Testing and Inference for Fixed Times of Discontinuity in Semimartingales

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We develop a nonparametric test for deciding whether a semimartingale process, modeling an asset price, contains a fixed time of discontinuity, i.e., a positive probability of a jump, at a given point in time, and we further propose a rate-optimal estimator of the jump distribution when this is the case. Itô semimartingales used commonly in applied work have absolutely continuous in time, with respect to Lebesgue measure, jump compensators, and this rules out fixed times of discontinuity in their paths. However, certain phenomena, such as scheduled economic announcements in finance, make the existence of such discontinuities a possibility. The inference in the paper is based on noisy observations of options written on the asset with different strikes and two different expiration dates. The asymptotics is joint in which the times to maturity of the options shrink to zero and the number of observed options increases to infinity. The test is based on estimates of the characteristic function of the increments of the semimartingale, constructed from the option data, and the fact that the asymptotic limit of the increments and their characteristic functions is different with and without fixed time of discontinuity. The limit distribution of the test statistic is derived and feasible inference is developed on the basis of wild bootstrap type techniques. A Monte Carlo and an empirical illustration show the applicability of the developed inference procedures.

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1. Introduction

Our interest in this paper is in the jump part of the following semimartingale process used to model the dynamics of an asset price X:

$$\frac{dX_t}{X_{t-}} = a_t dt + \sigma_t dW_t + \int_{\mathbb{R}} (e^z - 1)\widetilde{\mu}(ds, dz), \tag{1.1}$$

where a is a process with càdlàg paths, W is a Brownian motion, μ is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}$, counting the jumps in X, with compensator $\nu(dt, dz)$ and $\widetilde{\mu}$ is the martingale measure associated with μ. For the above process, we develop tests for deciding whether the realization of $\nu(\{t^*\}, \mathbb{R}) = \mathbb{P}_{t^*-}(\Delta X_{t^*} \neq 0)$ is positive, for some fixed time t^*. That is, we propose a test for deciding whether there is a positive probability of jump arrival at a given point in time, i.e., whether X contains fixed times of discontinuity. We further propose a rate-efficient estimator of the density of ν at t^*.

The standard models used in many applications, e.g., the class of Lévy processes, time-changed Lévy models with absolutely continuous time change, and more generally Itô semimartingales imply $\mathbb{P}_{t^*-}(\Delta X_{t^*} \neq 0) = 0$ for any deterministic time t^*. For example, the jump compensator of the classical Lévy process is of the form $dt \otimes F(dz)$, for some measure F satisfying $\int_{\mathbb{R}}(z^2 \wedge 1)F(dz) < \infty$, implying time homogeneity and hence no atoms in time. That is, for these processes the jump arrivals on a given time interval are uniformly distributed on the interval. However, for the modeling of various phenomena, it can be more natural for the process to have fixed times of discontinuity. For example, in finance for different reasons such as predetermined releases of information or periods of market closure, one might expect that jumps with fixed arrival time are built into asset prices.

How can we test whether $\mathbb{P}_{t^*-}(\Delta X_{t^*} \neq 0) > 0$? Obviously, from observing the jump times of X, we cannot decide whether these are times of fixed discontinuity. Similarly, if X does not jump at a given time on a given path, this does not mean that this time point is not one of fixed time of discontinuity. Our identification and testing for the presence of such times will be based on option prices written on the asset price X. The option prices will let us study directly the jump compensator of X in a way that we describe below, and hence they will allow us to identify whether the latter is strictly positive at a given fixed point in time.

More specifically, using nonparametric techniques and following the results of Carr and Madan (2001), we can recover from option prices written at time t and expiring at time $t+T$ the conditional characteristic function of the increment of $x = \log(X)$, i.e., $\mathbb{E}_t\left(e^{iu(x_{t+T}-x_t)}\right)$, for $u \in \mathbb{R}$. Then suppose that x is a semimartingale process with independent increments (Jacod and Shiryaev (2003), Definition II.4.1) with fixed time of discontinuity at $t^* \in (t, t+T)$, and further that the increments are stationary outside the jump time t^*. In this case, we can write (see Jacod and Shiryaev (2003), Theorem II.4.15):

$$\mathbb{E}_t\left(e^{iu(x_{t+T}-x_t)}\right) = e^{T\psi(u)}\left(1 + \int_{\mathbb{R}}(e^{iuz}-1)\nu(\{t^*\}, dz)\right), \tag{1.2}$$

for some function $\psi : \mathbb{R} \to \mathbb{C}$. This means that the characteristic exponent is proportional to the length of the time interval T whenever there is no fixed time of discontinuity in X. Therefore, we can construct a test for the latter scenario by using observations of options at time $t < t^*$ with two different times to maturity $T_1 < T_2$, such that $t^* < t + T_2$, and forming estimates of the difference $\mathbb{E}_t\left(e^{iu(x_{t+T_2}-x_t)}\right) - \left(\mathbb{E}_t\left(e^{iu(x_{t+T_1}-x_t)}\right)\right)^{T_2/T_1}$.

While the above discussion is for a semimartingale process with independent increments which are further stationary outside of the fixed time of discontinuity, it can be easily extended to the general semimartingale setting by letting T_1 and T_2 shrink to zero. This way, the effect of the variation of the semimartingale characteristics will be of higher asymptotic order (with the precise assumptions needed for this provided in the main text) and therefore the above-described statistics can still be used for the purposes of testing for fixed times of discontinuity in this more general context.

In our setup, the number of options with different strikes increases asymptotically at the same time as the maturities of the options shrink. The separation of the null from alternative hypothesis occurs because of the different asymptotic order of the options and the characteristic function of the price increments with and without fixed times of discontinuity in the underlying asset price. The characteristic function of the price increments is of asymptotic order proportional to the length of the increment (which is shrinking) under the null and it is of asymptotic order one under the alternative.

We derive a functional Central Limit Theorem (CLT) for our characteristic function estimates from the option data, and the associated test statistic, under the null of no fixed time of discontinuity in a weighted L_2 space. The rate of convergence of the statistic depends both on the length of the time to maturity of the options as well as on the mesh of the strike grid, both of which are asymptotically shrinking. The asymptotic variance of our estimators is determined by the diffusive volatility of the underlying asset price at the time of observing the option prices as well as the heteroskedastic volatility of the observation error, which is left unspecified. We develop an easy-to-implement wild bootstrap type method for doing feasible inference which consists of regenerating new option prices with error on the basis of noisy estimates of the heteroskedastic variance of the observation error. The developed limit theory should be of independent interest for conducting inference for models for the underlying asset and the option prices written on it.

We further propose a nonparametric rate-efficient estimator of the jump distribution at time t^*, when t^* is a point of fixed discontinuity in the underlying asset price. The estimator is based on recovering the characteristic function of the price increment from options which expire after t^*, and an appropriate bias correction formed from the options with the different times-to-maturity to correct for the Itô semimartingale component of the price outside of t^*. Unlike the case of the null hypothesis, now the options and the associated errors are not asymptotically shrinking in spite of the shrinking options' time-to-maturity. The error in the density recovery depends both on the shrinking maturity (because of the bias due to the Itô semimartingale component of the price increment) as well as the mesh of the observation grid (due to the observation error).

The current paper relates to several strands of existing work. First, Belomestny and Reiß (2006, 2015), Cont and Tankov (2004), Söhl (2014), Söhl and Trabs (2014) and Trabs (2014, 2015) propose rate-efficient estimators of the Lévy density from options with fixed time to maturity in exponential Lévy models and Qin and Todorov (2019) propose rate-efficient estimators of the Lévy density of Itô semimartingales in a setting with shrinking maturity of the options. Unlike these papers, we derive a functional CLT for estimates of the characteristic function of the underlying process and develop novel wild bootstrap for conducting feasible inference. Both of these results are nonstandard because of the different asymptotic order of the option prices that are used in the computation of the statistic. Another difference between the above-cited papers and the current work is that here we derive rate-efficient estimators at times of fixed discontinuity which is not allowed for in the setup of the above-cited work. Our estimator combines features of both asymptotic setups, with and without shrinking maturity of the options.

Second, our work is related to studies of the asymptotic behavior of the option price written on an underlying Itô semimartingale as their maturity shrinks, see e.g., Andersen et al. (2017), Bentata and Cont (2012), Euch et al. (2018), Figueroa-Lopez et al. (2012), Figueroa-Lopez and Olafsson (2016a,b), Fukasawa (2017), Medvedev and Scaillet (2006), Mijatović and Tankov (2016) and Muhle-Karbe and Nutz (2011), and the many references therein. Unlike this strand of work, we consider fixed times of discontinuity in the underlying process, allow for observation errors in the option prices (which drive our CLT), and integrate options with different strikes in the analysis (which is challenging because of their different asymptotic order as the maturity of the options shrinks).

The rest of the paper is organized as follows. In Section 2 we present the formal setup and state the assumptions. Section 3 formulates the test statistic and in Section 4 we analyze its asymptotic behavior. Section 5 presents a density estimator for the jump distribution at the fixed time of discontinuity. Sections 6 and 7 contain a Monte Carlo study and an empirical application, respectively. The proofs are given in Section 8.

2. Setup and Assumptions

The process X is defined on a filtered probability space $\big(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t^{(0)})_{t\geq 0}, \mathbb{P}^{(0)}\big)$. As noted in the introduction, our inference in this paper will be based on European style options written on X, and from finance theory, see e.g., Duffie (2001), and in the absence of arbitrage, their theoretical values equal their conditional expected future discounted payoffs (given in (2.1) below) under the so-called risk-neutral probability, which we henceforth denote with $\mathbb{Q}$. The latter is locally equivalent to the true probability measure and is of major interest both theoretically and for applications. The dynamics of X under $\mathbb{Q}$ is given in (1.1) above.

For ease of exposition, we will assume that the dividend yield of X and the risk-free interest rate are both zero. As in the introduction, henceforth we denote the logarithm of the underlying asset price with $x = \log(X)$. With these normalizations, the theoretical values of the option prices we will use in our analysis are given by

$$O_{t,T}(k) = \begin{cases} \mathbb{E}_t^{\mathbb{Q}}(e^k - e^{x_{t+T}})^+, & \text{if } k \leq x_t, \\ \mathbb{E}_t^{\mathbb{Q}}(e^{x_{t+T}} - e^k)^+, & \text{if } k > x_t, \end{cases} \tag{2.1}$$

where $K \equiv e^k$ and k are the strike and log-strike, respectively, of the option. $O_{t,T}(k)$ is the price of an out-of-the-money option, i.e., an option which will be worth zero if it were to expire today. This is a call contract (an option to buy the asset) if $k > x_t$ and a put contract (an option to sell the asset) if $k \leq x_t$.

Our data will consist of two sets of out-of-the-money options both observed at time t, with one set expiring at time $t + T_1$ and the other one at $t + T_2$, for some $0 < T_1 < T_2$. The log-strike grid of the observed options is given by

$$k_{l,1} < k_{l,2} < \cdots < k_{l,N_l}, \quad l = 1, 2. \tag{2.2}$$

We further denote

$$\underline{k} = k_{1,1} \wedge k_{2,1} \quad \text{and} \quad \overline{k} = k_{1,N_1} \vee k_{2,N_2}, \tag{2.3}$$

and we set $\underline{K} = \exp(\underline{k})$ and $\overline{K} = \exp(\overline{k})$. Finally, as common in empirical asset pricing, we allow for observation errors, i.e., we observe:

$$\widehat{O}_{t,T_l}(k_{l,j}) = O_{t,T_l}(k_{l,j}) + \epsilon_{t,T_l}(k_{l,j}), \quad j = 1, ..., N_l, \quad l = 1, 2, \tag{2.4}$$

where the sequence of observation errors is defined on a space $\Omega^{(1)} = \bigtimes_{k\in\mathbb{R}} \mathcal{A}_k$, for $\mathcal{A}_k = \mathbb{R}$. This space is equipped with the product Borel σ-field $\mathcal{F}^{(1)}$ and with transition probability $\mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)})$ from the original probability space $\Omega^{(0)}$ – on which X is defined – to $\Omega^{(1)}$. We further define,

$$\Omega = \Omega^{(0)} \times \Omega^{(1)}, \quad \mathcal{F} = \mathcal{F}^{(0)} \times \mathcal{F}^{(1)}, \quad \mathbb{P}(d\omega^{(0)}, d\omega^{(1)}) = \mathbb{P}^{(0)}(d\omega^{(0)})\mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)}).$$

Below we state our assumptions for the dynamics of X as well as the option observation scheme for which we make use the following additional notation. The compensator of the jumps for $t \neq t^*$ is of the form

$$\nu(dt, dz) = dt \otimes F_t(z)dz + \epsilon_{t^*}(dt)G_{t^*}(z)dz, \tag{2.5}$$

where ϵ_a denotes the Dirac measure at a, and F_t and G_{t^*} are some predictable functions.

A1. *The process σ has the following dynamics under $\mathbb{Q}$:*

$$\sigma_t = \sigma_0 + \int_0^t b_s ds + \int_0^t \eta_s dW_s + \int_0^t \widetilde{\eta}_s d\widetilde{W}_s + \int_0^t \int_{\mathbb{R}} \delta^\sigma(s, u)\mu^\sigma(ds, du), \tag{2.6}$$

where $\widetilde{W}$ is a Brownian motion independent of W; μ^σ is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\nu^\sigma(ds, du) = ds \otimes du + \epsilon_{t^*}(ds)\nu^\sigma_{t^*}(du)$, having arbitrary dependence with the random measures μ, and for some measure $\nu^\sigma_{t^*}$; b, η and $\widetilde{\eta}$ are processes with càdlàg paths and $\delta^\sigma(s, u) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is left-continuous in its first argument. In addition, $\inf_{t\in[t^*-\varepsilon,t^*]} \sigma_t > 0$, for some arbitrary small $\varepsilon > 0$ and $\nu^\sigma_{t^*}$ is identically zero if $\nu(\{t^*\}, \mathbb{R}) = 0$.

A2. *With the notation of A1, for $t \in [t^* - \varepsilon, t^*]$ with $\varepsilon > 0$ arbitrarily small, there exist $\mathcal{F}^{(0)}_t$-adapted random variables C_t and $\overline{t} > t$ such that for $s \in [t, \overline{t}]$:*

$$\mathbb{E}^{\mathbb{Q}}_t|a_s|^4 + \mathbb{E}^{\mathbb{Q}}_t|\sigma_s|^6 + \mathbb{E}^{\mathbb{Q}}_t(e^{4|x_s|}) + \mathbb{E}^{\mathbb{Q}}_t\left(\int_{\mathbb{R}}(e^{3|z|} - 1)F_s(z)dz\right)^4 < C_t, \tag{2.7}$$

and in addition for some $\iota > 0$

$$\mathbb{E}^{\mathbb{Q}}_t\left(\int_{\mathbb{R}}(|\delta^\sigma(s, z)|^4 \vee |\delta^\sigma(s, z)|)dz\right)^{1+\iota} \leq C_t. \tag{2.8}$$

Furthermore, we have $\sup_{t\in[t^*-\varepsilon,t^*]} C_t < \infty$ *and* $\inf_{t\in[t^*-\varepsilon,t^*]} \overline{t} > t^*$.

A3. *With the notation of A1, for* $t \in [t^*-\varepsilon, t^*]$ *with* $\varepsilon > 0$ *arbitrarily small, there exist* $\mathcal{F}_t^{(0)}$*-adapted random variables* C_t *and* $\overline{t} > t$ *such that if* $\nu(\{t^*\}, \mathbb{R}) = 0$*, we have for* $s, r \in [t, \overline{t}]$*:*

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}|a_s - a_r|^p + \mathbb{E}_t^{\mathbb{Q}}|\sigma_s - \sigma_r|^p + \mathbb{E}_t^{\mathbb{Q}}|\eta_s - \eta_r|^p + \mathbb{E}_t^{\mathbb{Q}}|\widetilde{\eta}_s - \widetilde{\eta}_r|^p \\ \leq C_t|s-r|, \ \forall p \in [2,4], \end{aligned} \tag{2.9}$$

and

$$\mathbb{E}_t^{\mathbb{Q}} \left(\int_{\mathbb{R}} (e^{z\vee 0}|z| \vee |z|^2)|F_s(z) - F_r(z)|dz \right)^p \leq C_t|s-r|, \ \forall p \in [2,3]. \tag{2.10}$$

Furthermore, we have $\sup_{t\in[t^*-\varepsilon,t^*]} C_t < \infty$ *and* $\inf_{t\in[t^*-\varepsilon,t^*]} \overline{t} > t^*$.

A4. *We have* $\int_{\mathbb{R}} (e^z - 1) G_{t^*}(z)dz = 0$*. For* $t \in [t^*-\varepsilon, t^*)$ *with* $\varepsilon > 0$ *arbitrarily small, there exists* $\mathcal{F}_{t-}^{(0)}$*-adapted random variable* C_t *such that for* $s, r \in [t, t^*]$ *with* $s < r \leq t^*$*:*

$$\mathbb{E}_{t-}^{\mathbb{Q}} \left| \mathbb{E}_{s-}^{\mathbb{Q}}(e^{iu\Delta x_{t^*}}) - \mathbb{E}_{r-}^{\mathbb{Q}}(e^{iu\Delta x_{t^*}}) \right|^2 \leq C_t(|u|^p \vee 1)|s-r|, \ \text{ for some } p > 0, \tag{2.11}$$

where $\sup_{t\in[t^*-\varepsilon,t^*]} C_t < \infty$*. In addition, for* $t \in [t^*-\varepsilon, t^*)$*, we have* $\mathbb{E}_t^{\mathbb{Q}}(e^{iu\Delta x_{t^*}}) = \mathbb{E}_{t-}^{\mathbb{Q}}(e^{iu\Delta x_{t^*}})$*, almost surely.*

A5. *The log-strike grids* $\{k_{l,j}\}_{j=1}^{N_l}$*, for* $l = 1, 2$*, are* $\mathcal{F}^{(0)}$*-adapted and we have*

$$c_t\Delta \leq k_{l,j} - k_{l,j-1} \leq C_t\Delta, \quad l = 1, 2, \ \text{ as } \Delta \downarrow 0, \tag{2.12}$$

where Δ is a deterministic sequence, and $\inf_{t\in[t^*-\varepsilon,t^*]} c_t > 0$ and $\sup_{t\in[t^*-\varepsilon,t^*]} C_t < \infty$, for some arbitrary small $\varepsilon > 0$. In addition, for some arbitrary small $\zeta > 0$:

$$\sup_{j:|k_{l,j}-x_t|<\zeta} \left| \frac{k_{l,j} - k_{l,j-1}}{\Delta} - \psi_l(k_{l,j-1} - x_t) \right| \ \xrightarrow{\mathbb{P}} \ 0, \ \ l = 1, 2, \ \text{ as } \Delta \downarrow 0, \tag{2.13}$$

where $\psi_l(k)$ *are* $\mathcal{F}^{(0)}$*-adapted functions which are continuous in* k *at* 0 *with* $\psi_l(0) > 0$.

