77}$$

From here,

$$A_T^{(2)} = -\frac{u_T^2}{2}\widetilde{\eta}_0^2\left(\frac{1}{2}e^{-\frac{u_T^2}{2}\sigma_0^2T}T^2 - \frac{1}{3}e^{-\frac{u_T^2}{2}\sigma_0^2T}u_T^2\sigma_0^2T^3\right). \tag{7.78}$$

Part III: Expansion of $A_T^{(3)}$. By exchanging the order of integration and expectation (as the integrand is always nonnegative), we have

$$\begin{aligned}A_T^{(3)} &= \int_0^T\int_{\mathbb{R}^2}\mathbb{E}_0\left(e^{iu_T\sigma_0W_T}\left(e^{iu_Tx+iu_T(W_T-W_s)y} - e^{iu_Tx}\right)\right)\nu_0(dx,dy)ds \\ &= \int_0^T\int_{\mathbb{R}^2}e^{iu_Tx}\left(e^{-\frac{u_T^2}{2}\sigma_0^2s-\frac{1}{2}(u_T\sigma_0+u_Ty)^2(T-s)} - e^{-\frac{u_T^2}{2}\sigma_0^2T}\right)\nu_0(dx,dy)ds \\ &= e^{-\frac{u_T^2}{2}\sigma_0^2T}\int_0^T\int_{\mathbb{R}^2}e^{iu_Tx}\left(e^{-u_T^2y\sigma_0(T-s)-\frac{u_T^2}{2}y^2(T-s)}-1\right)\nu_0(dx,dy)ds.\end{aligned} \tag{7.79}$$

From here, by a change of variable, we have

$$A_T^{(3)} = Te^{-\frac{u^2\sigma_0^2}{2}}\int_0^1\int_{\mathbb{R}^2}\left(e^{iu_Tx}\left(e^{-u^2y\sigma_0s-\frac{u^2}{2}y^2s}-1\right)\right)\nu_0(dx,dy)ds. \tag{7.80}$$

Part IV: Expansion of $A_T^{(4)}$*.* Using

$$\mathbb{E}(Ze^{iuZ}) = iu\sigma^2 e^{-\frac{u^2}{2}\sigma^2},\ \mathbb{E}(Z^3 e^{iuZ}) = ie^{-\frac{u^2}{2}\sigma^2}(3u\sigma^4 - u^3\sigma^6),\ Z \sim N(0,\sigma^2), \tag{7.81}$$

we have

$$A_T^{(4)} = \frac{1}{6}u_T^4 T^3 \sigma_0^\eta \sigma_0^3 e^{-\frac{u_T^2 T\sigma_0^2}{2}}. \tag{7.82}$$

Combining the expansion in (7.63) along with the ones in (7.76), (7.78), (7.80) and (7.82), we get the result in (7.2) to be proved. □

7.2 Proof of Theorem 2

We will make use of the following lemma in the proof:

Lemma 3 *Assume A3 holds. There exists* $\mathcal{F}_0^{(0)}$*-adapted random variables* C_0 *and* $\bar{t} > 0$ *that do not depend on* T *such that for* $T < \bar{t}$*, we have*

$$O_T(k) \le C_0 \left(Te^{3k} 1_{\{k<-1\}} + Te^{-k} 1_{\{k>1\}} + \left(\sqrt{T} \wedge \frac{T}{|k|} \right) 1_{\{|k|<1\}} \right), \tag{7.83}$$

$$|O_T(k_1) - O_T(k_2)| \le C_0 \left[\frac{T}{k_2^4} \bigwedge \frac{T}{k_2^2} \bigwedge 1 \right] |e^{k_1} - e^{k_2}|, \tag{7.84}$$

where $k_1 < k_2 < 0$ *or* $k_1 > k_2 > 0$*. In addition, for* $|k| \le \sqrt{T}|\log(T)|$*:*

$$\left| O_T(k) - \sqrt{T}\sigma_0 f\left(\frac{k}{\sqrt{T}\sigma_0} \right) - |k|\Phi\left(-\frac{k}{\sqrt{T}\sigma_0} \right) \right| \le C_0 T \log^2(T). \tag{7.85}$$

