SHORT-TIME EXPANSION OF CHARACTERISTIC FUNCTIONS IN A ROUGH VOLATILITY SETTING WITH APPLICATIONS

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We derive a higher-order asymptotic expansion of the conditional characteristic function of the increment of an Itô semimartingale over a shrinking time interval. The spot characteristics of the Itô semimartingale are allowed to have dynamics of general form. In particular, their paths can be rough, that is, exhibit local behavior like that of a fractional Brownian motion, while at the same time have jumps with arbitrary degree of activity. The expansion result shows the distinct roles played by the different features of the spot characteristics dynamics. As an application of our result, we construct a non-parametric estimator of the Hurst parameter of the diffusive volatility process from portfolios of short-dated options written on an underlying asset.

1. Introduction. Our interest in this paper is a small-time asymptotic expansion of the characteristic function of the increment of an Itô semimartingale, defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ satisfying the usual conditions, and given by

$$
\begin{aligned}
x_t = x_0 &+ \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\mu(ds,dz) \\
&+ \int_0^t \int_{\mathbb{R}} \delta(s,z)(\mu-\nu)(ds,dz),
\end{aligned}
\tag{1.1}
$$

where W is a Brownian motion, μ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator ν, α and σ are some processes with càdlàg paths, and $\gamma, \delta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are predictable functions. The technical assumptions for all quantities appearing in (1.1) are given in Section 2 below.

Under fairly weak assumptions on α, σ, γ and δ, one can show for $t \geq 0$ and $u \in \mathbb{R} \setminus \{0\}$ that

$$
\mathbb{E}_t\left[e^{iu(x_{t+T}-x_t)/\sqrt{T}}\right] = e^{-\frac{u^2}{2}\sigma_t^2} + o_p(1) \qquad \text{as } T \downarrow 0,
\tag{1.2}
$$

where $\mathbb{E}_t$ denotes the $\mathcal{F}_t$-conditional expectation with respect to $\mathbb{Q}$. This result is due to the fact that the variation in the spot semimartingale characteristics α, σ, γ and δ is of higher order when the length of the increment shrinks (see e.g., [16]) and the leading role the diffusion coefficient plays in the characteristic function of a Lévy process as the value of the characteristic exponent increases (see e.g., Theorem 8.1 in [26]). The result in (1.2) has been used for efficient volatility estimation from high-frequency observations of x by [18, 20] and from short-dated options written on x, when x is an asset price, by [27].

The goal of the current paper is to investigate how the higher-order terms in (1.2) depend on the dynamics of the spot semimartingale characteristics. This allows one to quantify their impact on volatility estimation, in which the $o_p(1)$-term in (1.2) is ignored, but more importantly, it allows one to obtain information about the spot characteristics themselves and their

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dynamics. Such a higher-order asymptotic expansion of the characteristic function has been derived in [28] in the case where x and its spot characteristics are governed by Itô semimartingales with finite variation jumps. In this paper, we consider an asymptotic expansion in a much more general setting, in which not only the jumps can be of infinite variation but also the volatility σ and the jump size functions γ and δ can have local behavior like that of a fractional Brownian motion. Models with the latter feature can generate rough volatility paths, with Hölder exponents (outside jump times) that are significantly smaller than $\frac{1}{2}$. Rough volatility models have been introduced recently by [15] and have been found to generate patterns like those observed in real financial data (see e.g., [2, 6, 7, 8, 14] among others). Models with rough dynamics have also been used by [4] to capture market microstructure noise in financial data with plausible dependence structure.

The setup we work in allows for rough dynamics of general form. In particular, all spot characteristics can be rough and their degree of roughness may differ. This generality allows us to determine whether the various model features have similar or different contributions to the asymptotic expansion. In particular, our higher-order asymptotic expansion result shows that the contribution of the jumps in x and the roughness of the volatility path to the characteristic function can be of the same asymptotic order, depending on the degree of jump activity and roughness of the volatility path. That is, for each value of volatility roughness, there is a degree of jump activity such that their contributions to the characteristic function will be of the same asymptotic order. However, our analysis also shows that the terms due to jumps in x and those due to roughness of volatility differ in how they depend on the characteristic function exponent, which in the end allows one to separate them. Compared to results in the constrained setting where the spot semimartingale characteristics are Itô semimartingales without rough components in their dynamics, higher-order terms play a much more prominent role in the presence of roughness. In other words, in rough settings the $o_p(1)$-term in (1.2) is much bigger, and hence ignoring it would lead to a much bigger error.

As an application of our expansion result, we derive a consistent estimator of the roughness parameter of volatility, using options written on an asset with short times to maturity. Following [24] and [27], we use a portfolio of options with different strikes and with the same expiration to estimate $\mathbb{E}_t[e^{iu(x_{t+T}-x_t)/\sqrt{T}}]$. We then infer the degree of roughness of volatility from the rate of decay of the argument of the characteristic function as the length of the price increment shrinks down to zero. This estimation approach works well when the underlying asset price does not contain jumps. If this is not the case, then one first needs to perform a bias correction of the argument of the conditional characteristic function by suitably differencing the latter computed over different short time horizons. While this differencing leads to loss of information, compared to the estimation with no price jumps, it shows that separation of the roughness parameter of volatility from price jumps is possible theoretically.

Our use of short-dated options in a rough setting can be compared with recent work on short-time expansions of option prices in various rough volatility specifications (see for example, [1, 6, 10, 11, 12, 13, 22]). This strand of work considers expansions of a single option, and typically these expansions only hold for a restricted class of models (e.g., the volatility only contains a rough component without jumps, and in addition there are no jumps in the asset price). By contrast, our model setting, as already explained, is very general and we consider a short-term asymptotic expansion for a portfolio of options that mimics the conditional characteristic function of the price increment.

The rest of the paper is organized as follows. We start in Section 2 with introducing our setting and stating the assumptions. The main result is given in Section 3. In Section 4, we introduce option-based estimators of the conditional characteristic function of the underlying asset price and derive their rate of convergence. Section 5 contains our application to estimation of the degree of volatility roughness from options. Section 6 contains a Monte Carlo study, while proofs are in Sections 7–9.

2. Setup and Assumptions. We start by introducing the setup and stating the assumptions needed for deriving our asymptotic expansion result presented in the next section. The dynamics of the volatility process σ are given by an Itô semimartingale plus a rough component:

$$
\begin{aligned}
\sigma_t = \sigma_0 &+ \int_0^t b_s ds + \int_0^t g_H(t-s)\eta_s dW_s + \int_0^t \widetilde{g}_H(t-s)\widetilde{\eta}_s d\widetilde{W}_s + \int_0^t \eta_s^\sigma dW_s \\
&+ \int_0^t \widetilde{\eta}_s^\sigma d\widetilde{W}_s + \int_0^t \overline{\eta}_s^\sigma d\overline{W}_s + \int_0^t \int_{\mathbb{R}} \gamma_\sigma(s,z)\mu(ds,dz) \\
&+ \int_0^t \int_{\mathbb{R}} \delta_\sigma(s,z)(\mu-\nu)(ds,dz).
\end{aligned} \tag{2.1}
$$

Except for the two integrals $\int_0^t g_H(t-s)\eta_s dW_s$ and $\int_0^t \widetilde{g}_H(t-s)\widetilde{\eta}_s d\widetilde{W}_s$, the dynamics of σ in (2.1) is that of a general Itô semimartingale. The integrals $\int_0^t g_H(t-s)\eta_s dW_s$ and $\int_0^t \widetilde{g}_H(t-s)\widetilde{\eta}_s d\widetilde{W}_s$, upon a suitable choice of the kernel functions g_H and $\widetilde{g}_H$, can exhibit local behavior like that of a fractional Brownian motion and therefore introduce roughness in the paths of σ. We note that in the above specification of σ we allow for jumps, and as will become clear later on, they can be of infinite variation. We also allow for dependence of general form between the diffusive and jump components of x and σ. Consequently, our model for σ is an extension of the class of mixed semimartingales from [5] that includes jumps.

In our asymptotic expansion, the dynamics of the processes η and $\widetilde{\eta}$ will also play a role. We make a similar assumption on their dynamics to the one for σ above. More specifically, we have

$$
\begin{aligned}
\eta_t = \eta_0 &+ \int_0^t g_H^\eta(t-s)\sigma_s^\eta dW_s + \int_0^t \widetilde{g}_H^\eta(t-s)\widetilde{\sigma}_s^\eta d\widetilde{W}_s + \int_0^t \overline{g}_H^\eta(t-s)\overline{\sigma}_s^\eta d\overline{W}_s \\
&+ \int_0^t \widehat{g}_H^\eta(t-s)\widehat{\sigma}_s^\eta d\widehat{W}_s + \theta_t
\end{aligned} \tag{2.2}
$$

and

$$
\begin{aligned}
\widetilde{\eta}_t = \widetilde{\eta}_0 &+ \int_0^t g_H^{\widetilde{\eta}}(t-s)\sigma_s^{\widetilde{\eta}} dW_s + \int_0^t \widetilde{g}_H^{\widetilde{\eta}}(t-s)\widetilde{\sigma}_s^{\widetilde{\eta}} d\widetilde{W}_s + \int_0^t \overline{g}_H^{\widetilde{\eta}}(t-s)\overline{\sigma}_s^{\widetilde{\eta}} d\overline{W}_s \\
&+ \int_0^t \widehat{g}_H^{\widetilde{\eta}}(t-s)\widehat{\sigma}_s^{\widetilde{\eta}} d\widehat{W}_s + \int_0^t \mathring{g}_H^{\widetilde{\eta}}(t-s)\mathring{\sigma}_s^{\widetilde{\eta}} d\mathring{W}_s + \widetilde{\theta}_t.
\end{aligned} \tag{2.3}
$$

Finally, the dynamics of the "small" jump size function δ will also play a role in our expansion. We assume that

$$
\begin{aligned}
\delta(t,z) = \delta(0,z) &+ \int_0^t g_{H_\delta}^\delta(t-s,z)\sigma^\delta(s,z)dW_s + \int_0^t \dot{g}_{H_\delta}^\delta(t-s,z)\dot{\sigma}^\delta(s,z)d\dot{W}_s \\
&+ \int_0^t \eta^\delta(s,z)dW_s + \int_0^t \ddot{\eta}^\delta(s,z)d\ddot{W}_s + \int_0^t \int_{\mathbb{R}} \delta_\delta(s,z,z')(\mu-\nu)(ds,dz') \\
&+ \theta_\delta(t,z).
\end{aligned} \tag{2.4}
$$

The specification in (2.4) is very general, allowing for both diffusive and jump shocks to determine changes over time in δ. We also note that δ can exhibit roughness in its paths through the integral $\int_0^t \dot{g}_{H_\delta}^\delta(t-s,z)\dot{\sigma}^\delta(s,z)d\dot{W}_s$.

The remaining ingredients of (1.1) and (2.1)–(2.4) are supposed to satisfy the following conditions, which depend on two roughness parameters $H, H_\delta \in (0, \frac{1}{2})$:

ASSUMPTION A. *We have (1.1) and (2.1)–(2.4), where*

1. x_0, σ_0, η_0 *and* $\widetilde{\eta}_0$ *are* $\mathcal{F}_0$*-measurable and* $\delta(0,\cdot)$ *is* $\mathcal{F}_0 \otimes \mathcal{B}(\mathbb{R})$*-measurable;*
2. α, b, σ^η, $\widetilde{\sigma}^\eta$, $\overline{\sigma}^\eta$, $\widehat{\sigma}^\eta$, $\sigma^{\widetilde{\eta}}$, $\widetilde{\sigma}^{\widetilde{\eta}}$, $\overline{\sigma}^{\widetilde{\eta}}$, $\widehat{\sigma}^{\widetilde{\eta}}$, $\mathring{\sigma}^{\widetilde{\eta}}$, σ^δ, $\dot{\sigma}^\delta$, η^σ, $\widetilde{\eta}^\sigma$, $\overline{\eta}^\sigma$, η^δ, $\ddot{\eta}^\delta$, θ, $\widetilde{\theta}$, θ_δ, γ, γ_σ, δ_σ *and* δ_δ *are predictable processes;*
3. W, $\widetilde{W}$, $\overline{W}$, $\widehat{W}$ *and* $\mathring{W}$ *are independent standard* $\mathbb{F}$*-Brownian motions;* $\dot{W}$ *and* $\ddot{W}$ *are standard* $\mathbb{F}$*-Brownian motions that are jointly Gaussian with* $(W,\widetilde{W},\overline{W},\widehat{W},\mathring{W})$*, independent of* W *but potentially dependent on each other and on* $(\widetilde{W},\overline{W},\widehat{W},\mathring{W})$*;*
4. μ *is an* $\mathbb{F}$*-Poisson random measure on* $[0,\infty)\times\mathbb{R}$ *with compensator* $\mathrm{Leb}\otimes\nu$ *that is independent of* W, $\widetilde{W}$, $\overline{W}$, $\widehat{W}$, $\mathring{W}$, $\dot{W}$ *and* $\ddot{W}$*;*
5. *each* $g\in\{g_H,\widetilde{g}_H,g^\eta_H,\widetilde{g}^\eta_H,\overline{g}^\eta_H,\widehat{g}^\eta_H,g^{\widetilde{\eta}}_H,\widetilde{g}^{\widetilde{\eta}}_H,\overline{g}^{\widetilde{\eta}}_H,\widehat{g}^{\widetilde{\eta}}_H,\mathring{g}^{\widetilde{\eta}}_H\}$ *is of the form* $g=k_H+\ell_g$ *and each* $g\in\{g^\delta_{H_\delta}(\cdot,z),\dot{g}^\delta_{H_\delta}(\cdot,z):z\in\mathbb{R}\}$ *is of the form* $g=k_{H_\delta}+\ell_g$*, where*

$$k_H(t)=\frac{1}{\Gamma(H+\frac12)}t^{H-1/2}\mathbf{1}_{\{t>0\}},\qquad k_{H_\delta}(t)=\frac{1}{\Gamma(H_\delta+\frac12)}t^{H_\delta-1/2}\mathbf{1}_{\{t>0\}} \tag{2.5}$$

   and ℓ_g *is a differentiable function with* $\ell_g(0)=0$ *and* $|\ell_g'(s)|\le L$ *for all* $s\in[0,t]$ *and some* $L\in(0,\infty)$ *that only depends on* t *(but not on* g*).*

Assumption A is a regularity condition with the exception of part 5 in which the rough components in the dynamics of the various processes are specified. We note that we allow different degrees of roughness for the processes driving the volatility process σ and the small jump size function δ, given by H and H_δ, respectively.

We will further impose the following moment and smoothness assumptions, which are parametrized by H and H_δ from Assumption A and three additional parameters: $q\in(0,1]$, $r\in[1,2)$ and $H_\gamma\in(0,\frac12)$. To simplify the notation, we write $o=o_p$ and $O=O_p$ in the following.

ASSUMPTION B. *For all* $t,T>0$*, there exists a positive finite* $\mathcal{F}_t$*-measurable random variable* C_t *and a positive* $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R})$*-measurable function* $C_t(z)$ *such that:*

1. *For* $s\in[t,t+T]$ *and* $v\in\{b,\eta^\sigma,\widetilde{\eta}^\sigma,\overline{\eta}^\sigma,\sigma^\eta,\widetilde{\sigma}^\eta,\overline{\sigma}^\eta,\widehat{\sigma}^\eta,\sigma^{\widetilde{\eta}},\widetilde{\sigma}^{\widetilde{\eta}},\overline{\sigma}^{\widetilde{\eta}},\widehat{\sigma}^{\widetilde{\eta}},\mathring{\sigma}^{\widetilde{\eta}},\theta,\widetilde{\theta}\}$*, we have*

$$\mathbb{E}_t[|\alpha_s|]+\mathbb{E}_t[v_s^2]^{1/2}<C_t. \tag{2.6}$$

2. *For any* $v\in\{\sigma^\eta,\widetilde{\sigma}^\eta,\overline{\sigma}^\eta,\widehat{\sigma}^\eta,\sigma^{\widetilde{\eta}},\widetilde{\sigma}^{\widetilde{\eta}},\overline{\sigma}^{\widetilde{\eta}},\widehat{\sigma}^{\widetilde{\eta}},\mathring{\sigma}^{\widetilde{\eta}}\}$*, we have*

$$\sup_{s,s'\in[t,t+T]}\mathbb{E}_t[(v_s-v_{s'})^2]^{1/2}=o(1) \tag{2.7}$$

   as $T\to0$*. Moreover, with the same* H *as in Assumption A, we have*

$$\sup_{s,s'\in[t,t+T]}\Big\{\mathbb{E}_t[(\theta_s-\theta_{s'})^2]^{1/2}+\mathbb{E}_t[(\widetilde{\theta}_s-\widetilde{\theta}_{s'})^2]^{1/2}\Big\}=o(T^H) \tag{2.8}$$

   and

$$\begin{aligned}\sup_{s,s'\in[t,t+T]}&\Big\{\mathbb{E}_t[|\alpha_s-\alpha_{s'}|]+\mathbb{E}_t[|\eta^\sigma_s-\eta^\sigma_{s'}|^2]^{1/2}+\mathbb{E}_t[|\widetilde{\eta}^\sigma_s-\widetilde{\eta}^\sigma_{s'}|^2]^{1/2}\\&+\mathbb{E}_t[|\overline{\eta}^\sigma_s-\overline{\eta}^\sigma_{s'}|^2]^{1/2}+\mathbb{E}_t\bigg[\int_{\mathbb{R}}(\delta_\sigma(s,z)-\delta_\sigma(s',z))^2\nu(dz)\bigg]^{1/2}\Big\}=O(T^H).\end{aligned} \tag{2.9}$$

