Intraday Volatility Patterns from Short-Dated Options

Viktor Todorov∗and Yang Zhang†

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Abstract

We propose a nonparametric estimator for the deterministic intraday periodic component of volatility from short-dated options within an in-fill asymptotic setting. The estimator uses options with zero and one day to expiration sampled at high-frequency during a trading day. At each point in time, we aggregate the options to form nonparametric estimates of conditional risk-neutral expectations of future integrated return variation for the two available option tenors. A suitable ratio of these estimates removes the stochastic components of the conditional expectations of future volatility, up to asymptotically higher-order terms, and allows to form estimates of the deterministic intraday volatility pattern. We derive a Central Limit Theorem for the estimator, with its rate of convergence determined from the mesh of the strike grid and the length of the time to expiration of the options. The newly-developed estimation procedure is applied to S&P 500 index options data.

Keywords: in-fill asymptotics, intraday volatility pattern, nonparametric volatility estimation, short-dated options.

JEL classification: C51, C52, G12.

∗Kellogg School of Management, Northwestern University; e-mail: v-todorov@northwestern.edu.
†Financial Market Infrastructure Institution; e-mail: zhangyang360@gmail.com. Yang Zhang completed this work in her personal time. The views expressed in the article are the author’s own and not those of her employer.
1 Introduction

A well-recognized and important feature of the volatility of most financial assets is its strong intraday periodic pattern, see e.g., Wood et al. (1985), Harris (1986), Admati and Pfleiderer (1988), Andersen and Bollerslev (1997, 1998) and Hong and Wang (2000), with volatility being much higher on average following market open and prior to market close than during the middle of the trading day. This intraday pattern of stochastic volatility is too large to be ignored and can have nontrivial effect on many estimation problems that involve the use of high-frequency data. For example, the nonparametric detection of jumps depends strongly on a good local estimator of volatility, which in turn depends on accounting for the intraday volatility pattern, see e.g., Boudt et al. (2011).

Earlier work has used both parametric and nonparametric methods for estimating the intraday periodic volatility component, see e.g., Andersen and Bollerslev (1997), Taylor and Xu (1997) and Boudt et al. (2011), among others. The nonparametric method, in particular, consists of forming local estimators of volatility during the day and then averaging the time series of these estimates. This estimation approach relies on a joint in-fill and long-span asymptotics. Intuitively, the high-frequency data is needed for recovering nonparametrically volatility from stock returns while a long time span of the data set is needed in order to disentangle the stochastic (and stationary) component of volatility from the intraday periodic one. However, in a recent work, Andersen et al. (2019) show using a nonparametric test that the periodic component of volatility can change over time and Andersen et al. (2023) estimate its average value.

Given this evidence, the goal of this paper is to develop a nonparametric method for identifying the intraday periodic component of volatility without making use of long-span asymptotics. Such a method will avoid the need to use long time series of data and avoid either making an assumption that the intraday periodic component remains constant over time or modeling this time series variation via a parametric model. Our method is based on short-dated options written on the asset, i.e., options that have short time to expiration and asymptotic expansions for risk-neutral variance measures constructed from them.

As documented in Andersen et al. (2017), short-dated options have increased in popularity among investors over the last decade. Trading in such options has been facilitated by the introduction of the so-called weekly options which have weekly expiration cycle. For example, many stocks and indices have options, traded on the CBOE options exchange, that expire every Friday. Moreover, for options written on the S&P 500 index, CBOE currently offers options expiring on

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1 One special situation in which long-span asymptotics will not be needed is if the stochastic component of volatility remains constant during the trading day, see Christensen et al. (2018).
every day of the trading week. In Figure 1, we show the percentage of daily volume of S&P 500 index options traded on CBOE grouped by time to expiration. We consider only days on which options that expire on the same day (left plot) or on the following trading day (right plot) are available. As seen from the figure, a rather nontrivial percent of the total volume of traded S&P 500 index options is for ones with zero or one day tenor. These are the options that we are going to use in our analysis.

Figure 1: S&P 500 Index Options Volume for 0-Day (left) and 1-Day (right) Tenor. The left/right plot corresponds to days with options expiring on the same/next day.

Our estimator is built from measures of conditional risk-neutral expectations of future return variation for two short horizons measured at high-frequency during a trading day. These measures are constructed from portfolios of options with zero- and one-day tenor. Using short-time asymptotics, i.e., asymptotics for shrinking tenor of the options, we can proceed as if the stochastic components of volatility and jump intensity remain constant over the life of the options and equal to their value at the time of observing the option prices. Thus, the ratio of the risk-neutral volatility estimates over the two horizons, up to higher-order bias terms, does not depend on the stochastic components of volatility and jump intensity. Instead, its asymptotic limit is determined by functions of the deterministic intraday periodic component of volatility we are after. By comparing such ratios at different parts of the trading day, we can estimate the latter. Importantly, such an estimator requires that the intraday periodic component of volatility is constant only over the short life of the options. This is in sharp contrast to a return-based nonparametric estimate of
We derive the rate of convergence of our estimator and provide an associated Central Limit Theorem (CLT). The asymptotic setup is of joint type: the mesh of the strike grids and the tenors of the options shrink simultaneously with the increase of the strike ranges of the options, which asymptotically converge to cover the whole positive part of the real line. The rate of convergence of the estimator is determined by both the mesh of the two strike grids as well as by the length of the two tenors. Its limiting behavior is governed by the options with strikes near the current underlying spot price, which in turn are dominated by the diffusive component of the spot price.

We evaluate the performance of our estimator on simulated data from a model that matches key empirical features of real option data. Our Monte Carlo results show that the precision of the estimation is affected by the point in time during the trading day that we try to estimate the intraday periodic function at and the overall level of the volatility. In particular, the precision of the estimation is lower for estimating the value of the periodic function at the earlier part of the trading day because the signal to noise ratio is weaker at that time. Similarly, the precision of the estimation is lower for periods of lower volatility. The reason is that for such levels of volatility, there is less averaging of the observation errors across the different strikes because there are fewer options observations per tenor.

We apply our estimation procedure to S&P 500 index options over the period 2018-2020. We use intraday option observations on Fridays with expiration on the same day and on the following trading day (typically Monday). Our estimation results show that the diurnal volatility pattern has become more right-skewed, with heightened level of volatility towards the very end of market trading.