A6. *We have* $\epsilon_{t,T_l}(k_{l,j}) = \xi_{t,l}(k_{l,j} - x_t)\overline{\epsilon}_{t,l,j}O_{t,T_l}(k_{l,j})$ *for* $l = 1, 2$*, where* $\xi_{t,l}(0)$ *is continuous in* t *at* t^* *and further for* k *in a neighborhood of zero, we have* $|\xi_{t,l}(k) - \xi_{t,l}(0)| \leq C_t|k|^\iota$*, for some* $\iota > 0$ *and* $\sup_{t\in[t^*-\varepsilon,t^*]} C_t < \infty$ *as well as* $\sup_{t\in[t^*-\varepsilon,t^*]} \sup_{k\in\mathbb{R}} |\xi_{t,l}| < \infty$*, with some arbitrary small* $\varepsilon > 0$*. For* $l = 1, 2$*,* $\overline{\epsilon}_{t,l,j} = \sum_{m=0}^{M} \psi_{t,l,m}\zeta_{t,l,j-m}$*, with* $\{\psi_{t,l,m}\}_{m=1}^{M}$ *being an* $\mathcal{F}^{(0)}$*-adapted sequence, with* $\psi_{t,l,m}$ *continuous in* t *at* t^**, and* $\{\zeta_{t,l,j}\}_{j=1-m}^{N_l}$ *being an i.i.d. sequence defined on an extension of* $\mathcal{F}^{(0)}$ *and independent of it, and for some nonnegative integer* M*. The sequences* $\{\zeta_{t,1,j}\}_{j=1-m}^{N_1}$ *and* $\{\zeta_{t,2,j}\}_{j=1-m}^{N_2}$ *are independent* *from each other and have arbitrary dependence on t. We further have $\mathbb{E}(\bar{\epsilon}_{t,l,j}|\mathcal{F}^{(0)}) = 0$, $\mathbb{E}((\bar{\epsilon}_{t,l,j})^2|\mathcal{F}^{(0)}) = 1$ and $\mathbb{E}(|\bar{\epsilon}_{t,l,j}|^\kappa|\mathcal{F}^{(0)}) < \infty$, for some $\kappa \geq 4$ and $l = 1, 2$.*

In the case $\nu(\{t^*\}, \mathbb{R}) = 0$, assumption A1 specifies that σ is an Itô semimartingale, which is the standard way of modeling stochastic volatility in applied work. In the case when x can has a fixed time of discontinuity at $t = t^*$ (i.e., when $\nu(\{t^*\}, \mathbb{R}) > 0$), assumption A1 allows the volatility process σ to have a fixed time of discontinuity at $t = t^*$ as well. We impose non-vanishing σ_t, for t in a neighborhood of t^*, which is satisfied in most applications and is important for characterizing the limiting distribution of our test statistic. Assumption A2 imposes existence of conditional moments and we note that the condition on C_t is automatically satisfied when C_t has càdlàg paths. Assumption A3 is a smoothness in expectation condition which holds when the processes involved in it are Itô semimartingales. This assumption is needed only in the case $\nu(\{t^*\}, \mathbb{R}) = 0$. Assumption A4 imposes existence of moments of the conditional jump distribution function $G_{t^*}(x)$ as well as a smoothness in expectation condition on the conditional expectation of the jump Δx_{t^*} which will hold if its dependence on time is through an Itô semimartingale process.

Assumption A5 is our regularity condition for the log-strike grid which imposes very mild smoothness of the denseness of the strike grid in a neighborhood of the current price. Finally, assumption A6 is about the observation error. We note that the errors are proportional to the option prices they are attached to, so that the relative errors remain $O_p(1)$ as the time to maturity of the options shrinks. As we will see later, this implies that the asymptotic order of the observation error depends on the distance of the strike to the current spot price. We allow for heteroskedasticity in the observation error and we only assume a very mild condition on the smoothness of the latter as a function of the strike which is needed for its nonparametric recovery from the observed options. In addition, we allow for $\mathcal{F}^{(0)}$-conditional dependence in the observation error. This dependence can change over time and can change from one sample path to another (the coefficients $\psi_{t,l,m}$ can be stochastic).

3. Formulation of the Test and Construction of the Test Statistic

We now state formally the null and alternative hypotheses and develop a test statistic to discriminate between the two. Our interest is in deciding whether the jump compensator is strictly positive at the fixed time t^*, i.e., whether t^* is a fixed time of discontinuity for X. Formally, we are trying to decide whether the realization of $\nu(\{t^*\}, \mathbb{R})$ is positive or not, i.e., in which of the following two subsets of the sample space Ω, the observed $\omega \in \Omega$ belongs to:

$$\Omega_0 = \{\omega \in \Omega : \nu(\{t^*\}, \mathbb{R}) = 0\}, \quad \Omega_A = \{\omega \in \Omega : \nu(\{t^*\}, \mathbb{R}) > 0\}. \tag{3.1}$$

The idea of the test we propose is the following. When T_1 and T_2 are small, then under the null hypothesis, the increments of X over the intervals $[t, t + T_1]$ and $[t, t + T_2]$ are approximately $\mathcal{F}_t^{(0)}$-conditionally like those from a Lévy process. Hence, by use of Lévy-Khintchine theorem, Theorem 8.1 in Sato (1999), their $\mathcal{F}_t^{(0)}$-conditional characteristic exponents are equal up to division by T_1 and T_2, respectively. This scaling of the characteristic exponents of the time increments with their length, however, does not work if X has a fixed time of discontinuity at t^*.

We follow Qin and Todorov (2019) and utilize results in Carr and Madan (2001) to propose the following estimator of the conditional characteristic function of the increments of the log-price x:

$$\widehat{\mathcal{L}}_{t,T}(u) = 1 - (u^2 + iu) \sum_{j=2}^{N_l} h_t(u, k_{l,j-1}, k_{l,j}) \widehat{O}_{t,T}(k_{l,j-1}), \quad u \in \mathbb{R}, \tag{3.2}$$

where

$$h_t(u, k_1, k_2) = e^{-iux_t} \frac{e^{(iu-1)k_2} - e^{(iu-1)k_1}}{iu - 1}, \quad u \in \mathbb{R}, \quad k_1,\, k_2 \in \mathbb{R}. \tag{3.3}$$

Since our test will be based on comparing $\widehat{\mathcal{L}}_{t,T_1}(u)$ and $\widehat{\mathcal{L}}_{t,T_2}(u)$ as functions in u, the convergence results that follow will be functional and will take place in the complex-valued Hilbert space $\mathcal{L}^2(w)$:

$$\mathcal{L}^2(w) = \left\{ f : \mathbb{R} \to \mathbb{C} \middle| \int_{\mathbb{R}} |f(u)|^2 w(u) du < \infty \right\}, \tag{3.4}$$

where w is some positive-valued and continuous weigth function with exponential tail decay. The inner product on $\mathcal{L}^2(w)$ is induced from the inner products of its real and imaginary parts, i.e., for f and g two elements of $\mathcal{L}^2(w)$, we set

$$\langle f, g \rangle = \int_{\mathbb{R}} f(z) \overline{g(z)} w(z) dz. \tag{3.5}$$

Next, for a random complex function Z taking values in $\mathcal{L}^2(w)$, we introduce the covariance operator $Kh = \mathbb{E}[(Z - \mathbb{E}(Z))\langle h, Z - \mathbb{E}(Z)\rangle]$ and the relation operator $Ch = \mathbb{E}[(Z - \mathbb{E}(Z))\langle h, \overline{Z - \mathbb{E}(Z)}\rangle]$, where $h \in \mathcal{L}^2(w)$. We recall that a Gaussian law on $\mathcal{L}^2(w)$ is uniquely identified by the mean, covariance and relation operators and we denote it with $\mathcal{CN}(\mu, K, C)$, for μ being the mean, K being the covariance and C being the relation operator.

As we will show later (see Lemma 1), under assumptions A1-A4, if $\nu(\{t^*\}, \mathbb{R}) = 0$, $t \uparrow t^*$ and $T \downarrow 0$,

$$\widehat{\mathcal{L}}_{t,T}(u) = \exp\left(iuT\widetilde{a}_t - T\frac{u^2}{2}\sigma_t^2 + T\int_{\mathbb{R}} (e^{iuz} - iuz - 1) F_t(z) dz \right) + O_p(T^{3/2}), \tag{3.6}$$

where we denote

$$\widetilde{a}_t = a_t - \frac{1}{2}\sigma_t^2 - \int_{\mathbb{R}} (e^z - 1 - z) F_t(z) dz, \tag{3.7}$$

and if $\nu(\{t^*\},\mathbb{R})>0$, $t\uparrow t^*$ and $T\downarrow 0$,

$$\widehat{\mathcal{L}}_{t,T}(u)=\int_{\mathbb{R}}e^{iuz}G_{t^*}(z)dz+o_p(1). \tag{3.8}$$

This motivates the following test statistic for discriminating the null from the alternative hypothesis:

$$\widehat{W}_{1,2}=||\widehat{\mathcal{L}}_{t,T_2}-\widehat{\mathcal{L}}_{t,T_1}^{T_2/T_1}||=\sqrt{\int_{\mathbb{R}}\left|\widehat{\mathcal{L}}_{t,T_2}(u)-\widehat{\mathcal{L}}_{t,T_1}(u)^{T_2/T_1}\right|^2 w(u)du}, \tag{3.9}$$

and the power in the above expression is uniquely defined by the principal value of the argument of the complex number.

4. Asymptotic Behavior of the Test Statistic

We proceed with charatcterising the limit behavior of our statistic under the null and alternative hypotheses. We start with a CLT result under the null.

Theorem 1. *Assume A1-A6 hold. Suppose $t<t^*<t+T_2$ and $T_2=\tau T_1$ for some $\tau\in(1,\kappa]$ (with κ being the constant in A6). Let $t\uparrow t^*$ together with $T_1\downarrow 0$, $\Delta\asymp T_1^{\alpha}$, $\underline{K}\asymp T_1^{\beta}$, $\overline{K}\asymp T_1^{-\gamma}$, for $\beta,\gamma>0$ and $\frac{1}{2}<\alpha<\frac{1}{2}+(1\wedge 4\beta\wedge 4\gamma)$. Then, under $\nu(\{t^*\},\mathbb{R})=0$, we have*

$$\frac{1}{T_1^{3/4}\sqrt{\Delta}}\left(\widehat{\mathcal{L}}_{t,T_2}-\widehat{\mathcal{L}}_{t,T_1}^{\tau}\right)\xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} Z, \tag{4.1}$$

with Z defined on an extension of the original probability space and having $\mathcal{F}^{(0)}$-conditional law of $\mathcal{CN}(0,K,C)$ where K and C are covariance and relation operators with integral representations,

$$Kh(z)=\int_{\mathbb{R}}k(z,u)\,h(u)\,w(u)\,du,\;\;Ch(z)=\int_{\mathbb{R}}c(z,u)\,h(u)\,w(u)\,du,\;\;\forall h\in\mathcal{L}^2(w), \tag{4.2}$$

and the kernels $k(z,u)$ and $c(z,u)$ are given in Section 8.2.

We provide several comments about the above result. First, the notation $\xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}}$ means convergence in probability of the conditional probability laws when the latter are considered as random variables taking values in the space of probability measures equipped with the weak topology, see e.g., VIII.5.26 of Jacod and Shiryaev (2003). Second, a similar CLT result holds for $\widehat{\mathcal{L}}_{t,T_1}-\mathbb{E}_t^{\mathbb{Q}}(e^{iu(x_{t+T_1}-x_t)})$ with the covariance and relation operators of the limit being the components of K and C above from the options with maturity $t+T_1$. Such a result should be of independent interest for making inference for the characteristic triplet of x at time t. Third, the asymptotic variance of the limit is determined by the diffusive part of X. This is because, by assumption A6, the option error is proportional to the true option price it is attached to and the latter is dominated by the diffusion in X for strikes that are close to the current stock price. Fourth, the conditions for α, β and γ in the theorem guarantee that the leading term in the asymptotic behavior of $\widehat{\mathcal{L}}_{t,T_2} - \widehat{\mathcal{L}}^{\tau}_{t,T_1}$ is due to the observation error. We note in that regard, that the condition for the strike range is rather weak and this condition can be satisfied even if only $\beta \vee \gamma > 0$ holds. Finally, even though the scaling in the CLT is using Δ (defined in assumption A5), which is not known, for performing feasible inference, its knowledge is not necessary.

From Theorem 1, the limit behavior of our test statistic $\widehat{W}_{t_1,t_2}$ follows by continuous mapping. For feasible inference, we will develop an easy-to-implement simulation-based approach which is reminiscent of the wild bootstrap, see e.g., Wu (1986). In an analogy to the applications of the bootstrap, here we have nontrivial heteroskedasticity in the two cross-sections of options in the strike domain. Indeed, the asymptotic order of magnitude of the observation errors varies across the strikes. To develop the feasible inference, we make use of the following estimate of the observation error:

$$\widehat{\epsilon}_{t,T_i}(k_{i,j}) = \sqrt{\frac{2}{3}} \left[\widehat{O}_{t,T_i}(k_{i,j}) - \frac{1}{2} \left(\widehat{O}_{t,T_i}(k_{i,j-1}) + \widehat{O}_{t,T_i}(k_{i,j+1}) \right) \right], \tag{4.3}$$

for $j = 2, ..., N_i - 1$ and $i = 1, 2$, and further $\widehat{\epsilon}_{t,T_i}(k_{i,1}) = \widehat{\epsilon}_{t,T_i}(k_{i,2})$ as well as $\widehat{\epsilon}_{t,T_i}(k_{i,T_i}) = \widehat{\epsilon}_{t,T_i}(k_{i,T_i-1})$ for $i = 1, 2$. We denote next with J_i^*, the smallest element of the set of integers $(1, 2, ..., N_i)$ such that

$$|k_{J_i^*} - x_t| \leq |k_{i,j} - x_t|, \; j = 1, .., N_i, \; i = 1, 2. \tag{4.4}$$

That is, $k_{J_i^*}$ is the available log-strike that is closest to the current log-price x_t. We then modify the estimate of the error corresponding to $k_{J_i^*}$ by replacing it with

$$\widehat{\epsilon}_{t,T_i}(k_{i,J_i^*}) = \frac{1}{2} \left(|\widehat{\epsilon}_{t,T_i}(k_{i,J_i^*-1})| + |\widehat{\epsilon}_{t,T_i}(k_{i,J_i^*+1})| \right), \quad i = 1, 2. \tag{4.5}$$

The construction of $\widehat{\epsilon}_{t,T_i}(k_{i,j})$ above makes use of the fact that the true option price is smooth as a function of its strike, and hence $\widehat{\epsilon}_{t,T_i}(k_{i,j})$ is dominated by the observation error in the options used in forming it. Given the smoothness in strike assumption for the $\mathcal{F}^{(0)}$-conditional volatility of the relative option error in A6, $|\widehat{\epsilon}_{t,T_i}(k_{i,j})|$ provides therefore an estimate of the $\mathcal{F}^{(0)}$-conditional volatility of the option error (albeit a very noisy one).

Since the observation error can have spatial dependence (recall assumption A6), we need to construct estimates for this dependence. Towards this end, we first form the sample spacial autocovariance of our estimates of the observation error:

$$\begin{aligned} \widehat{\chi}_{t,i}(h) = & \sum_{j=h+1}^{N_i} \left[\widehat{\epsilon}_{t,T_i}(k_{i,j}) \widehat{\epsilon}_{t,T_i}(k_{i,j-h}) 1_{\{j \neq J_i^*, \, j \neq J_i^*+h\}} \right] \\ & - \frac{1}{N_i - 1} \left(\sum_{j=1}^{N_i} \widehat{\epsilon}_{t,T_i}(k_{i,j}) 1_{\{j \neq J_i^*\}} \right)^2, \quad h = 0, ..., N_i - 1, \; i = 1, 2. \end{aligned} \tag{4.6}$$

The centering in (4.6), i.e., the inclusion of the second term on the right hand side of (4.6), is standard in covariance estimation but in the current context is not necessary as $\widehat{\epsilon}_{t,T_i}(k_{i,j})$ are, up to an asymptotically negligible error, mean zero. Nevertheless, we keep this term in (4.6) as in small samples it can help correcting biases in $\widehat{\epsilon}_{t,T_i}(k_{i,j})$ due to the change in the true option price across strikes.

The behavior of $\widehat{\chi}_{t,i}(h)$ is non-standard because the asymptotic order of the options used in its construction is different. As we show in the proofs (see Lemma 4), we have the following convergence in probability in the setting of Theorem 1 (where the null hypothesis holds):

$$\frac{T_i^{3/2}}{\Delta}\widehat{\chi}_{t,i}(h) \overset{\mathbb{P}}{\to} C_t\overline{\chi}_{t,i}(h),\ i=1,2, \tag{4.7}$$

where C_t is an $\mathcal{F}_t^{(0)}$-adapted random variable that does not depend on h, given explicitly in the statement of Lemma 4, and where $\overline{\chi}_{t,i}(h)$ is:

$$\overline{\chi}_{t,i}(h) = \frac{2}{3}\left(\frac{3}{2}\gamma_{t,i}(h) - \gamma_{t,i}(h+1) - \gamma_{t,i}(h-1) + \frac{1}{4}\gamma_{t,i}(h+2) + \frac{1}{4}\gamma_{t,i}(h-2)\right), \tag{4.8}$$

with $\gamma_{t,i}(h)$ denoting the $\mathcal{F}^{(0)}$-conditional covariances between $\overline{\epsilon}_{t,i,j}$ and $\overline{\epsilon}_{t,i,j-h}$:

$$\gamma_{t,i}(h) = \mathbb{E}\left(\overline{\epsilon}_{t,i,j}\overline{\epsilon}_{t,i,j-h}\middle|\mathcal{F}^{(0)}\right),\ i=1,2. \tag{4.9}$$

Since $\widehat{\epsilon}_{t,T_i}(k_{i,j})$ is formed by a second-order difference of the observed option price as a function of its strike, the error estimates $\widehat{\epsilon}_{t,T_i}(k_{i,j})$ have spatial dependence, which is generated by the overlap of the option observation errors contained in them. Using the $\mathcal{F}^{(0)}$-conditional M-dependence of the observation errors (assumed in A6), we have

$$\gamma_{t,i}(h) = 6\overline{\chi}_{t,i}(h+2)(1+L+...+L^M)^4, \quad h=0,1,..., \tag{4.10}$$

where L denotes the "spatial lag" operator that shifts the spatial autocovariance by one lag, i.e., $L\overline{\gamma}_{t,i}(h) = \overline{\gamma}_{t,i}(h+1)$. Using this relationship, we can solve iteratively for $\{\gamma_{t,i}(h)\}_{h=0,1,...,M}$ from $\{\widehat{\chi}_{t,i}(h)\}_{h=0,...,M+2}$ up to an $\mathcal{F}_t^{(0)}$-adapted scaling factor. We denote these estimates with $\{\widehat{\gamma}_{t,i}(h)\}_{h=0,1,...,M}$. We then introduce the following scaling factor

$$\widehat{S}_{t,i} = \sqrt{1+2\varkappa\left(\frac{\sum_{h=1}^{M}\widehat{\gamma}_{t,i}(h)}{\widehat{\gamma}_{t,i}(0)}1_{\{\widehat{\gamma}_{t,i}(0)>0\}};N_i\right)},\ i=1,2, \tag{4.11}$$

where the function $\varkappa(x,N)$ satisfies

$$\varkappa(x,N)\to x,\ \text{locally uniformly in } x\in\mathbb{R},\text{ as } N\to\infty. \tag{4.12}$$

The function $\varkappa(x,N)$ is a finite sample correction that guarantees that $\widehat{S}_{t,i}$ is a finite and positive number in finite samples. It reduces to a small-sample correction of the spatial autocovariances past lag zero. This correction is the counterpart in the current setting of the kernel-based estimators of the long-run asymptotic variance of sample averages of dependent sequences. The latter similarly dampen the autocovariance estimates past lag zero.