3. *For all* $s \in [t, t+T]$,

$$
\begin{aligned}
&|\delta(t,z)| + \mathbb{E}_t[\sigma^\delta(s,z)^2]^{1/2} + \mathbb{E}_t[\dot\sigma^\delta(s,z)^2]^{1/2} + \mathbb{E}_t[\eta^\delta(s,z)^2]^{1/2} \\
&\quad + \mathbb{E}_t[\ddot\eta^\delta(s,z)^2]^{1/2} + \mathbb{E}_t\left[\int_{\mathbb{R}} |\delta_\delta(s,z,z')|^r \nu(dz')\right]^{1/r} < C_t(z)
\end{aligned}
\tag{2.10}
$$

and

$$
\mathbb{E}_t\left[\int_{\mathbb{R}} (|\gamma(s,z)|^q + C_t(z)^r + |\gamma_\sigma(s,z)|^q + |\delta_\sigma(s,z)|^2)\nu(dz)\right] < C_t. \tag{2.11}
$$

4. *As* $T \to 0$,

$$
\sup_{s,s' \in [t,t+T]} \mathbb{E}_t\left[\int_{\mathbb{R}} |\gamma(s,z) - \gamma(s',z)|^q \nu(dz)\right]^{1/q} = O(T^{H_\gamma}). \tag{2.12}
$$

Moreover, for all $s, s' \in [t, t+T]$,

$$
\begin{aligned}
&\mathbb{E}_t[(\sigma^\delta(s,z) - \sigma^\delta(s',z))^2]^{1/2} + \mathbb{E}_t[(\dot\sigma^\delta(s,z) - \dot\sigma^\delta(s',z))^2]^{1/2} \\
&\quad + \mathbb{E}_t[(\eta^\delta(s,z) - \eta^\delta(s',z))^2]^{1/2} + \mathbb{E}_t[(\ddot\eta^\delta(s,z) - \ddot\eta^\delta(s',z))^2]^{1/2} \\
&\quad + \mathbb{E}_t\left[\int_{\mathbb{R}} |\delta_\delta(s,z,z') - \delta_\delta(s',z,z')|^r \nu(dz')\right]^{1/r} \le C_t(z) T^{H_\delta}
\end{aligned}
\tag{2.13}
$$

and

$$
\mathbb{E}_t[|\theta_\delta(s,z) - \theta_\delta(s',z)|^r]^{1/r} \le C_t(z) T^{2H_\delta}. \tag{2.14}
$$

5. *The three roughness parameters* H, H_γ *and* H_δ *satisfy the relations*

$$
H_\gamma > H - (\tfrac{2}{q} - 1)(\tfrac{1}{2} - H) \qquad \textit{and} \qquad H_\delta > H - (\tfrac{2}{r} - 1)[(\tfrac{1}{2} - H) \wedge \tfrac{1}{4}]. \tag{2.15}
$$

Several comments are in order regarding Assumption B. First, the constants q and r are upper bounds on the degree of activity of the finite and infinite variation jump components, respectively, of x. Second, the constant H_γ can be viewed as a bound on the smoothness in time of the "big" jump size function γ. Unlike the processes σ and δ, we do not model explicitly the dynamics of γ but only provide a bound for its smoothness in Assumption B-4. Both jumps in γ and rough components driven by Brownian motions, like the ones in σ and δ, will determine the value of H_γ. The reason for treating σ and δ on one hand and γ on the other hand differently is that for the same degree of smoothness in expectation of these processes, the contribution in the higher-order expansion of the characteristic function due to γ will be dominated by that of σ and δ. Finally, the restriction in Assumption B puts a lower bound on H_δ and H_γ (higher values of these parameters correspond to smoother paths), which are chosen in such a way that the variation of γ and certain components of the dynamics of δ can be neglected in the asymptotic expansion. This assumption is trivially satisfied in arguably the most typical case in which H, H_δ and H_γ are the same.

Overall, Assumptions A and B allow x to be an Itô semimartingale with very general spot semimartingale characteristics. In particular, both the standard situation in which the spot semimartingale characteristics are themselves Itô semimartingales (see e.g., [16]) and the pure rough setting in which σ has no jumps and no martingale component (see e.g., [15]) are nested in our model. We even allow for models in which the jump size function can have roughness in its paths, which to the best of our knowledge has not been considered in prior parametric modeling.

3. Short-Time Expansion of Characteristic Functions. In this section we present our main theoretical result on a small-time asymptotic expansion of the conditional characteristic function of an increment of x, which we denote with

$$\mathcal{L}_{t,T}(u) = \mathbb{E}_t[e^{iu(x_{t+T}-x_t)/\sqrt{T}}], \quad u \in \mathbb{R}. \tag{3.1}$$

As before, the above conditional expectation is taken under $\mathbb{Q}$. While $\mathcal{L}_{t,T}(u)$ is typically not known in closed form, it can be computed analytically for some specific parametric models such as the rough Heston volatility model, see [9].

To state our expansion result, we need some notation. For $t' \geq t \geq 0$ and $u \in \mathbb{R}$, we set

$$\begin{aligned}
\sigma_{t'|t} &= \sigma_t + \int_0^t (k_H(t'-s) - k_H(t-s))(\eta_s dW_s + \widetilde{\eta}_s d\widetilde{W}_s), \\
\eta_{t'|t} &= \eta_t + \int_0^t (k_H(t'-s) - k_H(t-s))(\sigma_s^\eta dW_s + \widetilde{\sigma}_s^\eta d\widetilde{W}_s + \overline{\sigma}_s^\eta d\overline{W}_s + \widehat{\sigma}_s^\eta d\widehat{W}_s), \\
\delta(t' \mid t, z) &= \delta(t,z) + \int_0^t (k_{H_\delta}(t'-s) - k_{H_\delta}(t-s))(\sigma^\delta(s,z) dW_s + \dot{\sigma}^\delta(s,z) d\dot{W}_s).
\end{aligned} \tag{3.2}$$

We note that

$$\mathbb{E}_t[\sigma_{t+T}] = \sigma_{t+T|t} + O(T) \quad \text{and} \quad \sigma_{t+T|t} = \sigma_t + O(T^H), \quad \text{as } T \to 0, \tag{3.3}$$

with similar expansions for $\mathbb{E}_t[\eta_{t+T}]$ and $\mathbb{E}_t[\delta(t+T,z)]$. In other words, up to a higher-order term, $\sigma_{t+T|t}$ is the $\mathcal{F}_t$-conditional expectation of σ_{t+T}. We note, however, that the gap between $\sigma_{t+T|t}$ and σ_t, which is mainly due to the rough component of σ_t, can be rather nontrivial, particularly for lower values of H. This is a major difference from a standard volatility model without a rough component.

We next introduce notation related to the characteristic exponents of the small and big jumps:

$$\begin{aligned}
&\phi_t(u) = \int_{\mathbb{R}} (e^{iu\gamma(t,z)} - 1)\nu(dz), \quad \varphi_t(u) = \int_{\mathbb{R}} (e^{iu\delta(t,z)} - 1 - iu\delta(t,z))\nu(dz), \\
&\psi_t(u) = \phi_t(u) + \varphi_t(u).
\end{aligned} \tag{3.4}$$

Finally, in our expansions, small jumps in x will play a key role, and we need the following additional notation for them:

$$\begin{aligned}
\chi_t^{(1)}(u) &= \int_{\mathbb{R}} \sigma^\delta(t,z)(e^{iu\delta(t,z)} - 1)\nu(dz), \\
\chi_{t'|t}^{(1)}(u) &= \int_{\mathbb{R}} (\delta(t' \mid t, z) - \delta(t,z))(e^{iu\delta(t,z)} - 1)\nu(dz), \\
\chi_t^{(2)}(u) &= \int_{\mathbb{R}} \eta^\delta(t,z)(e^{iu\delta(t,z)} - 1)\nu(dz), \\
\chi_t^{(3)}(u) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \delta_\delta(t,z,z')(e^{iu\delta(t,z)} - 1)(e^{iu\delta(t,z')} - 1)\nu(dz)\nu(dz'), \\
\chi_t^{(4)}(u) &= \int_{\mathbb{R}} \delta_\sigma(t,z)(e^{iu\delta(t,z)} - 1)\nu(dz).
\end{aligned} \tag{3.5}$$

The following theorem contains our main expansion result for $\mathcal{L}_{t,T}(u)$. In the statement of the theorem, the superscript in o^{uc} stands for "uniformly for u belonging to compact subsets of $\mathbb{R}$."

THEOREM 3.1. *If Assumptions A and B hold with $H, H_\gamma, H_\delta \in (0, \frac{1}{2})$, $q \in (0,1]$ and $r \in [1,2)$, then as $T \to 0$,*

$$\begin{aligned}\mathcal{L}_{t,T}(u) = \exp\bigg(& iu\alpha_t\sqrt{T} - \frac{1}{2}u^2\int_0^1 \sigma^2_{t+sT|t}ds + T\psi_t(\tfrac{u}{\sqrt{T}}) \\ & - iu^3\bigg(\frac{1}{2}\sigma_t^2\eta_t^\sigma T^{1/2} + \frac{\sigma_t^2\eta_t}{\Gamma(H+\frac{5}{2})}T^H + C'_{1,0}(T)_t T^{2H}\bigg)\bigg) \\ & + e^{-\frac{1}{2}u^2\sigma_t^2 + T\varphi_t(u/\sqrt{T})}\bigg(\sum_{i=1}^4 C_{1,i}(u,T)_t + C'_{1,1}(u,T)_t\bigg) \\ & + C_2(u)_t T^{2H} + o^{\mathrm{uc}}(T^{2H}),\end{aligned} \tag{3.6}$$

where

$$\begin{aligned} & C_{1,1}(u,T)_t = -\frac{u^2\sigma_t T^{H_\delta+1/2}\chi_t^{(1)}(\frac{u}{\sqrt{T}})}{\Gamma(H_\delta+\frac{5}{2})}, \quad C_{1,2}(u,T)_t = -\frac{1}{2}u^2\sigma_t T\chi_t^{(2)}(\tfrac{u}{\sqrt{T}}), \\ & C_{1,3}(u,T)_t = \frac{1}{2}iuT^{3/2}\chi_t^{(3)}(\tfrac{u}{\sqrt{T}}), \qquad\qquad C_{1,4}(u,T)_t = -\frac{1}{2}u^2\sigma_t T\chi_t^{(4)}(\tfrac{u}{\sqrt{T}})\end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & C'_{1,0}(T)_t = \frac{T^{-H}}{\Gamma(H+\frac{1}{2})}\int_0^1\int_0^s (s-r)^{H-1/2}(\sigma_{t+rT|t}\sigma_{t+sT|t}\eta_{t+rT|t} - \sigma_t^2\eta_t)drds, \\ & C'_{1,1}(u,T)_t = iuT^{1/2}\int_0^1 \chi^{(1)}_{t+sT|t}(\tfrac{u}{\sqrt{T}})ds\end{aligned} \tag{3.8}$$

and

$$\begin{aligned} C_2(u)_t = e^{-\frac{1}{2}u^2\sigma_t^2}\bigg(& \frac{\sigma_t^2 u^4(\sigma_t\sigma_t^\eta + \eta_t^2)}{\Gamma(2H+3)} + \frac{\sigma_t^2(\eta_t^2 + \widetilde{\eta}_t^2)u^4}{4(H+1)\Gamma(H+\frac{3}{2})^2} \\ & - \frac{(\eta_t^2 + \widetilde{\eta}_t^2)u^2}{8H\Gamma(H+\frac{1}{2})\Gamma(H+\frac{3}{2})}\bigg).\end{aligned} \tag{3.9}$$

REMARK 3.2. From the proof of Theorem 3.1, it follows that all processes in (1.1) and (2.1)–(2.4) are well defined under Assumptions A and B.

REMARK 3.3. The point $t = 0$ plays a special role in the Riemann–Liouville representation of rough processes. For example, $s = 0$ is the only time point at which for any $t > s$ we have $\mathbb{E}[\sigma_t \mid \mathcal{F}_s] = \sigma_s$, if $b \equiv \gamma_\sigma \equiv 0$. This has the consequence that $C'_{1,0}(T)_t = C'_{1,1}(u,T)_t = 0$ if (and only if) $t = 0$.

We make several comments about the above expansion result. First, compared with the corresponding result in [28] for the standard case when the spot semimartingale characteristics do not have rough components in their paths and x does not have jumps of infinite variation, we note that the higher-order terms in the expansion here are of much higher asymptotic order. This underscores the importance of deriving such a higher-order expansion in a rough setting. The leading term of $|\mathcal{L}_{t,T}(u)|$ is $\exp(-\frac{1}{2}u^2\int_0^1 \sigma^2_{t+sT|t}ds)$, which is $O(1)$, while that of the argument of the characteristic function is the highest among $\Im(T\psi_t(\frac{u}{\sqrt{T}}))$, $-u^3\frac{1}{2}\sigma_t^2\eta_t^\sigma T^{1/2}$ and $-u^3\sigma_t^2\eta_t\Gamma(H+\frac{5}{2})^{-1}T^H$. These three terms are of asymptotic order

$O(T^{1-r/2})$, $O(\sqrt{T})$ and $O(T^H)$, respectively. Therefore, which of them dominates will depend on the degree of roughness of volatility and the degree of jump activity, and of course which of these components are actually present in the dynamics of x.

Similar comments apply for the higher-order terms in the asymptotic expansion due to the dynamics of the spot semimartingale characteristics. In particular, $\{C_{1,i}(u,T)_t\}_{i=1,\ldots,4}$ and $C'_{1,1}(u,T)_t$ are all due to the dynamics of the infinite jump variation component of x. The leading ones among them are $C_{1,1}(u,T)_t$ and $C'_{1,1}(u,T)_t$, both of which are of order $O(T^{H_\delta+1-r/2})$ and depend on the rough component of the jump size function δ. We further have $C_{1,2}(u,T)_t + C_{1,3}(u,T)_t + C_{1,4}(u,T)_t = O(T^{3/2-r/2})$, and these three terms do not depend on the rough component in δ.

The higher-order term in the expansion due to the rough component of the volatility dynamics is the term $C_2(u)_t T^{2H}$, which is of order $O(T^{2H})$. Comparing this contribution with the one due to the variation of δ, which is of order $O(T^{H_\delta+1-r/2})$, if the small jumps have a rough component in their dynamics, we see that their asymptotic ranking in terms of size will depend on the values of the three parameters H, H_δ and r. Of course, the different components in the asymptotic expansion of Theorem 3.1 also typically depend in a different way on u, and this can be used to separate their contribution to the conditional characteristic function as we will illustrate in our application of the expansion result.

We finish this section with presenting an expansion result for $\mathbb{E}_t[x_{t+T} - x_t]$, whose proof is almost immediate from (1.1). We will make use of this result later on when we estimate the degree of volatility roughness.

PROPOSITION 3.4. *Suppose that the assumptions of Theorem 3.1 hold with $q = 1$ and $H_\gamma \geq H$. Then, as $T \to 0$, we have that*

$$\mathbb{E}_t[x_{t+T} - x_t] = \left(\alpha_t + \int_{\mathbb{R}} \gamma(t,z)\nu(dz)\right)T + O(T^{1+H}).$$

4. Option-Based Characteristic Function Portfolios. If x is an asset price, the conditional characteristic function of its price increments can be inferred from portfolios of short-dated options, as done in [24] and [27]. This allows for feasible inference and the development of model specification tests on the basis of the expansion results of the previous section.

More specifically, under the simplifying assumption of zero risk-free rate and dividend yield, we have the following option spanning result from [3]:

$$\mathbb{E}_t[e^{iu(x_{t+T}-x_t)/\sqrt{T}}] = 1 - \left(\frac{u^2}{T} + i\frac{u}{\sqrt{T}}\right)e^{-x_t}\int_{\mathbb{R}} e^{(iu/\sqrt{T}-1)(k-x_t)} O_{t,T}(k)dk \tag{4.1}$$

for $u \in \mathbb{R}$, where $O_{t,T}(k)$ denotes the price at time t of an European style out-of-the-money option expiring at $t+T$ and with log-strike of k. In agreement with the notation used so far, the expectation above is taken under $\mathbb{Q}$, which signifies the so-called risk-neutral probability measure. We remind the reader that $O_{t,T}(k)$ is a put if $k \leq \log(F_{t,T})$ and a call otherwise, where $F_{t,T}$ is the time-t futures price of the asset with expiration date $t+T$.

We can make a simple Riemann sum approximation of the integral in (4.1) using available options on a discrete log-strike grid. More specifically, for maturity T_ℓ, we denote the available log-strike grid with

$$\underline{k}_\ell \equiv k_{\ell,1} < k_{\ell,2} < \cdots < k_{\ell,N_\ell} \equiv \overline{k}_\ell, \qquad N_\ell \in \mathbb{N}_+. \tag{4.2}$$

The gap between the log-strikes is denoted $\Delta_{\ell,i} = k_{\ell,i} - k_{\ell,i-1}$ for $i = 2,\ldots,N_\ell$ and $\ell = 1,2,\ldots$ The log-strike grids need not be equidistant, that is, $\Delta_{\ell,i}$ may differ across i's. For simplicity, in the above notation of the log-strike grid, we have dropped the dependence on t, as t will be fixed throughout our applications.