The rest of the paper is organized as follows. In Section 2, we introduce our setting and the option observation scheme. In Section 3, we present our option-based estimate of the intraday periodic component of volatility and analyze its asymptotic properties. Section 4 contains a Monte Carlo study and Section 5 our empirical application. Section 6 concludes. Technical assumptions and the proofs are given in Section 7.
arbitrage-free asset prices, is locally equivalent to the true one. The significance of $Q$ stems from the fact that the discounted at the risk-free rate payoff process of any asset is a local martingale under $Q$. We will use this result to connect the value of derivatives written on the asset with its volatility. More specifically, the dynamics of $x$ under $Q$ is given by

$$dx_t = \alpha_t dt + \sqrt{\eta_t/\kappa - [t/\kappa]} \sigma_t dW_t + \int_{\R} x\mu(dt, dx),$$

where $W$ is a $Q$ Brownian motion and $\mu$ is an integer-valued random measure on $\R_+ \times \R$ with $Q$ jump compensator $dt \otimes \eta_t/\kappa - [t/\kappa] \nu_t(dx)$, for some predictable measure $\nu_t$ satisfying $\int_{\R} |x| \nu_t(dx) < \infty$. Finally, $\eta : [0, 1] \to \R_+$ is a deterministic function capturing the intraday periodicity in volatility. The length of the periodic cycle is an interval from market close on one day till market close on the next trading day. The length of that period is denoted with $\kappa$. Our interest in this paper is the study of the function $\eta$ from short-dated option observations.

We note that local equivalence of $P$ and $Q$ implies that $x$ obeys the same dynamics under $P$ but with a different drift coefficient $\alpha_t^P$ and a different jump compensator measure $\nu_t^P(dx)$. The diffusion coefficient $\sigma_t$ is the same under the two probability measures. Importantly, since $\eta_t$ is deterministic function, it does not change when switching from $Q$ to $P$.

**Remark 1.** In the specification in (1), we assumed that the intraday periodic component of the diffusive variance is the same as that of the jump intensity. This seems a natural assumption. If however, one is interested only in the intraday periodic component of diffusive variance, without making assumptions about the intraday behavior of jumps, then the analysis that follows should be modified by making use of jump-robust estimates of volatility from options data like the ones proposed in Todorov (2019).

We turn next to our option observation scheme. We denote with $O_{t,T}(K)$ the price at time $t$ of a European-style out-of-the-money option price written on the asset and expiring at time $t + T$. We recall that $O_{t,T}(K)$ is the minimum of the put and call option prices with strike $K$. The option prices corresponding to the pair $(t, T)$ are observed on the following discrete strike grid:

$$K_{t,T}(1) < ... < K_{t,T}(N_{t,T}), \quad N_{t,T} \in \N_+.$$  

Option prices are observed with error, i.e., we observe

$$\tilde{O}_{t,T}(K_{t,T}(j)) = O_{t,T}(K_{t,T}(j)) + \epsilon_{t,T}(j),$$

where the errors $\epsilon_{t,T}(j)$ are defined on a space $\Omega^{(1)} = \R^\R \times \R^\R$ which is equipped with the product Borel $\sigma$-field $\mathcal{F}^{(1)}$, and transition probability $P^{(1)}(\omega^{(0)}, d\omega^{(1)})$ from the probability space $\Omega^{(0)}$, on
which \( X \) is defined, to \( \Omega^{(1)} \). We further define,

\[
\Omega = \Omega^{(0)} \times \Omega^{(1)}, \quad \mathcal{F} = \mathcal{F}^{(0)} \times \mathcal{F}^{(1)},
\]

and

\[
P(d\omega^{(0)}, d\omega^{(1)}) = P^{(0)}(d\omega^{(0)}) P^{(1)}(\omega^{(0)}, d\omega^{(1)}).
\]

We assume that we observe the options on the following equidistant time grid:

\[
t_0 = 0 < t_1 < t_2 < \ldots < t_n = \left(\frac{K}{\Delta} - 3\right) \Delta,
\]

with \( t_i - t_{i-1} = \Delta \), for some \( 0 < \Delta < \kappa \). This corresponds to intraday sampling of option prices like in our application. At each sampling time \( t_i \), we will have observations of options with two tenors, \( T_i^0 \) and \( T_i^1 \). The shorter tenor will be \( T_i^0 = \kappa - t_i \) and the longer one will be \( T_i^1 = T_i^1 + \kappa \). That is, the shorter tenor will correspond to zero-day options expiring on the same trading day and the longer tenor to options expiring on the next trading day.

To simplify notation, we will use the following shorthand notation henceforth: \( N_{t_i} = N_{t_i,T_i^0} \), \( K_{t_i}(j) = K_{t_i,T_i^0}(j) \), \( \epsilon_{t_i}(j) = \epsilon_{t_i,T_i^0}(j) \) and \( \tilde{O}_{t_i}(j) = \tilde{O}_{t_i,T_i^0}(K_{t_i,T_i^0}(j)) \), for \( l = 0, 1 \) and \( j = 1, \ldots, N_{t_i,T_i^0} \).

**Remark 2.** In the specification of \( x \) in (1), \( \eta \) is assumed to be a deterministic function. However, as will become clear from the analysis below, we only need that \( \eta \) is \( \mathcal{F}_0 \)-adapted for the estimator that we propose. For notational simplicity, we will present our results for \( \eta \) deterministic.

### 3 Estimation of the Intraday Periodic Component of Volatility

Our strategy for recovering the intraday periodic volatility pattern is to form measures of conditional risk-neutral expectation of future return variation over the two tenors of the available options. These conditional expectations will depend both on the periodic (and deterministic) volatility component as well as on its stochastic component. By taking advantage of the two tenors and the fact that they are both short, we can suitably cancel out the stochastic components of the conditional expectations of future return variation and identify the periodic volatility component.

We now provide the details of our estimation strategy. Using the formula for computing the VIX index, see e.g., Britten-Jones and Neuberger (2000) and Carr and Wu (2009), we have

\[
QV_{t,T} \equiv E^Q_t \left( \int_t^{t+T} \eta_{s/\kappa} \left( s/\kappa \right) \left( \sigma^2 + 2 \int_0^{K} (e^x - 1 - x) \nu_s(dx) \right) ds \right) = 2 \int_0^\infty \frac{O_{t,T}(K)}{K^2} dK.
\]

We note that the integral in \( QV_{t,T} \) is not exactly the quadratic variation of the log-price, with the deviation from the latter being due to the price jumps. This difference, however, will play no role in our analysis henceforth.
Using the available options, a Riemann sum approximation of the integral in (5) leads to the following feasible counterpart of the risk-neutral conditional expectation of the quadratic variation:

\[ \hat{Q}V_{t,T} = 2 \sum_{j=2}^{N_{t,T}} \frac{\hat{O}_{j,T}(K_{t,T}(j-1))}{K_{t,T}(j-1)^2} \frac{K_{t,T}(j) - K_{t,T}(j-1)}{K_{t,T}(j)} \]  