Under the null hypothesis, the result of Lemma 4 implies

$$\widehat{S}_{t,i} \overset{\mathbb{P}}{\to} \sqrt{\frac{\sum_{h=-M}^{M}\gamma_{t,i}(h)}{\gamma_{t,i}(0)}},\ i=1,2. \tag{4.13}$$

The limit on the right-hand side of the above convergence captures the effect from the spatial dependence of the observation error on the limiting standard deviation of $\widehat{\mathcal{L}}_{t,T_i}$. In particular, if there is positive spatial dependence in the observation error, the asymptotic variance will naturally go up.

We are now ready to describe our bootstrap procedure. Using the above estimates, we add noise to the observed option prices and denote the new observations with

$$\widehat{O}^*_{t,T_i}(k_{i,j}) = \widehat{O}_{t,T_i}(k_{i,j}) + \widehat{\epsilon}_{t,T_i}(k_{i,j})\widehat{S}_{t,i}z_{i,j},\quad j=1,...,N_i,\quad i=1,2, \tag{4.14}$$

where $\{z_{1,j}\}_{j=1}^{N_1}$ and $\{z_{2,j}\}_{j=1}^{N_2}$ are two i.i.d. sequences of standard normal variables defined on an extension of the original probability space and independent from $\mathcal{F}$ and from each other. We then define $\widehat{\mathcal{L}}^*_{t,T}$ from $\widehat{\mathcal{L}}_{t,T}$ by replacing $\widehat{O}_{t,T_i}(k_{i,j})$ with $\widehat{O}^*_{t,T_i}(k_{i,j})$. With this notation, we set

$$\widehat{W}^*_{1,2} = \big|\big|\widehat{\mathcal{L}}^*_{t,T_2} - \widehat{\mathcal{L}}_{t,T_2} - \tau\widehat{\mathcal{L}}^{\tau-1}_{t,T_1}\big(\widehat{\mathcal{L}}^*_{t,T_1} - \widehat{\mathcal{L}}_{t,T_1}\big)\big|\big|. \tag{4.15}$$

We note that in defining $\widehat{W}^*_{1,2}$, we center $\widehat{\mathcal{L}}^*_{t,T_2}$ and $\widehat{\mathcal{L}}^*_{t,T_1}$ around $\widehat{\mathcal{L}}_{t,T_2}$ and $\widehat{\mathcal{L}}_{t,T_1}$, respectively. This guarantees that the $\mathcal{F}$-conditional limiting distribution of $\widehat{W}^*_{1,2}$ is the same as that of our statistic $\widehat{W}_{1,2}$ under the null and it has important implications also for the behavior of $\widehat{W}^*_{1,2}$ under the alternative hypothesis. The $\mathcal{F}$-conditional limit behavior of $\widehat{W}^*_{1,2}$ is given in the following theorem.

Theorem 2. *Under the conditions of Theorem 1, we have*

$$\frac{1}{T_1^{3/4}\sqrt{\Delta}}||\widehat{W}^*_{1,2}|| \overset{\mathcal{L}|\mathcal{F}}{\longrightarrow} ||Z||, \tag{4.16}$$

where Z is a random function in $\mathcal{L}^2(w)$, defined on extension of the original probability space with $\mathcal{F}$-conditional distribution of $\mathcal{CN}(0,K,C)$ where K and C are given in Theorem 1.

The above theorem allows for a very easy way of implementing our test. One needs only to compute the quantiles of $||\widehat{W}^*_{1,2}||$ via simulation. In particular, we avoid the need to estimate consistently the covariance and relation operators of the limiting distribution of Theorem 1. To fully characterize the asymptotic behavior of our test, we need to derive the behavior of $\widehat{W}_{1,2}$ and $\widehat{W}^*_{1,2}$ in the case of fixed time of discontinuity at time $t=t^*$. This is done in the following theorem.

Theorem 3. *Assume A1-A6 hold. Suppose* $t < t^* < t + T_2$ *and* $T_2 = \tau T_1$ *for some* $\tau \in (1, \kappa]$ *(with* κ *being the constant in A6). Let* $t \uparrow t^*$ *together with* $T_1 \downarrow 0$, $\Delta \asymp T_1^{\alpha}$, $\underline{K} \asymp T_1^{\beta}$, $\overline{K} \asymp T_1^{-\gamma}$ *with* α, β, $\gamma > 0$*. Then, under* $\nu(\{t^*\}, \mathbb{R}) > 0$*, we have*

$$\widehat{\mathcal{L}}_{t,T_1} \overset{\mathbb{P}}{\to} \mathbf{1}, \quad \widehat{\mathcal{L}}_{t,T_2} \overset{\mathbb{P}}{\to} \int_{\mathbb{R}} e^{iuz} G_{t^*}(z) dz, \quad \text{if } T_1 < t^*, \tag{4.17}$$

$$\widehat{\mathcal{L}}_{t,T_1} \overset{\mathbb{P}}{\to} \int_{\mathbb{R}} e^{iuz} G_{t^*}(z) dz, \quad \widehat{\mathcal{L}}_{t,T_2} \overset{\mathbb{P}}{\to} \int_{\mathbb{R}} e^{iuz} G_{t^*}(z) dz, \quad \text{if } t^* \geq T_1, \tag{4.18}$$

and

$$||\widehat{W}^*_{1,2}|| = O_p(\sqrt{\Delta}). \tag{4.19}$$

As seen from the above theorem, when X has a fixed time of discontinuity at $t = t^*$, then this becomes the leading component in the increments of x over time intervals that include t^*. Importantly, this holds regardless of whether σ and F have a fixed time of discontinuity at $t = t^*$. To convey the intuition for this, it is instructive to look at the special case when a is zero, F is zero and σ remains constant apart from the possible jump at $t = t^*$. In this case, for an interval $[t, t + T]$ that includes t^*, we can write (because $\sigma_t = \sigma_{t^*-}$):

$$x_{t+T} - x_t = \sigma_t(W_{t+T} - W_t) + \Delta\sigma_{t^*}(W_{t+T} - W_{t^*}) + \Delta x_{t^*}.$$

Then, in the asymptotic setting in which $t \uparrow t^*$ and $T \downarrow 0$, the first two terms are both $O_p(\sqrt{T})$ and the last term is $O_p(1)$. In particular, the fixed time of discontinuity in σ at $t = t^*$ plays an asymptotically negligible role in the behavior of $x_{t+T} - x_t$ as $T \downarrow 0$. This is because, $\Delta\sigma_{t^*}$ multiplies an increment of a Brownian motion which shrinks asymptotically as $T \downarrow 0$.

As a result, in the case of fixed time of discontinuity in x at $t = t^*$, the characteristic exponent of $x_{t+T_2} - x_t$ is no longer shrinking as T_2 goes down to zero. Instead, in this case $\widehat{\mathcal{L}}_{t,T_2}$ estimates the characteristic function of the jump distribution of x at t^*. This implies for our testing purposes that $||\widehat{W}_{1,2}|| = O_p(1)$. This, together with the fact that $||\widehat{W}^*_{1,2}|| = O_p(\sqrt{\Delta})$, means that our test will have asymptotic power to discriminate the null against the alternative.

To state formally the result for the asymptotic behavior of our test, we denote

$$\widehat{cv}_{\alpha} = Q_{1-\alpha}(\widehat{W}^*_{1,2}|\mathcal{F}), \quad \alpha \in (0, 1), \tag{4.20}$$

where $Q_{\alpha}(Z)$ denotes the α-quantile of the random variable Z. We can evaluate $\widehat{cv}_{\alpha}$ easily via simulation.

Corollary 4.1. *Assume A1-A6 hold. Suppose* $t < t^* < t + T_2$ *and* $T_2 = \tau T_1$ *for some* $\tau \in (1, \kappa]$ *(with* κ *being the constant in A6). Let* $t \uparrow t^*$ *together with* $T_1 \downarrow 0$, $\Delta \asymp T_1^{\alpha}$, $\underline{K} \asymp T_1^{\beta}$, $\overline{K} \asymp T_1^{-\gamma}$*, for* $\beta, \gamma > 0$ *and* $\frac{1}{2} < \alpha < \frac{1}{2} + (1 \wedge 4\beta \wedge 4\gamma)$*. Then, for* $\alpha \in (0, 1)$*, we have*

$$\mathbb{P}\left(\widehat{W}_{1,2} > \widehat{cv}_{\alpha}|\Omega_0\right) \longrightarrow \alpha, \quad \mathbb{P}\left(\widehat{W}_{1,2} > \widehat{cv}_{\alpha}|\Omega_A\right) \longrightarrow 1. \tag{4.21}$$

Remark 1. *Our theoretical results can be extended to testing, in a setting with no fixed time of discontinuity, whether the Lévy measure of the jumps in x is that of a time-changed Lévy process, i.e., whether $F_t(x)$ is of the form $\overline{a}_t \times \overline{F}(x)$, for $\overline{a}_t$ being a stochastic process with càglàd paths and $\overline{F}$ being a time-invariant Lévy measure (see e.g., Theorem 8.3 in Barndorff-Nielsen and Shiryaev (2010)). On an intuitive level, the above structure boils down to time-invariant jump distribution with all the variation of the jump compensator being through level shifts in the latter and no changes in its shape. This is the predominant approach of modeling jumps in applications. For example, the jumps in the popular affine class of models that are typically used in empirical asset pricing, see Duffie et al. (2000) and Duffie et al. (2003), are time-changes of Lévy processes (with additional restrictions on the jump compensator). See also Figueroa-López (2009), Belomestny (2011), Belomestny and Panov (2013a,b) and Bull (2014) for estimation of such types of models in various asymptotic setups.*

We can test the hypothesis of time-changed Lévy models by studying the change in the characteristic function of the increments of x at different points in time (which can be recovered from short-dated options at these time points). For this to be done, however, we will need estimators of the spot diffusive volatility (which can be obtained either from high-frequency record of X or the options themselves) as well as an estimate of the ratio of the time-change ($\overline{a}_t$ in the notation above) at the two different time points (which can be obtained from the estimates of the characteristic functions of the increments). We leave such an extension for future work.

5. Inference for the Jump Distribution at the Fixed Time of Discontinuity

When X contains a fixed time of discontinuity at $t = t^*$, we can use the options to recover the density of the jump distribution at t^*. As shown in Theorem 3, the Fourier transform of the latter is the dominant component of $\widehat{\mathcal{L}}_{t,T}(u)$. The estimator of the characteristic function of the jump at time t^* is therefore given by

$$\widehat{\mathcal{L}}_t^{fd}(u) = \begin{cases} \frac{1}{\widehat{\mathcal{B}_1}(u)}\widehat{\mathcal{L}}_{t,T_1}(u), & \text{if } \frac{\tau}{\tau-1}||\widehat{\mathcal{L}}_{t,T_2} - \widehat{\mathcal{L}}_{t,T_1}||/||\widehat{\mathcal{L}}_{t,T_2} - 1|| < 1, \\ \frac{1}{\widehat{\mathcal{B}_2}(u)}\widehat{\mathcal{L}}_{t,T_2}(u), & \text{if } \frac{\tau}{\tau-1}||\widehat{\mathcal{L}}_{t,T_2} - \widehat{\mathcal{L}}_{t,T_1}||/||\widehat{\mathcal{L}}_{t,T_2} - 1|| \geq 1, \end{cases} \tag{5.1}$$

where recall $\tau = T_2/T_1$ and we denote

$$\widehat{\mathcal{B}}_1(u) = \begin{cases} \left(\frac{\widehat{\mathcal{L}}_{t,T_2}(u)}{\widehat{\mathcal{L}}_{t,T_1}(u)}\right)^{\frac{1}{\tau-1}}, & \text{if } |\widehat{\mathcal{L}}_{t,T_2}(u)/\widehat{\mathcal{L}}_{t,T_1}(u) - 1| \leq (\tau-1)u^2\widehat{c}T_1, \\ (1 - u^2\widehat{c}T_1), & \text{if } |\widehat{\mathcal{L}}_{t,T_2}(u)/\widehat{\mathcal{L}}_{t,T_1}(u) - 1| > (\tau-1)u^2\widehat{c}T_1, \end{cases} \tag{5.2}$$

and

$$\widehat{\mathcal{B}}_2(u) = \begin{cases} \widehat{\mathcal{L}}_{t,T_1}(u)^{\tau}, & \text{if } |\widehat{\mathcal{L}}_{t,T_1}(u) - 1| \leq u^2\widehat{c}T_1, \\ 1 - u^2\widehat{c}T_2, & \text{if } |\widehat{\mathcal{L}}_{t,T_1}(u) - 1| > u^2\widehat{c}T_1, \end{cases} \tag{5.3}$$

with $\widehat{c}$ denoting a sequence of nonnegative-valued random variables which is $O_p(1)$ as $\Delta \to 0$ and $T_1 \to 0$. When the fixed time of discontinuity satisfies $t^* < t + T_1$, then we use the characteristic function from the shortest-dated options and the term $\widehat{\mathcal{B}}_1(u)$ in $\widehat{\mathcal{L}}_t^{fd}(u)$ corrects for the effect on $\widehat{\mathcal{L}}_{t,T_1}(u)$ due to $x_{t+T_1} - x_t - \Delta x_{t^*}$. Similarly, when $t+T_1 < t^* < t+T_2$, then we use $\widehat{\mathcal{L}}_{t,T_2}(u)$ and with $\widehat{\mathcal{B}}_2(u)$, we correct for the component in it that is due to $x_{t+T_2} - x_t - \Delta x_{t^*}$. In both cases, the bias correction terms will account fully for the biases stemming from $x_{t+T_1} - x_t - \Delta x_{t^*}$ only in the case when the semimartingale spot characteristics (a_t, σ_t and F_t) do not have a fixed time of discontinuity at $t = t^*$. Even if this is not the case, however, $\widehat{\mathcal{B}}_1(u)$ and $\widehat{\mathcal{B}}_2(u)$ will play only a higher order asymptotic role.

In general, the statistician might not know the location of t^* relative to $t+T_1$. However, the two alternative scenarios of $t^* < t+T_1$ and $t+T_1 < t^* < t+T_2$ can be easily separated by the value of $\frac{\tau}{\tau-1}||\widehat{\mathcal{L}}_{t,T_2} - \widehat{\mathcal{L}}_{t,T_1}||/||\widehat{\mathcal{L}}_{t,T_2} - 1||$ which converges to 0 in the former case and to $\frac{\tau}{\tau-1} > 1$ in the latter (and to 1 under the null of no fixed time to discontinuity).

Given $\widehat{\mathcal{L}}_t^{fd}(u)$, our estimator of $G_{t^*}(x)$ is then simply its Fourier inverse:

$$\widehat{G}_{t^*}(x) = \frac{1}{2\pi}\int_{-u_N}^{u_N} e^{-iux}\widehat{\mathcal{L}}_t^{fd}(u)du, \tag{5.4}$$

for some positive sequence $u_N \to \infty$ as $\Delta \to 0$ and $T_1 \to 0$. Our asymptotic result for $\widehat{G}_{t^*}(x)$ will be based on the following smoothness assumption for the density of the jump distribution:

A7. *The function $G_{t^*}(x)$ belongs to the class*

$$\mathcal{S}_r(C_{t^*}) = \left\{ f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) : \int_{\mathbb{R}} |\mathcal{F}f(x)|^2(1+x^2)^r dx \le C_{t^*} \right\},$$

for some positive constant r and some positive $\mathcal{F}_{t^*}$-adapted random variable C_{t^*}, and $\mathcal{F}f$ denoting the Fourier transform of f.

The next theorem derives the order of magnitude for the integrated squared error in recovering G_{t^*}.

Theorem 4. *Assume A1-A7 hold. Suppose $t < t^* < t + T_2$ and $T_2 = \tau T_1$ for some $\tau \in (1, \kappa]$ (with κ being the constant in A6). Let $t \uparrow t^*$ together with $T_1 \downarrow 0$, $\Delta \asymp T_1^{\alpha}$, $\underline{K} \asymp T_1^{\beta}$, $\overline{K} \asymp T_1^{-\gamma}$, for α, β, $\gamma > 0$. Let $\widehat{c}$ in (5.2)-(5.3) satisfy $\widehat{c} = O_p(1)$. If*

$$u_N \to \infty \text{ and } u_N^2 T_1 \to 0, \tag{5.5}$$

then, we have

$$\int_{\mathbb{R}} (\widehat{G}_{t^*}(x) - G_{t^*}(x))^2 dx = O_p\left(u_N^{-2r} \bigvee u_N^5 \left(\Delta \vee u_N^{2(p\vee 1)-4} T_1 \vee e^{-4(|\underline{k}\vee \overline{k}))}\right)\right), \tag{5.6}$$

where p is the constant in A4.

The above result combines features of related work on the recovery of Lévy density from options with fixed time to maturity (Belomestny and Reiß (2006, 2015)) and shrinking one (Qin and Todorov (2019)). Like Belomestny and Reiß (2006, 2015), the error in the density recovery that is due to observation error is of order $O_p(u_n^5\Delta)$. This is despite of the fact that here the maturity of the options is shrinking. The reason for this is that when there is a fixed time of discontinuity in X before the option expiration, then option prices are of asymptotic order $O_p(1)$, even though their time to maturity is shrinking. Since the observation error is proportional to the true option price, the observation errors also do not shrink when the time to maturity goes down. On the other hand, similar to Qin and Todorov (2019), the error in recovering G_{t^*} also depends on the time to maturity of the options, T_1. The reason for this is that the increment of X contains not only the jump at the fixed time of discontinuity, whose size distribution we are trying to recover, but also the other part of the process (the Itô semimartingale) which is of order $O_p(\sqrt{T_1})$. Also in our setting, we allow the jump size distribution at the fixed discontinuity time to depend also on time, and this in turn also generates error, the size of which depends on T_1.

The error in the density recovery naturally depends on the smoothness of the latter. Assumption A7 imposes power decay of the Fourier transform of G_{t^*}. A stronger assumption like for example G_{t^*} being supersmooth, see e.g., Fan (1991), which means that its Fourier transform has an exponential tail decay (which is satisfied for example by the density of the normal distribution), will lead to replacing u_N^{-2r} in (5.6) with a term that is of significantly smaller (exponential) asymptotic order. Finally, in the case when $u_N^{2(p\vee 1)-4}T_1 \vee e^{-4(|\underline{k}\vee\overline{k}))}$ is of smaller asymptotic order than Δ, using similar techniques as in Belomestny and Reiß (2006, 2015) and Qin and Todorov (2019), we can show that our estimator is rate-optimal.

