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A term structure model with preferences for the timing of resolution of uncertainty*

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Summary. In this paper we present a model of the term structure of interest rates with imperfect information and stochastic differential utility, a form of non-additive recursive utility. A principal feature of recursive utility, that distinguishes it from time-separable expected utility, is its dependence on the timing of resolution of uncertainty. In our model, we parametrize the non-linearity of recursive utility in a way that corresponds to preferences for the timing of resolution. This way we show explicitly the dependence of prices on the rate of information, as a consequence of the nature of utilities. State prices and the term structure of interest rates are obtained in closed form, and are shown to have a form in which derivative asset pricing is tractable. Comparative statics relating to the dependence of the term structure on the rate of information are also discussed.

JEL Classification Numbers: G12, D89, D99.

1 Introduction

In this paper, we consider a parametric model of a single, or homogeneous, agent equilibrium, much in the spirit of Lucas (1978), but with imperfect information and stochastic differential utility. This type of utility was introduced by Duffie and Epstein (1992), and can be thought of as a continuous-time version of the recursive utility of Kreps and Porteus (1978) or Epstein and Zin (1989). State prices and the term structure of interest rates are obtained in

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closed form in terms of parameters that correspond to the rate of information, and the curvature of the intertemporal aggregator in its utility argument, which is known to characterize preferences for the timing of resolution of uncertainty. This way, we obtain a characterization and interpretation of the role of non-linear intertemporal aggregation in asset pricing, in terms of the dependence of prices on the timing of resolution of uncertainty. Comparative statics show the dependence of the term structure on the rate of information, a phenomenon that is not present under separable preferences.

Time-separable expected utility is known to place overly severe independence restrictions across time and states of nature. A simple example, that is part of the folklore of the field, consists of two bets A and B. Bet A involves the repeated toss of a coin over a number of periods resulting in payoffs of, say, \$1000 or nothing, depending on the outcome in every period. On the other hand, B involves the toss of a single coin, resulting in a payoff of \$1000 in every period, or nothing. While bet A is clearly less risky, in some sense, any time-separable expected utility (even with time or state dependence) must assign the same utility to both A and B. Considerations such as this have led to an extensive literature on utility forms that involve some sort of non-additive intertemporal aggregation. Recursive utility is prominent in this literature, mainly due to the fact that, although recursive utility accommodates non-linear temporal aggregation, it retains a strong notion of dynamic consistency that allows much of the optimization technology for time-additive utilities to be still applicable. For further background and discussion of recursive utility the reader can consult the survey of Epstein (1992).

An important property of recursive utility, that has been studied by Kreps and Porteus (1978) and others, is that utility depends on the timing of resolution of uncertainty. Moreover, the functional form of the intertemporal aggregator is directly linked to preferences for the timing of resolution, with a convex aggregator favoring early resolution, and a concave aggregator favoring late resolution. An additive temporal aggregator then corresponds to indifference towards the timing of resolution. As an illustration, consider a third bet, C, that differs from bet A described above only in that all the coins are tossed at once in the first period, the timing of the payoffs being the same in both bets. Time-additivity implies indifference towards A and C, a convex temporal aggregator implies that C is preferred to A, while a concave temporal aggregator implies that A is preferred to C.

The role of the timing of resolution of uncertainty for the stochastic differential utility of Duffie and Epstein (1992) is analogous to that for the Kreps-Porteus utility, and is analyzed in Skiadas (1995). There preferences are defined over pairs of contingent consumption plans and information filtrations, and preferences for the timing of resolution are defined in terms of the monotonicity of the utility relative to the information filtration component. Monotonicity in the filtration is then characterized in terms of the curvature of the temporal aggregator in its utility argument. The main advantage of this approach over that of Kreps and Porteus is that it separates the role of information from that of an agent's beliefs as expressed by the distribution of consumption plans that Kreps and Porteus take as the primitive choice objects.