The true option prices are all defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the statistical (true) probability measure and is locally equivalent to $\mathbb{Q}$. The observed option prices contain errors, that is,

$$\widehat{O}_{t,T_\ell}(k_{\ell,j}) = O_{t,T_\ell}(k_{\ell,j}) + \epsilon_{t,T_\ell}(k_{\ell,j}), \quad j = 1, \dots, N_\ell, \quad \ell = 1, 2, \dots, \tag{4.3}$$

where the errors $\epsilon_{t,T_\ell}(k_{\ell,j})$ are defined on an auxiliary space $(\Omega^{(1)}, \mathcal{F}^{(1)})$ equipped with a transition probability $\mathbb{P}^{(1)}(\omega, d\omega^{(1)})$ from Ω, the probability space on which x is defined, to $\Omega^{(1)}$. We further define

$$\overline{\Omega} = \Omega \times \Omega^{(1)}, \quad \overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}^{(1)}, \quad \overline{\mathbb{P}}(d\omega, d\omega^{(1)}) = \mathbb{P}(d\omega)\, \mathbb{P}^{(1)}(\omega, d\omega^{(1)})\,.$$

Using the available options, our estimator of $\mathcal{L}_{t,T}(u)$ from (3.1) is given by

$$\widehat{\mathcal{L}}_{t,T_\ell}(u) = 1 - \left(\frac{u^2}{T} + i\frac{u}{\sqrt{T}}\right) e^{-x_t} \sum_{j=2}^{N_\ell} e^{(iu/\sqrt{T_\ell}-1)(k_{\ell,j-1}-x_t)} \widehat{O}_{T_\ell}(k_{\ell,j-1})\Delta_{\ell,j},\ u \in \mathbb{R}. \tag{4.4}$$

Our estimation procedure in Section 5 will be based on the argument of the characteristic function $\mathcal{L}_{t,T}(u)$. As seen from the expansion result in (3.6), the value α_t of the spot drift term at time t appears in it. We can estimate it from the option data. More specifically, similarly to (4.1), we have

$$\mathbb{E}_t[x_{t+T} - x_t] = -\int_{\mathbb{R}} e^{-k} O_{t,T}(k) dk. \tag{4.5}$$

The feasible (and standardized) counterpart of this is given by

$$\widehat{\mathcal{M}}_{t,T_\ell} = -\frac{1}{T_\ell} \sum_{j=2}^{N_\ell} e^{-k_{\ell,j-1}} \widehat{O}_{T_\ell}(k_{\ell,j-1})\Delta_{\ell,j}. \tag{4.6}$$

We note that $-2\widehat{\mathcal{M}}_{t,T_\ell}$ is an estimate of the conditional expected integrated variance of x (this follows by an application of Itô's lemma and the definition of the risk-neutral measure) and is used in the computation of the VIX volatility index by the CBOE options exchange.

Our goal in this section is to derive a bound for the error $\widehat{\mathcal{L}}_{T_\ell}(u) - \mathcal{L}_{T_\ell}(u)$ and $\widehat{\mathcal{M}}_{t,T_\ell} - \frac{1}{T_\ell}\mathbb{E}_t[x_{t+T_\ell} - x_t]$ as the mesh of the log-strike grid, $\sup_{i=2,\dots,N_\ell} \Delta_{\ell,i}$, shrinks towards zero. For deriving such a result, we need several assumptions concerning existence of conditional moments of x, the observation scheme as well as the observation errors. They are stated below. As before, if expectation is taken under $\mathbb{Q}$, we will not use superscript in the notation; if expectation is under the true probability or $\overline{\mathbb{P}}$, we put superscripts to signify this.

ASSUMPTION C. *There are $\mathcal{F}_t$-measurable random variables c_t and C_t such that the following holds:*

1. *We have $\sigma_t > 0$ and there exists $\bar{t} > t$ such that for $s \in [t, \bar{t}]$ we have*

$$\begin{aligned} &\mathbb{E}_t[|\sigma_s|^4] + \mathbb{E}_t[e^{2|x_s|}] + \mathbb{E}_t\left[\left(\int_{\mathbb{R}} (e^{|\gamma(s,z)|} - 1)\nu(dx)\right)^2\right] \\ &\quad + \mathbb{E}_t\left[\left(\int_{\mathbb{R}} (e^{|\delta(s,z)|} - 1 - |\delta(s,z)|)\nu(dx)\right)^2\right] < C_t. \end{aligned} \tag{4.7}$$

2. *The log-strike grids* $\{k_{\ell,j}\}_{j=1}^{N_\ell}$, *for* $\ell = 1, 2, \ldots$, *are* $\mathcal{F}_t$*-measurable and we have*

$$c_t\Delta \le k_{\ell,j} - k_{\ell,j-1} \le C_t\Delta, \qquad \ell = 1, 2, \ldots, \tag{4.8}$$

where Δ *is a deterministic sequence.*

3. *We have*

$$\epsilon_{t,T_\ell}(k_{\ell,j}) = \zeta_{t,\ell}(k_{\ell,j})\overline{\epsilon}_{t,\ell,j}O_{t,T_\ell}(k_{\ell,j}) \tag{4.9}$$

for some $\mathcal{F}_t$*-measurable* $\zeta_{t,\ell}(k_{\ell,j})$ *and some sequences of* $\mathcal{F}^{(1)}$*-measurable random variables* $\{\overline{\epsilon}_{t,\ell,j}\}_{j=1}^{N_\ell}$, *for* $\ell = 1, 2, \ldots$, *that are i.i.d. and independent of each other and of* $\mathcal{F}$. *We further have*

$$\mathbb{E}^{\overline{\mathbb{P}}}[\overline{\epsilon}_{t,\ell,j} \mid \mathcal{F}] = 0, \quad \mathbb{E}^{\overline{\mathbb{P}}}[(\overline{\epsilon}_{t,\ell,j})^2 \mid \mathcal{F}] = 1, \quad \mathbb{E}^{\overline{\mathbb{P}}}[|\overline{\epsilon}_{t,\ell,j}|^\kappa \mid \mathcal{F}] < \infty \tag{4.10}$$

for some $\kappa \ge 4$ *and all* $\ell = 1, 2, \ldots$

Under Assumption C, we have

$$\widehat{\mathcal{M}}_{t,T_\ell} \xrightarrow{\mathbb{P}} \frac{1}{T_\ell}\mathbb{E}_t[x_{t+T_\ell} - x_t] \quad \text{and} \quad \widehat{\mathcal{L}}_{T_\ell}(u) \xrightarrow{\mathbb{P}} \mathcal{L}_{T_\ell}(u),$$

locally uniformly in u for all $\ell = 1, 2, \ldots$ In the next theorem, we provide a bound for the rate of convergence of the feasible estimators.

THEOREM 4.1. *Suppose Assumptions A, B and C hold. For some finite* $L \in \mathbb{N}_+$, *assume that there are* $\beta, \gamma > 0$ *and* $\alpha > \frac{1}{2}$ *such that as* $T \to 0$, *we have* $T_\ell \asymp T$, $\Delta \asymp T^\alpha$, $e^{\underline{k}_\ell} \asymp T^\beta$, $e^{\overline{k}_\ell} \asymp T^{-\gamma}$ *for all* $\ell = 1, \ldots, L$. *Then, if* $\beta \wedge \gamma > \frac{\alpha}{4} - \frac{1}{8}$, *we have*

$$\left|\widehat{\mathcal{L}}_{t,T_\ell}(u) - \mathcal{L}_{t,T_\ell}(u)\right| + \left|\widehat{\mathcal{M}}_{t,T_\ell} - \frac{1}{T_\ell}\mathbb{E}_t[x_{t+T_\ell} - x_t]\right| = O_{\mathbb{P}}^{\mathrm{uc}}\left(\frac{\sqrt{\Delta}}{T^{1/4}}\right), \quad \ell = 1, 2, \ldots \tag{4.11}$$

The rate of convergence of the option-based estimators is determined by the mesh of the log-strike grid, Δ, and the asymptotic order of the maturities of the options used in the estimation T. We note that for consistent estimation, we need $\Delta/\sqrt{T} \to 0$. The reason for this is that if this is not the case the Riemann sum approximation error around-the-money, i.e., for log-strikes around x_t, will be too big relative to the quantity that is estimated (the conditional characteristic function or the conditional mean). In practice, this condition is expected to hold for the typical available strike grids and maturities, and this is reflected in the fact that the observed changes in option prices across consecutive strikes are typically not very big.

In the statement of the theorem, we further impose the requirement $\beta \wedge \gamma > \frac{\alpha}{4} - \frac{1}{8}$. This requirement guarantees that the error in $\widehat{\mathcal{L}}_{T_\ell}(u)$ and $\widehat{\mathcal{M}}_{t,T_\ell}$ due to the fact that Riemann sum is over a bounded (but asymptotically increasing) log-strike domain. This error is not expected to be binding in a typical application because we typically observe deep out-of-the-money option prices until their value falls below the minimum tick size. Finally, we note that with a slight strengthening of Assumption C, we can also derive a CLT for $\widehat{\mathcal{L}}_{t,T_\ell}(u)$ and $\widehat{\mathcal{M}}_{t,T_\ell}$. For brevity, we do not present such a result here.

5. Application: Estimating the Degree of Volatility Roughness. We will show in this section how one can estimate the volatility roughness parameter H using the expansion result of Theorem 3.1 and short-dated options. The parameter H enters both in the norm and the argument of the conditional characteristic function. Estimating H from the latter appears somewhat easier provided one knows that $\eta_t \neq 0$. Note that the process η, together with

η^σ, captures the dependence between the price and volatility diffusive moves, and there is a lot of empirical evidence for the existence of such dependence. Therefore, an assumption that $\eta_t \neq 0$ is not restrictive from an empirical point of view and we will work under such assumption in this section.

We first start by discussing the case where there are no jumps in x, that is,

$$\gamma(t,z) \equiv \delta(t,z) \equiv 0. \tag{5.1}$$

We might still have jumps in σ, η or $\widetilde{\eta}$, of course. For $u > 0$, we introduce

$$\begin{aligned} A_{t,T}(u) &= \operatorname{Arg}(\mathcal{L}_{t,T}(u)) - \frac{u}{\sqrt{T}}\mathbb{E}_t[x_{t+T} - x_t], \\ \widehat{A}_{t,T}(u) &= \operatorname{Arg}(\widehat{\mathcal{L}}_{t,T}(u)) - u\sqrt{T}\widehat{\mathcal{M}}_{t,T}, \end{aligned} \tag{5.2}$$

where as usual $\operatorname{Arg}(z)$ is the principal argument of the complex number z. Based on the expansions in Theorem 3.1 and Proposition 3.4, we derive the following result.

COROLLARY 5.1. *Under* (5.1) *and the assumptions of Proposition 3.4, we have*

$$A_{t,T}(u) = -\frac{\sigma_t^2 \eta_t u^3}{\Gamma(H+\frac{5}{2})} T^H + O^{\mathrm{uc}}(T^{\frac{1}{2}\wedge 2H}), \qquad T \to 0. \tag{5.3}$$

If furthermore $\eta^\sigma \equiv 0$, then this expansion improves to

$$A_{t,T}(u) = -\frac{\sigma_t^2 \eta_t u^3}{\Gamma(H+\frac{5}{2})} T^H + O^{\mathrm{uc}}(T^{2H}), \qquad T \to 0. \tag{5.4}$$

To make use of the corollary, we pick two short tenors $0 < T_1 < T_2$ and denote $\tau = T_2/T_1$. Then (5.3) (resp., (5.4)) implies that for any fixed $u > 0$,

$$\frac{A_{t,T_2}(u)}{A_{t,T_1}(u)} = \tau^H + O(T_1^{(\frac{1}{2}-H)\wedge H}) \quad (\text{resp., } O(T_1^H)),$$

from which one can easily estimate H with the same rate. To improve efficiency, we can use multiple u's in the estimation. We can also perform estimation on the basis of certain transformations of $A_{t,T}(u)$ that can help reduce finite-sample biases. More specifically, in a first step, we rewrite (5.3) (resp., (5.4)) as

$$A_{t,T}(u) = -\frac{\sigma_t^2 \eta_t u^3}{\Gamma(H+\frac{5}{2})} T^H + C_{1,5}(T)_t u^5 + O^{\mathrm{uc}}(T^{\frac{1}{2}\wedge 2H}) \quad (\text{resp., } O^{\mathrm{uc}}(T^{2H})),$$

where $C_{1,5}(T)_t = 4\sum_{j=1}^\infty \alpha_j^3 \beta_j^2$ in the notation of (7.32) in the proof of Theorem 3.1. While $C_{1,5}(T)_t = O(T^{3H})$, including it in the last display improves the performance of our estimators for given T_1 and T_2 in the simulation study. Also, we find that taking the inverse of $A_{t,T}(u)$ is more preferable, leading us to the expansion

$$\begin{aligned} -\frac{u^3}{A_{t,T}(u)} &= \frac{1}{\sigma_t^2 \eta_t \Gamma(H+\frac{5}{2})^{-1} T^H - C_{1,5}(T)_t u^2} + O(T^{(\frac{1}{2}-2H)\wedge 0}) \quad (\text{resp., } O(1)) \\ &= \frac{\Gamma(H+\frac{5}{2})}{\sigma_t^2 \eta_t} T^{-H} + \frac{\Gamma(H+\frac{5}{2})^2 C_{1,5}(T)_t}{\sigma_t^4 \eta_t^2 T^{2H}} u^2 + O(T^{(\frac{1}{2}-2H)\wedge 0}) \quad (\text{resp., } O(1)). \end{aligned}$$

Taking a grid of points $\underline{u} = (u_1, u_2, \dots, u_K)$ for $0 < u_1 < \cdots < u_K$ and some integer $K \geq 2$, we then define

$$A_{t,T}(\underline{u}) = \frac{\sum_{i=1}^K u_i^2 \sum_{i=1}^K A_{t,T}(u_i)^{-1} u_i^5 - \sum_{i=1}^K u_i^4 \sum_{i=1}^K u_i^3 A_{t,T}(u_i)^{-1}}{K\sum_{i=1}^K u_i^4 - \left(\sum_{i=1}^K u_i^2\right)^2}, \tag{5.5}$$

which is the intercept of a regression of $-u_i^3/A_{t,T}(u_i)$ on a constant and u_i^2. From the preceding discussion and the mean-value theorem, we obtain

$$\frac{A_{t,T_1}(\underline{u})}{A_{t,T_2}(\underline{u})} = \tau^H + O(T_1^{(\frac{1}{2}-H)\wedge H}) \quad \text{(resp., } O(T_1^H)\text{)}. \tag{5.6}$$

For a feasible estimator on the basis of the above result, we define $\widehat{A}_{t,T_j}(\underline{u})$, for $j = 1, 2$, in the same way as $A_{t,T_j}(\underline{u})$ in (5.5) but with $\widehat{A}_{t,T_j}(u_i)$ instead of $A_{t,T_j}(u_i)$. Then, the estimator of H is given by

$$\widehat{H}_n = \frac{\log \widehat{A}_{t,T_1}(\underline{u}) - \log \widehat{A}_{t,T_2}(\underline{u})}{\log \tau}. \tag{5.7}$$

The following theorem derives a bound for its rate of convergence.

THEOREM 5.2. *Assume (5.1) and the hypotheses of Proposition 3.4 and Theorem 4.1 hold. If $\sigma_t > 0$ and $\eta_t \neq 0$, then*

$$\widehat{H}_n = H + O_{\overline{\mathbb{P}}}(T_1^{(\frac{1}{2}-H)\wedge H} \vee \Delta^{\frac{1}{2}} T_1^{-\frac{1}{4}-H}). \tag{5.8}$$

If $\eta^\sigma \equiv 0$ in addition, this improves to

$$\widehat{H}_n = H + O_{\overline{\mathbb{P}}}(T_1^H \vee \Delta^{\frac{1}{2}} T_1^{-\frac{1}{4}-H}). \tag{5.9}$$

If $\sigma_t > 0$, $\eta \equiv 0$ but $\eta_t^\sigma \neq 0$, we have

$$\widehat{H}_n = \frac{1}{2} + O_{\overline{\mathbb{P}}}(T_1 \vee \Delta^{\frac{1}{2}} T_1^{-\frac{3}{4}}). \tag{5.10}$$

The last result in (5.10) shows that if σ does not contain a rough component, then our estimator converges to $\frac{1}{2}$, which is the value of the Hurst parameter of Brownian motion.

Next, we shall describe how one can construct an estimator of H in the presence of jumps that retains the same rate of convergence. For this, we need to take into account the fact that $A_{t,T}(u)$ is affected by jumps. To remove the effect due to them, we will use $A_{t,T}(u)$ for different T's and we will suitably difference them. This will eliminate the leading term in $A_{t,T}(u)$ due to the jumps but will result in a loss of signal for the value of H. This is the price to pay for the robustness of the estimation of H to jumps in x. For realistic time-to-maturity values, we expect such a jump-robust estimator of H to perform relatively poor due to the reduced signal about H and the consequent high sensitivity to even small finite-sample biases in the estimation. The situation here is reminiscent of the problem of estimating multiple jump activity indices, see [19]. Nevertheless, our subsequent analysis shows that the roughness parameter H can be recovered consistently, even in the presence of infinite variation jumps.