(6)

With the help of \( \hat{Q}V_{t,T} \), we can construct estimates of the function \( \eta \). Recall that we have observations of options at the discrete times \( t_i \), for \( i = 0, 1, \ldots, n \). Then, we set

\[ \hat{Q}V_{t_i,T_i}^{(0)} \quad \text{and} \quad \hat{Q}V_{t_i,T_i}^{(1)} \quad \text{for} \quad i = \left\lfloor \frac{\phi K}{\Delta} \right\rfloor \quad \text{and} \quad \phi \in [0, 1). \]  

(7)

With this notation, we introduce

\[ \hat{\eta}_\phi = \frac{\hat{Q}V_{t_i,T_i}^{(0)} - \hat{Q}V_{t_i,T_i}^{(1)}}{\hat{Q}V_{t_i,T_i}^{(1)}}, \quad \phi \in [0, 1). \]  

(8)

We will show that \( \hat{\eta}_\phi \) is an estimator of \( \int_{t_i}^{t_{i+1}} \eta_s ds / \int_0^{t_i} \eta_s ds \) (note that \( \eta \) is uniquely identified up to a constant only). From here, we can also define

\[ \hat{\eta}_{\phi_1,\phi_2} = \hat{\eta}_{\phi_2} - \hat{\eta}_{\phi_1}, \quad 0 \leq \phi_1 < \phi_2 < 1. \]  

(9)

which is an estimator of \( \int_{\phi_1}^{\phi_2} \eta_s ds / \int_0^{t_i} \eta_s ds \). In what follows, we will be interested in the asymptotic properties of \( \hat{\eta}_{\phi_1,\phi_2} \).

For this, we will first derive a CLT result for \( \hat{Q}V_{t_i,T_i}^{(0)} \) and \( \hat{Q}V_{t_i,T_i}^{(1)} \), for some fixed \( \phi \in [0, 1) \). The convergence in distribution result will hold \( \mathcal{F}^{(0)} \)-conditionally. This is denoted by \( \mathcal{L} \mathcal{F}^{(0)} \rightarrow \) and formally means convergence in probability of the conditional probability laws when the latter are considered as random variables taking values in the space of probability measures equipped with the weak topology, see e.g., VIII.5.26 of Jacod and Shiryaev (2003).

For stating the next theorem, we introduce the following additional notation

\[ \bar{\Phi}(k) = f(k) + |k| \Phi(-|k|), \quad k \in \mathbb{R}, \]  

(10)

where \( f \) and \( \Phi \) are the pdf and cdf, respectively, of a standard normal random variable. We also use the shorthand notation \( K = \max_{i=1,\ldots,n} K_{i,i}(1) \) and \( \overline{K} = \min_{i=1,\ldots,n} K_{i,i}(N_{i,i}) \). We note that \( K \) and \( \overline{K} \) will typically correspond to the minimum and maximum available strike, respectively, at the last observation time \( t_n \). Finally, in the theorem below, \( \delta \) denotes a reference “average” log-strike gap, formally defined in assumption A3.
Theorem 1. Suppose assumptions A1-A4 hold and fix \( \phi \in [0, 1) \). Let \( \kappa \downarrow 0 \) and \( \delta \asymp \kappa^\alpha \), \( K \asymp \kappa^\beta \), \( \overline{K} \asymp \kappa^{-\gamma} \), for \( \alpha > \frac{1}{2} \) and \( \beta, \gamma > \frac{\alpha}{4} - \frac{1}{8} \). In addition, let \( \Delta \sqrt{\delta \kappa} \rightarrow 0 \). We have
\[
\frac{1}{\sqrt{\delta \kappa}^{3/4}} \begin{pmatrix}
\overline{QV}_{\phi}^{(0)} - QV_{\phi_{\kappa},(1-\phi)\kappa} \\
\overline{QV}_{\phi}^{(1)} - QV_{\phi_{\kappa},(2-\phi)\kappa}
\end{pmatrix} \mathcal{L}_{\mathcal{F}^{(0)}} \rightarrow \begin{pmatrix}
Z_{\phi,0} \\
Z_{\phi,1}
\end{pmatrix},
\]
where \( \mathcal{F}^{(0)} \)-conditionally \( (Z_{\phi,0}, Z_{\phi,1}) \) is a centered Gaussian vector with \( \mathcal{F}^{(0)} \)-conditional variance given by \( \Sigma_{\phi} = \text{diag}(\Sigma_{\phi,0}, \Sigma_{\phi,1}) \), with
\[
\Sigma_{\phi,l} = \left( \int_{\phi}^{1} \eta_s ds + 1_{\{l=1\}} \int_{0}^{1} \eta_s ds \right)^{3/2} \sigma_0^3 \psi_{0,l}(0) \zeta_{0,l}^2(0) \int_{\mathbb{R}} \Phi^2(k) dk, \ l = 0, 1,
\]
where the functions \( \psi_{l,t} \) and \( \zeta_{t,t} \), for \( l = 0, 1 \), appear in assumption A4.

We make several observations about the above CLT result. First, the rate of convergence of the risk-neutral quadratic measures is determined from the options with strikes in the vicinity of the current spot price. The reason for this is that, these options dominate asymptotically the rest of the options with strikes away from the current spot price. We refer to Lemma 1 for a formal statement of this. In turn, the asymptotic behavior of the near-the-money options is dominated by the diffusive component of the underlying asset price. Thus, in spite of the fact that \( \overline{QV}_{l,T} \) is a measure of total return variation that includes jumps, its asymptotic behavior for \( T \) small is governed by the diffusive component of the underlying price. Second, the rate of convergence in the CLT is determined by the mesh of the log-strike grid \( \delta \) and the bound on the time-to-maturity of the options \( \kappa \). Third, the asymptotics here is of joint type as we require \( \delta \) and \( \kappa \) to go to zero simultaneously and also \( K \rightarrow 0 \) and \( \overline{K} \rightarrow \infty \).