The first two results of the lemma follow from Lemma 1 of Todorov (2019) and the last one from Lemma 2 of Todorov (2019). Using this lemma as well as the representation $\mathcal{L}_T(u) = 1 - \left(\frac{u^2}{T} + i\frac{u}{\sqrt{T}} \right) \int_{\mathbb{R}} e^{(iu/\sqrt{T}-1)k} O_T(k)dk$, we can decompose

$$\widehat{\mathcal{L}}_{T_l}(u) - \mathcal{L}_{T_l}(u) = \widehat{Z}_{T_l}(u) + O_p^{lu}\left(\frac{\Delta}{\sqrt{T_l}} \bigwedge e^{-2(|\underline{k}| \vee |\overline{k}|)} \right),\ l = 1,2, \tag{7.86}$$

where we denote

$$\widehat{Z}_{T_l}(u) = -\left(\frac{u^2}{T_l} + i\frac{u}{\sqrt{T_l}} \right) \sum_{j=2}^{N_l} e^{(iu/\sqrt{T_l}-1)k_{l,j-1}} \epsilon_{T_l}(k_{l,j-1})\Delta_{l,j},\ l = 1,2. \tag{7.87}$$

Application again of Lemma 3, and the assumption for the observation error, yields for fixed u:

$$\begin{aligned} \Im(\widehat{Z}_{T_l}(u)) = &-\frac{u^2}{T_l} \sum_{j=2}^{N_l} \sin\left(\frac{u}{\sqrt{T_l}} k_{l,j-1} \right) e^{-k_{l,j-1}} \epsilon_{T_l}(k_{l,j-1})\Delta_{l,j} \\ &+ O_P\left(\sqrt{\Delta} \right),\ l = 1,2, \end{aligned} \tag{7.88}$$

$$\Re(\widehat{Z}_{T_l}(u)) = -\frac{u^2}{T_l}\sum_{j=2}^{N_l}\cos\left(\frac{u}{\sqrt{T_l}}k_{l,j-1}\right)e^{-k_{l,j-1}}\epsilon_{T_l}(k_{l,j-1})\Delta_{l,j} + O_P\left(\sqrt{\Delta}\right),\ l=1,2. \tag{7.89}$$

Therefore, we will be done with establishing (4.15) if we can show

$$\frac{T_l^{1/4}}{\sqrt{\Delta}}\begin{pmatrix}\Im(\widehat{Z}_{T_l}(u))\\ \Re(\widehat{Z}_{T_l}(u))\end{pmatrix} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \begin{pmatrix}Z_{l,I}(u)\\ Z_{l,R}(u)\end{pmatrix},\ l=1,2, \tag{7.90}$$

where $(Z_{l,l}(u)\ Z_{l,R}(u))$ is the limiting process in the statement of the theorem. The finite-dimensional convergence follows by the same arguments as in the proof of Theorem 3 in Todorov (2019). For showing local uniformity, we can use the following bounds

$$\mathbb{E}\left(\left(\sum_{j=2}^{N_l}\left(\sin\left(\frac{u}{\sqrt{T_l}}k_{l,j-1}\right)-\sin\left(\frac{v}{\sqrt{T_l}}k_{l,j-1}\right)\right)e^{-k_{l,j-1}}\epsilon_{T_l}(k_{l,j-1})\Delta_{l,j}\right)^4\Bigg|\mathcal{F}^{(0)}\right) \le C_0|u-v|^{1+\iota}T_l^{3/2}\Delta,\ l=1,2, \tag{7.91}$$

$$\mathbb{E}\left(\left(\sum_{j=2}^{N_l}\left(\cos\left(\frac{u}{\sqrt{T_l}}k_{l,j-1}\right)-\cos\left(\frac{v}{\sqrt{T_l}}k_{l,j-1}\right)\right)e^{-k_{l,j-1}}\epsilon_{T_l}(k_{l,j-1})\Delta_{l,j}\right)^4\Bigg|\mathcal{F}^{(0)}\right) \le C_0|u-v|^{1+\iota}T_l^{3/2}\Delta,\ l=1,2, \tag{7.92}$$

for some arbitrary small $\iota > 0$, and a criterion for tightness given in Theorem 12.3 of Billingsley (1968). The above two bounds follow from A3 and A4 and an application of Lemma 3. More specifically, we make use of the fact that if $\{\zeta_i\}_{i=1,\dots,N}$ is a sequence of independent random variables, then $\mathbb{E}\left(\sum_{i=1}^N \zeta_i\right)^4 \le \left(\sum_{i=1}^N \mathbb{E}(\zeta_i^2)\right)^2 + \sum_{i=1}^N \mathbb{E}(\zeta_i^4)$.