To fix ideas, let us pick four (instead of two) short tenors $0 < T_1 < T_2 < T_3 < T_4$. For $j = 2, 3, 4$, let $\tau_j = T_j/T_1$ and

$$\begin{aligned} A_{t,T_1,T_j}(u) &= \frac{1}{u^3}\big(\mathrm{Arg}(\mathcal{L}_{t,T_1}(u)) - \mathrm{Arg}(\mathcal{L}_{t,T_j}(u\sqrt{\tau_j}))/\tau_j\big), \\ \widehat{A}_{t,T_1,T_j}(u) &= \frac{1}{u^3}\big(\mathrm{Arg}(\widehat{\mathcal{L}}_{t,T_1}(u)) - \mathrm{Arg}(\widehat{\mathcal{L}}_{t,T_j}(u\sqrt{\tau_j}))/\tau_j\big). \end{aligned} \tag{5.11}$$

With this differencing approach, the contribution of the drift α_t and of the jumps through $T\psi_t(u/\sqrt{T})$ to the argument of $\mathcal{L}_{t,T}(u)$ is canceled out identically. In order to further remove

$C_{1,j}(u,T)_t$ where $j=2,3,4$ as well as the term $-\frac{1}{2}u^3\sigma_t^2\eta_t^\sigma T^{1/2}$ in (3.6), we consider second-order differences of the form

$$
\begin{aligned}
A_{t,T_1,T_j,T_4}(u) &= A_{t,T_1,T_j}(u) - \frac{\tau_j-1}{\tau_4-1}A_{t,T_1,T_4}(u),\\
\widehat{A}_{t,T_1,T_j,T_4}(u) &= \widehat{A}_{t,T_1,T_j}(u) - \frac{\tau_j-1}{\tau_4-1}\widehat{A}_{t,T_1,T_4}(u), \qquad j=2,3.
\end{aligned} \tag{5.12}
$$

THEOREM 5.3. *Under the assumptions of Theorem 3.1, we have*

$$
\begin{aligned}
A_{t,T_1,T_2,T_4}(u) = {}& \frac{\sigma_t^2\eta_t}{\Gamma(H+\frac{5}{2})}\Big[(\tau_2^{\frac{1}{2}+H}-1) - \frac{\tau_2-1}{\tau_4-1}(\tau_4^{\frac{1}{2}+H}-1)\Big]T_1^H\\
&+ O(T_1^{(H_\delta+1-\frac{r}{2})\wedge 2H}).
\end{aligned} \tag{5.13}
$$

Moreover, if the conditions of Theorem 4.1 are satisfied and $\sigma_t>0$ and $\eta_t\neq 0$, then we have the following result for any $u>0$: if τ_2, τ_3 and τ_4 are such that the mapping

$$
F(H) = \Big[(\tau_2^{\frac{1}{2}+H}-1) - \frac{\tau_2-1}{\tau_4-1}(\tau_4^{\frac{1}{2}+H}-1)\Big]\Big/\Big[(\tau_3^{\frac{1}{2}+H}-1) - \frac{\tau_3-1}{\tau_4-1}(\tau_4^{\frac{1}{2}+H}-1)\Big]
$$

has an inverse F^{-1} and $\frac{d}{dH}F(H)\neq 0$, then the estimator

$$
\widehat{H}_n' = F^{-1}\left(\frac{\widehat{A}_{t,T_1,T_3,T_4}(u)}{\widehat{A}_{t,T_1,T_2,T_4}(u)}\right) \tag{5.14}
$$

satisfies

$$
\widehat{H}_n' = H + O_{\overline{\mathbb{P}}}(T_1^{(H_\delta-H+1-\frac{r}{2})\wedge H} \vee \Delta^{\frac{1}{2}}T_1^{-\frac{1}{4}-H}). \tag{5.15}
$$

As seen from the result of the theorem, $\widehat{H}_n'$ can achieve the same rate of convergence as $\widehat{H}_n$ even when jumps are present in x. The loss of information due to the robustification for jumps is easy to illustrate. Suppose, $\tau_2=2$, $\tau_3=3$ and $\tau_4=4$ as for a typical set of available option maturities. In this case, $F(0)=1.2370$ and $\lim_{H\to 1/2}F(H)=1.1525$. This implies a relatively small variation in the function $F(H)$ as H varies in $(0,\frac{1}{2}]$. By contrast, the counterpart of $F(H)$ for the estimator $\widehat{H}_n$ is τ_2^H and this quantity varies from 1 for $H=0$ to 1.4142 for $H=\frac{1}{2}$, which is a much larger spread.

We note that the estimator $\widehat{H}_n'$ should be used only if one knows that the volatility has a rough component, that is, $\eta_t\neq 0$. If this is not the case, one can first test for rough volatility (in the presence of jumps) using the ratio $\widehat{A}_{t,T_1,T_2}(u)/\widehat{A}_{t,T_1,T_3}(u)$. For brevity, we omit the details of such a test.

6. Monte Carlo Study. In this section we will illustrate the performance of our estimator $\widehat{H}_n$ using simulated data from the following model for the asset price X:

$$
\frac{dX_t}{X_t} = \sqrt{V_t}dW_t,\;\; V_t = V_0 + \frac{\nu}{\Gamma\left(H+\frac{1}{2}\right)}\int_0^t (t-s)^{H-\frac{1}{2}}\sqrt{V_s}(\rho dW_s + \sqrt{1-\rho^2}d\widetilde{W}_s), \tag{6.1}
$$

where W and $\widetilde{W}$ are independent Brownian motions, V_0 is an $\mathcal{F}_0$-measurable positive random variable, and $\nu>0$ and $\rho\in[-1,1]$ are some constants. This is the model considered by [8] and can be viewed as the limit of the rough Heston model of [7] with zero mean-reversion. We generate option prices from the model using the codes from the paper of [25].

We note that in this model, there are several restrictions to the general specification of $x = \log X$ in (1.1) and (2.1)–(2.4). First, there are no jumps, so that $\gamma(s,z) = \delta(s,z) = \gamma_\sigma(s,z) = \delta_\sigma(s,z) = 0$. Second, the variance only has a rough component and this implies $\eta_s^\sigma = \widetilde{\eta}_s^\sigma = \overline{\eta}_s^\sigma = 0$. These restrictions allow us to strengthen the asymptotic expansion result of Theorem 3.1 to the following one for this class of models (i.e., models with the above-mentioned restrictions):

$$\mathcal{L}_{0,T}(u) = \exp\left(iu\alpha_0\sqrt{T} - \frac{1}{2}u^2\sigma_0^2 - iu^3\frac{\sigma_0^2\eta_0}{\Gamma(H+\frac{5}{2})}T^H \right) + C_2(u)_0 T^{2H} + o^{\mathrm{uc}}(T^{2H}). \tag{6.2}$$

We set the parameters of the model as follows. The correlation parameter is set to $\rho = -0.9$, which allows for strong negative dependence between price and volatility innovations (also known as leverage effect). We put the volatility of volatility parameter to $\nu = 0.5$. This generates fat-tailed return distributions and consequently numbers of short-tenor options (for the empirically calibrated option observation scheme explained below) that are roughly consistent with those observed for market index options. Finally, we experiment with four values of the Hurst parameter: $H = 0.1$, 0.25, 0.4 and 0.49. In all considered cases, we set the starting value of the price at $X_0 = 3000$. For the starting variance, we consider three values: $V_0 = 0.015$ (low volatility), $V_0 = 0.03$ (medium volatility) and $V_0 = 0.06$ (high volatility). These numbers are calibrated to the level and volatility of the S&P 500 market index.

We turn next to the option observation scheme. We consider two short tenors: $T_1 = 3/252$ and $T_2 = 6/252$, corresponding to 3 and 6 business days to expiration, with a unit of time being one calendar year. The strike grid of observed options is equidistant at increments of 5. Starting from the current stock price, we keep adding lower and higher strikes until the true option price falls in value below 0.075. This setup mimics that of available options written on the S&P 500 index. Finally, option prices are observed with error that is proportional to the true option price. More specifically, we have

$$\widehat{O}_{0,T_\ell}(k_{\ell,j}) = (1 + 0.025 \times \epsilon_{t,T_\ell}(k_{\ell,j}))O_{0,T_\ell}(k_{\ell,j}), \quad \ell = 1,2, \tag{6.3}$$

where $\{\epsilon_{t,T_1}(k_{1,j}\}_{j\ge1}$ and $\{\epsilon_{t,T_2}(k_{2,j}\}_{j\ge1}$ are two independent i.i.d. sequences of standard normal random variables. We note that, since the option price varies a lot across strikes for a given tenor (in fact the option price asymptotic rate of decay differs depending on how far the strike is from the current price), the above option observation model implies nontrivial heteroskedasticity in the option observation errors.

For implementing the estimator of H, we need to set the grid of points of u on which we evaluate $\widehat{A}_{0,T_1}(u)$ and $\widehat{A}_{0,T_2}(u)$. We determine those in an adaptive way. More specifically, we set

$$\begin{aligned} &\widehat{u}_1 = \inf\{u \ge 0 : |\widehat{\mathcal{L}}_{0,T_2}(u)| < 0.9\}, \quad \widehat{u}_K = \inf\{u \ge 0 : |\widehat{\mathcal{L}}_{0,T_2}(u)| < 0.75\}, \\ &\widehat{u} = \inf\{u \ge 0 : |\widehat{A}_{0,T_2}(u)| > \pi/2\}, \end{aligned} \tag{6.4}$$

and from here $\underline{\widehat{u}} = \{\widehat{u}_1 \wedge \widehat{u} : 0.01 : \widehat{u}_K \wedge \widehat{u}\}$. Note that the probability that $\underline{\widehat{u}}$ contains less than two points converges to zero asymptotically. This also never happens in our simulations and for this reason we do not consider the situation when this is not the case as this is irrelevant from an asymptotic point of view.

The results from the Monte Carlo are reported in Table 1. They indicate good finite sample performance of the estimator across the different volatility regimes and the different values of the Hurst parameter. In most of the cases, there is a very small upward bias in the estimator. It is interesting to note also that the behavior of the estimator is not very sensitive to the different starting values of volatility. We note finally that in all considered cases the interquartile range is very small which indicates good precision.

TABLE 1
Monte Carlo Results

Case	Empirical Quantiles of $\widehat{H}$ 25-th	50-th	75-th
Low Volatility			
$V_0 = 0.015$, $H = 0.10$	0.0811	0.1191	0.1606
$V_0 = 0.015$, $H = 0.25$	0.2563	0.2630	0.2694
$V_0 = 0.015$, $H = 0.40$	0.4067	0.4163	0.4259
$V_0 = 0.015$, $H = 0.49$	0.4956	0.5101	0.5244
Medium Volatility			
$V_0 = 0.030$, $H = 0.10$	0.0905	0.1007	0.1111
$V_0 = 0.030$, $H = 0.25$	0.2690	0.2755	0.2812
$V_0 = 0.030$, $H = 0.40$	0.4060	0.4167	0.4276
$V_0 = 0.030$, $H = 0.49$	0.4741	0.4899	0.5067
High Volatility			
$V_0 = 0.060$, $H = 0.10$	0.1067	0.1112	0.1156
$V_0 = 0.060$, $H = 0.25$	0.2649	0.2713	0.2778
$V_0 = 0.060$, $H = 0.40$	0.3947	0.4069	0.4191
$V_0 = 0.060$, $H = 0.49$	0.4489	0.4680	0.4869

Results are based on $5{,}000$ Monte Carlo replications.

7. Proof of Theorem 3.1 and Proposition 3.4. In a first step, we consider a local expansion of the process x_{t+T} around $T=0$ that only depends on *a priori* estimates. Define

$$
\begin{aligned}
\check{x}_{t+T} &= x_t + \alpha_t T + \sigma_t (W_{t+T} - W_t) \\
&\quad + \sum_{*\in\{\emptyset,\sim\}} \int_t^{t+T} \int_0^s (\overset{*}{g}_H(s-r) - \overset{*}{g}_H(t-r)) \overset{*}{\eta}_{r\wedge t} d\overset{*}{W}_r dW_s \\
&\quad + \sum_{*\in\{\emptyset,\sim,-,\wedge\}} \int_t^{t+T} \int_t^s \int_0^r g_H(s-r) (\overset{*}{g}{}^{\eta}_H(r-v) - \overset{*}{g}{}^{\eta}_H(t-v)) \overset{*}{\sigma}{}^{\eta}_{v\wedge t} d\overset{*}{W}_v dW_r dW_s \\
&\quad + \sum_{*\in\{\emptyset,\sim,-,\wedge,\circ\}} \int_t^{t+T} \int_t^s \int_0^r \widetilde{g}_H(s-r) (\overset{*}{g}{}^{\widetilde{\eta}}_H(r-v) - \overset{*}{g}{}^{\widetilde{\eta}}_H(t-v)) \overset{*}{\sigma}{}^{\widetilde{\eta}}_{v\wedge t} d\overset{*}{W}_v d\widetilde{W}_r dW_s \\
&\quad + \sum_{*\in\{\emptyset,\sim,-\}} \overset{*}{\eta}{}^{\sigma}_t \int_t^{t+T} \int_t^s d\overset{*}{W}_r dW_s + \int_t^{t+T} \int_{\mathbb{R}} \gamma(t,z) \mu(ds,dz) \\
&\quad + \int_t^{t+T} \int_{\mathbb{R}} \delta(s,z)(\mu-\nu)(ds,dz) + \int_t^{t+T} \int_t^s \int_{\mathbb{R}} \delta_\sigma(t,z)(\mu-\nu)(dr,dz) dW_s,
\end{aligned}
\tag{7.1}
$$

where $\overset{*}{x}$ simply means x if $* = \emptyset$. In what follows, we use the notation $\iint_a^b = \int_a^b \int_{\mathbb{R}}$ and $A \lesssim B$ if there is a constant $C \in (0,\infty)$ that does not depend on any important parameter such that $A \leq CB$. Also, we abbreviate $u_T = u/\sqrt{T}$.

LEMMA 7.1. *Under the assumptions of Theorem 3.1,*

$$
\mathbb{E}_t[e^{iu_T(x_{t+T}-x_t)}] - \mathbb{E}_t[e^{iu_T(\check{x}_{t+T}-x_t)}] = o^{\mathrm{uc}}(T^{2H}), \quad T \to 0.
$$

PROOF. By the definition of x_{t+T} and $\check{x}_{t+T}$, their difference can be expressed as

$$\int_t^{t+T}(\alpha_s-\alpha_t)ds+\int_t^{t+T}\int_t^s b_r dr dW_s+\sum_{*\in\{\emptyset,\sim\}}\int_t^{t+T}\int_t^s \mathring{g}_H(s-r)(\overset{*}{\theta}_r-\overset{*}{\theta}_t)d\overset{*}{W}_r dW_s$$
$$+\sum_{*\in\{\emptyset,\sim,-,\wedge\}}\int_t^{t+T}\int_t^s\int_t^r g_H(s-r)\overset{*}{g}{}^{\eta}_H(r-v)(\overset{*}{\sigma}{}^{\eta}_v-\overset{*}{\sigma}{}^{\eta}_t)d\overset{*}{W}_v dW_r dW_s$$
$$+\sum_{*\in\{\emptyset,\sim,-,\wedge,\circ\}}\int_t^{t+T}\int_t^s\int_t^r \widetilde{g}_H(s-r)\overset{*}{g}{}^{\widetilde{\eta}}_H(r-v)(\overset{*}{\sigma}{}^{\widetilde{\eta}}_v-\overset{*}{\sigma}{}^{\widetilde{\eta}}_t)d\overset{*}{W}_v d\widetilde{W}_r dW_s$$
$$+\sum_{*\in\{\emptyset,\sim,-\}}\int_t^{t+T}\int_t^s(\overset{*}{\eta}{}^{\sigma}_r-\overset{*}{\eta}{}^{\sigma}_t)d\overset{*}{W}_r dW_s+\int_t^{t+T}\iint_t^s\gamma_\sigma(r,z)\mu(dr,dz)dW_s$$
$$+\int_t^{t+T}\iint_t^s(\delta_\sigma(r,z)-\delta_\sigma(t,z))(\mu-\nu)(dr,dz)dW_s$$
$$+\iint_t^T(\gamma(s,z)-\gamma(t,z))\mu(ds,dz).$$

This is a sum of nine terms, so let us denote them by $y_T^{(1)},\dots,y_T^{(9)}$. Since

$$\left|\mathbb{E}_t[e^{iu_T(x_{t+T}-x_t)}]-\mathbb{E}_t[e^{iu_T(\check{x}_{t+T}-x_t)}]\right|\le\mathbb{E}_t\left[uT^{-1/2}|x_{t+T}-\check{x}_{t+T}|\wedge 2\right]$$

as a consequence of the elementary inequality $|e^{ix}-e^{ix_0}|\le|x-x_0|\wedge 2$ and since the function $\varphi(x)=x\wedge 2$ is subadditive and for all $p\in[0,1]$ there is $C_p>0$ such that $\varphi(x)\le C_p x^p$ for all $x>0$, in order to prove the lemma it suffices to show that

$$\begin{cases}\mathbb{E}_t[|y_T^{(i)}|]=o(T^{2H+1/2}) & i=1,\dots,6,8,\\ \mathbb{E}_t[|y_T^{(i)}|^q]=o(T^{2H+q/2}) & i=7,9.\end{cases}\tag{7.2}$$

By (2.9), it is easy to verify that $\mathbb{E}_t[|y_T^{(1)}|]=O(T^{1+H})=o(T^{2H+1/2})$ since $H<\frac{1}{2}$. For $y_T^{(2)},\dots,y_T^{(6)}$, we use Jensen's inequality and bound the L^2-norm instead. Using Itô's isometry in the first step and Minkowski's integral inequality in the second step, we obtain

$$\mathbb{E}_t[(y_T^{(2)})^2]=\int_t^{t+T}\mathbb{E}_t\left[\left(\int_t^s b_r dr\right)^2\right]ds\le\int_t^{t+T}\left(\int_t^s\mathbb{E}_t[b_r^2]^{1/2}dr\right)^2 ds=O(T^3)$$

by (2.6), proving that $\mathbb{E}_t[|y_T^{(2)}|]=O(T^{3/2})=o(T^{H+1})=o(T^{2H+1/2})$. Similarly, by (2.9),

$$\mathbb{E}_t[|y_T^{(6)}|]\le\sum_{*\in\{\emptyset,\sim,-\}}\left(\int_t^{t+T}\int_t^s\mathbb{E}_t[(\overset{*}{\eta}{}^{\sigma}_r-\overset{*}{\eta}{}^{\sigma}_t)^2]drds\right)^{1/2}=O(T^{1+H})=o(T^{2H+1/2}).$$

For $y_T^{(3)}$, $y_T^{(4)}$ and $y_T^{(5)}$, note that for g as in Assumption A-5, we have that $\int_t^s g^2(s-r)dr\le 2(\int_t^s k_H(s-r)^2dr+\int_t^s\ell_g(s-r)^2dr)=O((s-t)^{2H})$. Therefore, using (2.8) for $y_T^{(3)}$ and (2.7) for $y_T^{(4)}$ and $y_T^{(5)}$, we derive

$$\mathbb{E}_t[|y_T^{(3)}|]+\mathbb{E}_t[|y_T^{(4)}|]+\mathbb{E}_t[|y_T^{(5)}|]=o(T^{1/2+2H}).$$

For $y_T^{(8)}$, we simply use Jensen's inequality and (2.9) to bound

$$\mathbb{E}_t[|y_T^{(8)}|] \le \left(\int_t^{t+T} \iint_t^s \mathbb{E}_t[(\delta_\sigma(r,z) - \delta_\sigma(t,z))^2] dr\nu(dz)ds \right)^{1/2},$$

which is $O(T^{1+H}) = o(T^{2H+1/2})$.