While we refer to Skiadas (1995) for more discussion of preferences for the timing of resolution, it is instructive to review here a main intuition that gives rise to such preferences, even when the agent's utility is ultimately derived purely out of state and time contingent consumption. With time-additive utility, today's felicity from consumption depends on today's consumption alone. With recursive utility, however, today's felicity from consumption depends on today's consumption, but also on the expected utility of future consumption. For example, one can feel elated at the prospect of high future consumption, without any present consumption, while the prospect of low future consumption may decrease the enjoyment of present consumption. The curvature of the intertemporal aggregator can then be thought of as representing the agent's risk attitude towards the impact of the expected utility of future consumption. For example, a risk-averse attitude implies that the agent would rather form a less informed expectation of the utility of future consumption, because of the "risk" of receiving bad news, thus reducing the enjoyment of present consumption. Not surprisingly, this type of "risk-aversion" corresponds to the concavity of the intertemporal aggregator in its utility argument. Conversely, convexity of the aggregator in its utility argument leads to preferences for early resolution.

The question of the impact of the timing of resolution of uncertainty on prices has been discussed in a variety of contexts. Steve Ross (1989) argued that the timing of resolution of uncertainty should not affect prices. The essence of his argument can be explained in terms of a complete-markets Arrow-Debreu equilibrium, with no production, where agents' preferences are described by increasing and concave Von Neumann-Morgenstern utility functions. For simplicity, assume that there is a terminal date T when all state-contingent payoffs and consumption occur. Also assume there is no information today, and there is perfect information at time T. The equilibrium price of a contingent claim making payment X at time T is given by $E(\pi X)$, where π represents the state-price density. The random variable π is given as the marginal utility of the representative agent (in the sense of Constantinides (1982) and Huang (1987)) at the aggregate endowment. It is clear then that the quantity $E(\pi X)$ does not depend on the manner in which information is revealed between times zero and T. In this sense, prices are independent of the timing of resolution of uncertainty. The case in which consumption and dividend payouts occur continuously over time is analogous, and the same conclusion can be drawn.

Ross also presented an anecdote involving the price reaction of a New York City bond issue to the news of a rescheduling of an audit describing the state of revenue collection. The issue rallied on the announcement that the news would be released earlier than originally planned. Ross pointed out that such a story is not compatible with his indifference result. There can be many reasons why the information structure may have a price impact. In the presence of some production technology, early resolution of uncertainty may lead to planning benefits (see Robichek and Myers (1966) and Epstein and Turnbull (1980)). The announcement of the rescheduling of the timing of the news release may be perceived in itself as an informative signal (see, for example, Chambers and Penman (1984) for a related discussion). In the presence of default, the payoff structure of a security may depend on the timing of resolution (see Duffie, Schroder, and Skiadas (1995)). A new information structure may have an effect because of market incompleteness, as in Berk and Uhlig (1993).

In the model of this paper, state prices depend on the timing of resolution because of the nature of the agents' utility function. For example, we find that when there are preferences for early resolution, discount-bond yields and forward rates are increasing with the quality and timeliness of the information in the economy. The opposite relationships hold with preferences for late resolution, while the effect disappears under resolution indifference, corresponding to standard additive-separable utilities. The sensitivity of forward rates to changes in the rate of information is shown to be increasing with maturity for "very short" maturities, and decreasing with maturity for maturities that are near the time of complete resolution of uncertainty. Moreover, the pricing framework of the paper is shown to be of a type that is analytically tractable for derivative asset pricing, and it can therefore be used to investigate the dependence on the rate of information of prices of options and other derivative securities. A number of authors have considered similar models, involving imperfect information, but time and state additive preferences: Dothan and Feldman (1986), Detemple (1986, 1987), Gennotte (1986), Feldman (1989), Apelfeld and Conze (1990), Karatzas and Xue (1991), and Kuwana (1993). In these models the rate of information has no price impact.