We are thus left with showing (4.19). First, using the first two bounds of Lemma 3 as well as assumptions A3 and A4 and similarly to the way of establishing tightness above, we have

$$\widehat{\Sigma}_{T_l}(u,v) - \mathbb{E}\left(\widehat{\Sigma}_{T_l}(u,v)\big|\mathcal{F}^{(0)}\right) = o_p\left(\frac{\Delta}{\sqrt{T_l}}\right),\quad l=1,2, \tag{7.93}$$

locally uniformly in u and v. Using Lemma 3 as well as assumptions A3 and A4, we can also show

$$\frac{\sqrt{T_l}}{\Delta}\mathbb{E}\left(\widehat{\Sigma}_{T_l}(u,v)\big|\mathcal{F}^{(0)}\right) \xrightarrow{\mathbb{P}} \Sigma_l(u,v),\quad l=1,2, \tag{7.94}$$

locally uniformly in u and v. Combining the above two bounds, we get (4.19).

7.3 Proof of Theorem 3

7.3.1 Case without jumps

We denote $x_T(\eta) = aT + \sigma W_T + \frac{1}{2}\eta W_T^2 - \frac{\eta}{2}T$, for some constants a and $\sigma > 0$, and for some $\eta > 0$. The OTM option price with log-strike k corresponding to terminal log-price of $x_T(\eta)$ will be denoted with $O_T(k;\eta)$. In the proof, we denote with $C(\eta)$ and $c(\eta)$ continuous functions of η, and with $C(\eta_1,\eta_2)$ a continuous function of η_1 and η_2. We also use the notation $a_n \gtrsim b_n$ and $a_n \lesssim b_n$ to mean the respective inequality up to a constant independent of the parameter n, for two sequences a_n and b_n.

The idea of the proof is to perturb locally η and then derive the order of magnitude of the Kullback-Leibler divergence of the resulting two probability distributions of the observed option prices. Applying Theorem 2.2 and the bound in (2.9) (see also (2.5)) in Tsybakov (2009), we then have

$$\inf_{\widehat{\eta}} \sup_{\mathcal{T}\in\mathcal{G}(R)} \mathbb{E}_{\mathcal{T}} \left(\frac{T^{3/2}\sqrt{\log(1/T)}}{\Delta} |\widehat{\eta}-\eta|^2 \right) \geq V(\varphi), \tag{7.95}$$

where φ is an upper bound on the Kullback-Leibler divergence of two probability distributions in $\mathcal{G}(R)$ with leverage parameters η_1 and η_2 such that $|\eta_1-\eta_2| = \frac{\sqrt{\Delta}}{T^{3/4}\log(1/T)^{1/4}}$, and $V(\varphi)$ is some strictly positive function of φ.

The KL divergence between the probability measures for the observed noisy option prices, corresponding to $\mathcal{T}$ with η_1 and η_2 and the same a and σ, both of which belong to $\mathcal{G}(R)$, is given by

$$\begin{aligned} KL(\eta_1,\eta_2) = & \sum_{i=1}^{N} \frac{(O_T(k_i;\eta_1)-O_T(k_i;\eta_2))^2}{2(O_T(k_i;\eta_2)\vee T)^2} \\ & + \frac{1}{2}\sum_{i=1}^{N}\left(\left(\frac{O_T(k_i;\eta_1)\vee T}{O_T(k_i;\eta_2)\vee T}\right)^2 - 1 - \log\left(\frac{O_T(k_i;\eta_1)\vee T}{O_T(k_i;\eta_2)\vee T}\right)^2 \right), \end{aligned} \tag{7.96}$$

Part I: Preliminary Estimates. We start with establishing some preliminary bounds for $O_T(k;\eta)$ and the difference $O_T(k;\eta_2) - O_T(k;\eta_1)$. First, for T sufficiently small (depending on the value of η) and using the tail behavior of a Gaussian random variable, we have

$$O_T(k;\eta) \leq C(\eta)T^p, \quad \text{for } k < -T^{1/2-\iota}, \tag{7.97}$$

for some arbitrary big $p > 0$ and some arbitrary small $\iota > 0$.