Next, we study $y_T^{(9)}$. Since $(\sum_{i=1}^\infty a_i)^q \le \sum_{i=1}^r a_i^q$ for any nonnegative numbers a_i, (2.12) and (2.15) yield

$$\mathbb{E}_t[|y_T^{(9)}|^q] \le \mathbb{E}_t\left[\iint_t^{t+T} |\gamma(s,z) - \gamma(t,z)|^q ds\nu(dz) \right] = O(T^{1+qH_\gamma}) = o(T^{2H+q/2}).$$

Finally, using the Burkholder–Davis–Gundy (BDG) inequality in the first step and (2.11) in the last step, we obtain

$$\begin{aligned}
\mathbb{E}_t[|y_T^{(7)}|^q] &\le C_q \mathbb{E}_t\left[\left(\int_t^{t+T} \left(\iint_t^s \gamma_\sigma(r,z)\mu(dr,dz) \right)^2 ds \right)^{q/2} \right] \\
&= C_q \mathbb{E}_t\left[\left(\int_t^{t+T} \iint_t^s \iint_t^s \gamma_\sigma(r,z)\gamma_\sigma(v,w)\mu(dr,dz)\mu(dv,dw)ds \right)^{q/2} \right] \\
&= C_q \mathbb{E}_t\left[\left(\iint_t^{t+T} \iint_t^{t+T} (t+T-r\vee v)\gamma_\sigma(r,z)\gamma_\sigma(v,w)\mu(dr,dz)\mu(dv,dw) \right)^{q/2} \right] \\
&\le C_q T^{q/2} \mathbb{E}_t\left[\left(\iint_t^{t+T} |\gamma_\sigma(r,z)|\mu(dr,dz) \right)^q \right] \le C_q T^{q/2+1} = o(T^{2H+q/2}). \qquad \square
\end{aligned}$$

Next, we define δ' in the same way as δ in (2.4), but with $g_H^\delta(\cdot,z)$ and $\dot{g}_H^\delta(\cdot,z)$ replaced by k_{H_δ} from (2.5) for all $z \in \mathbb{R}$. Similarly, we define x'_{t+T} in the same way as $\check{x}_{t+T}$ in (7.1), but with all kernels g as in Assumption A-5 replaced by k_H from (2.5) and with δ replaced by δ'.

LEMMA 7.2. *Under the assumptions of Theorem 3.1, one has*

$$\mathbb{E}_t[e^{iu_T(\check{x}_{t+T}-x_t)}] - \mathbb{E}_t[e^{iu_T(x'_{t+T}-x_t)}] = o^{\mathrm{uc}}(T^{2H}), \quad T \to 0.$$

PROOF. Since $|\mathbb{E}_t[e^{iu_T(\check{x}_{t+T}-x_t)}] - \mathbb{E}_t[e^{iu_T(x'_{t+T}-x_t)}]| \le uT^{-1/2}\mathbb{E}_t[|\check{x}_T - x'_T|]$, it suffices to show that for each term in (7.1) that contains one of the kernels g (or δ), if we replace any of the g's by ℓ_g (or δ by δ'), the resulting term is of order $o(T^{2H+1/2})$. For the substitutions of g, this is straightforward (and we actually have $O(T^2)$ as a bound) since ℓ_g is differentiable with $\ell_g(0) = 0$ and bounded derivative and therefore $(\int_t^s (\ell_g(s-r) - \ell_g(t-r))^2 dr)^{1/2} \lesssim (\int_t^s (s-t)^2 ds)^{1/2} = O(T^{3/2})$ as $T \to 0$. For the substitution of δ, let us only verify that

$$\mathbb{E}_t\left[\left| \iint_t^{t+T} \int_0^s (\ell(s-v,z) - \ell(t-v,z))\sigma^\delta(v,z)dW_v(\mu-\nu)(ds,dz) \right| \right] = o(T^{2H+1/2}) \tag{7.3}$$

for $\ell = \ell_{g_{H_\delta}^\delta}$. (The other expression, involving an integral with respect to $\dot{W}$, can be analyzed analogously.) We use Jensen's inequality and the BDG inequality in a first step and Hölder's inequality (with respect to ds) in a second step to bound the left-hand side of (7.3) by

$$
\begin{aligned}
&C_r\biggl(\iint_t^{t+T}\mathbb{E}_t\biggl[\biggl|\int_0^s(\ell(s-v,z)-\ell(t-v,z))\sigma^\delta(v,z)dW_v\biggr|^r\biggr]ds\nu(dz)\biggr)^{1/r}\\
&\quad\le C_r'\biggl(\iint_t^{t+T}\mathbb{E}_t\biggl[\biggl(\int_0^s(\ell(s-v,z)-\ell(t-v,z))^2\sigma^\delta(v,z)^2dv\biggr)^{r/2}\biggr]ds\nu(dz)\biggr)^{1/r}\\
&\quad\le C_r'T^{1/r-1/2}\biggl(\int_{\mathbb{R}}\biggl(\int_0^{t+T}\biggl(\int_{v\vee t}^{t+T}(\ell(s-v,z)\\
&\qquad\qquad -\ell(t-v,z))^2ds\biggr)\mathbb{E}_t[\sigma^\delta(v,z)^2]dv\biggr)^{r/2}\nu(dz)\biggr)^{1/r}.
\end{aligned}
$$

As seen above, $\int_{v\vee t}^{t+T}(\ell(s-v,z)-\ell(t-v,z))^2ds=O(T^3)$. Therefore, by (2.10), the last display is $o(T^{3/2})=o(T^{H+1})=o(T^{2H+1/2})$. □

The next two approximation results are more subtle, as they do not follow from *a priori* estimates. Here it is crucial to exploit the independence properties of μ and the different Brownian motions that appear in (1.1) and (2.1)–(2.4). Let

$$
x''_{t+T}=x'_{t+T}+\iint_t^{t+T}(\delta(t,z)-\delta'(s,z))(\mu-\nu)(ds,dz) \tag{7.4}
$$

and

$$
\begin{aligned}
\overline{x}_{t+T}&=x_t+\alpha_tT+\sigma_t(W_{t+T}-W_t)+\eta_t^\sigma\int_t^{t+T}\int_t^s dW_rdW_s\\
&\quad+\sum_{*\in\{\emptyset,\sim\}}\int_t^{t+T}\int_0^s(k_H(s-r)-k_H(t-r))\overset{*}{\eta}_{r\wedge t}d\overset{*}{W}_rdW_s\\
&\quad+\sum_{*\in\{\emptyset,\sim,-,\wedge\}}\int_t^{t+T}\int_t^s k_H(s-r)\int_0^t(k_H(r-v)-k_H(t-v))\overset{*}{\sigma}{}_v^\eta d\overset{*}{W}_vdW_rdW_s\\
&\quad+\sigma_t^\eta\int_t^{t+T}\int_t^s\int_t^r k_H(s-r)k_H(r-v)dW_vdW_rdW_s+\iint_t^{t+T}\gamma(t,z)\mu(ds,dz)\\
&\quad+\iint_t^{t+T}\delta(t,z)(\mu-\nu)(ds,dz)+\int_t^{t+T}\iint_t^s\delta_\sigma(t,z)(\mu-\nu)(dr,dz)dW_s.
\end{aligned}
\tag{7.5}
$$

LEMMA 7.3. *Under the assumptions of Theorem 3.1, we have*

$$
\begin{aligned}
&\mathbb{E}_t[e^{iu_T(x'_{t+T}-x_t)}]-\mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}]\\
&\qquad=e^{-\frac12u^2\sigma_t^2+T\varphi_t(u_T)}\biggl(\sum_{i=1}^3C_{1,i}(u,T)_t+C'_{1,1}(u,T)_t\biggr)+o^{\mathrm{uc}}(T^{2H}),\qquad T\to0.
\end{aligned}
$$

LEMMA 7.4. *Under the assumptions of Theorem 3.1, we have*

$$
\mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}]-\mathbb{E}_t[e^{iu_T(\overline{x}_{t+T}-x_t)}]=o^{\mathrm{uc}}(T^{2H}),\quad T\to0.
$$

PROOF OF LEMMA 7.3. For any $r \in [1,2)$, there is a constant $C_r \in (0,\infty)$ such that $|e^{ix} - e^{ix_0} - ie^{ix_0}(x-x_0)| \le C_r|x-x_0|^r$ for all $x_0, x \in \mathbb{R}$. Thus,

$$\begin{aligned}&\left|\mathbb{E}_t[e^{iu_T(x'_{t+T}-x_t)}] - \mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}] - iu_T\mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}(x'_{t+T}-x''_{t+T})]\right| \\ &\qquad \le C_r|u|^rT^{-r/2}\mathbb{E}_t[|x'_{t+T}-x''_{t+T}|^r],\end{aligned} \tag{7.6}$$

By the BDG inequality and (2.13),

$$|u|^rT^{-r/2}\mathbb{E}_t[|x'_{t+T}-x''_{t+T}|^r] \le C'_r|u|^rT^{-r/2}\mathbb{E}_t\left[\iint_t^{t+T}|\delta'(s,z)-\delta(t,z)|^r\nu(dz)ds\right]$$

(the difference between δ and δ' can be neglected, as seen before). It is straightforward to show that (2.4) (together with Assumptions A and B) implies

$$\mathbb{E}_t\left[\int_{\mathbb{R}}|\delta'(s,z)-\delta(t,z)|^r\nu(dz)\right] = O((s-t)^{rH_\delta}). \tag{7.7}$$

Therefore, (7.6) is of order $O^{\mathrm{uc}}(T^{1-r/2+rH_\delta})$, which is $o^{\mathrm{uc}}(T^{2H})$ by (2.15).

It therefore remains to evaluate $\mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}(x'_{t+T}-x''_{t+T})]$. Since $T^{-1/2}\mathbb{E}_t[|x'_{t+T} - x''_{t+T}|] \le T^{-1/2}\mathbb{E}_t[|x'_{t+T}-x''_{t+T}|^r]^{1/r} = O(T^{1/r-1/2+H_\delta}) = o(T^{2H/r}) = o(T^H)$ by (2.15), if we expand $e^{iu_T(x''_{t+T}-x_t)}$ around $T=0$, any term coming from $x''_{t+T}-x_t$ that is of order $O(T^{1/2+H})$ becomes negligible for Theorem 3.1. As a result,

$$\begin{aligned}&T^{-1/2}\mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}(x'_{t+T}-x''_{t+T})] \\ &= T^{-1/2}\mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)+iu_T\iint_t^{t+T}\delta(t,z)(\mu-\nu)(ds,dz)}(x'_{t+T}-x''_{t+T})] + o^{\mathrm{uc}}(T^{2H}).\end{aligned} \tag{7.8}$$

Our next claim is that

$$\mathbb{E}_t\left[\left|x'_{t+T}-x''_{t+T}-\sum_{i=1}^5 x^{(i)}_{t+T}\right|^r\right]^{1/r} = o(T^{1/2+2H}), \tag{7.9}$$

where

$$\begin{aligned}
x^{(1)}_{t+T} &= \iint_t^{t+T}\int_0^s (k_{H_\delta}(s-v)-k_{H_\delta}(t-v))\sigma^\delta(v\wedge t,z)dW_v(\mu-\nu)(ds,dz),\\
x^{(2)}_{t+T} &= \iint_t^{t+T}\int_0^s (k_{H_\delta}(s-v)-k_{H_\delta}(t-v))\dot\sigma^\delta(v\wedge t,z)d\dot W_v(\mu-\nu)(ds,dz),\\
x^{(3)}_{t+T} &= \iint_t^{t+T}\int_t^s \eta^\delta(t,z)dW_v(\mu-\nu)(ds,dz),\\
x^{(4)}_{t+T} &= \iint_t^{t+T}\int_t^s \ddot\eta^\delta(t,z)d\ddot W_v(\mu-\nu)(ds,dz),\\
x^{(5)}_{t+T} &= \iint_t^{t+T}\iint_t^s \delta_\delta(t,z,z')(\mu-\nu)(dv,dz')(\mu-\nu)(ds,dz).
\end{aligned} \tag{7.10}$$

To prove this claim, we only consider one particular contribution (the largest one) to the difference in (7.9), namely

$$\mathbb{E}_t\left[\left|\iint_t^{t+T}\int_t^s k_{H_\delta}(s-v)(\sigma^\delta(v,z)-\sigma^\delta(t,z))dW_v(\mu-\nu)(ds,dz)\right|^r\right]^{1/r},$$

and leave the remaining terms (which can be treated analogously) to the reader. Similarly to how we estimated (7.3), we can bound the term in the previous display by a constant times

$$T^{1/r-1/2}\bigg(\int_{\mathbb{R}}\bigg(\int_t^{t+T}\bigg(\int_{v\vee t}^{t+T}k_{H_\delta}(s-v)^2ds\bigg)\mathbb{E}_t[(\sigma^\delta(v,z)-\sigma^\delta(t,z))^2]dv\bigg)^{r/2}\nu(dz)\bigg)^{1/r},$$

which by (2.13) and the fact that $\int_v^{t+T}k_{H_\delta}(s-v)^2ds=O(T^{2H_\delta})$ is of order $O(T^{1/r+2H_\delta})$. This in turn is $o(T^{1/2+2H})$ because of (2.15).

Thanks to (7.8) and (7.9), upon defining $M_\tau=\sigma_t(W_\tau-W_t)+\iint_t^\tau\delta(t,z)(\mu-\nu)(ds,dz)$ for $\tau\geq t$, we have that

$$T^{-1/2}\mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}(x'_{t+T}-x''_{t+T})]=T^{-1/2}\mathbb{E}_t\bigg[e^{iu_TM_{t+T}}\sum_{i=1}^5x_{t+T}^{(i)}\bigg]+o^{\mathrm{uc}}(T^{2H}).\tag{7.11}$$

By Itô's formula (see [17, Chapter I, Theorem 4.57]), the process $V(u)_\tau=e^{iuM_\tau}$ satisfies the stochastic differential equation (SDE)

$$\begin{aligned}dV(u)_\tau&=iu\sigma_tV(u)_\tau dW_\tau+\int_{\mathbb{R}}V(u)_{\tau-}(e^{iu\delta(t,z)}-1)(\mu-\nu)(d\tau,dz)\\&\qquad+b(u)V(u)_\tau d\tau,\quad\tau\geq t,\\V(u)_t&=1,\end{aligned}\tag{7.12}$$

where

$$b(u)=-\frac{1}{2}u^2\sigma_t^2+\int_{\mathbb{R}}(e^{iu\delta(t,z)}-1-iu\delta(t,z))\nu(dz)=-\frac{1}{2}u^2\sigma_t^2+\varphi_t(u).\tag{7.13}$$

We are now in the position to compute $\mathbb{E}_t[e^{iu_TM_{t+T}}\sum_{i=1}^5x_{t+T}^{(i)}]$ for each $i=1,\dots,5$. In a first step, note that $V(u)_\tau$ is measurable with respect to the σ-algebra $\mathcal{G}$ generated by $\mathcal{F}_t$, W and μ. Since we can rewrite

$$x_{t+T}^{(4)}=\int_t^{t+T}\iint_v^{t+T}\ddot{\eta}^\delta(t,z)(\mu-\nu)(ds,dz)d\ddot{W}_v,$$

and $d\ddot{W}_\tau$ for $\tau\geq t$ is independent of $\mathcal{G}$, it follows that

$$\mathbb{E}_t[e^{iu_TM_{t+T}}x_{t+T}^{(4)}]=\mathbb{E}_t[e^{iu_TM_{t+T}}\mathbb{E}[x_{t+T}^{(4)}\mid\mathcal{G}]]=0.\tag{7.14}$$

Next, we consider $i=1$. Using integration by parts, we have that

$$\begin{aligned}\mathbb{E}_t[e^{iu_TM_{t+T}}x_{t+T}^{(1)}]&=\iint_t^{t+T}\mathbb{E}_t\bigg[V(u_T)_s\int_0^s(k_{H_\delta}(s-v)-k_{H_\delta}(t-v))\sigma^\delta(v\wedge t,z)dW_v\bigg]\\&\qquad\times(e^{iu_T\delta(t,z)}-1)\nu(dz)ds+b(u_T)\int_t^{t+T}\mathbb{E}_t[e^{iu_TM_s}x_s^{(1)}]ds.\end{aligned}$$

So if we denote $m(u,z)_{r,s}=\mathbb{E}_t[V(u)_r\int_0^r(k_{H_\delta}(s-v)-k_{H_\delta}(t-v))\sigma^\delta(v\wedge t,z)dW_v]$ for $t\leq r\leq s$, then

$$\mathbb{E}_t[e^{iu_TM_{t+T}}x_{t+T}^{(1)}]=\iint_t^{t+T}e^{b(u_T)(t+T-s)}m(u_T,z)_{s,s}(e^{iu_T\delta(t,z)}-1)\nu(dz)ds.\tag{7.15}$$

Using integration by parts one more time, we derive the identity

$$\begin{aligned}m(u_T,z)_{r,s}&=\mathbb{E}_t[V(u_T)_t]\int_0^t(k_{H_\delta}(s-v)-k_{H_\delta}(t-v))\sigma^\delta(v,z)dW_v\\&\qquad+iu_T\sigma_t\sigma^\delta(t,z)\int_t^rk_{H_\delta}(s-v)\mathbb{E}_t[V(u_T)_v]dv+b(u_T)\int_t^rm(u_T,z)_{v,s}dv.\end{aligned}$$

Since $\mathbb{E}_t[V(u_T)_v] = e^{b(u_T)(v-t)}$, it follows that

$$\begin{aligned} m(u_T,z)_{r,s} &= e^{b(u_T)(r-t)}\int_0^t (k_{H_\delta}(s-v)-k_{H_\delta}(t-v))\sigma^\delta(v,z)dW_v \\ &\quad + iu_T\sigma_t\sigma^\delta(t,z)e^{b(u_T)(r-t)}\int_t^r k_{H_\delta}(s-v)dv \\ &= e^{b(u_T)(r-t)}\int_0^t (k_{H_\delta}(s-v)-k_{H_\delta}(t-v))\sigma^\delta(v,z)dW_v \\ &\quad + \frac{iu_T\sigma_t\sigma^\delta(t,z)e^{b(u_T)(r-t)}}{\Gamma(H_\delta+\frac{3}{2})}[(s-t)^{H_\delta+1/2}-(s-r)^{H_\delta+1/2}]. \end{aligned}$$