6. Monte Carlo Study

We now present results for the performance of our test for fixed time of discontinuity on simulated data from the following model for the risk-neutral dynamics of X:

$$\frac{dX_t}{X_{t-}} = \sqrt{V_t}dW_t + \int_{\mathbb{R}} (e^z - 1)\widetilde{\mu}(ds, dz) + \eta Y_t, \tag{6.1}$$

with W being a Brownian motion and V having the dynamics

$$dV_t = 3.6(0.02 - V_t)dt - 0.1\sqrt{V_t}dW_t + 0.1732\sqrt{V_t}d\widetilde{W}_t, \tag{6.2}$$

where $\widetilde{W}$ is a Brownian motion orthogonal to W. The jump measure μ has a compensator $F_t(x)dt \otimes dx$ with

$$F_t(x) = V_t\left(91.75\frac{e^{-20|x|}}{|x|^{1.5}}1_{\{x<0\}} + 102.58\frac{e^{-100|x|}}{|x|^{1.5}}1_{\{x>0\}}\right). \tag{6.3}$$

Finally,

$$Y_t = \begin{cases} 0, & \text{for } t < t^*, \\ e^{\Delta x_{t^*}} - 1, & \text{for } t \geq t^* \end{cases}, \quad \text{with} \quad \Delta x_{t^*} = 0.0546\sqrt{V_0} \times Z - 0.0546^2 V_0/2, \quad (6.4)$$

where Z is a standard normal variable independent from W, $\widetilde{W}$ and μ. The case $\eta \neq 0$ corresponds to a process with fixed time of discontinuity. When $\eta = 0$, the specification in (6.1)-(6.3) belongs to the affine class of models of Duffie et al. (2000) commonly used in empirical option pricing work. The jumps outside of t^* have time-varying jump intensity, are of infinite activity, and their distribution is like that of a tempered stable process, see e.g., Carr et al. (2002), which is found to provide good fit to option data. The jump size at time t^* is drawn from a $\mathcal{F}_0$-conditionally normal distribution with volatility proportional to the level of diffusive volatility at time zero.

The model parameters are set in a way that results in option prices similar to observed equity index option data. In particular, the unconditional mean of volatility is similar to that inferred from S&P 500 index return and option data. The model, as in the data, allows for a negative correlation between return and volatility innovations (so called leverage effect). Jump tails have exponential tail decay, with tail decay parameters yielding out-of-the-money short-maturity option decays (as the strikes moves further from the current stock price) like those observed in S&P 500 index options, see e.g., Andersen et al. (2017). Finally, the jump at time t^* has $\mathcal{F}_0$-conditional variance that is only 1/8-th of the $\mathcal{F}_0$-conditional variance of the increment $x_{T_1} - x_0 - \Delta x_{t^*}$. This is rather challenging for our asymptotics under the alternative hypothesis as this way $x_{T_1} - x_0 - \Delta x_{t^*}$, although asymptotically negligible relative to Δx_{t^*}, is rather nontrivial for the current values of T_1 and T_2.

Options written on X are observed at time $t = 0$ with maturities of 3 and 5 business days and we set $X_0 = 2500$. The strike grid and range of the options are calibrated to match roughly the data we use in the empirical application. In particular, for each of the maturities the strike grid is equidistant with increments of 5. The strike range is determined by the requirement that the true option prices should be at least 0.05 in value. Finally, the option observation error is set to $\epsilon_{0,T_i}(k_{i,j}) = \xi_{0,T_i}(k_{i,j}) O_{0,T_i}(k_{i,j}) Z_{i,j}$, for $i = 1, 2$ and where $\{Z_{1,j}\}_{j\geq 1}$ and $\{Z_{2,j}\}_{j\geq 1}$ are two independent sequences of i.i.d. standard normal random variables, and

$$\xi_{0,T_i}(k_{i,j}) = \begin{cases} \sqrt{0.05^2 + 0.0046875[(x_0 - k_{i,j})/(\sqrt{T_i}\sigma_{t,i}^{ATM}) \wedge 1]}, & \text{if } x_0 \geq k_{i,j}, \\ \sqrt{0.05^2 + 0.009375[(k_{i,j} - x_0)/(\sqrt{T_i}\sigma_{t,i}^{ATM}) \wedge 1]}, & \text{if } x_0 < k_{i,j}, \end{cases} \quad (6.5)$$

with $\sigma_{t,i}^{ATM}$ denoting the at-the-money Black-Scholes implied volatility computed from the option with time to maturity T_i and log-strike $k = x_t$. This specification implies (as in the data) smallest relative error for the options with strikes closest to the current stock price, and larger observation error for out-of-the-money calls versus puts with strikes equally distant from X_0.

In the Monte Carlo study, we consider three cases for the starting value of volatility: low, median and high, corresponding to 25th, 50th and 75th quantiles, respectively, of the unconditional distribution of V. For simplicity we assume that the statistical and risk-neutral probabilities for the volatility dynamics coincide. Finally, we analyze two alternative scenarios: in alternative 1 we have $t^* < T_1$ while in alternative 2 we have $T_1 < t^* < T_2$.

For the implementation of the test, we set w equal to the pdf of a mean zero normal variable with variance of $0.5u_{max}^2$, and we then approximate the integral in our test statistic by a Riemann sum over the interval $[-u_{max}, u_{max}]$, split into increments of length 0.25, and we set $u_{max} = 30$. Finally, for the calculation of the scaling factor $\widehat{S}_{t,i}$ that accounts for the potential dependence in the observation error, we set $M = 1$ and we use the following function $\varkappa$:

$$\varkappa(x, N) = \begin{cases} -(1 - \frac{30}{N})\frac{N}{400}, & \text{if } x \leq -\frac{N}{400}, \\ (1 - \frac{30}{N})x, & \text{if } x \in \left(-\frac{N}{400}, \frac{N}{400}\right), \\ (1 - \frac{30}{N})\frac{N}{400}, & \text{if } x \geq \frac{N}{400}. \end{cases} \tag{6.6}$$

This choice of $\varkappa(x, N)$ corresponds to tempering the value of the ratio $\frac{\sum_{h=1}^M \widehat{\gamma}_{t,i}(h)}{\widehat{\gamma}_{t,i}(0)}$ in the two tails, and further dampening this ratio when it takes moderate values. This way we modify slightly $\frac{\sum_{h=1}^M \widehat{\gamma}_{t,i}(h)}{\widehat{\gamma}_{t,i}(0)}$ around zero and we modify it more severely when it takes big values. The resulting estimate of $\frac{\sum_{h=1}^M \gamma_{t,i}(h)}{\gamma_{t,i}(0)}$ can be viewed as the counterpart in the current context of the estimate of the long-run asymptotic variance of sample averages from sequences with dependence based on a Bartlett kernel.

The results for the performance of the test in the Monte Carlo study are summarized in Table 1. The finite sample behavior of the test under the null hypothesis is satisfactory with only minor deviations from the nominal size in all considered cases for the starting level of volatility. Table 1 also shows that the test has very good power against the considered alternatives. Not surprisingly, the power against alternative 2 in which $T_1 < t^* < T_2$ is higher. This is because, under alternative 2, the difference in the distributions of $x_{T_1} - x_0$, which does not contain the jump at time t^*, and $x_{T_2} - x_0$, which contains the jump at time t^*, is bigger. We also note that we have less power to reject the alternative hypothesis when volatility starts at a low level. The reason for this is that in the low volatility regime, options are cheaper and since we keep only options with price of at least 0.05, this leads to fewer options and hence less power.

7. Empirical Application

We now apply the inference techniques developed above to options written on the S&P 500 market index in the year of 2017. More specifically, we use data at market close on each of the Mondays in our sample period for the weeks for which there is no public holiday. The maturity of the options in our empirical analysis expire on the Wednesday and Friday of the same week in which the option prices are recorded. The tuning parameters of the test are set exactly as in our Monte Carlo experiment. In particular, we use $\varkappa$ given in (6.6) and set $M = 1$. In this regard, we note that our sample estimates of

Scenario	**Vol Regime**	**Test Size**		
		10%	5%	1%
$\eta = 0$ (Null)	Low	10.0	5.1	1.0
$\eta = 0$ (Null)	Median	9.2	4.8	1.2
$\eta = 0$ (Null)	High	9.2	4.7	1.0
$\eta = 1$, $t^* < T_1$ (Alt. 1)	Low	63.8	50.5	26.8
$\eta = 1$, $t^* < T_1$ (Alt. 1)	Median	76.8	63.7	38.5
$\eta = 1$, $t^* < T_1$ (Alt. 1)	High	81.9	70.7	44.8
$\eta = 1$, $T_1 < t^* < T_2$ (Alt. 2)	Low	96.7	92.6	78.6
$\eta = 1$, $T_1 < t^* < T_2$ (Alt. 2)	Median	95.9	91.9	76.2
$\eta = 1$, $T_1 < t^* < T_2$ (Alt. 2)	High	95.9	91.5	75.0

Table 1. Monte Carlo Results for the Test. Critical values are based on $3,000$ simulations.

$\widehat{\chi}_{t,i}(4)$, for $i = 1, 2$, are both very small in absolute value (smaller than their sampling standard deviation), suggesting M-dependence with $M = 1$ for the option observation errors.

The p-values of the test are displayed in Figure 1. As seen from the figure, for the majority of the weeks in our sample, there is no statistical evidence for fixed times of discontinuity. That said, our test rejects the null hypothesis at 1% level in 7 weeks. Many of these rejections can be associated with pre-scheduled economic announcements such as those after the Federal Open Market Committee (FOMC) meetings, unemployment reports etc. However, it is interesting to note that not all of these events trigger fixed times of discontinuity in the market index.

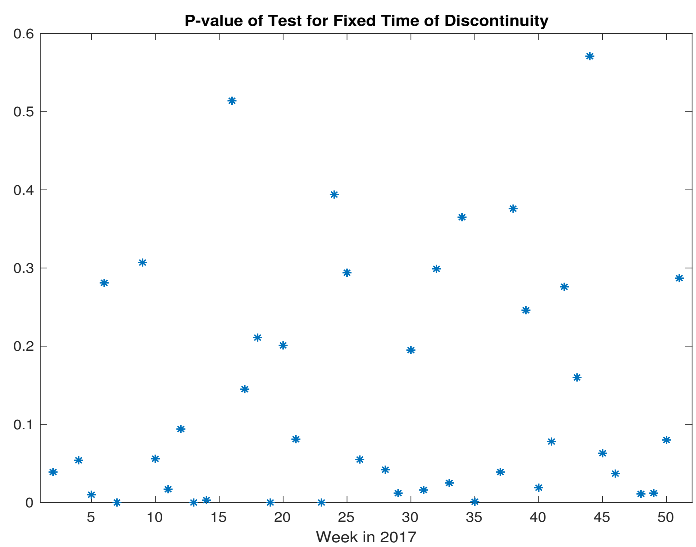

Figure 1. P-values of Test for Presence of Fixed Time of Discontinuity for S&P 500 Index Options.

We next recover the jump distribution at a fixed time of discontinuity for one of the weeks in the sample where our test rejects, mainly the week of June 5, 2017. The Friday expiration options for that week are much more expensive than the ones expiring Wednesday (both recorded at market close on Monday), reflecting the potential effect on the markets from the UK elections on Thursday, May 8 and the European Central Bank announcement on the same day as well as the release of important information about the technology sector on Friday, May 9. Thus, our shortest-dated options on June 5, 2017 do not include the fixed-arrival jump events while the longer-dated ones do include them. We set the maximum Fourier frequency in the density recovery to the smallest value of u for which $|\widehat{\mathcal{L}}_{t,T_2}(u)|$ falls below 0.25 and $\widehat{c}$ to $(\sigma_{t,1}^{ATM})^2$. The recovered jump density is displayed on the left plot of Figure 2. As seen from the figure, the jump distribution appears symmetric, reflecting the fact that the fixed jump events created uncertainty on the market. The median size of the jump is around 0.5% which is nontrivial given the low volatility at the time. On the right plot of Figure 2, we show the S&P 500 Index futures price (sampled at 1-minute frequency) for the period June 5, 2017 till June 9, 2017. As seen from this plot, the S&P 500 Index is significantly more volatile on Thursday and Friday when the fixed arrival jump events occur. This increased volatility also suggests that the price jumps are accompanied by volatility jumps.

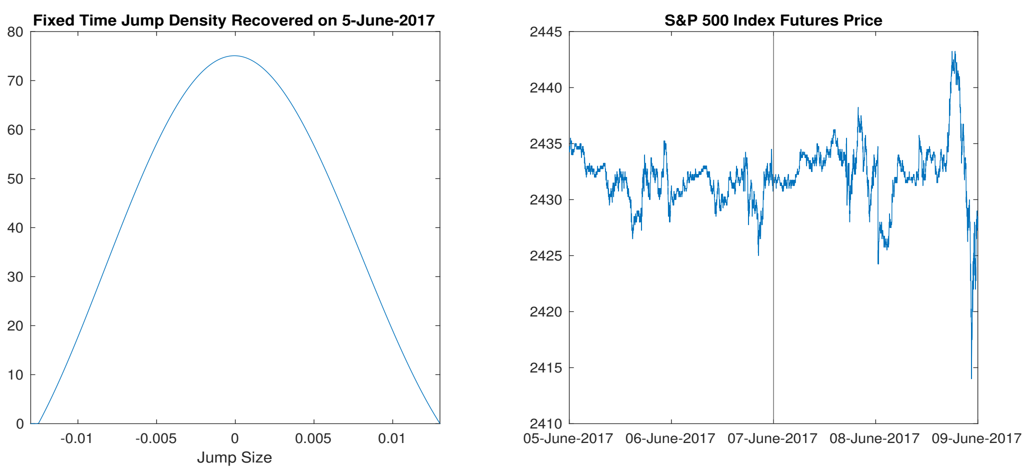

Figure 2. Fixed Time Jumps in the S&P 500 Index. Left plot displays recovered Fixed Time Jump Distribution in the S&P 500 Index on June 5, 2017. Right plot displays the S&P 500 Index futures price sampled at 1-minute for the period June 5, 2017 till June 9, 2017. The ticks on the x-axis of the right plot correspond to the market close on each day (3.15pm CST).

8. Proofs

8.1. Decomposition and Notation

The jump part of the process x_t, excluding the jump at time t^*, can be represented as an integral with respect to Poisson random measure (see Qin and Todorov (2019)):

$$\int_{[0,t]\setminus t^*}\int_{\mathbb{R}} z\widetilde{\mu}(ds,dz) \equiv \int_0^t \int_E \delta^x(s,z)\widetilde{\mu}^x(ds,dz), \quad t \geq 0, \quad \mathbb{Q}-a.s., \tag{8.1}$$

where $\mu^x(ds,dz)$ is a Poisson random measure on $\mathbb{R}_+ \times E$ with compensator $dt \otimes \lambda(dz)$, for some Polish space E and σ-finite measure on it, $\widetilde{\mu}^x$ is the martingale counterpart of μ^x, and δ^x is a predictable and $\mathbb{R}$-valued function on $\Omega \times \mathbb{R}_+ \times E$ such that $F_t(z)dz$ is the image of the measure λ under the map $z \to \delta^x(t,z)$ on the set $\{z : \delta^x(\omega,t,z) \neq 0\}$.

With this notation, we can split $x_s - \Delta x_{t^*}1_{\{s\geq t^*\}}$ into

$$x_s^c = x_t + \int_t^s \widetilde{a}_u du + \int_t^s \sigma_u dW_u, \quad x_s^d = \int_t^s \int_E \delta^x(u,z)\widetilde{\mu}^x(du,dz), \quad s \geq t, \tag{8.2}$$

where recall $\widetilde{a}_s = a_s - \frac{1}{2}\sigma_s^2 - \int_{\mathbb{R}}(e^z - 1 - z)F_s(z)dz$, and this follows from the dynamics of X in (1.1) and an application of Itô's formula. We can approximate $x_s - \Delta x_{t^*}1_{\{s\geq t^*\}} - x_t$ with $\widetilde{x}_s = \widetilde{x}_s^c + \widetilde{x}_s^d$, where for $s \geq t$:

$$\widetilde{x}_s^c = \widetilde{a}_t(s-t) + \sigma_t(W_s - W_t), \quad \widetilde{x}_s^d = \int_t^s \int_E \delta^x(t,z)\widetilde{\mu}^x(du,dz). \tag{8.3}$$

The option prices at time t associated with terminal value $x_t + \widetilde{x}_{t+T}$ are denoted with $\widetilde{O}_{t,T}(k)$.

Using the above notation, we set for $u \in \mathbb{R}$ and $t \leq t^*$:

$$\mathcal{L}_{t,T}(u) = \mathbb{E}_t^{\mathbb{Q}}(e^{iu(x_{t+T}-x_t)}), \quad \widetilde{\mathcal{L}}_{t,T}(u) = \mathbb{E}_t^{\mathbb{Q}}(e^{iu\widetilde{x}_{t+T}}), \quad \widetilde{\mathcal{L}}_t^{fd}(u) = \mathbb{E}_{t-}^{\mathbb{Q}}(e^{iu\Delta x_{t^*}}). \tag{8.4}$$

Finally, we will make use of the following decomposition

$$\widehat{\mathcal{L}}_{t,T_l}(u) - \mathcal{L}_{t,T_l}(u) = \sum_{j=1}^{3} \eta_{t,T_l}^{(j)}(u), \quad l = 1,2, \tag{8.5}$$

with $\eta_{t,T_l}^{(j)}(u) = -(u^2 + iu)\overline{\eta}_{t,T_l}^{(j)}(u)$, and where

$$\overline{\eta}_{t,T_l}^{(1)}(u) = \sum_{j=2}^{N_l} h_t(u, k_{l,j-1}, k_{l,j})\epsilon_{t,T_l}(k_{l,j}), \tag{8.6}$$

$$\overline{\eta}_{t,T_l}^{(2)}(u) = \sum_{j=2}^{N_l} \int_{k_{l,j-1}}^{k_{l,j}} e^{(iu-1)k-iux_t}(O_{t,T_l}(k_{l,j-1}) - O_{t,T_l}(k))dk, \tag{8.7}$$

$$\overline{\eta}_{t,T_l}^{(3)}(u) = \int_{k<k_{l,1}\,\cup\, k>k_{l,N_l}} e^{(iu-1)k-iux_t}O_{t,T_l}(k)dk. \tag{8.8}$$

8.2. Notation for the K and C operators of Theorem 1

We denote

$$\zeta(u) = -(u^2 + iu), \quad u \in \mathbb{R}, \tag{8.9}$$

$$\widetilde{\Phi}(k) = f(k) + |k|\Phi(-|k|), \quad k \in \mathbb{R}, \tag{8.10}$$

where f and Φ are the pdf and cdf of a standard normal random variable. We further set

$$\overline{\gamma}_{t,i} = \sum_{h=-M}^{M} \gamma_{t,i}(h), \; i = 1, 2, \tag{8.11}$$

where $\gamma_{t,i}(h)$ is defined in (4.9). With this notation, the kernels of the operators K and C are given by

$$k(z,u) = \sum_{l=1}^{2} (1_{\{l=1\}} + \tau^2 1_{\{l=2\}})\zeta(z)\zeta(u)\overline{\gamma}_{t^*,l}\sigma_{t^*}^3 \xi_{t^*,l}^2(0)\psi_l(0) \int_{\mathbb{R}} \widetilde{\Phi}^2(k)dk, \tag{8.12}$$

$$c(z,u) = \sum_{l=1}^{2} (1_{\{l=1\}} + \tau^2 1_{\{l=2\}})\zeta(z)\overline{\zeta(u)}\overline{\gamma}_{t^*,l}\sigma_{t^*}^3 \xi_{t^*,l}^2(0)\psi_l(0) \int_{\mathbb{R}} \widetilde{\Phi}^2(k)dk. \tag{8.13}$$

8.3. Preliminary Results

Lemma 1. *Assume A1-A4 hold. For $t \in [t^* - \varepsilon, t^*)$ with $\varepsilon > 0$ arbitrary small, there exist $\mathcal{F}_t^{(0)}$-adapted random variables C_t and $\bar{t} > t$, that do not depend on u and T and satisfy $\sup_{t \in [t^*-\varepsilon, t^*)} C_t < \infty$ and $\inf_{t \in [t^*-\varepsilon, t^*)} \bar{t} > t^*$, such that for $T < \bar{t} - t$ we have*

$$|\widetilde{\mathcal{L}}_{t,T}(u) - 1| \le C_t(|u|^2 \vee 1)T, \tag{8.14}$$

$$|\mathcal{L}_{t,T}(u) - \widetilde{\mathcal{L}}_t^{fd}(u)| \le C_t(|u| \vee 1)\sqrt{T}, \quad \textit{almost surely}, \tag{8.15}$$

and if in addition $\nu(\{t^*\}, \mathbb{R}) = 0$, then

$$\left|\mathcal{L}_{t,T}(u) - \widetilde{\mathcal{L}}_{t,T}(u)\right| \le C_t(|u|^2 \vee 1)T^{3/2}. \tag{8.16}$$

Proof of Lemma 1. Throughout the proof, C_t will denote an $\mathcal{F}_t^{(0)}$-adapted random variable satisfying the conditions in the statement of the lemma, and it can further change from line to line. We will assume in addition, without loss of generality, that $T < \bar{t} - t$, for $\bar{t}$ being the random variable in assumptions A2-A4.