Next, using integration by parts, we have:

$$O_T(k;\eta) = \int_{-\infty}^{k} e^u \mathbb{Q}_0\left(x_T(\eta) < u\right) du, \quad k \leq 0. \tag{7.98}$$

Note further that we can write

$$O_T(k_1;\eta) = \int_{k_2}^{k_1} e^u \mathbb{Q}_0\left(x_T(\eta) < u\right) du + O_T(k_2;\eta), \text{ for } k_2 \le k_1 \le 0. \tag{7.99}$$

Therefore, by taking into account the bound in (7.97), we have

$$\left| O_T(k;\eta) - \int_{-T^{1/2-\iota}}^{k} e^u \mathbb{Q}_0\left(x_T(\eta) < u\right) du \right| \le C(\eta) T^2, \text{ for } k \in [-T^{1/2-\iota}, 0]. \tag{7.100}$$

Next, by denoting $Z = W_T/\sqrt{T}$, we have that $x_T(\eta) < u$ is equivalent to

$$\left(Z + \frac{\sigma}{\eta\sqrt{T}} \right)^2 < \frac{2u}{\eta T} + \frac{\sigma^2}{\eta^2 T} + \left(1 - \frac{2a}{\eta}\right). \tag{7.101}$$

Now, using Taylor expansion for the function $\sqrt{1+x}$ around $x = 0$, we have for $|u| \le T^{1/2-\iota}$ and T sufficiently small (so that x in the function that is Taylor expanded is less than a half) which depends on the value of η:

$$-\frac{\sigma}{\eta\sqrt{T}} - \zeta(u,T;\eta) - T^{2-\iota}C(\eta) < Z + \frac{\sigma}{\eta\sqrt{T}} < \frac{\sigma}{\eta\sqrt{T}} + \zeta(u,T;\eta) + T^{2-\iota}C(\eta), \tag{7.102}$$

where

$$\zeta(u,T;\eta) = \frac{u}{\sigma\sqrt{T}} + \widetilde{\zeta}(u,T;\eta), \tag{7.103}$$

and

$$\begin{aligned} \widetilde{\zeta}(u,T;\eta) = & \left(1 - \frac{2a}{\eta}\right) \frac{\sqrt{T}\eta}{2\sigma} - \frac{1}{8}\frac{\sigma}{\eta\sqrt{T}} \left(\frac{2u\eta}{\sigma^2} + \left(1 - \frac{2a}{\eta}\right)\frac{\eta^2 T}{\sigma^2} \right)^2 \\ & + \frac{1}{16}\frac{\sigma}{\eta\sqrt{T}} \left(\frac{2u\eta}{\sigma^2} + \left(1 - \frac{2a}{\eta}\right)\frac{\eta^2 T}{\sigma^2} \right)^3 \\ & - \frac{5}{128}\frac{\sigma}{\eta\sqrt{T}} \left(\frac{2u\eta}{\sigma^2} + \left(1 - \frac{2a}{\eta}\right)\frac{\eta^2 T}{\sigma^2} \right)^4 . \end{aligned} \tag{7.104}$$

Therefore, using the fact that $|\Phi(x) - \Phi(y)| \le |x-y|/\sqrt{2\pi}$, we have for T sufficiently small and for $k \in [-T^{1/2-\iota}, 0]$:

$$\left| O_T(k;\eta) - \int_{-T^{1/2-\iota}}^{k} e^u \Phi\left(\zeta(u,T;\eta)\right) du \right| \le C(\eta) T^{2-\iota}. \tag{7.105}$$

Using first-order Taylor expansion for $\Phi(x)$, and for T sufficiently small such that $|\widetilde{\zeta}(u,T;\eta)| \le T^{1/2-3\iota}$, we have for $k \in [-T^{1/2-\iota}, 0]$:

$$\begin{aligned} & \left| O_T(k;\eta) - \int_{-\infty}^{k} e^u \Phi\left(\frac{u}{\sigma\sqrt{T}}\right) du \right| \\ & \le C(\eta) T^{2-\iota} + C(\eta) \int_{-\infty}^{k} e^u f\left(\frac{u}{\sigma\sqrt{T}} + T^{1/2-3\iota} \right) |\widetilde{\zeta}(u,T;\eta)| du. \end{aligned} \tag{7.106}$$