In the statement of the theorem, we impose various rate conditions that ensure that biases that arise in the estimation are of higher asymptotic order. In particular, the rate condition for \( K \) and \( \overline{K} \) should hold in real data as the available strike range of traded options typically cover the “effective” support of the return distribution. The requirement \( \delta \asymp \kappa^\alpha \) with \( \alpha > 1/2 \) guarantees that error due to the Riemann sum approximation of the integral in (5) is of higher order. If this condition does not hold, then the change of the option price as the strike moves on the strike grid in the vicinity of the current stock price will be too big which in turn will cause the Riemann sum approximation error to be big. From a practical point of view, the strike grid is fixed by the exchange (for example for the S&P 500 index options used in our empirical analysis it is set to 5), so the above rate condition involving \( \delta \) and \( \kappa \) puts a limit on how short the tenor of the options can be for our asymptotics to work well. This will be discussed later on in the Monte Carlo analysis. Finally, the requirement \( \Delta \sqrt{\delta \kappa}^{3/4} \rightarrow 0 \) is due to the fact that the option observations are
on a discrete grid while our interest is estimation at a fixed point in time (which is not necessarily on the observation time grid). If the centering of $\tilde{QV}_\phi^{(0)}$ and $\tilde{QV}_\phi^{(1)}$ is with their limits at the observation time, then this condition will not be necessary.

**Remark 3.** The CLT result of the theorem is derived under the assumption that the ratio of the option observation error to the true (latent) option price is $O_p(1)$, see assumption A4. Empirically, the ratio of option bid-ask spread to the mid-quote, which can be viewed as a proxy for the relative option observation error, is a small number, see e.g., Andersen et al. (2015a). We can generalize the result of the above theorem by allowing for the relative option pricing error to shrink to zero. The result of Theorem 1 can be extended to cover such a situation, with the rate of decay of the option observation error showing up in the rate of convergence of the estimator.

The limit quantities $QV_{\phi \kappa,(1-\phi)\kappa}$ and $QV_{\phi \kappa,(2-\phi)\kappa}$ can be used to estimate the intraday periodic component of volatility by taking advantage of the fact that $\kappa$ is small and that the stochastic component of the diffusive volatility and of the jump intensity is smooth in expectation by our assumption A1. We note in this regard that we do not require an assumption regarding the smoothness of the periodic component $\eta$. The next theorem presents the formal result.

**Theorem 2.** Suppose assumptions A1-A2 hold and fix $\phi \in [0,1)$. We have

$$ QV_{\phi \kappa,(1-\phi)\kappa} - QV_{\phi \kappa,(2-\phi)\kappa} - \frac{1}{\int_0^1 \eta_s ds} = O_p(\kappa). $$

(13)

The bound on the error in the above theorem is sharp and is directly linked to the assumed degree of smoothness of the stochastic component of diffusive volatility and jump intensity given in assumption A1. This assumption is satisfied when diffusive volatility and jump intensity are modeled via Itô semimartingales, which is the case for most asset pricing models used in applied work.

By combining the results of the above two theorems, we can arrive at a feasible estimate of tail integrals of the periodic component of volatility. The result is stated in the following corollary.

**Corollary 1.** Suppose the conditions in Theorem 1 hold. We then have

$$ \frac{\kappa^{1/4}}{\sqrt{\delta}} \left( \tilde{\eta}_\phi - \int_0^1 \eta_s ds \right) \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \frac{1}{\sqrt{v_0}} \left( \int_0^1 \eta_s ds + \int_0^1 \eta_s ds \right) Z_{\phi,0} - \frac{1}{\sqrt{v_0}} \left( \int_0^1 \eta_s ds \right)^2 Z_{\phi,1}, $$

(14)

for $Z_{\phi,0}$ and $Z_{\phi,1}$ being the limit random variables defined in Theorem 1 and

$$ v_0 = \sigma_0^2 + 2 \int \left( e^x - 1 - x \right) \nu_0(dx). $$

(15)
Given the above result, it is trivial to state also a CLT for the estimator \( \hat{\eta}_{\phi_1,\phi_2} \) of the integrated periodic component for brevity. We note also that in the above Corollary, we do not need any additional requirement on the option observation scheme than what is already assumed in Theorem 1.

Although feasible estimates of the asymptotic variance of the estimator are theoretically possible, see e.g., Todorov (2019), we will not provide such here as we conjecture that reliable fully nonparametric estimation of the asymptotic variance will be difficult given the very short tenor of the options used in the analysis (which results in nontrivial changes in the true option prices around-the-money).

### 4 Monte Carlo Study

In this section, we evaluate the performance of the option-based estimator of the periodic volatility component on simulated data.

#### 4.1 Setup

We use the following model for the underlying asset dynamics, under the risk-neutral probability, to generate the true option prices:

\[
\frac{dX_t}{X_{t^-}} = \sqrt{V_t}dW_t + \int_{\mathbb{R}} (e^x - 1) \mu(dt, dx),
\]

\[
dV_t = \kappa_v(\theta_v - V_t)dt + \sigma_v \sqrt{V_t}dB_t,
\]

where \( W_t \) and \( B_t \) are \( \mathbb{Q} \) Brownian motions with \( \text{corr}(dW_t, dB_t) = \rho dt \), and \( \mu \) is an integer-valued random measure with \( \mathbb{Q} \) compensator \( dt \otimes \nu_t(dx) \), for \( \nu_t \) given by

\[
\nu_t(dx) = V_t \times \nu^g(x) dx, \quad \nu^g(x) = c_- \frac{e^{-\lambda_- |x|}}{|x|} 1_{\{x < 0\}} + c_+ \frac{e^{-\lambda_+ |x|}}{|x|} 1_{\{x > 0\}}.
\]

In the above specification for \( X \), the stochastic variance is modeled as a square-root diffusion process like in the popular model of Heston (1993). The jump intensity is affine in the level of diffusive variance like in the stochastic volatility model of Duffie et al. (2000) and subsequent empirical option pricing work. Our jump specification is a time-changed generalized variance gamma process, see Madan and Seneta (1990), with the time-change being the integrated diffusive variance. We remind the reader that the variance gamma jump process is a process of infinity jump activity. We allow for different parameters to control the negative and positive jumps. Throughout, we set \( \lambda_- = 30 \) and \( \lambda_+ = 100 \). This choice implies tail decays of out-of-the-money puts and calls similar to those
of observed options written on the S&P 500 index, see e.g., Andersen et al. (2015b). We set $c_\pm$ according to

$$c_- = 0.9 \times \lambda_-^2 \quad \text{and} \quad c_+ = 0.1 \times \lambda_+^2,$$

which implies that risk-neutral spot jump variation is equal to spot diffusive variance, and further that 90% of the jump variation is due to negative jumps. This separation of the risk-neutral variation into diffusive and one due to positive and negative jumps is similar to that implied from parametric models fitted to observed S&P 500 index options, see e.g., Andersen et al. (2015b).