The first bound follows trivially upon noticing that by application of Lévy-Khintchine formula (Theorem 8.1 in Sato (1999)), we can write

$$\widetilde{\mathcal{L}}_{t,T}(u) = \exp\left(iuT\widetilde{a}_t - T\frac{u^2}{2}\sigma_t^2 + T\int_{\mathbb{R}} (e^{iuz} - iuz - 1)F_t(z)dz\right), \tag{8.17}$$

and using the fact that $\widetilde{a}_t$, σ_t and $\int_{\mathbb{R}} |z|^2 F_t(z)dz$ have càdlàg paths.

We continue with (8.16). Using Taylor series expansion, we have:

$$\begin{aligned}&\left|\mathbb{E}_t^{\mathbb{Q}}(e^{iu(x_{t+T}-x_t)} - e^{iu\widetilde{x}_{t+T}}) - iu\mathbb{E}_t^{\mathbb{Q}}(x_{t+T} - x_t - \widetilde{x}_{t+T})\right| \\ &\quad \leq C_t(|u|^2 \vee 1)\mathbb{E}_t^{\mathbb{Q}}\left[|x_{t+T} - x_t - \widetilde{x}_{t+T}|^2 + |x_{t+T} - x_t - \widetilde{x}_{t+T}||\widetilde{x}_{t+T}|\right].\end{aligned} \tag{8.18}$$

From here the result follows by applying Cauchy-Schwarz inequality, the integrability assumption for the the components of x as well as the following results

$$\left|\mathbb{E}_t^{\mathbb{Q}}(x_{t+T} - x_t - \widetilde{x}_{t+T})\right| \leq C_t T^{3/2},\ \mathbb{E}_t^{\mathbb{Q}}(x_{t+T} - x_t - \widetilde{x}_{t+T})^2 \leq C_t T^2, \tag{8.19}$$

which in turn follows from assumption A3. The last bound of the lemma then follows from applying the above two inequalities.

We finish with the bound in (8.15). First, using the second bound in (8.19), we have

$$\left|\mathcal{L}_{t,T}(u) - \mathbb{E}_t^{\mathbb{Q}}(e^{iu\Delta x_{t^*}})\right| \leq C_t(|u| \vee 1)\sqrt{T}. \tag{8.20}$$

From here, the bound in (8.15) follows by taking into account that $\mathbb{E}_t^{\mathbb{Q}}(e^{iu\Delta x_{t^*}}) = \widetilde{\mathcal{L}}_t^{fd}(u)$ almost surely.

□

Lemma 2. *Assume A1-A3 hold and $\nu(\{t^*, \mathbb{R}\}) = 0$. There exist $\mathcal{F}_t^{(0)}$-adapted random variables C_t and $\overline{t} > t$, that do not depend on T and satisfy $\sup_{t\in[t^*-\varepsilon,t^*]} C_t < \infty$ and $\inf_{t\in[t^*-\varepsilon,t^*]} \overline{t} > t^*$ with some arbitrary small $\varepsilon > 0$, such that for $T < \overline{t} - t$ we have*

$$O_{t,T}(k) \leq C_t \begin{cases} Te^{3(k-x_t)}, & \text{if } k - x_t < -1, \\ Te^{-(k-x_t)}, & \text{if } k - x_t > 1, \\ \sqrt{T} \bigwedge \frac{T}{|k-x_t|}, & \text{if } |k - x_t| \leq 1, \end{cases} \tag{8.21}$$

$$\left|O_{t,T}(k) - \widetilde{O}_{t,T}(k)\right| \leq C_t|\log(T)| \left(T^{3/2} \bigvee \left(\frac{T^{3/2}}{|e^{k-x_t} - 1|} \bigwedge T\right)\right), \tag{8.22}$$

$$|O_{t,T}(k_1) - O_{t,T}(k_2)| \leq C_t \left[\frac{T}{(k_2 - x_t)^4} \bigwedge \frac{T}{(k_2 - x_t)^2} \bigwedge 1\right] |e^{k_1} - e^{k_2}|, \tag{8.23}$$

where $k_1 < k_2 < x_t$ or $k_1 > k_2 > x_t$. In addition, for $|k - x_t| \leq \sqrt{T}|\log(T)|$, we have

$$\left|O_{t,T}(k) - e^{x_t}\sqrt{T}\sigma_t\widetilde{\Phi}\left(\frac{k - x_t}{\sqrt{T}\sigma_t}\right)\right| \leq C_t T \log^2(T). \tag{8.24}$$

Proof of Lemma 2. The first three bounds follow from Lemmas 2-7 in Qin and Todorov (2019) plus the fact that the bounding $\mathcal{F}_t^{(0)}$-adapted processes in these bounds have càdlàg paths. We show the last one here. Throughout the proof, C_t will denote an $\mathcal{F}_t^{(0)}$-adapted random variable satisfying the conditions in the statement of the lemma, and it can further change from line to line. We will assume in addition, without loss of generality, that $T < \bar{t} - t$, for $\bar{t}$ being the random variable in assumptions A2-A3.

By dividing the strike of the option and the option price by X_t, we can reduce the analysis to the case $x_t = 0$ and we do so in the rest of this proof. From the second result of the lemma, it is clear that it suffices to show the last inequality with $O_{t,T}(k)$ replaced by $\widetilde{O}_{t,T}(k)$. Now, if we denote with $\widetilde{O}^c_{t,T}(k)$, the counterpart of $\widetilde{O}_{t,T}(k)$ in which $\widetilde{x}_{t+T}$ is replaced with $\widetilde{x}^c_{t+T}$, then upon using assumption A1, we have

$$|\widetilde{O}_{t,T}(k) - \widetilde{O}^c_{t,T}(k)| \leq \mathbb{E}^{\mathbb{Q}}_t |e^{\widetilde{x}^d_{t+T}} - 1| \leq C_t T. \tag{8.25}$$

Next, direct calculation yields for $k > x_t = 0$:

$$\widetilde{O}^c_{t,T}(k) = e^{\widetilde{a}_t T + \sigma_t^2 T/2}\left(1 - \Phi\left(\frac{k - \widetilde{a}_t T}{\sqrt{T}\sigma_t} - \sqrt{T}\sigma_t\right)\right) - e^k\left(1 - \Phi\left(\frac{k - \widetilde{a}_t T}{\sqrt{T}\sigma_t}\right)\right), \tag{8.26}$$

with a similar result holding for the case $k \leq x_t = 0$. From here, using Taylor expansion, we get the result in (8.24). □

Lemma 3. *Assume A1-A4 hold and $\nu(\{t^*, \mathbb{R}\}) > 0$. There exist $\mathcal{F}^{(0)}_t$-adapted random variables C_t and $\bar{t} > t$, that do not depend on T and satisfy $\sup_{t\in[t^*-\varepsilon,t^*]} C_t < \infty$ and $\inf_{t\in[t^*-\varepsilon,t^*]} \bar{t} > t^*$ for some arbitrary small $\varepsilon > 0$, such that for $T < \bar{t} - t$ we have*

$$O_{t,T}(k) \leq C_t \begin{cases} e^{-3|k-x_t|}, & \text{if } k \leq x_t, \\ e^{-|k-x_t|}, & \text{if } k > x_t, \end{cases} \tag{8.27}$$

$$|O_{t,T}(k_1) - O_{t,T}(k_2)| \leq C_t |e^{k_1} - e^{k_2}| e^{-4|k_2 - x_t|}, \tag{8.28}$$

where $k_1 < k_2 < x_t$ or $k_1 > k_2 > x_t$.

Proof of Lemma 3. The first of the two bounds follows from the following algebraic inequalities (similar bounds are used in the proof of Theorem 2.1 in Lee (2004))

$$(e^k - e^{x_{t+T}})^+ \leq 2e^{x_t} e^{3(k-x_t)} e^{-2(x_{t+T}-x_t)},\ (e^{x_{t+T}} - e^k)^+ \leq 2e^{x_t} e^{-(k-x_t)} e^{2(x_{t+T}-x_t)}, \tag{8.29}$$

and assumption A2 for the existence of conditional moments of x_{t+T} combined with the fact that the process x_t has càdlàg paths. The second bound follows trivially from the following algebraic inequalities

$$|(e^x - e^{k_1})^+ - (e^x - e^{k_2})^+| \leq |e^{k_1} - e^{k_2}| 1_{\{x \geq \min\{k_1,k_2\}\}}, \tag{8.30}$$

$$|(e^{k_1} - e^x)^+ - (e^{k_2} - e^x)^+| \leq |e^{k_1} - e^{k_2}| 1_{\{x \leq \max\{k_1,k_2\}\}}, \tag{8.31}$$

for x, k_1, $k_2 \in \mathbb{R}$ together with assumption A2 for the existence of conditional moments of x_{t+T}. □

Lemma 4. *Assume A1-A6 hold and fix $h \in \mathbb{N}$. Suppose $t < t^* < t + T_2$ and $T_2 = \tau T_1$ for some $\tau \in (1, \kappa]$ (with κ being the constant in A6). Let $t \uparrow t^*$ together with $T_1 \downarrow 0$, $\Delta \asymp T_1^{\alpha}$, $\underline{K} \asymp T_1^{\beta}$, $\overline{K} \asymp T_1^{-\gamma}$, for $\beta, \gamma > 0$ and $\frac{1}{2} < \alpha < \frac{1}{2} + (1 \wedge 4\beta \wedge 4\gamma)$.*

If $\nu(\{t^*, \mathbb{R}\}) = 0$:

$$T_i^{-3/2} \Delta \widehat{\chi}_{t,i}(h) \xrightarrow{\mathbb{P}} \frac{e^{2x_t}}{\psi_i(0)} \xi^2_{t^*,i}(0) \sigma^3_{t^*} \overline{\chi}_{t^*,i}(h) \int_{\mathbb{R}} \widetilde{\Phi}^2(k) dk, \;\; i = 1, 2, \tag{8.32}$$

and if $\nu(\{t^*, \mathbb{R}\}) > 0$:

$$\widehat{\chi}_{t,i}(h) = O_p\left(\frac{1}{\Delta}\right), \text{ and } \Delta^{-1} \widehat{\chi}_{t,i}^{-1}(0) = O_p(1), \;\; i = 1, 2, \tag{8.33}$$

where $\overline{\chi}_{t,i}(h)$ is given in (4.8).

Proof of Lemma 4. In the proof, we will assume that $t > t^* - \varepsilon$ and $T_2 < \inf_{t \in [t^* - \varepsilon, t^*]} \overline{t} - t$, with ε and $\overline{t}$ being the ones that appear in the statements of Lemmas 1 and 2. Note that $\overline{t}$ is $\mathcal{F}^{(0)}$-adapted random variable which satisfies $\overline{t} > t^*$, so the above conditions for t and T_2 will hold for each $\omega^{(0)}$ in $\mathcal{F}^{(0)}$ when Δ and T_2 are sufficiently small (the values for Δ and T_1 for which this happens will depend on $\omega^{(0)}$). Throughout the proof, c_t and C_t will denote an $\mathcal{F}_t^{(0)}$-adapted random variables satisfying the conditions in the statement of Lemmas 1 and 2, and they can further change from line to line.

We can decompose

$$\widehat{\chi}_{t,i}(h) = \widehat{\chi}^{(1)}_{t,i}(h) + \widehat{\chi}^{(2)}_{t,i}(h) + \widehat{\chi}^{(3)}_{t,i}(h) + \widehat{\chi}^{(4)}_{t,i}(h), \;\; i = 1, 2, \tag{8.34}$$

where

$$\widehat{\chi}^{(1)}_{t,i}(h) = -\frac{1}{N_i - 1} \left(\sum_{j=1}^{N_i} \widehat{\epsilon}_{t,T_i}(k_{i,j}) 1_{\{j \neq J_i^*\}} \right)^2 \tag{8.35}$$

$$\begin{aligned} \widehat{\chi}^{(2)}_{t,i}(h) = \sum_{j=h+1}^{N_i} \Bigg[& \left(O_{t,T_i}(k_{i,j}) - \frac{1}{2} O_{t,T_i}(k_{i,j-1}) - \frac{1}{2} O_{t,T_i}(k_{i,j+1}) \right) \\ & \times \left(O_{t,T_i}(k_{i,j-h}) - \frac{1}{2} O_{t,T_i}(k_{i,j-h-1}) - \frac{1}{2} O_{t,T_i}(k_{i,j-h+1}) \right) 1_{\{j \neq J_i^*, j \neq J_i^* + h\}} \Bigg], \end{aligned} \tag{8.36}$$

$$\begin{aligned} \widehat{\chi}^{(3)}_{t,i}(h) = \sum_{j=h+1}^{N_i} \Bigg[& \left(\epsilon_{t,T_i}(k_{i,j}) - \frac{1}{2} \epsilon_{t,T_i}(k_{i,j-1}) - \frac{1}{2} \epsilon_{t,T_i}(k_{i,j+1}) \right) \\ & \times \left(\epsilon_{t,T_i}(k_{i,j-h}) - \frac{1}{2} \epsilon_{t,T_i}(k_{i,j-h-1}) - \frac{1}{2} \epsilon_{t,T_i}(k_{i,j-h+1}) \right) 1_{\{j \neq J_i^*, j \neq J_i^* + h\}} \Bigg], \end{aligned} \tag{8.37}$$

and $\widehat{\chi}^{(4)}_{t,i}(h)$ is defined as the residual term $\widehat{\chi}_{t,i}(h) - \widehat{\chi}^{(1)}_{t,i}(h) - \widehat{\chi}^{(2)}_{t,i}(h) - \widehat{\chi}^{(3)}_{t,i}(h)$. Starting with $\widehat{\chi}^{(1)}_{t,i}(h)$, we have the following inequality

$$\begin{aligned}&\Big|\sum_{j=1}^{N_i}\widehat{\epsilon}_{t,T_i}(k_{i,j})1_{\{j\neq J_i^*\}}\Big|\\&\leq C\sum_{j=2}^{N_i}|O_{t,T_i}(k_{i,j})-O_{t,T_i}(k_{i,j-1})|+C\sum_{j=1,2,J^*-1,J^*,J^*+1}|\epsilon_{t,T_i}(k_{i,j})|,\end{aligned}\tag{8.38}$$

for $i = 1, 2$ and some constant C. From here, using assumptions A5 and A6, the growth condition for $|\underline{k}|$ and $\overline{k}$ as well as Lemma 2, we have

$$\widehat{\chi}^{(1)}_{t,i}(h) = O_p\left(\frac{T_i\Delta}{|\log(T_i)|}\right),\ \text{if } \nu(\{t^*,\mathbb{R}\}) = 0,\ i = 1, 2,\tag{8.39}$$

and upon using further Lemma 3, we have

$$\widehat{\chi}^{(1)}_{t,i}(h) = O_p(\Delta|\log(T_i)|),\ \text{if } \nu(\{t^*,\mathbb{R}\}) > 0,\ i = 1, 2.\tag{8.40}$$

For $\widehat{\chi}^{(2)}_{t,i}(h)$, using Lemma 2, we have

$$\widehat{\chi}^{(2)}_{t,i}(h) = O_p(\sqrt{T_i}\Delta),\ \text{if } \nu(\{t^*,\mathbb{R}\}) = 0,\ i = 1, 2,\tag{8.41}$$

and using Lemma 3, we have

$$\widehat{\chi}^{(2)}_{t,i}(h) = O_p(\Delta),\ \text{if } \nu(\{t^*,\mathbb{R}\}) > 0,\ i = 1, 2.\tag{8.42}$$

Using next the M-dependence assumption for the observation error in A6 as well as the algebraic inequality $2xy \leq x^2 + y^2$ for $x, y \in \mathbb{R}$, we have for $i = 1, 2$:

$$\mathbb{E}\left(\left(\widehat{\chi}^{(4)}_{t,i}(h)\right)^2|\mathcal{F}^{(0)}\right) \leq C_t\sum_{l=-h-2}^{h+2}\sum_{j=2}^{N_i}(O_{t,T_i}(k_{i,j}) - O_{t,T_i}(k_{i,j-1}))^2O^2_{t,T_i}(k_{i,j+l}),\tag{8.43}$$

where C_t is some $\mathcal{F}^{(0)}_t$-adapted random variable and we set to zero $O_{t,T_i}(k_{i,j})$ for $j < 1$ and $j > N_i$. From here by using Lemma 2, we have

$$\widehat{\chi}^{(4)}_{t,i}(h) = O_p(T_i^{3/4}\sqrt{\Delta}),\ \text{if } \nu(\{t^*,\mathbb{R}\}) = 0,\ i = 1, 2,\tag{8.44}$$

and using Lemma 3, we have

$$\widehat{\chi}^{(4)}_{t,i}(h) = O_p(\sqrt{\Delta}),\ \text{if } \nu(\{t^*,\mathbb{R}\}) > 0,\ i = 1, 2.\tag{8.45}$$

Altogether, using the restriction $\alpha > \frac{1}{2}$, we have

$$\widehat{\chi}^{(1)}_{t,i}(h) + \widehat{\chi}^{(2)}_{t,i}(h) + \widehat{\chi}^{(4)}_{t,i}(h) = o_p\left(\frac{T_i^{3/2}}{\Delta}\right),\ \text{if } \nu(\{t^*,\mathbb{R}\}) = 0,\ i = 1, 2,\tag{8.46}$$

and

$$\widehat{\chi}^{(1)}_{t,i}(h) + \widehat{\chi}^{(2)}_{t,i}(h) + \widehat{\chi}^{(4)}_{t,i}(h) = o_p\left(\frac{1}{\Delta}\right),\ \text{if } \nu(\{t^*,\mathbb{R}\}) > 0,\ i = 1,2. \tag{8.47}$$