The remainder of the paper is organized in four sections. Section 2 describes the structure of the endowment, state, and signal processes, as well as the information observed by the agent. The dynamics of the three processes are recast in terms of the conditional mean (or "filter") and the conditional variance, using standard filtering theory. In Section 3 we introduce the agent's utility function and discuss its properties. We compute the utility in closed form for the endowment process and information structure of Section 2. The state prices and term structure of interest rates are computed explicitly in Section 4. The dependence of the term structure on the timing of the resolution of uncertainty is discussed in Section 5. An appendix contains the proofs not presented in the main paper.

2 Consumption and information structure

We begin with a filtered probability space $(\Omega, \mathscr{F}, \mathbf{F}, P)$ over a finite time-horizon, $[0, T]$, that supports[1] a three-dimensional Brownian motion B. There is also a subfiltration of $\mathbf{F} = \{\mathscr{F}_t : t \geq 0\}$, denoted $\mathbf{I} = \{\mathscr{I}_t : t \geq 0\}$, that is

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[1] In the sense that $\mathbf{F}$ contains the augmented filtration generated by B. The process of filtration augmentation is described, for example, by Karatzas and Shreve (1988).

stochastically independent of the Brownian motion B. The filtration $\mathbf{I}$ should be thought of as available information that is irrelevant to consumption. On the other hand, B reveals information that is all relevant to consumption, but is not directly observable. The reason for introducing $\mathbf{I}$ at this point is that it will later serve as a source of "noise" with which the agent's signal can be contaminated, thus altering the rate at which information is revealed over time.

There is a single agent in the economy who consumes an endowment e, adapted to $\mathbf{F}$, that satisfies[2]

$$\frac{de_t}{e_t} = (a_e + b_e x_t)dt + v_e' dB_t, \quad 0 \leq t \leq T.$$

Here a_e and b_e are (real) constants, v_e is a given (column) vector[3] in $\mathbb{R}^3$, and x is a state variable process not observed by the agent. The initial value e_0 is assumed to be a given constant. The unobservable state process x evolves according to

$$dx_t = (a_x + b_x x_t)dt + v_x' dB_t, \quad 0 \leq t \leq T,$$

where $a_x, b_x \in \mathbb{R}$, $v_x \in \mathbb{R}^3$, and the initial value x_0 is a constant. We assume that $b_x < 0$, so that the state process exhibits mean reversion. In addition to the endowment process e, the agent observes a signal process s that is governed by the equation

$$ds_t = (a_s + b_s x_t)dt + v_s' dB_t, \quad 0 \leq t \leq T,$$

where $a_s, b_s \in \mathbb{R}$, $v_s \in \mathbb{R}^3$, and $s_0 = x_0$.

Let $\mathbf{F}^{e,s} = \{\mathscr{F}_t^{e,s} : 0 \leq t \leq T\}$ be the filtration generated by the endowment process e and the signal process s. The total information observed by the agent is given by the filtration $\mathbf{F}^o = \{\mathscr{F}_t^o : 0 \leq t \leq T\}$, where $\mathscr{F}_t^o = \mathscr{F}_t^{e,s} \vee \mathscr{I}_t$. That is, the agent observes e, s, as well as the "irrelevant" information stream $\mathbf{I}$.

To analyze the agent's state-estimation problem, we need an additional assumption and some notation. We define the matrix $v = [v_x, v_e, v_s]$, and we assume throughout that $v'v$ is positive definite. We also define the matrix

$$\Sigma = [v_e, v_s]'[v_e, v_s] = \begin{pmatrix} v_e' v_e & v_e' v_s \\ v_s' v_e & v_s' v_s \end{pmatrix},$$

which is necessarily also positive definite. Finally, we introduce the notation

$$m_t = E[x_t | \mathscr{F}_t^o] = E[x_t | \mathscr{F}_t^{e,s}], \quad 0 \leq t \leq T,$$

and

$$\gamma_t = E[(x_t - m_t)^2 | \mathscr{F}_t^o] = E[(x_t - m_t)^2 | \mathscr{F}_t^{e,s}], \quad 0 \leq t \leq T,$$

to denote the conditional mean and variance of the state process x, respectively.