In our model in the Monte Carlo, the periodic function is the identity. This allows us to obtain option prices in semi-closed form, which will not be the case otherwise.\(^2\) Our goal in the Monte Carlo is to investigate the effect from the mean reversion in the stochastic component of volatility, the discreteness and finite range of the strike grids and the option observation error on the precision in recovering the intraday periodic component of volatility. For this, a Monte Carlo setup with $\eta$ being the identity function should suffice.

We consider two parameter settings for the diffusive variance dynamics. In both of them, we set the mean of the variance to $\theta_v = 0.02$ and the correlation between the two Brownian motions to $\rho = -0.9$. These parameters match roughly estimates from prior empirical work for the risk-neutral mean of variance and the leverage effect. In the first of our two specifications, case S, we set $\kappa_v$ so that the half-life of a shock to variance is one month. In the second specification, case F, we increase the speed of mean reversion by setting $\kappa_v$ so that the half-life of a shock to variance is only one week. In both cases, we set the volatility of volatility parameter $\sigma_v$ so that the coefficient of variation of $V$ (which is given by $\sigma_v / \sqrt{2 \kappa_v \theta_v}$) is equal approximately to 0.35. The parameter values for the two cases are given in Table 1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Variance Parameters</th>
<th>Jump Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta_v$</td>
<td>$\kappa_v$</td>
</tr>
<tr>
<td>S</td>
<td>0.02</td>
<td>8.3</td>
</tr>
<tr>
<td>F</td>
<td>0.02</td>
<td>34.9</td>
</tr>
</tbody>
</table>

We turn next to the option observation scheme. Observed options are given by

$$\hat{O}_{t,T}(k_{t,T}(j)) = O_{t,T}(k_{t,T}(j))(1 + 0.015 \times z_{t,T}(j)), \quad j = 1, \ldots, N_{t,T},$$

(19)

\(^2\)We are grateful to Nicola Fusari for providing the option pricing codes used in the Monte Carlo section.
where \( \{z_{t,T}(j)\}_{j=1}^{N_{t,T}} \) are sequences of i.i.d. standard normal variables which are independent of each other. The size of the observation error is calibrated to match roughly bid-ask spreads of index option data around the money. The initial level of the stock price at the start of the day is set to 3500. For each pair \((t, T)\), the strikes are multiples of 5. The strikes below and above the current price are extended in both directions by increments of 5 until the true out-of-the-money option price falls below 0.075. This specification of the strike grid mimics that of available of S&P 500 index options. The starting value of the diffusive variance at the beginning of the day is set to 25-th, 50-th or 75-th quantile of its marginal distribution. Option prices are observed at 77 equidistant times in a trading day. This corresponds to sampling at a five-minute frequency over the period 9.35-15.55 EST in a trading day. Note that the option trading day at CBOE, where S&P 500 index options are traded, lasts over the period 9.30 – 16.15 EST. Therefore, in our case \( \Delta = 1/(252 \times 82) \). Finally, the time to maturities \( T_1^1 \) and \( T_1^2 \) are equal to \( 81/(252 \times 82) \) and \( 1/252 + 81/(252 \times 82) \), respectively.

### 4.2 Results

The results from the Monte Carlo are summarized in Figures 2 and 3. For both parameter settings and for all starting values of volatility, we can notice a monotone increase in the precision of the estimation, with the noisiest estimation being for the value of the periodic volatility function at the start of the trading day. The explanation for this is that the signal about the value of \( \eta \) at market open from the available options at the time is weakest relative to the noise in them. The signal is given by \( \int_{\phi_1}^{\phi_2} \eta_s ds \), which is the same for \( \phi_2 - \phi_1 \) constant because in our simulation \( \eta \) is the identity function. On the other hand, \( \int_{\phi_1}^{1} \eta_s ds \) increases when \( \phi_1 \) goes down, i.e., when we look at earlier parts of the trading day. Since option observation errors are proportional to option prices, this automatically translates in larger noise (in absolute terms) in \( \overline{QV}_{\phi \kappa,(1-\phi)} \kappa \) and \( \overline{QV}_{\phi \kappa,(2-\phi)} \kappa \) for lower values of \( \phi \). These two facts combined together can explain the observed time-of-day pattern in the precision in the recovery of \( \eta \). This strong dependence of the precision of the estimation on the time-of-day will carry over to the situation where one uses longer-tenor options than the ones used here to recover \( \eta \). That is, if one attempts estimation of \( \eta \) on the basis of options \( \overline{QV}_{\phi \kappa,(k-\phi)} \kappa \), for some \( k > 1 \), then the precision of such estimation would drop significantly relative to our estimator based on zero- and one-day tenor options.

Another observation from the reported Monte Carlo results is that the precision of the estimation appears slightly higher for higher values of the volatility. This difference is very small at can be seen at the end of the trading day. The explanation for that effect is that when volatility is higher,
Figure 2: Monte Carlo Results for Simulation Scenario S. For each time of day, we report 25-th, 50-th and 75-th quantile of the estimate of $\tilde{\eta}_{\phi_1,\phi_2}$ for $\phi_1\kappa$ and $\phi_2\kappa$ corresponding to consecutive observations on a 5-minute grid covering the time interval 9:35-15:55. The estimates are standardized by dividing by their average value across the times-of-day. The number of Monte Carlo replications is 3000.
Figure 3: Monte Carlo Results for Simulation Scenario F. For each time of day, we report 25-th, 50-th and 75-th quantile of the estimate of \( \tilde{b}_{\phi_1, \phi_2} \), for \( \phi_1 \kappa \) and \( \phi_2 \kappa \) corresponding to consecutive observations on a 5-minute grid covering the time interval 9:35-15:55. The estimates are standardized by dividing by their average value across the times-of-day. The number of Monte Carlo replications is 3000.
we have more option observations on average. Thus, in this case the option observation error can be more effectively “diversified away” in the option portfolios $\hat{QV}_{\phi\kappa,(1-\phi)\kappa}$ and $\hat{QV}_{\phi\kappa,(2-\phi)\kappa}$. This effect of the level of the volatility on the precision of $\hat{\eta}_\phi$ can be also seen by from the expression for its asymptotic variance given in Corollary 1. In fact, if volatility is very low, then the number of available options towards the end of the trading day for computing $\hat{QV}_{\phi\kappa,(1-\phi)\kappa}$ can become very low.

Finally, comparing the results in the two figures, we can detect little differences in the estimation in the two parameter settings. In particular, in both cases the medians of the estimates are not very different from their true values (recall in the Monte Carlo $\eta$ is the identity function). This suggests that the effect from mean-reversion in the stochastic component of volatility on the recovery of $\eta$ is negligible. This is, of course, due to the very short tenor of the options used in the estimation.