We are left with $\widehat{\chi}^{(3)}_{t,i}(h)$. Using again the M-dependence assumption for the observation error in A6, we have

$$\mathbb{E}\left[\left(\widehat{\chi}^{(3)}_{t,i}(h) - \mathbb{E}\left(\widehat{\chi}^{(3)}_{t,i}(h)\big|\mathcal{F}^{(0)}\right)\right)^2\bigg|\mathcal{F}^{(0)}\right] \le C_t \sum_{j=1}^{N_i} O^4_{t,T_i}(k_{i,j}),\ i = 1,2. \tag{8.48}$$

From here by using Lemma 2, we have

$$\widehat{\chi}^{(3)}_{t,i}(h) - \mathbb{E}\left(\widehat{\chi}^{(3)}_{t,i}(h)\big|\mathcal{F}^{(0)}\right) = O_p\left(\frac{T_i^{5/4}}{\sqrt{\Delta}}\right),\ \text{if } \nu(\{t^*,\mathbb{R}\}) = 0,\ i = 1,2, \tag{8.49}$$

and using Lemma 3, we have

$$\widehat{\chi}^{(3)}_{t,i}(h) - \mathbb{E}\left(\widehat{\chi}^{(3)}_{t,i}(h)\big|\mathcal{F}^{(0)}\right) = O_p\left(\frac{1}{\sqrt{\Delta}}\right),\ \text{if } \nu(\{t^*,\mathbb{R}\}) > 0,\ i = 1,2. \tag{8.50}$$

We next expand $\mathbb{E}\left(\widehat{\chi}^{(3)}_{t,i}(h)\big|\mathcal{F}^{(0)}\right)$ in the case $\nu(\{t^*,\mathbb{R}\}) = 0$. First, using A6, we have

$$\begin{aligned}&\mathbb{E}\left(\epsilon_{t,T_i}(k_{i,j})\epsilon_{t,T_i}(k_{i,j-h})\big|\mathcal{F}^{(0)}\right)\\ &\quad = \xi_{t,T_i}(k_{i,j}-x_t)O_{t,T_i}(k_{i,j})\xi_{t,T_i}(k_{i,j-h}-x_t)O_{t,T_i}(k_{i,j-h})\gamma_{t,i}(h),\quad i = 1,2.\end{aligned} \tag{8.51}$$

This result and the first bound in Lemma 2 imply

$$\left|\mathbb{E}\left(\epsilon_{t,T_i}(k_{i,j})\epsilon_{t,T_i}(k_{i,j-h})\big|\mathcal{F}^{(0)}\right)\right| \le C_t T_i,\quad i = 1,2. \tag{8.52}$$

Further, for $|k_{i,j} - x_t| \le \sqrt{T_i}|\log(T_i)|$ and T_i sufficiently small, using the first and third bounds in Lemma 2 as well as the smoothness assumption for $\zeta_{t,T_i}(k)$ as a function of k in A6, we have

$$\begin{aligned}&\left|\mathbb{E}\left(\epsilon_{t,T_i}(k_{i,j})\epsilon_{t,T_i}(k_{i,j-h})\big|\mathcal{F}^{(0)}\right) - \xi_{t,T_i}(k_{i,j}-x_t)^2 O^2_{t,T_i}(k_{i,j})\gamma_{t,i}(h)\right|\\ &\quad \le C_t\sqrt{T_i}\Delta + C_t T_i \Delta^{\iota},\ \text{for } |k_{i,j}-x_t| \le \sqrt{T_i}|\log(T_i)|,\ i = 1,2,\end{aligned} \tag{8.53}$$

where ι is the constant of assumption A6. The smoothness assumption for $\xi_{t,T_i}(k)$ as a function of k and the first bound in Lemma 2 further imply

$$\begin{aligned}&|(\xi_{t,T_i}(k_{i,j}-x_t)^2 - \xi_{t,T_i}(0)^2)O^2_{t,T_i}(k_{i,j})|\\ &\quad \le C_t T_i(\sqrt{T_i}|\log(T_i)|)^{\iota},\ \text{for } |k_{i,j}-x_t| \le \sqrt{T_i}|\log(T_i)|,\ i = 1,2.\end{aligned} \tag{8.54}$$

Finally, for $|k_{i,j}-x_t| > \sqrt{T_i}|\log(T_i)|$, an application of the first bound in Lemma 2 implies

$$O_{t,T_i}(k_{i,j}) \leq C_t e^{-\alpha|k_{i,j}-x_t|}\frac{T_i}{|k_{i,j}-x_t|},\ \text{for } |k_{i,j}-x_t| > \sqrt{T_i}|\log(T_i)|,\ i=1,2, \tag{8.55}$$

and some $\alpha \in (0,1)$. Combining the results in (8.51)-(8.55), using A5 and taking into account the asymptotic order of $\underline{k}$ and $\overline{k}$ assumed in the statement of the lemma, we have altogether for the case $\nu(\{t^*,\mathbb{R}\})=0$:

$$\begin{aligned}\mathbb{E}\left(\widehat{\chi}^{(3)}_{t,i}(h)\big|\mathcal{F}^{(0)}\right) &= \xi^2_{t,i}(0)\overline{\chi}_{t,i}(h)\sum_{j:|k_{i,j}-x_t|\leq\sqrt{T_i}|\log(T_i)|} O^2_{t,T_i}(k_{i,j}) \\ &\quad + O_p\left(\frac{T_i^{3/2}}{\Delta|\log(T_i)|}\bigvee\frac{T_i^{3/2+\iota/2}|\log(T_i)|^{1+\iota}}{\Delta}\bigvee T_i|\log(T_i)|\right),\ i=1,2,\end{aligned} \tag{8.56}$$

where recall that $\overline{\chi}_{t,i}(h)$ is defined in (4.8). Next, using the second and fourth bounds in Lemma 2, and denoting the set

$$\overline{I}^{(i)}_l := \{j=2,...,N_i : |k_{i,j}-x_t| \leq |\log(T_i)|\sqrt{T_i}\},\quad i=1,2, \tag{8.57}$$

we have for $\nu(\{t^*,\mathbb{R}\})=0$:

$$\sum_{j\in\overline{I}^{(i)}_l} O^2_{t,T_i}(k_{i,j}) = T_i e^{2x_t}\sigma_t^2\sum_{j\in\overline{I}^{(i)}_l}\widetilde{\Phi}^2\left(\frac{k_{i,j}-x_t}{\sqrt{T_i}\sigma_t}\right) + O_p\left(\frac{T_i^2|\log(T_i)|^3}{\Delta}\right),\ i=1,2, \tag{8.58}$$

where the function $\widetilde{\Phi}$ is defined in (8.10). Using assumption A5, we have

$$\begin{aligned}\sum_{j\in\overline{I}^{(i)}_l}\widetilde{\Phi}^2\left(\frac{k_{i,j}-x_t}{\sqrt{T_i}\sigma_t}\right) &= \frac{1}{\psi_i(0)\Delta}\sum_{j\in\overline{I}^{(i)}_l}\widetilde{\Phi}^2\left(\frac{k_{i,j}-x_t}{\sqrt{T_i}\sigma_t}\right)(k_{i,j}-k_{i,j-1}) \\ &\quad + o_p(1)\times\sum_{j\in\overline{I}^{(i)}_l}\widetilde{\Phi}^2\left(\frac{k_{i,j}-x_t}{\sqrt{T_i}\sigma_t}\right),\ i=1,2,\end{aligned} \tag{8.59}$$

and using the smoothness of $\widetilde{\Phi}(k)$ and upon changing variable of integration, we have

$$\begin{aligned}\sum_{j\in\overline{I}^{(i)}_l}\widetilde{\Phi}^2\left(\frac{k_{i,j}-x_t}{\sqrt{T_i}\sigma_t}\right)(k_{i,j}-k_{i,j-1}) &= \sqrt{T_i}\int_{-|\log(T_i)|}^{|\log(T_i)|}\widetilde{\Phi}^2\left(\frac{k}{\sigma_t}\right)dk + O_p(\Delta|\log(T_i)|) \\ &= \sqrt{T_i}\sigma_t\int_{\mathbb{R}}\widetilde{\Phi}^2(k)dk + O_p(\Delta|\log(T_i)|\vee T_i^2),\ i=1,2.\end{aligned} \tag{8.60}$$

Combining the results in (8.58)-(8.60), we get for the case $\nu(\{t^*, \mathbb{R}\}) = 0$ and for $i = 1, 2$:

$$\sum_{j:|k_{i,j}-x_t|\leq\sqrt{T_i}|\log(T_i)|} O^2_{t,T_i}(k_{i,j}) = \frac{T_i^{3/2}}{\psi_i(0)\Delta} e^{2x_t}\sigma_t^3 \int_{\mathbb{R}} \widetilde{\Phi}^2(k)dk + o_p\left(\frac{T_i^{3/2}}{\Delta}\right). \quad (8.61)$$

For the case $\nu(\{t^*, \mathbb{R}\}) > 0$, we can use assumption A5 and Lemma 3 and conclude

$$\frac{c_t}{\Delta}\int_{\mathbb{R}} O^2_{t,T_i}(k)dk \leq \mathbb{E}\left(\widehat{\chi}^{(3)}_{t,i}(h)\big|\mathcal{F}^{(0)}\right) \leq \frac{C_t}{\Delta}\int_{\mathbb{R}} O^2_{t,T_i}(k)dk,\ \ i = 1, 2, \quad (8.62)$$

for some $\mathcal{F}^{(0)}_t$-adapted random variables $c_t > 0$ and $C_t > 0$. We note further that

$$c_t \leq \int_{\mathbb{R}} O^2_{t,T_i}(k)dk \leq C_t,\ \ i = 1, 2, \quad (8.63)$$

again for some $\mathcal{F}^{(0)}_t$-adapted random variables $c_t > 0$ and $C_t > 0$. The upper bound in the above inequality follows from Lemma 3. For the lower bound, we can first use the fact that $\mathbb{E}^{\mathbb{Q}}_t(X_{t+T_i} - X_t)^2 = 2\int_{\mathbb{R}} e^k O_{t,T_i}(k)dk > 0$. This is because $\nu(\{t^*, \mathbb{R}\}) > 0$. Therefore, there exists a region for the log-strike with positive Lebesgue measure where $O_{t,T_i}(k)$ is bounded from below by an $\mathcal{F}^{(0)}_t$-adapted strictly positive random variable. From here, the lower bound for $\int_{\mathbb{R}} O^2_{t,T_i}(k)dk$ follows.

Combining the bounds in (8.46)-(8.63) and upon making use of the fact that when $\nu(\{t^*, \mathbb{R}\}) = 0$, σ_t, $\xi_{t,i}$ and $\overline{\chi}_{t,i}(h)$ are almost surely continuous at $t = t^*$, we get the two results of the lemma. □

8.4. Proof of Theorem 1

In the proof (including of Lemma 5 below), we will assume that $t > t^* - \varepsilon$ and $T_2 < \inf_{t\in[t^*-\varepsilon,t^*]} \bar{t} - t$, with ε and $\bar{t}$ being the ones that appear in the statements of Lemmas 1 and 2. Throughout the proof, C_t will denote an $\mathcal{F}^{(0)}_t$-adapted random variable satisfying the conditions in the statement of Lemmas 1 and 2, and it can further change from line to line.

We start with establishing a few preliminary results. First, using Lemma 2, we have

$$\sup_{u\in\mathbb{R}} |\overline{\eta}^{(2)}_{t,T_l}(u)| \leq C_t\sqrt{T_l}\Delta,\quad \sup_{u\in\mathbb{R}} |\overline{\eta}^{(3)}_{t,T_l}(u)| \leq C_t T_l e^{-2(|\underline{k}|\vee\bar{k})},\quad l = 1, 2. \quad (8.64)$$

Next, making use of assumption A6 as well as Lemma 2, Burkholder-Davis-Gundy inequality, inequality in means as well as Tonelli's inequality, and taking into account that $\alpha > \frac{1}{2}$, we have

$$\mathbb{E}\left(||G(u)|\eta^{(1)}_{t,T_l}(u)|^p||^2\big|\mathcal{F}^{(0)}\right) = O_p(T_1^{p+1/2}\Delta^{2p-1}),\quad l = 1, 2, \quad (8.65)$$

for any deterministic function G with at most polynomial growth for $|u|$ increasing and any $1 \leq p < \kappa$, for κ being the constant in assumption A6. We further have by making use of assumption A5 and Lemma 2, and taking into account again that $\alpha > \frac{1}{2}$:

$$\begin{aligned}\mathbb{E}\left(\sup_{u\in\mathbb{R}}|\overline{\eta}^{(1)}_{t,T_l}(u)|\big|\mathcal{F}^{(0)}\right) &\leq C_t\sum_{j=2}^{N_l} e^{-k_{l,j-1}}|\xi_{t,l}(k_{l,j-1}-x_t)|O_{t,T_l}(k_{l,j-1})\Delta \\ &\leq C_t|\log(T_1)|T_1, \quad l=1,2.\end{aligned} \tag{8.66}$$

With these bounds, we can now turn to the proof of the theorem. We denote

$$\widehat{Z} = \sum_{j=2}^{N_2}\widehat{Z}^{(2)}_j - \tau\sum_{j=2}^{N_1}\widehat{Z}^{(1)}_j, \tag{8.67}$$

where

$$\widehat{Z}^{(i)}_j = -(u^2+iu)h_t(u,k_{i,j-1},k_{i,j})\epsilon_{t,T_i}(k_{i,j}), \quad j=1,...,N_i, \ i=1,2, \tag{8.68}$$

and we split the difference $\widehat{\mathcal{L}}_{t,T_2}(u) - \widehat{\mathcal{L}}_{t,T_1}(u)^\tau - \widehat{Z}$ into the following components:

$$R_1(u) = \tau\mathcal{L}_{t,T_1}(u)^{\tau-1}(\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)) - (\widehat{\mathcal{L}}_{t,T_1}(u)^\tau - \mathcal{L}_{t,T_1}(u)^\tau), \tag{8.69}$$

$$R_2(u) = -\tau(\mathcal{L}_{t,T_1}(u)^{\tau-1}-1)(\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)), \tag{8.70}$$

$$R_3(u) = \mathcal{L}_{t,T_2}(u) - \mathcal{L}_{t,T_1}(u)^\tau, \quad R_4(u) = \sum_{j=2}^{3}\eta^{(j)}_{t,T_2}(u) - \tau\sum_{j=2}^{3}\eta^{(j)}_{t,T_1}(u). \tag{8.71}$$

For R_1 and R_2, using Taylor expansion (note that the power function of a complex variable is analytic outside the negative real axis) and making use of the fact that $|\mathcal{L}_{t,T_1}(u)| \leq 1$, we have

$$|R_1(u)| \leq C\left[|\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)|^\tau + |\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)|^2\right], \tag{8.72}$$

$$|R_2(u)| \leq C|\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)||\mathcal{L}_{t,T_1}(u)-1|^{(\tau-1)\wedge 1}, \tag{8.73}$$

for some positive constant $C > 0$. From here, using the bounds for $\{\eta^{(j)}_{t,T}(u)\}_{j=1,2,3}$ in (8.64)-(8.66) as well as Lemma 1 (note that $\tau \in (1,\kappa]$), we have

$$||R_1|| = O_p\left(T_1^{\frac{\tau\wedge 2}{2}+\frac{1}{4}}\Delta^{\tau\wedge 2-\frac{1}{2}}\bigvee\left(\sqrt{T_1}\Delta\right)^{\tau\wedge 2}\bigvee\left(T_1e^{-2(|\underline{k}|\vee\overline{k})}\right)^{\tau\wedge 2}\right), \tag{8.74}$$

$$||R_2|| = O_p\left(\left[T_1^{3/4}\sqrt{\Delta}\bigvee\sqrt{T_1}\Delta\bigvee T_1e^{-2(|\underline{k}|\vee\overline{k})}\right]T_1^{(\tau-1)\wedge 1}\right), \tag{8.75}$$

Using again the bounds for $\{\eta^{(j)}_{t,T}(u)\}_{j=1,2,3}$ as well as Lemma 1, we have

$$||R_3|| = O_p(T_1^{3/2}), \quad ||R_4|| = O_p\left(T_1e^{-2(|\underline{k}|\vee\overline{k})}\bigvee\sqrt{T_1}\Delta\right). \tag{8.76}$$

Altogether, we get,

$$\begin{aligned}&\widehat{\mathcal{L}}_{t,T_2}(u) - \widehat{\mathcal{L}}_{t,T_1}(u)^\tau - \widehat{Z} \\ &= O_p\left(T_1^{\frac{\tau\wedge 2}{2}+\frac{1}{4}}\Delta^{\tau\wedge 2-\frac{1}{2}} \bigvee T_1^{\frac{3}{4}+(\tau-1)\wedge 1}\sqrt{\Delta} \bigvee \sqrt{T_1}\Delta \bigvee T_1 e^{-2(|\underline{k}|\vee\overline{k})} \bigvee T_1^{3/2}\right),\end{aligned} \tag{8.77}$$

and we note that $\tau > 1$, and under the conditions of the theorem for the rate of growth of Δ and $|\underline{k}| \vee \overline{k}$, we can therefore write

$$\widehat{\mathcal{L}}_{t,T_2}(u) - \widehat{\mathcal{L}}_{t,T_1}(u)^\tau - \widehat{Z} = o_p\left(T_1^{3/4}\sqrt{\Delta}\right), \tag{8.78}$$

and hence the result of the theorem will follow upon showing the following result.