Inserting this with $r=s$ in (7.15) yields

$$\begin{aligned} \mathbb{E}_t[e^{iu_TM_{t+T}}x^{(1)}_{t+T}] &= e^{b(u_T)T}\int_t^{t+T}\int_0^t [k_{H_\delta}(s-v)-k_{H_\delta}(t-v)] \\ &\qquad \times \int_{\mathbb{R}} \sigma^\delta(v,z)(e^{iu_T\delta(t,z)}-1)\nu(dz)dW_v ds \\ &\quad + \frac{iu_T\sigma_te^{b(u_T)T}}{\Gamma(H_\delta+\frac{5}{2})}T^{H_\delta+3/2}\int_{\mathbb{R}}\sigma^\delta(t,z)(e^{iu_T\delta(t,z)}-1)\nu(dz). \end{aligned}$$

Thus, in the notation of (3.4) and (3.5), the contribution of $x^{(1)}_{t+T}$ to (7.11) is

$$\begin{aligned} T^{-1/2}\mathbb{E}[e^{iu_TM_{t+T}}x^{(1)}_{t+T}] &= e^{-\frac{1}{2}\sigma_t^2u^2+T\varphi_t(u_T)}T^{1/2}\int_0^1\int_0^t [k_{H_\delta}(t+sT-v)-k_{H_\delta}(t-v)] \\ &\qquad \times \int_{\mathbb{R}} \sigma^\delta(v,z)(e^{iu_T\delta(t,z)}-1)\nu(dz)dW_v ds \\ &\quad + \frac{iu\sigma_te^{-\frac{1}{2}\sigma_t^2u^2+T\varphi_t(u_T)}}{\Gamma(H_\delta+\frac{5}{2})}T^{H_\delta+1/2}\chi^{(1)}_t(u_T). \end{aligned} \tag{7.16}$$

A similar argument can be employed to analyze $x^{(2)}_{t+T}$, $x^{(3)}_{t+T}$ and $x^{(5)}_{t+T}$. As a result,

$$\begin{aligned} T^{-1/2}\mathbb{E}[e^{iu_TM_{t+T}}x^{(2)}_{t+T}] &= e^{-\frac{1}{2}\sigma_t^2u^2+T\varphi_t(u_T)}T^{1/2}\int_0^1\int_0^t [k_{H_\delta}(t+sT-v)-k_{H_\delta}(t-v)] \\ &\qquad \times \int_{\mathbb{R}} \dot\sigma^\delta(v,z)(e^{iu_T\delta(t,z)}-1)\nu(dz)d\dot W_v ds \end{aligned} \tag{7.17}$$

and

$$T^{-1/2}\mathbb{E}[e^{iu_TM_{t+T}}x^{(3)}_{t+T}] = \frac{1}{2}iu\sigma_te^{-\frac{1}{2}\sigma_t^2u^2+T\varphi_t(u_T)}T\chi^{(2)}_t(u_T), \tag{7.18}$$

$$T^{-1/2}\mathbb{E}[e^{iu_TM_{t+T}}x^{(5)}_{t+T}] = \frac{1}{2}e^{-\frac{1}{2}\sigma_t^2u^2+T\varphi_t(u_T)}T^{3/2}\chi^{(3)}_t(u_T). \tag{7.19}$$

Recalling (3.7), we deduce the lemma by combining (7.11), (7.14) and (7.16)–(7.19). $\square$

PROOF OF LEMMA 7.4. By first conditioning on W and $\widetilde{W}$, we obtain

$$\mathbb{E}_t\Big[e^{iu_T(x''_{t+T}-x_t)}\mid W,\widetilde{W}\Big] = \prod_{i=1}^5 A^{(i)}_T,$$

where

$$
\begin{aligned}
A_T^{(1)} &= \exp\Bigg(iu_T\Bigg(\alpha_t T + \sigma_t(W_{t+T}-W_t) + \sum_{*\in\{\emptyset,\sim\}} \overset{*}{\eta}{}_t^{\sigma} \int_t^{t+T}\int_t^s d\overset{*}{W}_r dW_s \\
&\quad + \sum_{*\in\{\emptyset,\sim\}} \int_t^{t+T}\int_0^s (k_H(s-r)-k_H(t-r))\overset{*}{\eta}_{r\wedge t} d\overset{*}{W}_r dW_s \\
&\quad + \sum_{*\in\{\emptyset,\sim\}} \int_t^{t+T}\int_t^s\int_0^r k_H(s-r)(k_H(r-v)-k_H(t-v))\overset{*}{\sigma}{}^{\eta}_{v\wedge t} d\overset{*}{W}_v dW_r dW_s \\
&\quad + \sum_{*\in\{\emptyset,\sim\}} \int_t^{t+T}\int_t^s\int_0^r k_H(s-r)(k_H(r-v)-k_H(t-v))\overset{*}{\sigma}{}^{\widetilde{\eta}}_{v\wedge t} d\overset{*}{W}_v d\widetilde{W}_r dW_s \\
&\quad + \sum_{*\in\{-,\wedge\}} \int_t^{t+T}\int_t^s\int_0^t k_H(s-r)(k_H(r-v)-k_H(t-v))\overset{*}{\sigma}{}^{\eta}_{v} d\overset{*}{W}_v dW_r dW_s \\
&\quad + \sum_{*\in\{-,\wedge,\circ\}} \int_t^{t+T}\int_t^s\int_0^t k_H(s-r)(k_H(r-v)-k_H(t-v))\overset{*}{\sigma}{}^{\widetilde{\eta}}_{v} d\overset{*}{W}_v d\widetilde{W}_r dW_s\Bigg)\Bigg), \\
A_T^{(2)} &= \exp\Bigg(T\phi_t(u_T) + \iint_t^{t+T}\Big(e^{iu_T(\delta(t,z)+\delta_\sigma(t,z)(W_{t+T}-W_s))} - 1 \\
&\quad - iu_T(\delta(t,z)+\delta_\sigma(t,z)(W_{t+T}-W_s))\Big)\nu(dz)ds\Bigg), \\
A_T^{(3)} &= \exp\Bigg(-\frac{1}{2}u_T^2\int_t^{t+T}\Bigg(\overline{\eta}{}_t^{\sigma}(W_{t+T}-W_v) + \overline{\sigma}{}_t^{\eta}\int_v^{t+T}\int_v^s k_H(s-r)k_H(r-v)dW_r dW_s \\
&\quad + \overline{\sigma}{}_t^{\widetilde{\eta}}\int_v^{t+T}\int_v^s k_H(s-r)k_H(r-v)d\widetilde{W}_r dW_s\Bigg)^2 dv\Bigg), \\
A_T^{(4)} &= \exp\Bigg(-\frac{1}{2}u_T^2\int_t^{t+T}\Bigg(\widehat{\sigma}{}_t^{\eta}\int_v^{t+T}\int_v^s k_H(s-r)k_H(r-v)dW_r dW_s \\
&\quad + \widehat{\sigma}{}_t^{\widetilde{\eta}}\int_v^{t+T}\int_v^s k_H(s-r)k_H(r-v)d\widetilde{W}_r dW_s\Bigg)^2 dv\Bigg), \\
A_T^{(5)} &= \exp\Bigg(-\frac{1}{2}u_T^2\int_t^{t+T}\Bigg(\mathring{\sigma}{}_t^{\widetilde{\eta}}\int_v^{t+T}\int_v^s k_H(s-r)k_H(r-v)d\widetilde{W}_r dW_s\Bigg)^2 dv\Bigg).
\end{aligned}
$$

Since $|A_T^{(i)}|\leq 1$ for all i and $|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i| \leq \sum_{i=1}^n |a_i - b_i|$ for any complex numbers $a_1,\dots,a_n$ and $b_1,\dots,b_n$ in the unit disk, we can replace any $A_T^{(i)}$ by 1 if $\mathbb{E}_t[|A_T^{(i)}-1|] = o^{\mathrm{uc}}(T^{2H})$. We shall apply this to $i=3,4,5$. Indeed, using the bound $|1-e^{-x}|\leq x$, we have $\mathbb{E}_t[|A_T^{(3)}-1|] = O^{\mathrm{uc}}(T\vee T^{4H})$, $\mathbb{E}_t[|A_T^{(4)}-1|] = O^{\mathrm{uc}}(T^{4H})$ and $\mathbb{E}_t[|A_T^{(5)}-1|] = O^{\mathrm{uc}}(T^{4H})$. This shows that

$$
\mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}] = \mathbb{E}_t[A_T^{(1)}A_T^{(2)}] + o^{\mathrm{uc}}(T^{2H}). \tag{7.20}
$$

Next, we take expectation with respect to $\widetilde{W}$, still conditioning on W, which amounts to computing $\mathbb{E}_t[A_T^{(1)} \mid W]$. Conditionally on W and $\mathcal{F}_t$, $A_T^{(1)}$ belongs to the direct sum of

Wiener chaoses (with respect to $(\widetilde{W}_\tau - \widetilde{W}_t)_{\tau \geq t}$) up to order 2. Hence, by Theorem A.1,

$$\mathbb{E}_t[A_T^{(1)} \mid W] = A_T^{(11)} A_T^{(12)}$$

where

$$\begin{aligned}
A_T^{(11)} = \exp\Bigg(& iu_T\Bigg(\alpha_t T + \sigma_t(W_{t+T} - W_t) + \eta_t^\sigma \int_t^{t+T}\int_t^s dW_r dW_s \\
& + \int_t^{t+T}\int_0^s (k_H(s-r) - k_H(t-r))\eta_{r\wedge t} dW_r dW_s \\
& + \int_t^{t+T}\int_0^t (k_H(s-r) - k_H(t-r))\widetilde{\eta}_r d\widetilde{W}_r dW_s \\
& + \int_t^{t+T}\int_t^s\int_0^r k_H(s-r)(k_H(r-v) - k_H(t-v))\sigma^\eta_{v\wedge t} dW_v dW_r dW_s \\
& + \sum_{*\in\{\sim,-,\wedge\}} \int_t^{t+T}\int_t^s\int_0^t k_H(s-r)(k_H(r-v) - k_H(t-v))\overset{*}{\sigma}{}^\eta_v d\overset{*}{W}_v dW_r dW_s\Bigg)\Bigg), \\
A_T^{(12)} = \exp\Bigg(& -\frac{1}{2}\sum_{j=1}^\infty \Big[\log(1-2i\alpha_j u_T) + 2i\alpha_j u_T + \frac{\beta_j^2 u_T^2}{1-2i\alpha_j u_T}\Big]\Bigg)
\end{aligned}$$

and $(\alpha_j)_{j\geq 1}$ and $(\beta_j)_{j\geq 1}$ are $(\mathcal{F}_t \vee \sigma(W))$-measurable random variables. If $Z_T^{(1)}$ and $Z_T^{(2)}$ denote the projections of $A_T^{(1)}$ onto the first- and second-order Wiener chaos, respectively, then

$$\begin{aligned}
\sum_{j=1}^\infty \alpha_j^2 &= \frac{1}{2}\mathbb{E}_t[(Z_T^{(2)})^2 \mid W] = \frac{1}{2}\widetilde{\sigma}_t^{\widetilde{\eta}} \int_t^{t+T}\int_t^r \left(\int_r^{t+T} k_H(s-r)dW_s\right)^2 k_H^2(r-v)dvdr, \\
\sum_{j=1}^\infty \beta_j^2 &= \mathbb{E}_t[(Z_T^{(1)})^2 \mid W] = \int_t^{t+T}\Bigg(\widetilde{\eta}_t^\sigma(W_{t+T} - W_v) + \widetilde{\eta}_t \int_v^{t+T} k_H(s-v)dW_s \\
&\quad + \widetilde{\sigma}_t^\eta \int_v^{t+T}\int_v^s k_H(s-r)k_H(r-v)dW_r dW_s \\
&\quad + \int_v^{t+T}\int_0^v k_H(s-v)(k_H(v-r) - k_H(t-r))\sigma^{\widetilde{\eta}}_{r\wedge t} dW_r dW_s \\
&\quad + \int_v^{t+T}\int_0^v k_H(s-v)(k_H(v-r) - k_H(t-r))\widetilde{\sigma}^{\widetilde{\eta}}_r d\widetilde{W}_r dW_s \\
&\quad + \sum_{*\in\{-,\wedge,\circ\}} \int_v^{t+T}\int_0^t k_H(s-r)(k_H(r-v) - k_H(t-v))\overset{*}{\sigma}{}^{\widetilde{\eta}}_v d\overset{*}{W}_v dW_s\Bigg)^2 dv.
\end{aligned}$$

Since $\log(1+x) = x + O(x^2)$ and $(1-x)^{-1} = 1 + O(x)$,

$$A_T^{(12)} = \exp\left(-\frac{1}{2}u_T^2\sum_{j=1}^\infty \beta_j^2 + O^{\mathrm{uc}}\left(T^{-1}\sum_{j=1}^\infty \alpha_j^2\right) + O^{\mathrm{uc}}\left(T^{-3/2}\sum_{j=1}^\infty \alpha_j\beta_j^2\right)\right).$$

Clearly, $\mathbb{E}_t[\sum_{j=1}^\infty \alpha_j^2] = O^{\mathrm{uc}}(T^{1+4H})$ and

$$\mathbb{E}_t\left[\sum_{j=1}^\infty \alpha_j\beta_j^2\right] \leq \mathbb{E}_t\left[\left(\max_{j\geq 1}\alpha_j\right)\sum_{j=1}^\infty \beta_j^2\right] \leq \mathbb{E}_t\left[\left(\sum_{j=1}^\infty \alpha_j^2\right)^{1/2}\sum_{j=1}^\infty \beta_j^2\right]$$

$$\leq \mathbb{E}_t\left[\sum_{j=1}^{\infty}\alpha_j^2\right]^{1/2}\mathbb{E}_t\left[\left(\sum_{j=1}^{\infty}\beta_j^2\right)^2\right]^{1/2} = O^{\mathrm{uc}}(T^{3/2+4H}),$$

which, combined with (7.20) and the fact that

$$\mathbb{E}_t\left[\left|\sum_{j=1}^{\infty}\beta_j^2 - \widetilde{\eta}_t^2\int_t^{t+T}\left(\int_v^{t+T}k_H(s-v)dW_s\right)^2 dv\right|\right] = O^{\mathrm{uc}}(T(T^{H+1/2}\vee T^{3H})),$$

proves that

$$\mathbb{E}_t[e^{iu_T(x''_{t+T}-x_t)}] = \mathbb{E}_t[A_T^{(11)}A_T^{(2)}e^{-\frac{1}{2}u_T^2\widetilde{\eta}_t^2\int_t^{t+T}(\int_v^{t+T}k_H(s-v)dW_s)^2dv}] + o^{\mathrm{uc}}(T^{2H}). \tag{7.21}$$

The last expectation is exactly $\mathbb{E}_t[e^{iu_T(\overline{x}_{t+T}-x_t)}]$, as the reader can easily confirm. □

Next, let us define

$$X_T = X_T^{(0)} + X_T^{(1)} + X_T^{(2)}, \qquad X'_T = X_T + X_T^{(3)}, \tag{7.22}$$

where $X_T^{(0)} = \alpha_t\sqrt{T}$ and

$$\begin{aligned} X_T^{(1)} &= T^{-1/2}\int_t^{t+T}\left(\sigma_t + \sum_{*\in\{\emptyset,\sim\}}\int_0^t (k_H(s-r)-k_H(t-r))\overset{*}{\eta}_r d\overset{*}{W}_r\right)dW_s \\ &= T^{-1/2}\int_t^{t+T}\sigma_{s|t}dW_s, \\ X_T^{(2)} &= T^{-1/2}\int_t^{t+T}\int_t^s\Bigg(\eta_t^\sigma + \eta_t k_H(s-r) \\ &\quad + \sum_{*\in\{\emptyset,\sim,-,\wedge\}}k_H(s-r)\int_0^t (k_H(r-v)-k_H(t-v))\overset{*}{\sigma}{}_v^\eta d\overset{*}{W}_v\Bigg)dW_r dW_s \\ &= T^{-1/2}\int_t^{t+T}\int_t^s(\eta_t^\sigma + k_H(s-r)\eta_{r|t})dW_r dW_s, \\ X_T^{(3)} &= \sigma_t^\eta T^{-1/2}\int_t^{t+T}\int_t^s\int_t^r k_H(s-r)k_H(r-v)dW_v dW_r dW_s. \end{aligned}$$

Furthermore, define

$$Y_T = T^{-1}\widetilde{\eta}_t^2\int_t^{t+T}\left(\int_v^{t+T}k_H(s-v)dW_s\right)^2 dv. \tag{7.23}$$

In particular, $A_T^{(11)} = e^{iuX'_T}$ and by (7.21),

$$\mathbb{E}_t[e^{iu_T(\overline{x}_{t+T}-x_t)}] = \mathbb{E}_t[e^{iuX'_T}A_T^{(2)}e^{-\frac{1}{2}u^2Y_T}] + o^{\mathrm{uc}}(T^{2H}). \tag{7.24}$$

LEMMA 7.5. *Under the assumptions of Theorem 3.1, we have*

$$\begin{aligned} \mathbb{E}_t[e^{iu_T(\overline{x}_{t+T}-x_t)}] &= e^{T\psi_t(u_T)}\mathbb{E}_t[e^{iuX_T}] + e^{-\frac{1}{2}u^2\sigma_t^2+T\varphi_t(u_T)}C_{1,4}(u,T)_t \\ &\quad + \widetilde{C}_2(u,T)_t + o^{\mathrm{uc}}(T^{2H}) \end{aligned} \tag{7.25}$$

as $T\to 0$, where

$$\widetilde{C}_2(u,T)_t = iu\mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)}X_T^{(3)}] - \frac{1}{2}u^2\mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)}Y_T]. \tag{7.26}$$