Overall the reported Monte Carlo results show that the intraday periodic component of volatility can be reliably estimated from short-dated options. While estimation using a single day of data is considerably noisy, pooling option data across several trading days should significantly improve the precision.

5 Empirical Application

5.1 Data

We apply the newly-developed estimator of the intraday diurnal volatility pattern to short-dated options written on the S&P 500 market index. Our sample covers the period from January 2, 2018 until December 31, 2020. The data is obtained from CBOE Data Shop and consists of best bid and best ask option quotes. Over our sample period, CBOE issued weeklies expiring on Monday, Wednesday and Friday of the week. Since in our analysis we use zero- and one-day tenor options (counting in business days), we thus look at intraday option data on every Friday which is not a holiday and consider the options on that day that expire either on the same day or on the next trading day. At each point in time and for each tenor, we determine an implied forward rate by using put-call parity for three distinct strikes with the smallest gap between call and put mid-quotes. Using the implied forward, we determine the moneyness of the options (defined as strike over forward) and keep only out-of-the-money options with non-zero bid quotes. We compliment the option data with intraday price records of the SPY exchange traded fund on the S&P 500 Index obtained from the NYSE Trade and Quote (TAQ) database. Our sampling frequency for both option and price data is every five minutes over the interval 9:35-15:55 EST.

In Figures 4 and 5, we provide key summary statistics for the option data used in our analysis.
As seen from Figure 4, the number of strikes of the out-of-the-money short-dated options declines over the trading day as the options get closer to expiration. This effect is very pronounced for the zero-day options and is barely noticeable for the one-day options. Nevertheless, the median number of available strikes at 15:55 for the zero-day options is still nontrivial making possible the calculation of $QV_{t,T}$. Not surprisingly, the number of available strikes for the one-day options is significantly higher as they have one more day to expiration.

In Figure 5, we plot the volatility-adjusted moneyness range of the available options. We use at-the-money Black-Scholes implied volatility for computing the volatility adjustment of the moneyness range. As seen from the figure, the moneyness range for the one-day options is stable during the day. It is pretty wide on the negative side due to the well-known pricing of short-term left tail risk. The available range for the zero-day options is somewhat smaller and we note that it starts gradually shrinking around 14:30, with most of the change due to reduction in the available strikes for out-of-the-money puts. This change can be in part rationalized with the fact that, while the minimum tick size for the options is constant in absolute value, it is nevertheless increasing in terms of total asset volatility until maturity as the options get closer to expiration. Nevertheless, the median strike range even at 15:55 is still relatively large, particularly when one takes into account the fact that Black-Scholes at-the-money implied volatility is higher than true volatility due to the risk premium embedded in it.

5.2 Empirical Results

Using the available data, we compute the risk-neutral variance measures and form our estimates of the intraday volatility pattern. In order to minimize the effect of the Riemann sum approximation error, we conduct linear interpolation in Black-Scholes implied volatility space to create option prices on equidistant strike grid with mesh of five (which is the typical gap between observed strikes) and use these options in the calculation of $\widehat{QV}_{t,T}$. We further remove from the analysis pairs $(t, T)$ with less than five observed out-of-the-money options. The estimates of $\eta$ are plotted in Figure 6. To minimize the effect from measurement error, we average the estimates across the trading days and report time series averages of $\widehat{\eta}_{\phi_1, \phi_2}$.

We compare our estimator of the diurnal intraday volatility pattern with one computed from high-frequency returns of the underlying asset. The later is defined as

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$^3$Another possible explanation for the observed shrinkage of the strike range of the available out-of-the-money puts towards the end of the day is that the degree of left-skew of the returns decreases with the horizon.
Figure 4: Number of Strikes of S&P 500 Index Options for 0-Day (upper) and 1-Day (lower) Tenor. For each time of day, we report 25-th, 50-th and 75-th quantile of the out-of-the-money strike counts across all days in the sample on a 5-minute grid covering the time interval 9:35-15:55.

\[
\hat{\eta}_{\theta_1, \theta_2}^{ret} = \sum_{i=\lfloor \phi_1 \kappa / \Delta \rfloor + 1}^{\lfloor \phi_2 \kappa / \Delta \rfloor} \sum_{j=1}^{T} \left( x(j-1)_{\kappa} + i \Delta - x(j-1)_{\kappa} + (i-1) \Delta \right)^2 \div \sum_{i=1}^{\lfloor \kappa / \Delta \rfloor} \sum_{j=1}^{T} \left( x(j-1)_{\kappa} + i \Delta - x(j-1)_{\kappa} + (i-1) \Delta \right)^2, \tag{20}
\]
Figure 5: **Moneyness Range of S&P 500 Index Options for 0-Day (upper) and 1-Day (lower) Tenor.** For each time of day, we report 25-th, 50-th and 75-th quantile of the volatility-adjusted moneyness across all days in the sample on a 5-minute grid covering the time interval 9:35-15:55. The moneyness is volatility-adjusted using Black-Scholes at-the-money implied volatility.

where $0 \leq \phi_1 < \phi_2 < 1$ and $\mathcal{T}$ is some integer denoting the number of trading days used in the estimation. We note that we do not perform truncation of the returns because the diffusive and jump volatility share the same intraday diurnal pattern per our assumption in Section 2. We use
Figure 6: **Intraday Periodic Volatility Pattern of S&P 500 Index Volatility.** For each time of day, we report the time series average of the estimate of $\hat{\eta}_{1,\kappa}$ and $\hat{\phi}_{1,\kappa}$ corresponding to consecutive observations on a 5-minute grid covering the time interval 9:35-15:55. Top panel corresponds to the option-based estimate and the bottom one to the return-based one. The estimates are standardized by dividing by their average value across the times-of-day. The solid lines correspond to a tenth-order polynomial fit to the time-of-day estimates.
the days with option observations for computing $\tilde{\eta}_{\phi_1, \phi_2}^{ret}$.

We can make several observations about the reported estimates. First, consistent with our simulation results, the option-based estimates at the beginning of the day are noisier than the ones for later on in the trading day. Second, the estimated intraday volatility pattern has the familiar U-shape. Relative to prior estimates, however, see e.g., Figure 1 in Andersen et al. (2019), the option-implied estimate of $\eta$ is right-skewed. This is mostly due to the high estimated value of volatility at 15:55. Third, the return-based estimate of the intraday volatility pattern has similar shape as that recovered from the options but it appears noisier. Indeed, the realized quadratic variation based on the estimates of $\tilde{\eta}_{\phi_1, \phi_2}^{ret}$ is twice that of $\tilde{\eta}_{\phi_1, \phi_2}$. Similarly, with the return-based estimate of $\eta$, the evidence for the increased volatility towards the end of the trading day is harder to detect. Indeed, the estimated level of volatility at 15:50 is high but this is followed by a much lower estimate of volatility at 15:55, which is likely due to noise.