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[2] Standard theory guarantees the existence and uniqueness of strong solutions to all of the stochastic differential equations of this section (see, for example, Karatzas and Shreve (1988)).

[3] For any matrix z, z' denotes the transpose of z.

As is well known from filtering theory (see Lipster and Shiryayev (1978) and the Appendix), the conditional distribution of the state process x up to any time t, given the agent's information $\mathscr{F}_t^o$, is Gaussian, and is therefore completely determined by the conditional mean and variance of x:

Proposition 1. *(a) The conditional variance γ is a deterministic function of time that is given explicitly in the Appendix. (b) There exists a process W such that $(W, \mathbf{F}^{e,s})$ is a standard Brownian motion in $\mathbb{R}^2$, and the following dynamics hold for $t \in [0, T]$:*

$$dm_t = (a_x + b_x m_t)dt + [b_e \gamma_t + v_x' v_e, b_s \gamma_t + v_x' v_s]\Sigma^{-1/2} dW_t, \quad m_0 = x_0,$$

$$\begin{pmatrix} de_t/e_t \\ ds_t \end{pmatrix} = \begin{pmatrix} a_e + b_e m_t \\ a_s + b_s m_t \end{pmatrix} dt + \Sigma^{1/2} dW_t.$$

(c) The filtration generated by W is $\mathbf{F}^{e,s}$.

This result essentially reduces the dynamics of the problem to one of complete information, and has been utilized by Dothan and Feldman (1986) and related papers in order to apply standard asset pricing methodology in settings of incomplete information. In our setting, however, the signal process will continue to play an active role, since it will directly affect the agent's utility.

The setup of this section was deliberately kept simple for purposes of exposition. Similar results are true in greater generality. The coefficients of the stochastic differential equations of e, x, and s, can, subject to restrictions in Section 12.3 of Lipster and Shiryayev (1978), depend on time as well as the history of the observed processes e and s. In the case in which the coefficients are deterministic functions of time only, the joint process $\{(x_t, \log(e_t), s_t): t \in [0, T]\}$ is Gaussian, and the conditional variance, γ, is a deterministic function of time. The initial conditions of the processes can be made stochastic. Increasing the dimensionality of the processes, or introducing an infinite horizon, presents no complications.

3 The agent's utility process

As discussed in the Introduction, the agent derives utility both from consumption and from the information filtration. That is, the agent cares not only about the distribution of the sample path for consumption, but also about the manner in which the conditional distribution of the consumption path evolves over time. For example, conditions given below imply preferences for early resolution of information, or for late resolution of information, just two of many possibilities. A special case is additive utility, under which an agent has no preferences over the information structure.

Given any sub-filtration $\mathbf{G} = \{\mathscr{G}_t: t \in [0, T]\}$ of $\mathbf{F}$, and any measurable consumption process $c = \{c_t: t \in [0, T]\}$ adapted to $\mathbf{G}$, the agent's utility process, $V(c, \mathbf{G})$, is defined (under technical conditions) as the unique solution of the backward integral equation:

$$V_t(c,\mathbf{G}) = E\left[\int_t^T f(c_u, V_u(c,\mathbf{G}))du \,\middle|\, \mathscr{G}_t\right], \quad 0 \le t \le T, \tag{1}$$

where $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is the primitive function determining preferences. The agent's utility at time zero is defined as $U(c,\mathbf{G}) = V_0(c,\mathbf{G})$, while $V_t(c,\mathbf{G})$ should be thought of as the time-t utility of the remaining consumption and filtration, conditional on the information available at time t. The existence of a unique integrable solution to equation (1), under appropriate assumptions on f, is discussed by Duffie and Epstein (1992a), Duffie and Lions (1992), and Antonelli (1993). Here we will assume a specific parametric form of f and obtain an expression for the utility process $V(e,\mathbf{F}^o)$ in closed form. For simplicity, we write V instead of $V(e,\mathbf{F}^o)$.