Next, for p a nonnegative integer, we have

$$\begin{aligned} \int_{-\infty}^{k} f(u)|u|^p du \ &\sim \ f(k)|k|^{p-1}, \ \int_{-\infty}^{k} \Phi(u) du \ \sim \ f(k)|k|^{-2}, \\ \int_{-\infty}^{k} |u|\Phi(u) du \ &\sim \ f(k)|k|^{-1}, \ \text{as } k \downarrow -\infty, \end{aligned} \tag{7.107}$$

and therefore since $f(k)$ and $\Phi(k)$ are continuous and bounded functions taking positive values, we have

$$\begin{aligned} cf(k)(|k| \vee 1)^{p-1} &\leq \int_{-\infty}^{k} f(u)|u|^p du \leq Cf(k)(|k| \vee 1)^{p-1}, \ \text{for } k \leq 0, \\ cf(k)(|k| \vee 1)^{-2} &\leq \int_{-\infty}^{k} \Phi(u) du \leq Cf(k)(|k| \vee 1)^{-2}, \ \text{for } k \leq 0, \\ cf(k)(|k| \vee 1)^{-1} &\leq \int_{-\infty}^{k} |u|\Phi(u) du \leq Cf(k)(|k| \vee 1)^{-1}, \ \text{for } k \leq 0, \end{aligned} \tag{7.108}$$

for some $0 < c < C < \infty$ that depend on the value of the nonnegative integer p. Therefore, provided ι is sufficiently small, we have for sufficiently small T, we have

$$k \in [-T^{1/2-\iota}, 0] \Longrightarrow \left| \int_{-\infty}^{k} f\left(\frac{u}{\sigma\sqrt{T}} + T^{1/2-3\iota}\right) |\widetilde{\zeta}(u, T; \eta)| du \Big/ \int_{-\infty}^{k} \Phi\left(\frac{u}{\sigma\sqrt{T}}\right) du \right| \leq \varepsilon, \tag{7.109}$$

$$k \in [-T^{1/2-\iota}, 0] \Longrightarrow \left| \int_{-\infty}^{k} (1 - e^u) \Phi\left(\frac{u}{\sigma\sqrt{T}}\right) du \Big/ \int_{-\infty}^{k} \Phi\left(\frac{u}{\sigma\sqrt{T}}\right) du \right| \leq \varepsilon, \tag{7.110}$$

for some arbitrary small $\varepsilon > 0$. Altogether, we can write

$$\left| O_T(k; \eta) - \int_{-\infty}^{k} e^u \Phi\left(\frac{u}{\sigma\sqrt{T}}\right) du \right| \leq C(\eta) T^{2-\iota} + \frac{1}{2} \int_{-\infty}^{k} e^u \Phi\left(\frac{u}{\sigma\sqrt{T}}\right) du, \tag{7.111}$$

for $k \in [-T^{1/2-\iota}, 0]$ and provided T is sufficiently small. Next, we can use a second-order Taylor expansion and write

$$|\Phi(u_2) - \Phi(u_1) - f(u_1)(u_2 - u_1)| \leq \frac{1}{2} f(|u_1| \wedge |u_2|)(|u_1| \vee |u_2|)(u_2 - u_1)^2, \tag{7.112}$$

for $u_1, u_2 < 0$. Using this result as well as (7.105), we have for T sufficiently small and for $k \in [T^{1/2-\iota}, 0]$:

$$\begin{aligned} &\left| O_T(k; \eta_2) - O_T(k; \eta_1) - \int_{-T^{1/2-\iota}}^{k} e^u f\left(\zeta(u, T; \eta_1)\right) \left(\zeta(u, T; \eta_2) - \zeta(u, T; \eta_1)\right) du \right| \\ &\leq C(\eta_1, \eta_2) T^{2-\iota} + C(\eta_1, \eta_2) |\eta_2 - \eta_1|^2 T^{1-4\iota} \int_{-\infty}^{k} f\left(\frac{u}{\sigma\sqrt{T}} + T^{1/2-3\iota}\right) du, \end{aligned} \tag{7.113}$$

for two positive η_1 and η_2 with $|\eta_2 - \eta_1| \leq 1$. Next, for T sufficiently small and η_1 and η_2 with $|\eta_2 - \eta_1| \leq 1$, we have