Overall, the reported estimates $\eta$ from the short-dated options illustrate the ability of such data to extract information for the intraday behavior of volatility. The recent introduction of more very short-tenor options by exchanges should facilitate the application of such an analysis and in particular the study of variation in the intraday volatility pattern over time.

6 Conclusion

In this paper we propose an option-based nonparametric estimator of the deterministic intraday periodic volatility pattern. The estimator is based on short-dated options written on the underlying asset, expiring at the end of the day option prices are observed at and at the end of the following trading day. Unlike existing nonparametric methods for estimating intraday diurnal volatility patterns from high-frequency returns, the option-based method does not require the use of long time series of data (and the associated long-span asymptotics and assumptions needed for it). Instead, it relies on short-time asymptotics and options with two different times-to-maturity to disentangle the stochastic component of volatility from its deterministic intraday periodic component. We derive the rate of convergence and an associated CLT for our estimator. A Monte Carlo study and an empirical application using S&P 500 index options show the applicability of the newly-developed estimation procedure.
7 Assumptions and Proofs

7.1 Assumptions

The process \( x \) is defined on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q}) \). We now state our assumptions for the dynamics of \( x \) which we need for the asymptotic analysis. Since we use different probability measures, to avoid confusion in the statements that follow, we will denote with \( \mathbb{E}^\mathbb{Q} \) the expectation under \( \mathbb{Q} \) and similarly \( \mathbb{E}^\mathbb{Q}_t \) will be the \( \mathcal{F}_t \)-conditional expectation under \( \mathbb{Q} \). We will not use superscripts for the counterparts of these expectations under \( \mathbb{P} \). Our assumptions are as follows:

\textbf{A1.} For \( s \leq t \) in a neighborhood of 0, we have
\[
\mathbb{E}^\mathbb{Q}_s |\sigma_t^2 - \sigma_s^2|^2 \leq C_s |t - s|,
\]
and
\[
\mathbb{E}^\mathbb{Q}_s \left| \int_\mathbb{R} (e^x - 1 - x) \nu_s(dx) - \int_\mathbb{R} (e^x - 1 - x) \nu_s(dx) \right|^2 \leq C_s |t - s|,
\]
for some process \( C_t \) with càdlàg paths.

\textbf{A2.} For \( t \) in a neighborhood of 0, there exist \( \mathcal{F}^{(0)}_t \)-adapted random variables \( C_t \) and \( \overline{t} > t \) such that for \( s \in [t, \overline{t}] \):
\[
\mathbb{E}^\mathbb{Q}_t |\alpha_s|^4 + \mathbb{E}^\mathbb{Q}_t |\sigma_s|^6 + \mathbb{E}^\mathbb{Q}_t (e^{4|x|}) + \mathbb{E}^\mathbb{Q}_t \left( \int_\mathbb{R} (e^{3|x|} - 1) \nu_s(dx) \right)^4 < C_t.
\]

\textbf{A3.} For \( i = 1, \ldots, n \), the log-strike grids \( \{k_{i,0}(j)\}_{j=1}^{N_{i,0}} \) and \( \{k_{i,1}(j)\}_{j=1}^{N_{i,1}} \) are \( \mathcal{F}^{(0)} \)-adapted and we have
\[
c_{i,t} \Delta \leq k_{i,l}(j) - k_{i,l}(j - 1) \leq C_{i,t} \delta, \quad l = 0, 1, \quad \text{as } \delta \downarrow 0,
\]
where \( \delta \) is a deterministic sequence, and \( c_t \) and \( C_t \) are \( \mathcal{F}^{(0)} \)-adapted processes with càdlàg paths with \( \inf_t c_t > 0 \). In addition, for some arbitrary small \( \zeta > 0 \):
\[
\sup_{j:|k_{i,l}(j) - x_{t_i}| < \zeta} \left| \frac{k_{i,l}(j) - k_{i,l}(j - 1)}{\delta} - \psi_{i,l}(k_{i,l}(j - 1) - x_{t_l}) \right| \xrightarrow{\mathbb{P}} 0, \quad l = 0, 1, \quad \text{as } \delta \downarrow 0,
\]
where \( \psi_{i,l}(k) \) are \( \mathcal{F}^{(0)} \)-adapted functions which are continuous in \( k \) at 0.

\textbf{A4.} We have \( \epsilon_{i,l}(k_{i,l}(j)) = \zeta_{i,l}(k_{i,l}(j) - x_{t_{i,l}}) \tau_{i,l,j} O_{i,l}(j) \) for \( l = 0, 1 \), where for \( k \) in a neighborhood of zero, we have \( |\zeta_{i,l}(k) - \zeta_{i,l}(0)| \leq C_{i,l} |k|^4 \), for some \( \iota > 0 \) and \( C_t < \infty \) being an \( \mathcal{F}^{(0)} \)-adapted process with càdlàg paths. The two sequences \( \{\tau_{i,0,j}\}_{j=1}^{N_{i,0}} \) and \( \{\tau_{i,1,j}\}_{j=1}^{N_{i,1}} \) are defined on \( \mathcal{F}^{(1)} \), are i.i.d. and...
independent of each other and of $\mathcal{F}^{(0)}$. We further have $\mathbb{E}(\bar{\tau}_{i,l,j}|\mathcal{F}^{(0)}) = 0$, $\mathbb{E}((\bar{\tau}_{i,l,j})^2|\mathcal{F}^{(0)}) = 1$ and $\mathbb{E}(|\bar{\tau}_{i,l,j}|^\kappa|\mathcal{F}^{(0)}) < \infty$, for some $\kappa \geq 4$ and $l = 0,1$.