Lemma 5. *Assume A1-A5 hold. Suppose $t < t^* < t + T_2$ and $T_2 = \tau T_1$ for some $\tau \in (1, \kappa]$ (with κ being the constant in A6). Let $t \uparrow t^*$ together with $T_1 \downarrow 0$, $\Delta \asymp T_1^\alpha$, where $\alpha > \frac{1}{2}$. Then, we have*

$$\frac{1}{T_1^{3/4}\sqrt{\Delta}}\widehat{Z} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \mathcal{CN}(0, K, C). \tag{8.79}$$

Proof of Lemma 5. Using a subsequence criterion for convergence in probability, we need to show that for all $\omega^{(0)}$ and every subsequence, there is a further subsequence along which we have $\frac{1}{T_1^{3/4}\sqrt{\Delta}}\widehat{Z}(\omega^{(0)})$ converge in distribution to $\mathcal{CN}(0, K(\omega^{(0)}), C(\omega^{(0)}))$. Using Bessel's inequality, dominated convergence and the bounds of Lemma 2, we have under the conditions of the lemma

$$\limsup_{T_1\downarrow 0} \frac{1}{T_1^{3/2}\Delta}\mathbb{E}\left(\sum_{j>J}\langle\widehat{Z}, e_j\rangle^2 \middle| \mathcal{F}^{(0)}\right) \to 0, \quad \text{as } J \to \infty, \tag{8.80}$$

where $\{e_j\}_{j\geq 1}$ denotes an orthonormal basis in $\mathcal{L}^2(w)$. This means that the sequence is asymptotically finite-dimensional, see 1.8 in Vaart and Wellner (1996). Therefore, the limit result of the theorem will follow from Theorem 1.8.4 in Vaart and Wellner (1996) if we can establish

$$\frac{1}{T_1^{3/4}\sqrt{\Delta}}\langle\widehat{Z}, h\rangle \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \langle Z, h\rangle, \tag{8.81}$$

for Z denoting the limit of Theorem 1 and h an arbitrary element in $\mathcal{L}^2(w)$. Since Z is $\mathcal{F}^{(0)}$-conditionally $\mathcal{CN}(0, K, C)$, we have

$$\mathbb{E}\left(\langle Z, h\rangle^2 \middle| \mathcal{F}^{(0)}\right) = \langle Kh, h\rangle, \quad \mathbb{E}\left(\langle Z, h\rangle\overline{\langle Z, h\rangle} \middle| \mathcal{F}^{(0)}\right) = \langle h, C\overline{h}\rangle. \tag{8.82}$$

Since the observation errors have $\mathcal{F}^{(0)}$-conditional dependence, to establish finite-dimensional convergence, we will apply a big block-small block approach. More specifically, we denote with $l_T^{(i)}$, for $i = 1, 2$, two $\mathcal{F}^{(0)}$-adapted sequences of integers increasing to infinity. We further set $r_T^{(i)} = \lfloor N_i/l_T^{(i)} \rfloor$, for $i = 1, 2$. Since $l_T^{(i)}$ are increasing, without loss of generality, we will assume $l_T^{(i)} > 2M$, for $i = 1, 2$, almost surely. We then split

$$\begin{aligned}\langle \widehat{Z}, h\rangle = & \sum_{j=1}^{r_T^{(2)}} \widehat{A}_j^{(2)} - \tau \sum_{j=1}^{r_T^{(1)}} \widehat{A}_j^{(1)} + \sum_{j=1}^{r_T^{(2)}} \widehat{B}_j^{(2)} - \tau \sum_{j=1}^{r_T^{(1)}} \widehat{B}_j^{(1)} \\ & + \sum_{k=r_T^{(2)} l_T^{(2)}+1}^{N_2} \langle \widehat{Z}_k^{(2)}, h\rangle - \tau \sum_{k=r_T^{(1)} l_T^{(1)}+1}^{N_1} \langle \widehat{Z}_k^{(1)}, h\rangle,\end{aligned} \tag{8.83}$$

where we set

$$\widehat{A}_j^{(i)} = \sum_{k \in L_j^{(i)}} \langle \widehat{Z}_k^{(i)}, h\rangle, \ \widehat{B}_j^{(i)} = \sum_{k \in S_j^{(i)}} \langle \widehat{Z}_k^{(i)}, h\rangle, \ j = 1, ..., r_T^{(i)}, \ i = 1, 2, \tag{8.84}$$

and for $j = 1, ..., r_T^{(i)}$ and $i = 1, 2$, we denote the sets

$$S_j^{(i)} = \{(j-1)l_T^{(i)} + 1, ..., (j-1)l_T^{(i)} + M\}, \ L_j^{(i)} = \{(j-1)l_T^{(i)} + M + 1, ..., j l_T^{(i)}\}. \tag{8.85}$$

We choose $l_T^{(i)}$ such that the following rate conditions are satisfied:

$$\frac{|\log(T_i)|^2}{\sqrt{T_i} l_T^{(i)}} \to \infty \text{ and } \frac{|\log(T_i)|}{\sqrt{T_i} l_T^{(i)}} \to 0, \ i = 1, 2, \ a.s. \tag{8.86}$$

We note that because of our rate conditions on the asymptotic order of T_1, T_2, Δ, $\underline{k}$ and $\overline{k}$ as well as assumption A5 for the strike grid, this will imply $\frac{N_i}{l_T^{(i)}} \to \infty$ almost surely.

Using Lemma 2 and the fact that $l_T^{(i)}\Delta \to 0$ almost surely (because of the above condition on $l_T^{(i)}$), we have

$$\sum_{k=r_T^{(i)} l_T^{(i)}+1}^{N_i} \langle \widehat{Z}_k^{(i)}, h\rangle = O_p(l_T^{(i)} T_i \Delta) = o_p(T_i^{3/4}\sqrt{\Delta}), \ i = 1, 2, \tag{8.87}$$

where the last equality follows because of the rate condition on $l_T^{(i)}$ as well as the restriction $\alpha > 1/2$ of the theorem. Next, using Lemma 2, the $\mathcal{F}^{(0)}$-conditional M-dependence of the observations errors from assumption A6 as well as the fact that $l_T^{(i)} > 2M$, we have

$$\mathbb{E}\left(\left|\sum_{j=1}^{r_T^{(2)}} \widehat{B}_j^{(i)}\right|^2 \middle| \mathcal{F}^{(0)}\right) = \mathbb{E}\left(\sum_{j=1}^{r_T^{(2)}} \left|\widehat{B}_j^{(i)}\right|^2 \middle| \mathcal{F}^{(0)}\right) \le C_t \frac{N_i T_i \Delta^2}{l_T^{(i)}} = o_p(T_i^{3/2}\Delta), \tag{8.88}$$

for $i = 1, 2$ and an $\mathcal{F}^{(0)}$-adapted random variable C_t. The last inequality in (8.88) follows from the second rate condition for $l_T^{(i)}$ in (8.86) as well as the assumption in the theorem regarding the asymptotic size of $\underline{k}$ and $\overline{k}$.

Given the above two results in (8.87) and (8.88), the finite-dimensional CLT result in (8.81) will be shown if we can establish:

$$\frac{1}{T_1^{3/4}\sqrt{\Delta}}\widehat{A} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \langle Z, h\rangle, \tag{8.89}$$

where

$$\widehat{A} = \sum_{j=1}^{r_T^{(2)}} \widehat{A}_j^{(2)} - \tau \sum_{j=1}^{r_T^{(1)}} \widehat{A}_j^{(1)}. \tag{8.90}$$

This CLT, in turn, will hold by application of Theorem VIII.5.25 of Jacod and Shiryaev (2003), if we can establish the following convergence results:

$$\frac{1}{T_1^{3/2}\Delta}\mathbb{E}\left(\widehat{A}^2\big|\mathcal{F}^{(0)}\right) \overset{\mathbb{P}}{\to} \langle Kh, h\rangle, \quad \frac{1}{T_1^{3/2}\Delta}\mathbb{E}\left(\widehat{A}\overline{\widehat{A}}\big|\mathcal{F}^{(0)}\right) \overset{\mathbb{P}}{\to} \langle h, C\overline{h}\rangle, \tag{8.91}$$

$$\frac{1}{T_1^{3/2+3\varepsilon/4}\Delta^{1+\varepsilon/2}}\mathbb{E}\left(|\widehat{A}|^{2+\varepsilon}\big|\mathcal{F}^{(0)}\right) \overset{\mathbb{P}}{\to} 0, \quad \text{for some } \varepsilon \in (0,1). \tag{8.92}$$

Starting with the last convergence in (8.92), we can apply Burkholder-Davis-Gundy inequality, inequality in means as well as assumptions A5 and A6 for the log-strike grid and the observation error, and get

$$\frac{1}{T_1^{3/2+3\varepsilon/4}\Delta^{1+\varepsilon/2}}\mathbb{E}\left(|\widehat{A}|^{2+\varepsilon}\big|\mathcal{F}^{(0)}\right) \le \frac{C_t\Delta^{1+\varepsilon/2}}{T_1^{3/2+3\varepsilon/4}}\sum_{l=1}^{2}\sum_{j=2}^{N_l} O_{t,T_l}^{2+\varepsilon}(k_{l,j-1}). \tag{8.93}$$

From here, we can make use of Lemma 2 and get

$$\frac{1}{T_1^{3/2+3\varepsilon/4}\Delta^{1+\varepsilon/2}}\mathbb{E}\left(|\widehat{A}|^{2+\varepsilon}\big|\mathcal{F}^{(0)}\right) = O_p\left(\Delta^{\varepsilon/2}T_1^{-\varepsilon/4}|\log(T_1)|\right) = o_p(1), \tag{8.94}$$

where for the last equality, we made use of the fact that $\alpha > \frac{1}{2}$. Next, using the $\mathcal{F}^{(0)}$-conditional M-dependence of the observations errors from assumption A6, the $\mathcal{F}^{(0)}$-conditional independence of the errors across the two maturities, as well as the fact that $l_T^{(1)} > 2M$ and $l_T^{(2)} > 2M$, we have

$$\mathbb{E}\left(\widehat{A}^2\big|\mathcal{F}^{(0)}\right) = \sum_{j=1}^{r_T^{(2)}}\mathbb{E}\left((\widehat{A}_j^{(2)})^2\big|\mathcal{F}^{(0)}\right) + \tau^2\sum_{j=1}^{r_T^{(1)}}\mathbb{E}\left((\widehat{A}_j^{(1)})^2\big|\mathcal{F}^{(0)}\right). \tag{8.95}$$

Using assumption A6 for the observation error, we have for $l = 1, 2$:

$$\begin{aligned}\mathbb{E}\left((\widehat{A}_j^{(l)})^2\big|\mathcal{F}^{(0)}\right) = &\sum_{k,k'\in L_j^{(l)}:|k-k'|\le M}\big[\langle\widetilde{h}_t(u,k_{l,k-1},k_{l,k}),h(u)\rangle\langle\widetilde{h}_t(u,k_{l,k'-1},k_{l,k'}),h(u)\rangle\\ &\times \xi_{t,l}(k_{l,k-1}-x_t)\xi_{t,l}(k_{l,k'-1}-x_t)O_{t,T_l}(k_{l,k-1})O_{t,T_l}(k_{l,k'-1})\gamma_{t,l}(k-k')\big],\end{aligned} \tag{8.96}$$

where $\gamma_{t,l}(h)$ is defined in (4.9) and we further denote

$$\widetilde{h}_t(u,k_1,k_2) = -(u^2+iu)h_t(u,k_1,k_2), \quad u \in \mathbb{R},\ k_1,k_2 \in \mathbb{R}. \tag{8.97}$$

Form here, by an application of Lemma 2, we have

$$\begin{aligned}\mathbb{E}\left(\widehat{A}^2\big|\mathcal{F}^{(0)}\right) = & \sum_{j:L_j^{(2)}\subseteq \overline{I}_l^{(2)}} \mathbb{E}\left((\widehat{A}_j^{(2)})^2\big|\mathcal{F}^{(0)}\right) + \tau^2 \sum_{j:L_j^{(1)}\subseteq \overline{I}_l^{(1)}} \mathbb{E}\left((\widehat{A}_j^{(1)})^2\big|\mathcal{F}^{(0)}\right) \\ & + O_p\left(\frac{T_1^{3/2}\Delta}{|\log(T_1)|}\right),\end{aligned} \tag{8.98}$$

where as in the proof of Lemma 4, we denote the sets

$$\overline{I}_l^{(i)} := \{j=2,...,N_i : |k_{i,j}-x_t| \le |\log(T_i)|\sqrt{T_i}\}, \quad i=1,2. \tag{8.99}$$

Using (8.96), the first and third bound in Lemma 2 as well as the smoothness assumptions for ξ in assumption A6, we next have for j such that $L_j^{(l)} \subseteq \overline{I}_l^{(l)}$:

$$\begin{aligned}\mathbb{E}\left((\widehat{A}_j^{(l)})^2\big|\mathcal{F}^{(0)}\right) = & \sum_{k\in L_j^{(l)}} \left[\langle \widetilde{h}_t(u,k_{l,k-1},k_{l,k}),h(u)\rangle^2 \xi_{t,l}^2(k_{l,k-1}-x_t) O_{t,T_l}^2(k_{l,k-1})\right]\gamma_{t,l}(0) \\ & + 2\sum_{h=1}^{M} \sum_{k\in L_j^{(l),h}} \left[\langle \widetilde{h}_t(u,k_{l,k-1},k_{l,k}),h(u)\rangle^2 \xi_{t,l}^2(k_{l,k-1}-x_t) O_{t,T_l}^2(k_{l,k-1})\right]\gamma_{t,l}(h) + \widehat{\mathcal{R}}_j^{(l)} \\ = & \sum_{k\in L_j^{(l)}} \left[\langle \widetilde{h}_t(u,k_{l,k-1},k_{l,k}),h(u)\rangle^2 \xi_{t,l}^2(k_{l,k-1}-x_t) O_{t,T_l}^2(k_{l,k-1})\right] \\ & \times \left(\gamma_{t,l}(0) + 2\sum_{h=1}^{M}\gamma_{t,l}(h)\right) + \widehat{\mathcal{R}}_j^{(l)},\ l=1,2,\end{aligned} \tag{8.100}$$

where $L_j^{(l),h}$ is the same as the set $L_j^{(l)}$ but with the last h elements removed, and the residual term $\widehat{\mathcal{R}}_j^{(l)}$ satisfies

$$\begin{aligned}|\widehat{\mathcal{R}}_j^{(l)}| \le & C_t l_T^{(l)}\sqrt{T_l}\Delta^2(\Delta^\iota\sqrt{T_l}+\Delta) + C_t T_l \Delta^2 \\ & + C_t \overline{\psi}_T^{(l)}\Delta^2 \sum_{k\in L_j^{(l)}} O_{t,T_l}^2(k_{l,k-1}),\ l=1,2,\end{aligned} \tag{8.101}$$

for $\iota > 0$ being the constant appearing in assumption A6, C_t being an $\mathcal{F}^{(0)}$-adapted random variable, and $\overline{\psi}_T^{(l)}$ given by

$$\overline{\psi}_T^{(l)} = \sup_{j:|k_{l,j}-k_{l,j-1}|\le\sqrt{T_l}|\log(T_l)|} \left|\frac{k_{l,j}-k_{l,j-1}}{\Delta} - \psi_l(0)\right|,\ l=1,2. \tag{8.102}$$

Using the first bound of Lemma 2, we have

$$\Delta \sum_{k\in \overline{I}_j^{(l)}} O_{t,T_l}^2(k_{l,k-1}) \leq C_t T_l^{3/2},\ l=1,2. \tag{8.103}$$

Using the results in (8.100) and (8.103), the bound $O_{t,T}(k) \leq C_t\sqrt{T}$ from Lemma 2, the fact that $\overline{\psi}_T^{(l)} = o_p(1)$ by assumption A5 as well as the second condition for $l_T^{(l)}$ in (8.86), we have altogether (recall notation in (8.11)):

$$\begin{aligned}\mathbb{E}\left(\widehat{A}^2\middle|\mathcal{F}^{(0)}\right) &= \overline{\gamma}_{t,2} \sum_{k\in\overline{I}_l^{(2)}} \left[\langle \widetilde{h}_t(u,k_{2,k-1},k_{2,k}), h(u)\rangle^2 \xi_{t,2}^2(k_{2,k-1}-x_t) O_{t,T_2}^2(k_{2,k-1})\right] \\ &+ \tau^2\overline{\gamma}_{t,1} \sum_{k\in\overline{I}_l^{(1)}} \left[\langle \widetilde{h}_t(u,k_{1,k-1},k_{1,k}), h(u)\rangle^2 \xi_{t,1}^2(k_{1,k-1}-x_t) O_{t,T_1}^2(k_{1,k-1})\right] \\ &+ o_p\left(T_1^{3/2}\Delta\right).\end{aligned} \tag{8.104}$$

Next, using the bounds in (8.22) and (8.24) of Lemma 2, we have for $i=1,2$:

$$\begin{aligned}&\sum_{j\in\overline{I}_l^{(i)}} \langle \widetilde{h}_t(u,k_{i,j-1},k_{i,j}), h(u)\rangle^2 \xi_{t,i}^2(k_{i,j-1}-x_t) O_{t,T_i}^2(k_{i,j-1}) \\ &= T_i \sum_{j\in\overline{I}_l^{(i)}} \langle \widetilde{h}_t(u,k_{i,j-1},k_{i,j}), h(u)\rangle^2 \xi_{t,i}^2(k_{i,j-1}-x_t) \overline{O}_{t,T_i}^2(k_{i,j-1}) + o_p\left(T_i^{3/2}\Delta\right),\end{aligned} \tag{8.105}$$

where we denote

$$\overline{O}_{t,T}(k) = e^{x_t} f\left(\frac{k-x_t}{\sqrt{T}\sigma_t}\right)\sigma_t + e^{x_t}\frac{|k-x_t|}{\sqrt{T}}\Phi\left(-\frac{|k-x_t|}{\sqrt{T}\sigma_t}\right). \tag{8.106}$$

Now, we note that

$$\langle \widetilde{h}_t(u,k_{i,j-1},k_{i,j}), h(u)\rangle = e^{-x_t}\int_{k_{i,j-1}}^{k_{i,j}} \langle \zeta_t(u,k), h(u)\rangle dk,\ i=1,2, \tag{8.107}$$

where we use the notation $\zeta_t(u,k) = \zeta(u)e^{(iu-1)(k-x_t)}$ (recall (8.9)). By Taylor expansion, we have

$$\left|\overline{O}_{t,T}(k_1) - \overline{O}_{t,T}(k_2)\right| \leq C_t\frac{|k_1-k_2|}{\sqrt{T}},\quad \text{for } k_1<k_2\leq x_t \text{ or } k_1>k_2\geq x_t, \tag{8.108}$$

and therefore

$$\begin{aligned}&\left|\langle \widetilde{h}_t(u,k_{i,j-1},k_{i,j}), h(u)\rangle^2 \overline{O}_{t,T_i}^2(k_{i,j-1}) - \Delta_{i,j}e^{-2x_t}\int_{k_{i,j-1}}^{k_{i,j}} \langle \zeta_t(u,k), h(u)\rangle^2 \overline{O}_{t,T_i}^2(k)dk\right| \\ &\leq C_t\frac{\Delta^3}{\sqrt{T_i}},\ i=1,2,\end{aligned} \tag{8.109}$$

where in the above expression we use the shorthand $\Delta_{i,j} = k_{i,j} - k_{i,j-1}$. Taking into account that $\alpha > \frac{1}{2}$, the results in (8.104), (8.105) and (8.109) lead to

$$\begin{aligned}\mathbb{E}\left(\widehat{A}^2\middle|\mathcal{F}^{(0)}\right) &= \overline{\gamma}_{t,2}T_2e^{-2x_t}\sum_{j\in\overline{I}_l^{(2)}}\Delta_{2,j}\int_{k_{2,j-1}}^{k_{2,j}}\langle\zeta_t(u,k),h(u)\rangle^2\overline{O}_{t,T_2}^2(k)\xi_{t,2}^2(k_{2,j-1}-x_t)dk\\&+\tau^2\overline{\gamma}_{t,1}T_1e^{-2x_t}\sum_{j\in\overline{I}_l^{(1)}}\Delta_{1,j}\int_{k_{1,j-1}}^{k_{1,j}}\langle\zeta_t(u,k),h(u)\rangle^2\overline{O}_{t,T_1}^2(k)\xi_{t,1}^2(k_{1,j-1}-x_t)dk\\&+o_p\left(T_1^{3/2}\Delta\right).\end{aligned} \tag{8.110}$$