PROOF. Since $X_T^{(3)}, Y_T = O(T^{2H})$, $X_T = \sigma_t T^{-1/2}(W_{t+T} - W_t) + O(T^H)$ and $A_T^{(2)} = 1 + o^{\mathrm{uc}}(1)$—we will show the latter at the end of this proof—Taylor's theorem and (7.24) imply that

$$\begin{aligned}\mathbb{E}_t[e^{iu_T(\overline{x}_{t+T}-x_t)}] &= \mathbb{E}_t[e^{iuX_T}A_T^{(2)}] + \mathbb{E}_t[e^{iuX_T}A_T^{(2)}(iuX_T^{(3)} - \tfrac{1}{2}u^2 Y_T)] + o^{\mathrm{uc}}(T^{2H})\\ &= \mathbb{E}_t[e^{iuX_T}A_T^{(2)}] + \mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)}(iuX_T^{(3)} - \tfrac{1}{2}u^2 Y_T)] + o^{\mathrm{uc}}(T^{2H}).\end{aligned}$$

Note that $\mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)}(iuX_T^{(3)} - \frac{1}{2}u^2 Y_T)] = \widetilde{C}_2(u,T)_t$, so we are left to analyze $\mathbb{E}_t[e^{iuX_T}A_T^{(2)}]$. To this end, observe that $e^{z+w} - e^z - w = (e^z - 1)w + O(e^{z+w}w^2)$ and therefore (recall (2.11), (3.4) and (3.5)),

$$\begin{aligned}A_T^{(2)} &= e^{T\psi_t(u_T)} \exp\biggl(\iint_t^{t+T}\Bigl(e^{iu_T\delta(t,z)+iu_T\delta_\sigma(t,z)(W_{t+T}-W_s)}\\ &\qquad - e^{iu_T\delta(t,z)} - iu_T\delta_\sigma(t,z)(W_{t+T}-W_s)\Bigr)\nu(dz)ds\biggr) \qquad (7.27)\\ &= e^{T\psi_t(u_T)} \exp\biggl(iu_T\chi_t^{(4)}(u_T)\int_t^{t+T}(W_{t+T}-W_s)ds + O^{\mathrm{uc}}(T)\biggr).\end{aligned}$$

Next, we note that by the Cauchy–Schwarz inequality and (2.10) and (2.11),

$$|\chi_t^{(4)}(u_T)| \le u_T \int_{\mathbb{R}} \delta(t,z)\delta_\sigma(t,z)\nu(dz). \tag{7.28}$$

As a consequence, using the fact that $X_T = \sigma_t T^{-1/2}(W_{t+T} - W_t) + O(T^H)$ in the second step, we obtain

$$\begin{aligned}&\mathbb{E}_t[e^{iuX_T}A_T^{(2)}]\\ &\quad = e^{T\psi_t(u_T)}\biggl(\mathbb{E}_t[e^{iuX_T}] + iu_T\chi_t^{(4)}(u_T)\mathbb{E}_t\biggl[e^{iuX_T}\int_t^{t+T}(W_{t+T}-W_s)ds\biggr] + O^{\mathrm{uc}}(T)\biggr)\\ &\quad = e^{T\psi_t(u_T)}\biggl(\mathbb{E}_t[e^{iuX_T}] + iu_T\chi_t^{(4)}(u_T)\\ &\qquad \times \mathbb{E}_t\biggl[e^{iu_T\sigma_t(W_{t+T}-W_t)}\int_t^{t+T}(W_{t+T}-W_s)ds\biggr]\biggr) + O^{\mathrm{uc}}(T^{1/2+H}).\end{aligned}$$

Now observe that $e^{T\psi_t(u_T)}T\chi_t^{(4)}(u_T) = e^{T\varphi_t(u_T)}T\chi_t^{(4)}(u_T) + O^{\mathrm{uc}}(T)$, which follows from (7.28) and the bound

$$|\phi_t(u_T)| \le \frac{|u|^q}{T^{q/2}}\int_{\mathbb{R}}|\gamma(t,z)|^q\nu(dz). \tag{7.29}$$

Moreover, since $\mathbb{E}[e^{iuX}X] = i^{-1}\frac{d}{du}\mathbb{E}[e^{iuX}] = -i^{-1}uve^{-\frac{1}{2}u^2v}$ for $X \sim N(0,v)$, we have

$$\begin{aligned}&\mathbb{E}_t\biggl[e^{iu_T\sigma_t(W_{t+T}-W_t)}\int_t^{t+T}(W_{t+T}-W_s)ds\biggr]\\ &\qquad = \int_t^{t+T}\mathbb{E}_t\Bigl[e^{iu_T\sigma_t(W_s-W_t)}\mathbb{E}_s[e^{iu_T\sigma_t(W_{t+T}-W_s)}(W_{t+T}-W_s)]\Bigr]ds\\ &\qquad = -\frac{u\sigma_t T^{3/2}}{2i}e^{-\frac{1}{2}u^2\sigma_t^2},\end{aligned}$$

which proves the expansion in the lemma.

Finally, we have to justify the approximation of $A_T^{(2)}$ used above. By (7.27) and (7.28), we only need to show that

$$\mathbb{E}_t[|T\phi_t(u_T)|] = \mathbb{E}_t\left[\left|\int_{\mathbb{R}} T(e^{iu_T\gamma(t,z)}-1)\nu(dz)\right|\right] = O^{\mathrm{uc}}(T^{1/2}),$$
$$\mathbb{E}_t[|T\varphi_t(u_T)|] = \mathbb{E}_t\left[\left|\int_{\mathbb{R}} T(e^{iu_T\delta(t,z)}-1-iu_T\delta(t,z))\nu(dz)\right|\right] = o^{\mathrm{uc}}(1)$$

as $T\to 0$. The first statement follows from (2.11) and the bound

$$\mathbb{E}_t[|T\phi_t(u_T)|] \le T^{1-q/2}u^q\mathbb{E}_t\left[\int_{\mathbb{R}}|\gamma(t,z)|^q\nu(dz)\right] = O^{\mathrm{uc}}(T^{1/2}),$$

while the second statement follows from (2.10) and the dominated convergence theorem. □

To complete the proof of Theorem 3.1, it remains to obtain explicit expressions for $\mathbb{E}_t[e^{iuX_T}]$ and $\widetilde{C}_2(u,T)_t$.

LEMMA 7.6. *In the notation of* (7.22) *and* (7.26), *we have that*

$$\begin{aligned}\mathbb{E}_t[e^{iuX_T}] &= \exp\biggl(iu\alpha_t\sqrt{T}-\frac{1}{2}u^2\int_0^1\sigma^2_{t+sT|t}ds - iu^3\biggl(\frac{1}{2}\sigma_t^2\eta_t^\sigma T^{1/2}+\frac{\sigma_t^2\eta_t}{\Gamma(H+\frac{5}{2})}T^H \\ &\quad + C'_{1,0}(T)_tT^{2H}\biggr)\biggr) - e^{-\frac{1}{2}\sigma_t^2u^2}\biggl(\frac{u^2}{8H\Gamma(H+\frac{3}{2})\Gamma(H+\frac{1}{2})}-\frac{u^4\sigma_t^2}{\Gamma(2H+3)} \\ &\quad - \frac{u^4\sigma_t^2}{4(H+1)\Gamma(H+\frac{3}{2})^2}\biggr)\eta_t^2T^{2H}+o^{\mathrm{uc}}(T^{2H})\end{aligned}\tag{7.30}$$

and

$$\begin{aligned}\widetilde{C}_2(u,T)_t &= e^{-\frac{1}{2}u^2\sigma_t^2}\biggl(\frac{u^4\sigma_t^3\sigma_t^\eta}{\Gamma(2H+3)} \\ &\quad + \frac{u^4\sigma_t^2\widetilde{\eta}_t^2}{4(H+1)\Gamma(H+\frac{3}{2})^2}-\frac{u^2\widetilde{\eta}_t^2}{8H\Gamma(H+\frac{1}{2})\Gamma(H+\frac{3}{2})}\biggr)T^{2H}.\end{aligned}\tag{7.31}$$

PROOF. By (7.22), X_T belongs to the direct sum of Wiener chaoses up to order 2, where $X_T^{(0)}$, $X_T^{(1)}$ and $X_T^{(2)}$ belong to the zeroth, first and second Wiener chaos, respectively (see [23, Proposition 1.1.4]). Thus, we can use Theorem A.1 to evaluate $\mathbb{E}_t[e^{iuX_T}]$. There are numbers $(\alpha_j)_{j\ge1}$ and $(\beta_j)_{j\ge1}$ (not related to those in the proof of Lemma 7.4, even though we use the same notation) such that

$$\begin{aligned}\mathbb{E}_t[e^{iuX_T}] &= \exp\biggl(iu\alpha_t\sqrt{T}-\frac{1}{2}\sum_{j=1}^\infty\Bigl[\log(1-2i\alpha_ju)+2i\alpha_ju+\frac{\beta_j^2u^2}{1-2i\alpha_ju}\Bigr]\biggr) \\ &= \exp\biggl(iu\alpha_t\sqrt{T}-u^2\sum_{j=1}^\infty\alpha_j^2-\frac{1}{2}u^2\sum_{j=1}^\infty\beta_j^2-iu^3\sum_{j=1}^\infty\alpha_j\beta_j^2 \\ &\quad + 2u^4\sum_{j=1}^\infty\alpha_j^2\beta_j^2+4iu^5\sum_{j=1}^\infty\alpha_j^3\beta_j^2\biggr)+O^{\mathrm{uc}}\biggl(\sum_{j=1}^\infty\alpha_j^3\biggr).\end{aligned}\tag{7.32}$$

The last term, that is, $4iu^5\sum_{j=1}^\infty \alpha_j^3\beta_j^2$ can be included in the $O^{\mathrm{uc}}(\sum_{j=1}^\infty \alpha_j^3)$-term; we only spell it out because we referred to this in Section 5. By (A.4),

$$\sum_{j=1}^\infty \alpha_j^2 = \frac{1}{2}\mathbb{E}_t[(X_T^{(2)})^2] = \frac{\eta_t^2}{2\Gamma(H+\frac{1}{2})^2 T}\int_t^{t+T}\int_t^s (s-r)^{2H-1}drds + O(T^{H+1/2}\vee T^{3H})$$
$$= \frac{1}{8H\Gamma(H+\frac{3}{2})\Gamma(H+\frac{1}{2})}\eta_t^2 T^{2H} + O(T^{H+1/2}\vee T^{3H})$$

and

$$\sum_{j=1}^\infty \beta_j^2 = \mathbb{E}_t[(X_T^{(1)})^2] = \frac{1}{T}\int_t^{t+T}\sigma_{s|t}^2 ds = \int_0^1 \sigma_{t+sT|t}^2 ds.$$

In particular, $\sum_{j=1}^\infty \alpha_j^3 \le (\sum_{j=1}^\infty \alpha_j^2)^{3/2} = O(T^{3H})$. Moreover, by means of (A.4) and some tedious (but entirely straightforward) computations, one can show that

$$\sum_{j=1}^\infty \alpha_j\beta_j^2 = \frac{1}{2}T^{1/2}\eta_t^\sigma\left(\int_0^1 \sigma_{t+sT|t}ds\right)^2$$
$$+ \frac{1}{\Gamma(H+\frac{1}{2})}T^H\int_0^1\int_0^s (s-r)^{H-1/2}\sigma_{t+rT|t}\eta_{t+rT|t}dr\sigma_{t+sT|t}ds$$
$$= \frac{1}{2}\sigma_t^2 T^{1/2}\eta_t^\sigma + \frac{\sigma_t^2\eta_t}{\Gamma(H+\frac{5}{2})}T^H + C_{1,0}'(T)_t T^{2H} + O(T^{1/2+H}),$$

$$\sum_{j=1}^\infty \alpha_j^2\beta_j^2 = \frac{1}{2T^2}\int_t^{t+T}\int_t^s \sigma_{r|t}\sigma_{s|t}\bigg(\int_t^r (\eta_t^\sigma + k_H(s-v)\eta_{v|t})(\eta_t^\sigma + k_H(r-v)\eta_{v|t})dv$$
$$+ \int_r^s (\eta_t^\sigma + k_H(s-v)\eta_{v|t})(\eta_t^\sigma + k_H(v-r)\eta_{r|t})dv\bigg)drds$$
$$= \frac{\sigma_t^2\eta_t^2}{2T^2}\int_t^{t+T}\int_t^s\bigg(\int_t^r k_H(s-v)k_H(r-v)dv$$
$$+ \int_r^s k_H(s-v)k_H(v-r)dv\bigg)drds + O(T^{1/2+H}\vee T^{3H})$$
$$= \left(\frac{1}{2\Gamma(2H+3)} + \frac{1}{8(H+1)\Gamma(H+\frac{3}{2})^2}\right)\sigma_t^2\eta_t^2 T^{2H} + O(T^{1/2+H}\vee T^{3H}).$$

Consequently,

$$\mathbb{E}_t[e^{iuX_T}] = \exp\bigg(iu\alpha_t\sqrt{T} - \frac{1}{2}u^2\int_0^1\sigma_{t+sT|t}^2 ds - iu^3\bigg(\frac{1}{2}\sigma_t^2\eta_t^\sigma T^{1/2} + \frac{\sigma_t^2\eta_t}{\Gamma(H+\frac{5}{2})}T^H$$
$$+ C_{1,0}'(T)_t T^{2H}\bigg)\bigg)\bigg(1 - \frac{u^2\eta_t^2 T^{2H}}{8H\Gamma(H+\frac{3}{2})\Gamma(H+\frac{1}{2})} + \bigg(\frac{1}{\Gamma(2H+3)}$$
$$+ \frac{1}{4(H+1)\Gamma(H+\frac{3}{2})^2}\bigg)u^4\sigma_t^2\eta_t^2 T^{2H}\bigg) + O^{\mathrm{uc}}(T^{H+1/2}\vee T^{3H})$$
$$= \exp\bigg(iu\alpha_t\sqrt{T} - \frac{1}{2}u^2\int_0^1\sigma_{t+sT|t}^2 ds - iu^3\bigg(\frac{1}{2}\sigma_t^2\eta_t^\sigma T^{1/2} + \frac{\sigma_t^2\eta_t}{\Gamma(H+\frac{5}{2})}T^H$$

$$+ C'_{1,0}(T)_t T^{2H}\Big)\Big) - e^{-\frac{1}{2}\sigma_t^2 u^2}\left(\frac{u^2}{8H\Gamma(H+\frac{3}{2})\Gamma(H+\frac{1}{2})} - \left(\frac{1}{2\Gamma(2H+3)}\right.\right.$$
$$\left.\left. + \frac{1}{4(H+1)\Gamma(H+\frac{3}{2})^2}\right) u^4 \sigma_t^2 \eta_t^2 T^{2H} + o^{\mathrm{uc}}(T^{2H}),\right.$$

which proves (7.30).

In order to find $\widetilde{C}_2(u,T)_t$, we first determine $\mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)}X_T^{(3)}]$. To this end, let $M_\tau = e^{iu_T\sigma_t(W_\tau - W_t) - \frac{1}{2}(iu_T\sigma_t)^2(\tau - t)}$, which is an exponential martingale and solves the SDE

$$dM_\tau = iu_T\sigma_t M_\tau dW_\tau, \quad \tau \geq t, \quad M_t = 1.$$

In integral form, this becomes

$$M_\tau = 1 + iu_T\sigma_t \int_t^\tau M_s dW_s.$$

So upon iterating this equation, we obtain the chaos expansion of M as

$$M_\tau = 1 + \sum_{n=1}^\infty (iu_T\sigma_t)^n \int_t^\tau \int_t^{t_1} \cdots \int_t^{t_{n-1}} dW_{t_n} \cdots dW_{t_2} dW_{t_1}.$$

Now observe that $\mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)}X_T^{(3)}] = e^{-\frac{1}{2}u^2\sigma_t^2}\mathbb{E}_t[M_{t+T}X_T^{(3)}]$. Because Wiener chaoses are orthogonal to each other (see (A.2)), to compute the last expectation, only the projection of M_{t+T} on the third-order Wiener chaos has a non-zero contribution. Therefore,

$$\begin{aligned}
\mathbb{E}_t[M_{t+T}X_1^{(3)}] &= (iu_T\sigma_t)^3 \mathbb{E}_t\Bigg[\int_t^{t+T}\int_t^s\int_t^r dW_v dW_r dW_s \\
&\qquad \times \sigma_t^\eta T^{-1/2}\int_t^{t+T}\int_t^s\int_t^r k_H(s-r)k_H(r-v)dW_v dW_r dW_s\Bigg] \\
&= (iu_T\sigma_t)^3\sigma_t^\eta T^{-1/2}\int_t^{t+T}\int_t^s\int_t^r k_H(s-r)k_H(r-v)dv dr ds \\
&= \frac{-iu^3\sigma_t^3\sigma_t^\eta}{\Gamma(2H+3)}T^{2H}.
\end{aligned}$$

In a similar fashion, we can find $\mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)}Y_T]$. First, note that $Y_T = Y_T^{(0)} + Y_T^{(2)}$, where

$$\begin{aligned}
Y_T^{(0)} &= T^{-1}\widetilde{\eta}_t^2 \int_t^{t+T}\int_v^{t+T} k_H^2(s-v) ds dv = \frac{\widetilde{\eta}_t^2}{4H\Gamma(H+\frac{1}{2})\Gamma(H+\frac{3}{2})}T^{2H}, \\
Y_T^{(2)} &= 2T^{-1}\widetilde{\eta}_t^2 \int_t^{t+T}\int_v^{t+T}\int_v^s k_H(s-v)k_H(r-v) dW_r dW_s dv \\
&= \frac{2T^{-1}\widetilde{\eta}_t^2}{\Gamma(H+\frac{1}{2})^2}\int_t^{t+T}\int_t^s\int_t^r (s-v)^{H-1/2}(r-v)^{H-1/2} dv dW_r dW_s.
\end{aligned}$$