7.2 Proof of Theorem 1

We will make use of the following lemma in the proof:

Lemma 1. Assume $A$ and $C1$ hold and let $x_t = 0$. There exists $\mathcal{F}^{(0)}_t$-adapted random variables $C_t$ and $\bar{t} > 0$ that do not depend on $T$ such that for $T < \bar{t}$, we have

$$O_{t,T}(K) \leq C_t \left( TK^21_{\{K<e^{-1}\}} + TK^{-1}1_{\{K>e\}} + \left( \sqrt{T} \wedge \frac{T}{\log(K)} \right) 1_{\{\log(K) < 1\}} \right),$$

$$|O_{t,T}(K_1) - O_{t,T}(K_2)| \leq C_t \left[ \frac{T}{\log(K_2)^4} \wedge \frac{T}{\log(K_2)^{4/3} \wedge 1} \right] |K_1 - K_2|,$$

where $K_1 < K_2 < 1$ or $K_1 > K_2 > 1$. In addition, for $|\log(K)| \leq \sqrt{T} |\log(T)|$ and some $\iota > 0$ arbitrary small:

$$|O_{t,T}(K) - \sqrt{\eta_{T,T}}\sigma_t f \left( \frac{\log(K)}{\sqrt{\eta_{T,T}}\sigma_t} \right) - |k|\Phi \left( -\frac{\log(K)}{\sqrt{\eta_{T,T}}\sigma_t} \right) | \leq C_t T^{1-\iota},$$

where $\eta_{T,T} = \int_t^{t+T} \eta_{s/\kappa - [s/\kappa]} ds$.

Proof of Lemma 1 The first two results of the lemma follow from Lemma 1 of Todorov (2019). We establish the last one. Denote with $O_{t,T}^c(K)$ corresponding to log-price $x_t^c = \sigma_t \int_t^s \sqrt{\eta_{u/\kappa - [u/\kappa]}} dW_u$, for $s \in [t, t + T]$, instead of $x_t$. Then, using Hölder inequality, we have that $|O_{t,T}^c(K) - O_{t,T}(K)| \leq C_t T^{1-\iota}$. Now, we note that $\eta$ is a deterministic function. Hence, $x_{t+T}^c - x_t^c$ is $\mathcal{F}_t$-conditionally normal with mean zero and variance $\sigma_t^2 \int_t^{t+T} \eta_{u/\kappa - [u/\kappa]} du$. From here, the proof follows exactly as that of the last one from Lemma 2 of Todorov (2019).

We proceed with the proof of the theorem. We can make the decomposition

$$\widehat{QV}_{t,T} - QV_{t,T} = \widehat{Z}_{t,T} + R_{t,T},$$

where

$$\widehat{Z}_{t,T} = \sum_{j=2}^{N_{t,T}} \frac{2}{K_{t,T}^2(j-1)} \bar{\tau}_{t,T}(j-1)(K_{t,T}(j) - K_{t,T}(j-1)),$$

and

$$R_{t,T} = -2 \int_0^{K_{t,T}(1)} \frac{O_{t,T}(K)}{K^2} dK - 2 \int_{K_{t,T}(N_{t,T})}^\infty \frac{O_{t,T}(K)}{K^2} dK$$

$$+ 2 \sum_{j=2}^{N_{t,T}} \int_{K_{t,T}(j-1)}^{K_{t,T}(j)} \left( \frac{O_{t,T}(K)}{K_{t,T}^2(j-1)} - \frac{O_{t,T}(K)}{K_{t,T}^2(j)} \right) dK.$$

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Using the first two bounds of Lemma 1, we have

$$R_{t,T} = O_p \left( (K_{t,T}^2(1) + K_{t,T}^{-2}(N_{t,T}))\kappa + \delta \sqrt{\kappa} \right).$$  \hspace{1cm} (32)$$

Next, let us set \( \hat{Z}_{\phi}^{(l)} = \hat{Z}_{t_l,T_l}^{\phi} \) for \( i = \left\lceil \frac{\phi \kappa}{\Delta} \right\rceil \) and \( l = 0, 1 \), and write

$$\hat{Z}_{\phi}^{(l)} = \sum_{i=2}^{N_{\phi}} z_{i,\phi}^{(l)}. \hspace{1cm} (33)$$

Using our assumption for the observation error and Lemma 1, we have

$$E \left( z_{l,\phi}^{(l)} | \mathcal{F}^{(0)} \right) = 0, \sum_{i=2}^{N_{\phi}} E \left( |z_{l,\phi}^{(l)}|^{2+i} | \mathcal{F}^{(0)} \right) = O_p \left( \kappa^{3/2+i/2} \delta^{1+i} \right), \hspace{1cm} (34)$$

for some \( \iota \in (0, 1) \) and \( l = 0, 1 \). Therefore, we will have \( \frac{1}{\sqrt{\delta \kappa^{3/4}}} (\hat{Z}_{\phi}^{(1)}, \hat{Z}_{\phi}^{(2)}) \overset{L^{[\mathcal{F}^{(0)}]}}{\rightarrow} (Z_{\phi,1}, Z_{\phi,2}) \), by Theorem VIII.5.25 of Jacod and Shiryaev (2003), if we can establish the following convergence

$$\frac{1}{\delta \kappa^{3/2}} \sum_{i=2}^{N_{\phi}} E \left( |z_{l,\phi}^{(l)}|^{2} | \mathcal{F}^{(0)} \right) \overset{p}{\rightarrow} \Sigma_{\phi,l}, \ l = 0, 1. \hspace{1cm} (35)$$

This convergence can be shown by applying Lemma 1 and using assumptions A1 and A4.

In order to establish the result of the theorem, we are left with evaluating the difference \( QV_{[\phi \kappa / \Delta] \Delta, (l+1) \kappa - [\phi \kappa / \Delta] \Delta} - QV_{\phi \kappa, (l+1 - \phi) \kappa} \). We trivially have

$$QV_{[\phi \kappa / \Delta] \Delta, (l+1) \kappa - [\phi \kappa / \Delta] \Delta} - QV_{\phi \kappa, (l+1 - \phi) \kappa} = O_p(\Delta), \ l = 0, 1. \hspace{1cm} (36)$$

Taking into account the rate condition in the theorem involving \( \Delta \), we get the result of the theorem.

### 7.3 Proof of Theorem 2

Using assumption A1 and the fact that the function \( \eta \) is deterministic, we have

$$E_{\phi \kappa}^{Q} \left( \int_{\phi \kappa}^{\kappa} \eta_{s/s-\kappa} \kappa \left( \sigma_{s}^{2} + 2 \int_{\mathbb{R}} (e^{x} - 1 - x) \nu_{s}(dx) \right) ds \right) - \kappa \int_{\phi \kappa}^{1} \eta_{s} ds \left( \sigma_{\phi \kappa}^{2} + \int_{\mathbb{R}} (e^{x} - 1 - x) \nu_{\phi \kappa}(dx) \right) = O_p(\kappa^{2}). \hspace{1cm} (37)$$

Similar result holds for \( QV_{\phi \kappa, (2 - \phi) \kappa} \). From here, the result of the theorem follows.