$$\begin{aligned}&\left|\int_{-T^{1/2-\iota}}^{k} e^u f\left(\zeta(u,T;\eta_1)\right)\left(\zeta(u,T;\eta_2) - \zeta(u,T;\eta_1)\right)du\right| \\ &\leq C(\eta_1,\eta_2)|\eta_2-\eta_1| \int_{-T^{1/2-\iota}}^{k} f\left(\frac{u}{\sigma\sqrt{T}} + T^{1/2-3\iota}\right)\left(\sqrt{T} + \frac{u^2}{\sqrt{T}}\right)du \\ &+ C(\eta_1,\eta_2)|\eta_2-\eta_1| \int_{-T^{1/2-\iota}}^{k} f\left(\frac{u}{\sigma\sqrt{T}} + T^{1/2-3\iota}\right)\left(|u|\sqrt{T} + \frac{|u|^3}{\sqrt{T}}\right)du,\end{aligned} \tag{7.114}$$

for $k \in [T^{1/2-\iota}, 0]$. To proceed further, we make use of

$$\int_{-\infty}^{k} u^2 f(u)du \;\sim\; f(k)|k|, \;\; \int_{-\infty}^{k} |u| f(u)du \;\sim\; f(k), \;\; \Phi(k) \;\sim\; f(k)/|k|, \tag{7.115}$$

as $k \downarrow -\infty$. Moreover, all of the above integrals are smooth and bounded functions. This implies

$$\int_{-\infty}^{k} u^2 f(u)du + \int_{-\infty}^{k} |u| f(u)du + \Phi(k) \leq C f(k)(|k| \vee 1), \quad k \leq 0, \tag{7.116}$$

for some constant that does not depend on k. Therefore, for $k \in [-T^{1/2-\iota}, 0]$ and $|\eta_2 - \eta_1| \leq 1$, we can write

$$\begin{aligned}&|O_T(k;\eta_2) - O_T(k;\eta_1)| \\ &\leq C(\eta_1,\eta_2)T^{2-\iota} + C(\eta_1,\eta_2)T|\eta_2-\eta_1| f\left(\frac{k}{\sqrt{T}\sigma} + T^{1/2-3\iota}\right)\left(\frac{|k|}{\sqrt{T}} \bigvee 1\right) \\ &+ C(\eta_1,\eta_2)T^{3/2-2\iota}|\eta_2-\eta_1| f\left(\frac{k}{\sqrt{T}\sigma} + T^{1/2-3\iota}\right)\left(\frac{|k|}{\sqrt{T}} \bigvee 1\right).\end{aligned} \tag{7.117}$$

Part II: Bounding the KL distance. Analogous bounds to the ones in (7.97), (7.111) and (7.117) can be shown in exactly the same way as above for $k \geq 0$. We also note that $\int_{-\infty}^{k} e^u \Phi\left(\frac{u}{\sigma\sqrt{T}}\right) du$ corresponds to an option price for terminal log-price of $x_T(0)$, i.e., with $\eta = 0$ (and with $a = 0$ in addition). Therefore, we can proceed exactly as in the proof of Theorem 5 of Todorov (2019), and conclude

$$KL(\eta_1,\eta_2) \lesssim \sum_{i=1}^{N} \frac{(O_T(k_i;\eta_1) - O_T(k_i;\eta_2))^2}{O_T^2(k_i;\eta_2) \vee T^2}. \tag{7.118}$$

Now, using (7.97) above, we have

$$\sum_{i:k_i<-T^{1/2-\iota}} \frac{(O_T(k;\eta_2) - O_T(k;\eta_1))^2}{O_T^2(k;\eta_2) \vee T^2} \leq C(\eta_1,\eta_2)\frac{T^p}{\Delta}, \tag{7.119}$$

for some arbitrary big $p > 0$. Next, exactly as in the proof of Theorem 5 of Todorov (2019) and given our bound in (7.111), we have for T sufficiently small,

$$k < -\sqrt{2T}\sigma\sqrt{\log(1/\sqrt{T})} \implies O_T(k;\eta_2) \vee T = T, \tag{7.120}$$

and

$$k \geq -\sqrt{2T}\sigma\sqrt{\log(1/\sqrt{T})} \implies O_T(k;\eta_2) \geq C(\eta_2)\sqrt{T} f\left(\frac{k}{\sqrt{T}\sigma}\right)\left(\frac{|k|}{\sqrt{T}} \bigvee 1\right)^{-2}, \tag{7.121}$$