Thus,

$$\mathbb{E}_t[e^{iu_T\sigma_t(W_{t+T}-W_t)}Y_T] = \widetilde{\eta}_t^2 e^{-\frac{1}{2}u^2\sigma_t^2}\Bigg(\frac{T^{2H}}{4H\Gamma(H+\frac{1}{2})\Gamma(H+\frac{3}{2})}$$

$$- \frac{2T^{-2}u^2\sigma_t^2}{\Gamma(H+\frac{1}{2})^2}\int_t^{t+T}\int_t^s\int_t^r (s-v)^{H-1/2}(r-v)^{H-1/2}dvdrds\Bigg)$$

$$= \widetilde{\eta}_t^2 e^{-\frac{1}{2}u^2\sigma_t^2}\left(\frac{T^{2H}}{4H\Gamma(H+\frac{1}{2})\Gamma(H+\frac{3}{2})} - \frac{u^2\sigma_t^2T^{2H}}{2(H+1)\Gamma(H+\frac{3}{2})^2}\right).$$

Gathering the formulas derived above, we obtain (7.31). □

PROOF OF THEOREM 3.1. By Lemmas 7.1–7.6, we have that

$$\begin{aligned}
&\mathbb{E}_t[e^{iu_T(x_{t+T}-x_t)}]\\
&= e^{T\psi_t(u_T)}\Bigg\{\exp\Bigg(iu\alpha_t\sqrt{T} - \frac{1}{2}u^2\int_0^1 \sigma^2_{t+sT|t}ds - iu^3\Bigg(\frac{1}{2}\sigma_t^2\eta_t^\sigma T^{1/2} + \frac{\sigma_t^2\eta_t}{\Gamma(H+\frac{5}{2})}T^H\\
&\quad + C'_{1,0}(T)_t T^{2H}\Bigg)\Bigg) - e^{-\frac{1}{2}\sigma_t^2u^2}\Bigg(\frac{u^2}{8H\Gamma(H+\frac{3}{2})\Gamma(H+\frac{1}{2})} - \frac{u^4\sigma_t^2}{\Gamma(2H+3)}\\
&\quad - \frac{u^4\sigma_t^2}{4(H+1)\Gamma(H+\frac{3}{2})^2}\Bigg)\eta_t^2T^{2H}\Bigg\}\\
&\quad + e^{-\frac{1}{2}\sigma_t^2u^2+T\varphi_t(u_T)}\Bigg(\sum_{i=1}^4 C_{1,i}(u,T)_t + C'_{1,1}(u,T)_t\Bigg) + \widetilde{C}_2(u,T)_t + o^{\mathrm{uc}}(T^{2H})\\
&= \exp\Bigg(iu\alpha_t\sqrt{T} - \frac{1}{2}u^2\int_0^1 \sigma^2_{t+sT|t}ds + T\psi_t(u_T) - iu^3\Bigg(\frac{1}{2}\sigma_t^2\eta_t^\sigma T^{1/2} + \frac{\sigma_t^2\eta_t}{\Gamma(H+\frac{5}{2})}T^H\\
&\quad + C'_{1,0}(T)_tT^{2H}\Bigg)\Bigg) + e^{-\frac{1}{2}\sigma_t^2u^2+T\varphi_t(u_T)}\Bigg(\sum_{i=1}^4 C_{1,i}(u,T)_t + C'_{1,1}(u,T)_t\Bigg)\\
&\quad + C_2(u)_tT^{2H} + o^{\mathrm{uc}}(T^{2H}),
\end{aligned}$$

which shows Theorem 3.1. □

PROOF OF PROPOSITION 3.4. By (1.1), we have that

$$\begin{aligned}
\mathbb{E}_t[x_{t+T}-x_t] &= \left(\alpha_t + \int_{\mathbb{R}}\gamma(t,z)\nu(dz)\right)T\\
&\quad + \mathbb{E}_t\left[\int_t^{t+T}(\alpha_s-\alpha_t)ds + \iint_t^{t+T}(\gamma(s,z)-\gamma(t,z))ds\nu(dz)\right].
\end{aligned}$$

Both α and γ are H-Hölder continuous in L^1 by (2.9) and (2.12) (and our hypotheses on q and H_γ). Therefore, the conditional expectation on the right-hand side above is $O(T^{1+H})$, which gives the desired result. □

8. Proof of Theorem 4.1. Throughout the proof, we will set $t=0$ and $x_0=0$ for simplicity of notation. We will also drop t from the notation. We will make use of the following lemma in the proof:

LEMMA 8.1. *Suppose that Assumptions A, B and C-1 hold. There exists $\mathcal{F}_0^{(0)}$-measurable random variables C_0 and $\bar{t}>0$ that do not depend on T such that for $T<\bar{t}$, we have*

$$O_T(k) \le C_0\left(Te^{3k}1_{\{k<-1\}} + Te^{-k}1_{\{k>1\}} + \left(\sqrt{T}\wedge\frac{T}{|k|}\right)1_{\{|k|<1\}}\right), \tag{8.1}$$

$$|O_T(k_1)-O_T(k_2)|\le C_0\left[\frac{T}{k_2^4}\wedge\frac{T}{k_2^2}\wedge 1\right]|e^{k_1}-e^{k_2}|, \tag{8.2}$$

where $k_1<k_2<0$ *or* $k_1>k_2>0$.

PROOF OF LEMMA 8.1. By an application of Itô's lemma, we have

$$e^{x_t}-1=\int_0^t e^{x_s}\alpha_s ds+\int_0^t e^{x_s}\sigma_s dW_s+\int_0^t\int_{\mathbb{R}} e^{x_{s-}}(e^{\gamma(s,z)}-1)\mu(ds,dz)$$
$$+\int_0^t\int_{\mathbb{R}} e^{x_{s-}}(e^{\delta(s,z)}-1-\delta(s,z))(\mu-\nu)(ds,dz)+\frac{1}{2}\int_0^t e^{x_s}\sigma_s^2 ds,$$

and we have an analogous expression for $e^{-x_t}-1$. By the Cauchy–Schwarz inequality and the integrability conditions in Assumptions B and C-1, we then have

$$\mathbb{E}_0[(e^{x_s}-1)^2]+\mathbb{E}_0[(e^{-x_s}-1)^2]\le C_0T,\quad s\in[0,T\wedge\bar{t}]. \tag{8.3}$$

To proceed further, we make use of the following algebraic inequalities:

$$(e^k-e^x)^+\le 2e^{2k}\frac{|e^{-x}-1|^2}{|e^{-k}-1|},\ k<0,\ x\in\mathbb{R}, \tag{8.4}$$

$$(e^x-e^k)^+\le 2\frac{|e^{x}-1|^2}{|e^{k}-1|},\ k>0,\ x\in\mathbb{R}, \tag{8.5}$$

with the notation $x^+=\max(x,0)$. Using this result and (8.3), we get the bound in (8.1). We proceed with showing (8.2). For this, we make use of the following inequalities:

$$|(X-K_1)^+-(X-K_2)^+|\le|K_1-K_2|\mathbf{1}_{\{X>K_2\}},\quad X\in\mathbb{R},\quad K_1\ge K_2, \tag{8.6}$$

$$|(K_1-X)^+-(K_2-X)^+|\le|K_1-K_2|\mathbf{1}_{\{X<K_2\}},\quad X\in\mathbb{R},\quad K_1\le K_2. \tag{8.7}$$

From here, to show (8.2), we need to bound $\mathbb{Q}_0(|x_T|>k)$ for any $k>0$. An application of (8.3) yields

$$\mathbb{E}_0[|x_s|^4]\le C_0T,\quad s\in[0,T\wedge\bar{t}]. \tag{8.8}$$

Combining (8.8) with the inequalities in (8.6)–(8.7), we get (8.2). □

We are now ready to prove Theorem 4.1. We can make the following decomposition:

$$\widehat{\mathcal{L}}_T(u)-\mathcal{L}_T(u)=\zeta_T^{(1)}(u)+\zeta_T^{(2)}(u)+\zeta_T^{(3)}(u), \tag{8.9}$$

where

$$\zeta_T^{(1)}(u)=\left(\frac{u^2}{T}+i\frac{u}{\sqrt{T}}\right)\left(\int_{-\infty}^{\underline{k}} e^{(iu/\sqrt{T}-1)k}O_T(k)dk+\int_{\overline{k}}^{\infty} e^{(iu/\sqrt{T}-1)k}O_T(k)dk\right),$$

$$\zeta_T^{(2)}(u)=-\left(\frac{u^2}{T}+i\frac{u}{\sqrt{T}}\right)\sum_{j=2}^N\int_{k_{j-1}}^{k_j}\left(e^{(iu/\sqrt{T}-1)k_{j-1}}O_T(k_{j-1})-e^{(iu/\sqrt{T}-1)k}O_T(k)\right)dk,$$

$$\zeta_T^{(3)}(u)=-\left(\frac{u^2}{T}+i\frac{u}{\sqrt{T}}\right)\sum_{j=2}^N e^{(iu/\sqrt{T}-1)k_{j-1}}\epsilon_T(k_{j-1})\Delta_j.$$

Using (8.1), we have

$$\zeta_T^{(1)}(u) = O_{\overline{\mathbb{P}}}^{\mathrm{uc}}\left(e^{2\underline{k}} + e^{-2\overline{k}}\right). \tag{8.10}$$

Next, using (8.1) and (8.2), we get

$$\zeta_T^{(2)}(u) = O_{\overline{\mathbb{P}}}^{\mathrm{uc}}\left(\frac{\Delta}{\sqrt{T}}\log T\right). \tag{8.11}$$

Finally, using our assumption for $\mathcal{F}$-conditional independence of the option observation errors and the bounds in (8.1) and (8.2), we have

$$\begin{aligned}
&\mathbb{E}_0^{\overline{\mathbb{P}}}[|\zeta_T^{(3)}(u)|^2] \le C_0(|u|^3 \vee 1)\frac{\Delta}{\sqrt{T}},\\
&\mathbb{E}_0^{\overline{\mathbb{P}}}[|\zeta_T^{(2)}(u) - \zeta_T^{(2)}(v)|^4] \le C_0(|u| \vee |v| \vee 1)^8(|u-v|^4 \vee |u-v|^8)\frac{\Delta^2}{T},
\end{aligned} \tag{8.12}$$

for some $\mathcal{F}_0$-adapted random variable C_0 (that does not depend on u and v). From here, we have the tightness of $\frac{T^{1/4}}{\sqrt{\Delta}}\zeta_T^{(3)}(u)$ in the space of continuous functions of u equipped with the local uniform topology. Combining the above three bounds, we have the result of the theorem about $\widehat{\mathcal{L}}_T(u) - \mathcal{L}_T(u)$. The result for $\widehat{\mathcal{M}}_T - \mathcal{M}_T$ in exactly the same way.

9. Proofs for Section 5.

PROOF OF COROLLARY 5.1. Since there are no jumps, ψ_t as well as the terms $C_{1,j}(u,T)_t$, for $j = 1, \dots, 4$, and $C'_{1,1}(u,T)_t$ are identically zero. Therefore, by (3.6),

$$\operatorname{Arg}(\mathcal{L}_{t,T}(u)) = u\alpha_t T^{1/2} - \frac{\sigma_t^2 \eta_t u^3}{\Gamma(H+\frac{5}{2})}T^H - \frac{1}{2}u^3\sigma_t^2\eta_t^\sigma T^{1/2} + O^{\mathrm{uc}}(T^{2H}).$$

By Proposition 3.4, we thus have

$$A_{t,T}(u) = -\frac{\sigma_t^2 \eta_t u^3}{\Gamma(H+\frac{5}{2})}T^H - \frac{1}{2}u^3\sigma_t^2\eta_t^\sigma T^{1/2} + O^{\mathrm{uc}}(T^{2H}),$$

from which all statements of the corollary follow. □

PROOF OF THEOREM 5.2. By Theorem 4.1 and the mean-value theorem, we have

$$\widehat{A}_{t,T}(\underline{u}) = A_{t,T}(\underline{u}) + O(\Delta^{1/2}T^{-1/4-2H}).$$

Thus, by another application of the mean-value theorem, we deduce from (5.6) that

$$\begin{aligned}
\widehat{H}_n &= \frac{\log A_{t,T_1}(\underline{u}) - \log A_{t,T_2}(\underline{u})}{\log \tau} + O_{\overline{\mathbb{P}}}(\Delta^{\frac{1}{2}}T_1^{-\frac{1}{4}-H})\\
&= H + O_{\overline{\mathbb{P}}}(T^{(\frac{1}{2}-H)\wedge H} \vee \Delta^{\frac{1}{2}}T_1^{-\frac{1}{4}-H}),
\end{aligned}$$

which proves (5.8). Equation (5.10) is proved analogously. □

PROOF OF THEOREM 5.3. The expansion (5.13) follows from (3.6) and the fact that the second-order differences are constructed in such a way that the drift $\alpha_t\sqrt{T}$, the jump component $T\psi_t(u/\sqrt{T})$, the leverage component $-\frac{1}{2}iu^3\sigma_t^2\eta_t^\sigma T^{1/2}$ and the terms $C_{1,j}(u,T)_t$ for $j = 2, 3, 4$ are canceled out perfectly. What remains in the argument are therefore terms of order T^{2H} and $C_{1,1}(u,T)_t$ and $C'_{1,1}(u,T)_t$, both of which are $O^{\mathrm{uc}}(T^{H_\delta+1-r/2})$ as the following estimates show:

$$\begin{aligned}
\mathbb{E}_t[|\chi_t^{(1)}(\tfrac{u}{\sqrt{T}})|] &\le |\tfrac{u}{\sqrt{T}}|^{r-1}\int_{\mathbb{R}}|\sigma^\delta(t,z)||\delta(t,z)|^{r-1}\nu(dz) \le |\tfrac{u}{\sqrt{T}}|^{r-1}\int_{\mathbb{R}} C_t(z)^r\nu(dz),\\
\mathbb{E}_T[|\chi_{t'|t}^{(1)}(\tfrac{u}{\sqrt{T}})|] &\le |\tfrac{u}{\sqrt{T}}|^{r-1}\int_{\mathbb{R}}|\delta(t'\mid t,z)-\delta(t,z)||\delta(t,z)|^{r-1}\nu(dz)\\
&\lesssim |t'-t|^{H_\delta}|\tfrac{u}{\sqrt{T}}|^{r-1}\int_{\mathbb{R}} C_t(z)^r\nu(dz).
\end{aligned}$$

The second claim is an easy consequence of (5.13) and the mean-value theorem. □

APPENDIX A: SOME ELEMENTS OF WIENER SPACE THEORY

Let $(W_t)_{t\in[0,1]}$ be a one-dimensional standard Brownian motion and consider the Gaussian Hilbert space $\mathcal{H}=\{\int_0^1 f(t)dW_t : f\in L^2([0,1])\}\subset L^2(\Omega)$. Let $H_0(x)=1$ and, for $n\ge 1$, $H_n(x)=\frac{(-1)^n}{n!}e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$ be the *nth Hermite polynomial*. The first four Hermite polynomials are

$$H_0(x)=1,\quad H_1(x)=x,\quad H_2(x)=\frac{1}{2}(x^2-1),\quad H_3(x)=\frac{1}{6}x^3-\frac{1}{2}x. \tag{A.1}$$

For $n\ge 0$, define the *nth Wiener chaos* $\mathcal{H}_n$ as the L^2-closure of the linear subspace $\{H_n(\int_0^1 f(t)dW_t) : h\in L^2([0,1]), \int_0^1 h^2(t)dt=1\}$ and further let $\mathcal{P}_n=\bigoplus_{i=0}^n\mathcal{H}_i$. It is well-known that Wiener chaoses of different order are orthogonal to each other:

$$X\in\mathcal{H}_n,\ Y\in\mathcal{H}_m,\ n\ne m \implies \mathbb{E}[XY]=0; \tag{A.2}$$

cf. [23, Theorem 1.1.1]. While there is no general characterization of the distribution of elements in $\mathcal{H}_n$, the case $n=2$ is special. In fact, the distribution of elements in $\mathcal{P}_2$ is explicitly known:

THEOREM A.1. *If $X\in\mathcal{P}_2$, then are real numbers $(\alpha_j)_{j\ge1}$ and $(\beta_j)_{j\ge1}$ satisfying $\sum_{j=1}^\infty\alpha_j^2<\infty$ and $\sum_{j=1}^\infty\beta_j^2<\infty$ such that*

$$\mathbb{E}[e^{iuX}]=\exp\Bigg(i\mathbb{E}[X]u-\frac{1}{2}\sum_{j=1}^\infty\Big[\log(1-2i\alpha_j u)+2i\alpha_j u+\frac{\beta_j^2u^2}{1-2i\alpha_j u}\Big]\Bigg). \tag{A.3}$$

Moreover, there exists a sequence of independent standard normal random variables $(\xi_n)_{n\ge1}$ such that if we write $X=X_0+X_1+X_2$ with $X_0\in\mathcal{H}_0$, $X_1\in\mathcal{H}_1$ and $X_2\in\mathcal{H}_2$, then $X_0=\mathbb{E}[X]$, $X_1=\sum_{j=1}^\infty\beta_j\xi_j$ and $X_2=\sum_{j=1}^\infty\alpha_j(\xi_j^2-1)$. In particular, we have that

$$\begin{aligned}
&\mathbb{E}[X_1^2]=\sum_{j=1}^\infty\beta_j^2,\quad \mathbb{E}[X_2^2]=2\sum_{j=1}^\infty\alpha_j^2,\quad \mathbb{E}[X_2^3]=8\sum_{j=1}^\infty\alpha_j^3,\quad \mathbb{E}[X_1^2X_2]=2\sum_{j=1}^\infty\alpha_j\beta_j^2,\\
&\mathrm{Cov}(X_1^2,X_2^2)=8\sum_{j=1}^\infty\alpha_j^2\beta_j^2,\quad \mathrm{Cov}(X_1^2,X_2^3)-3\mathbb{E}[X_1^2X_2]\mathbb{E}[X_2^2]=48\sum_{j=1}^\infty\alpha_j^3\beta_j^2.
\end{aligned}\tag{A.4}$$

PROOF. The first two claims and the first two formulas in (A.4) are shown in [21, Theorem 6.1]. The last two formulas in (A.4) follow from a straightforward calculation based on the series representation of X_1 and X_2. □

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