Finally, we can note by change of variable of integration that $\int_{\mathbb{R}}\overline{O}_{t,T}^2(k)dk = O_p(\sqrt{T})$. Therefore, taking into account the smoothness assumption for $\Delta_{l,j}$ in A5 and for $\xi_{t,l}^2(k)$ in A6, and by changing the variable of integration, we can write

$$\begin{aligned}\mathbb{E}\left(\widehat{A}^2\middle|\mathcal{F}^{(0)}\right) &= \overline{\gamma}_{t,2}\Delta T_2^{3/2}\int_{\mathbb{R}}\langle\zeta(u),h(u)\rangle^2\sigma_t^3\widetilde{\Phi}^2(k)\xi_{t,2}^2(0)\psi_2(0)dk\\&+\overline{\gamma}_{t,1}\Delta T_1^{3/2}\int_{\mathbb{R}}\langle\zeta(u),h(u)\rangle^2\sigma_t^3\widetilde{\Phi}^2(k)\xi_{t,1}^2(0)\psi_1(0)dk+o_p\left(T_1^{3/2}\Delta\right).\end{aligned} \tag{8.111}$$

From here, the first convergence result in (8.91) follows by taking into account that $\xi_{t,l}(0)$ and $\overline{\gamma}_{t,l}$ are continuous in t and σ_t does not have fixed time of discontinuity at t^*. The second convergence in (8.91) can be shown in an analogous way, and from here (8.81) follows and hence the result of the lemma. □

8.5. Proof of Theorem 2

In the proof, we will assume that $t > t^* - \varepsilon$ and $T_2 < \inf_{t\in[t^*-\varepsilon,t^*]}\overline{t} - t$, with ε and $\overline{t}$ being the ones in Lemmas 1 and 2. Throughout the proof, C_t will denote an $\mathcal{F}_t^{(0)}$-adapted random variable satisfying the conditions in the statement of Lemmas 1 and 2, and it can change from line to line. We denote

$$\widehat{\mathcal{L}}_{t,T}^*(u) - \widehat{\mathcal{L}}_{t,T}(u) = \eta_{t,T}^{(4)}(u) = -(u^2+iu)\overline{\eta}_{t,T}^{(4)}(u). \tag{8.112}$$

The analysis of $\overline{\eta}_{t,T}^{(4)}(u)$ is similar to that of $\overline{\eta}_{t,T}^{(1)}(u)$. In particular, using Lemma 2 as well as the bounds for $\{\overline{\eta}_{t,T}^{(j)}(u)\}_{j=1,2,3}$, we have

$$\mathbb{E}\left(||G(u)\overline{\eta}_{t,T}^{(4)}(u)||^2\middle|\mathcal{F}\right) = O_p\left(T\sqrt{T}\Delta\right), \tag{8.113}$$

for any deterministic function $G:\mathbb{R}\to\mathbb{R}$ with polynomial growth. Using Taylor expansion, taking into account $|\mathcal{L}_{t,T_1}(u)|\leq 1$ and $\tau>1$, we have

$$\begin{aligned}\left|\widehat{\mathcal{L}}_{t,T_1}(u)^{\tau-1}-1\right| \leq C|\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)|^{\tau-1}+C|\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)| \\ +C|\mathcal{L}_{t,T_1}(u)-1|^{\tau-1}+C|\mathcal{L}_{t,T_1}(u)-1|,\end{aligned} \tag{8.114}$$

for some positive constant. From here, using the bounds for $\{\eta_{t,T}^{(j)}(u)\}_{j=1,\dots,4}$, derived in the proof of Theorem 1 and above, and taking into account that $\tau>1$, we have

$$\begin{aligned}&\widehat{\mathcal{L}}^*_{t,T_2}(u)-\widehat{\mathcal{L}}_{t,T_2}(u)-\tau\widehat{\mathcal{L}}_{t,T_1}(u)^{\tau-1}(\widehat{\mathcal{L}}^*_{t,T_1}(u)-\widehat{\mathcal{L}}_{t,T_1}(u)) \\ &\quad=\widehat{Z}^*+o_p(T_1^{3/4}\sqrt{\Delta}),\end{aligned} \tag{8.115}$$

where

$$\begin{aligned}\widehat{Z}^* &= \widehat{S}_{t,2}\sum_{j=2}^{N_2}\widetilde{h}_t(u,k_{2,j-1},k_{2,j})\widehat{\epsilon}_{t,T_2}(k_{2,j-1})z_{2,j-1} \\ &\quad-\tau\widehat{S}_{t,1}\sum_{j=2}^{N_1}\widetilde{h}_t(u,k_{1,j-1},k_{1,j})\widehat{\epsilon}_{t,T_1}(k_{1,j-1})z_{1,j-1},\end{aligned} \tag{8.116}$$

and the function $\widetilde{h}_t$ is defined in (8.97) above. Hence, we will be done if we can show

$$\frac{1}{T_1^{3/4}\sqrt{\Delta}}\widehat{Z}^* \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{CN}(0,K,C). \tag{8.117}$$

First, exactly as in the proof of Lemma 5, we can show

$$\limsup_{T_1\downarrow 0}\frac{1}{T_1^{3/2}\Delta}\mathbb{E}\left(\sum_{j>J}\langle\widehat{Z}^*,e_j\rangle^2\middle|\mathcal{F}\right)\to 0,\quad \text{as } J\to\infty, \tag{8.118}$$

where $\{e_j\}_{j\geq 1}$ denotes an orthonormal basis in $\mathcal{L}^2(w)$. Therefore, the limit result of the theorem will follow from Theorem 1.8.4 in Vaart and Wellner (1996) if we can establish

$$\frac{1}{T_1^{3/4}\sqrt{\Delta}}\langle\widehat{Z}^*,h\rangle \xrightarrow{\mathcal{L}|\mathcal{F}} \langle Z,h\rangle, \tag{8.119}$$

for Z denoting the limit of Theorem 1 and h an arbitrary element in $\mathcal{L}^2(w)$, and for the latter we need to show

$$\frac{1}{T_1^{3/2}\Delta}\mathbb{E}\left(\langle\widehat{Z}^*,h\rangle^2\middle|\mathcal{F}\right) \xrightarrow{\mathbb{P}} \langle Kh,h\rangle,\quad \frac{1}{T_1^{3/2}\Delta}\mathbb{E}\left(\langle\widehat{Z}^*,h\rangle\overline{\langle\widehat{Z}^*,h\rangle}\middle|\mathcal{F}\right) \xrightarrow{\mathbb{P}} \langle h,C\overline{h}\rangle, \tag{8.120}$$

$$\frac{1}{T_1^{3/2+3\varepsilon/4}\Delta^{1+\varepsilon/2}}\mathbb{E}\left(|\langle\widehat{Z}^*,h\rangle|^{2+\varepsilon}\middle|\mathcal{F}\right) \xrightarrow{\mathbb{P}} 0,\quad \text{for some } \varepsilon\in(0,1). \tag{8.121}$$

The last of the above results follows by making use of Lemma 2. Below we show the first of the above convergences, with the second one being established in analogous way. Using the fact that both sequences $\{z_{1,j}\}_{j=1}^{N_1}$ and $\{z_{2,j}\}_{j=1}^{N_2}$ are i.i.d, we have

$$\begin{aligned}\mathbb{E}\left(\langle \widehat{Z}^*, h\rangle^2 \big| \mathcal{F}\right) &= \widehat{S}_{t,2}^2 \sum_{j=2}^{N_2} \langle \widetilde{h}_t(u, k_{2,j-1}, k_{2,j}), h(u)\rangle^2 \widehat{\epsilon}_{t,T_2}^2(k_{2,j-1}) \\ &\quad + \tau^2 \widehat{S}_{t,1}^2 \sum_{j=2}^{N_1} \langle \widetilde{h}_t(u, k_{1,j-1}, k_{1,j}), h(u)\rangle^2 \widehat{\epsilon}_{t,T_1}^2(k_{1,j-1}),\end{aligned} \tag{8.122}$$

and using assumption A6 for the observation errors, Lemma 2 and the restriction $\alpha > 1/2$, we have for $l = 1, 2$:

$$\begin{aligned}&\sum_{j=2}^{N_l} \langle \widetilde{h}_t(u, k_{l,j-1}, k_{l,j}), h(u)\rangle^2 \widehat{\epsilon}_{t,T_l}^2(k_{l,j-1}) \\ &= \sum_{j \in \{2,\ldots,N_l\}\setminus J_l^*} \langle \widetilde{h}_t(u, k_{l,j-1}, k_{l,j}), h(u)\rangle^2 \overline{\xi}_{t,l}^2(k_{l,j-1}) O_{t,T_l}^2(k_{l,j-1}) + O_p(T_1^{5/4}\Delta^{3/2}),\end{aligned} \tag{8.123}$$

where $\overline{\xi}_{t,l}^2(k_{l,j}) = \frac{2}{3}\xi_{t,l}^2(k_{l,j}) + \frac{1}{6}\xi_{t,l}^2(k_{l,j-1}) + \frac{1}{6}\xi_{t,l}^2(k_{l,j+1})$, for $j = 2, ..., N_l - 1$, and $\overline{\xi}_{t,l}^2(k_{l,N_l}) = \overline{\xi}_{t,l}^2(k_{l,N_l-1})$, and J_l^* is defined in (4.4). Note that $\sqrt{\Delta}/T_1^{1/4} \to 0$, and hence $O_p(T_1^{5/4}\Delta^{3/2}) = o_p(T_1^{3/2}\Delta)$. From here, we can use the fact that $\widehat{S}_{t,l}^2 \xrightarrow{\mathbb{P}} \overline{\gamma}_{t^*,l}$, for $l = 1, 2$ from Lemma 4 and the local uniform convergence result in (4.12), and then proceed exactly as in the proof of Lemma 5 to show (8.120).

8.6. Proof of Theorem 3

In the proof, we will assume that $t > t^* - \varepsilon$ and $T_2 < \inf_{t \in [t^*-\varepsilon, t^*]} \bar{t} - t$, with ε and $\bar{t}$ being the ones in Lemmas 1 and 2. Throughout the proof, C_t will denote an $\mathcal{F}_t^{(0)}$-adapted random variable satisfying the conditions in the statement of Lemmas 1 and 2, and it can further change from line to line.

Using Lemma 3, we have

$$|\eta_{t,T}^{(2)}(u)| + |\eta_{t,T}^{(3)}(u)| \le C_t(u^2 \wedge 1)\left(\Delta \bigvee e^{-2(|\underline{k}| \vee \overline{k})}\right). \tag{8.124}$$

Furthermore, using assumptions A5 and A6 for the mesh of the log-strike grid and the structure of the observation error and Lemma 3 we have

$$\mathbb{E}\left(|\eta_{t,T}^{(1)}(u)|^2 \Big| \mathcal{F}^{(0)}\right) \le C_t(u^4 \vee 1)\Delta. \tag{8.125}$$

Combining these results with the bound in (8.15) of Lemma 1, we get altogether

$$||\widehat{\mathcal{L}}_{t,T}(u) - \widetilde{\mathcal{L}}^{fd}_t(u)|| = O_p\left(\sqrt{\Delta} \bigvee e^{-2(|\underline{k}| \vee \bar{k})} \bigvee \sqrt{T}\right). \tag{8.126}$$

From assumption A4, we further have

$$||\widetilde{\mathcal{L}}^{fd}_t(u) - \widetilde{\mathcal{L}}^{fd}_{t^*}(u)|| = O_p\left(\sqrt{T}\right). \tag{8.127}$$

The results in (8.126)-(8.127) imply the first result of the theorem.

Next, with the notation for $\eta^{(4)}_{t,T}(u)$ in the proof of Theorem 2, exactly as for $\eta^{(1)}_{t,T}(u)$ above, we can show

$$\mathbb{E}\left(|\eta^{(4)}_{t,T}(u)|^2 \middle| \mathcal{F}\right) \leq C_t(u^4 \vee 1)\Delta. \tag{8.128}$$

Therefore, since by Lemma 4, $\widehat{S}_{t,l} = O_p(1)$, for $l = 1, 2$, we have

$$||\widehat{\mathcal{L}}^*_{t,T}(u) - \widehat{\mathcal{L}}_{t,T}(u)|| = O_p(\sqrt{\Delta}), \tag{8.129}$$

and from here the second result of the theorem follows.

8.7. Proof of Corollary 4.1

We start with the result under the null hypothesis. By Theorem 2 and an application of portmanteau theorem, we have $\frac{1}{T_1^{3/4}\sqrt{\Delta}}\widehat{cv}_\alpha \xrightarrow{\mathbb{P}} cv_\alpha$, with cv_α denoting the $\mathcal{F}^{(0)}$-conditional $(1-\alpha)$-quantile of $||Z||$ and Z being the limit variable in Theorem 1, because Z has $\mathcal{F}^{(0)}$-conditional continuous distribution. From Theorem 1, we have that $\mathbb{P}(\widehat{W}_{1,2} > cv_\alpha|\Omega_0)$ converges to α, and this convergence holds locally uniformly in α as $\mathbb{P}(\widehat{W}_{1,2} > cv_\alpha|\Omega_0)$ is a monotone function of α. From here, the result to be proved under the null hypothesis follows.

We turn next to showing the result under the alternative. We denote the set $\mathcal{U}_\epsilon = \{u : |\mathcal{L}^{fd}_{t^*}(u) - 1| < \epsilon\}$, for some $\epsilon \in (0, 1)$. This set is of positive Lebesgue measure, since $\mathcal{L}^{fd}_{t^*}(u)$ is continuous and converges to zero as $|u|$ approaches infinity by Riemann-Lebesgue lemma. Then, using the bounds (8.124)-(8.125) as well as Lemma 1, we have

$$\int_{u\in\mathcal{U}_\epsilon} \left|\widehat{\mathcal{L}}_{t,T_2}(u) - \widehat{\mathcal{L}}_{t,T_1}(u)^\tau\right|^2 w(u)du \xrightarrow{\mathbb{P}} \int_{u\in\mathcal{U}_\epsilon} \left|\widehat{\mathcal{L}}^{fd}_{t^*}(u) - 1\right|^2 w(u)du > 0, \tag{8.130}$$

if $T_1 < t^*$ and

$$\int_{u\in\mathcal{U}_\epsilon} \left|\widehat{\mathcal{L}}_{t,T_2}(u) - \widehat{\mathcal{L}}_{t,T_1}(u)^\tau\right|^2 w(u)du \xrightarrow{\mathbb{P}} \int_{u\in\mathcal{U}_\epsilon} \left|\widehat{\mathcal{L}}^{fd}_{t^*}(u) - \widehat{\mathcal{L}}^{fd}_{t^*}(u)^\tau\right|^2 w(u)du > 0, \tag{8.131}$$

if $t^* \geq T_1$. This together with the order of magnitude of $\widehat{W}^*_{1,2}$ derived in Theorem 3 (which means that $\widehat{cv}_\alpha = O_p(\sqrt{\Delta})$) implies the result of the corollary under the alternative hypothesis.

8.8. Proof of Theorem 4

Applying the Plancherel's identity, we can write

$$\int_{\mathbb{R}}\left(\widehat{G}_{t^*}(x)-G_{t^*}(x)\right)^2dx \\ =\frac{1}{2\pi}\int_{|u|\le u_N}\left|\mathcal{F}\widehat{G}_{t^*}(u)-\mathcal{F}G_{t^*}(u)\right|^2du+\frac{1}{2\pi}\int_{|u|>u_N}|\mathcal{F}G_{t^*}(u)|^2du, \tag{8.132}$$

whee $\mathcal{F}\widehat{G}_{t^*}$ and $\mathcal{F}G_{t^*}$ are the Fourier transforms of $\widehat{G}_{t^*}$ and G_{t^*}, respectively. By assumption A7, the first term on the right-hand side of the above decomposition is of order $O(u_N^{-2r})$.

We continue with the second term in that decomposition. First, we note that regardless of whether $t^*<t+T_1$ or $t^*\ge t+T_1$, we have that $||\widehat{\mathcal{L}}_{t,T_2}-1||$ converges in probability to a strictly positive number. Also, $||\widehat{\mathcal{L}}_{t,T_2}-\widehat{\mathcal{L}}_{t,T_1}||=||\widehat{\mathcal{L}}_{t,T_2}-1||+o_p(1)$ in the asymptotic setup with $t^*\ge t+T_1$ and $||\widehat{\mathcal{L}}_{t,T_2}-\widehat{\mathcal{L}}_{t,T_1}||=o_p(1)$ in the asymptotic setup with $t^*<t+T_1$. These results follow from the bounds in (8.124)-(8.125). Therefore,

$$\frac{\tau}{\tau-1}\frac{||\widehat{\mathcal{L}}_{t,T_2}-\widehat{\mathcal{L}}_{t,T_1}||}{||\widehat{\mathcal{L}}_{t,T_2}-1||}\stackrel{\mathbb{P}}{\to}\begin{cases}\frac{\tau}{\tau-1}>1, & \text{in the asymp. setup with } t^*\ge t+T_1,\\ 0, & \text{in the asymp. setup with } t^*<t+T_1.\end{cases} \tag{8.133}$$

This implies that on a set with probability approaching one $\frac{\tau}{\tau-1}\frac{||\widehat{\mathcal{L}}_{t,T_2}-\widehat{\mathcal{L}}_{t,T_1}||}{||\widehat{\mathcal{L}}_{t,T_2}-1||}<1$ will correctly identify the setup with $t^*<t+T_1$ and similarly $\frac{\tau}{\tau-1}\frac{||\widehat{\mathcal{L}}_{t,T_2}-\widehat{\mathcal{L}}_{t,T_1}||}{||\widehat{\mathcal{L}}_{t,T_2}-1||}\ge 1$ will correctly identify the setup with $t^*\ge t+T_1$. Hence, we can proceed in the proof assuming that this is the case.

Taking into account that $\widehat{c}=O_p(1)$ and $u_N^2T_1\to 0$, we have for T_1 sufficiently small on a set with probability approaching one and for $|u|\le u_N$:

$$\left|\frac{1}{\widehat{\mathcal{B}}_1(u)}-1\right|\le C_t\left[\frac{1}{(1-(u_N^2\widehat{c}T_1))}\bigvee\frac{1}{((1-(\tau-1)u_N^2\widehat{c}T_1))^{1/(\tau-1)}}\right] \\ \times|\widehat{\mathcal{B}}_1(u)-1|, \tag{8.134}$$

$$\left|\frac{1}{\widehat{\mathcal{B}}_2(u)}-1\right|\le C_t\left[\frac{1}{1-u_N^2\widehat{c}T_2}\bigvee\frac{1}{|1-u_N^2\widehat{c}T_1|^\tau}\right]|\widehat{\mathcal{B}}_2(u)-1|, \tag{8.135}$$

for some variable C_t that depends only on a_t, σ_t and F_t. We further have for T_1 sufficiently small on a set with probability approaching one

$$|\widehat{\mathcal{B}}_j(u)-1|\le Cu^2\widehat{c}T_1,\quad j=1,2, \tag{8.136}$$

for some positive constant C. From here, we need to study only the distance between the second term in $\widehat{\mathcal{L}}_t^{fd}(u)$ (i.e., $\widehat{\mathcal{L}}_{t,T_1}(u)$ or $\widehat{\mathcal{L}}_{t,T_2}(u)$) and $\mathcal{L}_{t^*}^{fd}(u)$. Its analysis follows from the bounds for the terms $\{\eta_{t,T}^{(j)}(u)\}_{j=1,2,3}$ as well as (8.127) in the proof of Theorem 3.

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