for some positive $C(\eta_2)$. Therefore, for T sufficiently small and ι sufficiently small and upon using (7.117), we have

$$\begin{aligned}
&\sum_{i:-T^{1/2-\iota}\leq k_i \leq -\sqrt{2T}\sigma\sqrt{\log(1/\sqrt{T})}} \frac{(O_T(k_i;\eta_1)-O_T(k_i;\eta_2))^2}{O_T^2(k_i;\eta_2)\vee T^2} \\
&\lesssim \frac{T^{5/2-3\iota}}{\Delta} + |\eta_2-\eta_1|^2 \sum_{i:-T^{1/2-\iota}\leq k_i \leq -\sqrt{2T}\sigma\sqrt{\log(1/\sqrt{T})}} f^2\left(\frac{k_i}{\sqrt{T}\sigma} + T^{1/2-3\iota}\right)\frac{k_i^2}{T} \\
&\lesssim \frac{T^{5/2-3\iota}}{\Delta} + \frac{\sqrt{T}}{\Delta}|\eta_2-\eta_1|^2 \int_{-T^{-\iota}-\Delta}^{-\sqrt{2}\sigma\sqrt{\log(1/\sqrt{T})}} f^2\left(\frac{k}{\sigma} + T^{1/2-3\iota}\right)k^2 dk \\
&\lesssim \frac{T^{5/2-3\iota}}{\Delta} + \frac{T^{3/2}}{\Delta}\sqrt{\log(1/T)}|\eta_2-\eta_1|^2,
\end{aligned} \tag{7.122}$$

where for the second inequality we made use of the fact that $f(k)|k|$ is decreasing for $|k| \geq 1$. Next, using the lower bound for $O_T(k;\eta)$ in (7.121) as well as the bound in (7.117), we have

$$\begin{aligned}
&\sum_{i:-\sqrt{2T}\sigma\sqrt{\log(1/\sqrt{T})}\leq k_i \leq 0} \frac{(O_T(k_i;\eta_1)-O_T(k_i;\eta_2))^2}{O_T^2(k_i;\eta_2)\vee T^2} \lesssim \frac{T^{5/2-2\iota}}{\Delta}\log(1/T) \\
&+ T|\eta_2-\eta_1|^2 \sum_{i:-\sqrt{2T}\sigma\sqrt{\log(1/\sqrt{T})}\leq k_i \leq 0} \left(\frac{|k_i|}{\sqrt{T}} \bigvee 1\right)^6 \lesssim \frac{T^{5/2-3\iota}}{\Delta} + |\eta_2-\eta_1|^2 \frac{T^{3/2}(\log(1/T))^{7/2}}{\Delta}.
\end{aligned} \tag{7.123}$$

The same bounds hold also when summing over k positive, and therefore altogether, we have

$$KL(\eta_1,\eta_2) \lesssim \frac{T^{5/2-3\iota}}{\Delta} + |\eta_2-\eta_1|^2 \frac{T^{3/2}}{\Delta}(\log(1/T))^{7/2}. \tag{7.124}$$

Evaluating the above bound with $\eta_2 - \eta_1 = \frac{\sqrt{\Delta}}{T^{3/4}(\log(1/T))^{7/4}}$ (and making use of the fact that $\alpha < 5/2$ by assumption), we get the result of the theorem.

7.3.2 Case with jumps

Denoting with $O_T^c(k;\eta)$ the option price corresponding to terminal payoff of $x_T - J_T^x$ and using the fact that J_t^x is compound Poisson with unit intensity and deterministic jump size of J, we have for $k \leq 0$

$$\begin{aligned}
O_T(k;\eta) &= e^{-T}O_T^c(k;\eta) + O_T^r(k;\eta) \\
&\quad + Te^{-T}e^J\left\{O_T^c(k-J;\eta)1_{\{k\leq J\}} + (O_T^c(k-J;\eta) - 1 + e^{k-J})1_{\{k>J\}}\right\},
\end{aligned} \tag{7.125}$$

for some non-negative $O^r_T(k;\eta)$ satisfying

$$|O^r_T(k;\eta)| \le C(\eta)T^2. \tag{7.126}$$

Similar result holds also for the case $k > 0$. From here, bounding the KL distance follows similar steps as in the case of no jumps in x.

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