Duffie and Epstein (1992a) first defined the utility process of a consumption plan as the unique solution to equation (1), for a given fixed filtration $\mathbf{G}$, and called it stochastic differential utility. They motivated this definition as a limiting case of the discrete-time recursive utility of Epstein and Zin (1989), and they showed some basic properties of the utility. Skiadas (1995) extended and axiomatized stochastic differential utility to spaces of consumption processes and filtrations, and defined preferences for the timing of resolution of uncertainty in this context. The result most relevant here is the monotonicity of the utility process in the filtration argument, that we now briefly review.

An information stream (filtration) is "smaller" than another information stream, if at any given moment in time the former reveals no more information than the latter. This definition does not rule out the possibility of identical final information in both information streams, in which case the ranking of the two streams is simply in terms of the timing of the resolution of uncertainty. Formally, given filtrations $\mathbf{G} = \{\mathscr{G}_t : t\in[0,T]\}$ and $\mathbf{H} = \{\mathscr{H}_t : t\in[0,T]\}$, we define $\mathbf{G}\le\mathbf{H}$ to denote that $\mathscr{G}_t\subseteq\mathscr{H}_t$, for all t in $[0,T]$. The utility U is *increasing in information* if for any filtrations $\mathbf{G}$ and $\mathbf{H}$, and any consumption plan c (measurable with respect to $\mathbf{G}$ and $\mathbf{H}$), $\mathbf{G}\le\mathbf{H}$ implies $U(c,\mathbf{G})\le U(c,\mathbf{H})$. The utility U is *decreasing in information* if $-U$ is increasing in information. Clearly, an agent whose preferences are described by a utility increasing (decreasing) in information has preferences for early (late) resolution of uncertainty. It is shown in Skiadas (1995) that if $f(c,\cdot)$ is convex (concave) for any value of c, then U is increasing (decreasing) in information. In the special case where $f(c,\cdot)$ is linear for every c, U reduces to the standard form of a time and state additive utility, and U does not depend at all on the filtration argument. In the context of this paper we will see in Section 5 that the observed filtration can be made smaller or larger essentially by varying the parameter v_s. The explicit formula for V will then make clear the dependence of the utility on information.

Another nice property of the agent's utility is that, although in general it depends on the underlying filtration, it does not depend on the "irrelevant" filtration $\mathbf{I}$. The agent's utility depends only on information that reveals something about the distribution of future consumption. To see the result formally, notice that if V is the unique solution to (1) with $c = e$ and $\mathbf{G} = \mathbf{F}^{e,s}$, then it is also a solution to (1) with $c = e$ and $\mathbf{G} = \mathbf{F}^{o}$, since $\mathbf{I}$ is jointly stochastically independent of (e, s) and the integral in the conditional expectation. Assuming that (1) always has a unique solution, it follows that $V = V(\mathbf{F}^{o}) = V(\mathbf{F}^{e,s})$.

The particular parametric form of f we adopt in this model is a special case of that considered by Duffie and Epstein (1992b):

$$f(c, v) = \beta(1 + \alpha v)\left(\log(c) - \frac{\log(1 + \alpha v)}{\alpha}\right),$$

for some constants α and β satisfying $\beta > 0$ and $0 \neq \alpha < 1$. Notice that as $\alpha \to 0$, $f(c, v) \to f^{0}(c, v) = \beta(\log(c) - v)$. For $\alpha = 0$, we therefore define $f = f^{0}$, and the resulting utility process takes the familiar additive form:

$$U(c, \mathbf{G}) = \beta E\left[\int_{0}^{T} e^{-\beta u} \log(c_u)\, du\right], \quad \alpha = 0.$$

The utility process V can be computed in closed form, after characterizing it as a solution to a partial differential equation:

Proposition 2. *The process $V = V(e, \mathbf{F}^{o})$ is given by*

$$V_t = \frac{1}{\alpha}(\exp[\alpha(q_t \log e_t + h_t m_t + k_t)] - 1),$$

where q, h, and k are deterministic processes, given by

$$q_t = 1 - e^{-\beta(T-t)}$$

$$h_t = \frac{\beta b_e}{-b_x}\left(\frac{1 - e^{(b_x - \beta)(T-t)}}{b_x - \beta} - \frac{1 - e^{-\beta(T-t)}}{-\beta}\right)$$

$$k_t = \int_{t}^{T} e^{-\beta(u-t)} \psi_u\, du + \frac{1}{2}\alpha h_t^2 \gamma_t,$$

where

$$\psi_t = a_x h_t + \left(a_e - \frac{v_e' v_e}{2}\right) q_t + \frac{\alpha}{2}(\beta h_t^2 \gamma_t + v_e' v_e q_t^2 + v_x' v_x h_t^2 + 2 v_x' v_e q_t h_t).$$

The proof in the Appendix also shows that V as a function of e is monotonically increasing, and V as a function of m is increasing if $b_e > 0$, and decreasing if $b_e < 0$. In Section 5 we show that, within the class of filtrations that can be observed by the agent of our model, U is increasing in information for negative α, and decreasing in information for positive α. For $\alpha = 0$, the utility does not depend on the filtration argument. Finally, notice that the expression for the utility confirms that V does not depend on the extraneous information $\mathbf{I}$.

4 Equilibrium security prices

Equilibrium prices in our setting can be described in terms of a state-price process π, a strictly positive Ito process with the property that the price process of any security with cumulative dividend process D (adapted and of integrable variation) is given by

$$S_t = \frac{1}{\pi_t} E\left(\int_t^T \pi_u dD_u + \pi_T S_T \,\middle|\, \mathscr{F}_t^o \right).$$

A complete description of the equilibrium notion, and the relationship between π and the utility gradient can be found in Duffie and Skiadas (1994). Alternatively, the Markovian approach of Duffie and Epstein (1992a, b) can be employed. The fact that the agent in our setting receives incomplete information does not present any problem, since we have already shown how to reduce the agent's problem to an equivalent one with complete information. For our purposes, we only need the state-pricing formula:

$$\pi_t = \exp\left(\int_0^t f_v(e_u, V_u) du \right) f_c(e_t, V_t),$$

where f_c and f_v denote the two partial derivatives of f.

An application of Ito's lemma implies the following characterization of π:

Proposition 3. *The state-price process π satisfies*

$$\frac{d\pi_t}{\pi_t} = -r_t dt + \sigma_\pi(t) dW_t, \quad t \in [0, T],$$

where

$$r_t = a_e + b_e m_t - v_e' v_e + \alpha[h_t(b_e \gamma_t + v_x' v_e) + q_t v_e' v_e] + \beta,$$

and

$$\sigma_\pi(t) = (\alpha q_t - 1)[1, 0]\Sigma^{1/2} + \alpha h_t [b_e \gamma_t + v_x' v_e, b_s \gamma_t + v_x' v_s] \Sigma^{-1/2}.$$

The process r is the short rate process, defined as the dividend rate of a security whose equilibrium price is always equal to one.

The formula for r shows that the short rate contains all the information relevant to estimating the state variable, that is, m_t is determined uniquely given r_t. Also, the instantaneous variance of r is proportional to the instantaneous variance of m_t. (Analogous results were derived in related papers cited in the Introduction.)

Since r can be expressed as an Ornstein-Uhlenbeck (O-U) process, it is Gaussian, and the same is true under the equivalent martingale measure.[4]

[4] The equivalent martingale measure, Q, is defined by its Radon-Nikodym derivative, $dQ/dP = \exp(-(1/2)\int_0^T \sigma_\pi(u)^2 du + \int_0^T \sigma_\pi(u) dW_t)$. By Girsanov's theorem, the process $\tilde{W}_t = W_t - \int_0^t \sigma_\pi(u) du$ is standard Brownian motion under Q. It is then clear that the SDE for r is of the O-U type when expressed in terms of $\tilde{W}$ if and only if the same is true of the SDE for r when expressed in terms of W.

Asset pricing in this setting is tractable because of the Gaussian property of r under the equivalent martingale measure, and because r does not depend on the endowment process. Vasicek (1977), Jamshidian (1989), El Karoui and Rochet (1989), and Hull and White (1990) have examined bond and option pricing when the short-rate process is O-U under the equivalent martingale measure.

In order to examine the role of the filtration on the term structure of interest rates, we now give explicit expressions for discount bond prices. According to the state-pricing formula, the price process of a discount bond that matures at time τ is given by $P_t(\tau) = E[\pi_\tau | \mathscr{F}_t^o]/\pi_t$, $t \leq \tau$. A calculation provided in the Appendix, using only simple properties of the normal distribution, shows the following result:

Proposition 4. *Let* $K(t) = -(1 - \exp(b_x t))/b_x$ *and*

$$\xi(t) = b_e a_x - b_x a_e - \beta b_x + v_e' v_e(b_x - \alpha b_x q(t) - \alpha\beta(1 - q(t))) + v_x' v_x \alpha b_e h(t) + v_x' v_e(\alpha\beta h(t) - b_e).$$

Then

$$\log P_t(\tau) = -r(t)K(\tau - t) - \int_t^\tau K(\tau - u)\left(\xi(u) - \frac{1}{2}K(\tau - u)b_e^2 v_x' v_x\right)du + \frac{1}{2}b_e^2 K^2(\tau - t)\gamma_t - \alpha\beta b_e \int_t^\tau K(\tau - u)\gamma(u)h(u)du.$$

5 The effect of a change in the filtration

Inspection of the formula for the state-price process shows that a shift in $\mathbf{F}^o$ will, in general, change state prices, since V is a function of the observed filtration. In this section we show how a change in the filtration observed by the agent can be accomplished within our parametric setup, essentially by varying the variance of the signal. The pricing formulas of the last section can then be interpreted in terms of the timing and amount of the observed information.

For the purposes of this section, we assume that the "irrelevant" information $\mathbf{I}$ is generated by a three-dimensional standard Brownian motion Z, that is necessarily independent of B. Let $\lambda \in \mathbb{R}^3$ be orthogonal to both v_e and v_x, and define the "contaminated" signal $\bar{s} = s + \lambda Z$. (The reader may choose to consider only the special case in which the third component of v_e and v_x, and the first two components of λ are all zero.) So far we have assumed that the agent observes the filtration $\mathbf{I}$, but finds it irrelevant to consumption, and hence $\mathbf{I}$ does not affect prices in equilibrium. Consider now an economy that is identical to the one described so far, except that the agent does not observe $\mathbf{I}$ and s. Instead, the agent observes the "contaminated" signal $\bar{s}$. We refer to this economy as the *high-noise economy*. The total information observed by the agent of the high-noise economy is represented by the filtration $\mathbf{F}^{e,\bar{s}} = \{\mathscr{F}_t^{e,\bar{s}} : t \geq 0\}$, generated by the endowment process e and the signal $\bar{s}$.

Clearly,

$$\mathscr{F}_t^{e,\bar{s}} \subset \mathscr{F}_t^o, \quad t \in [0, T].$$

That is, the agent of the high-noise economy receives coarser, or less timely, information. The inclusion relationship is strict, except for the special case $\lambda = 0$. In fact, the set of high-noise economies is ordered in this way by λ.

Solving for the prices in the high-noise economy amounts to simply adjusting the parameter v_s in the original economy to $v_s + \lambda$. This is because $(e, x, \bar{s})$ and $(e, x, s + \lambda B)$ are Gaussian with the same variance-covariance matrix, and therefore are identically distributed. Formally, we can treat $\sigma = v_s' v_s$ as a parameter representing the timing and quality of the information available to the agent. The parameter σ can be varied independently of all other parameters in the model, without changing the covariance terms $v_e' v_s$ and $v_x' v_s$, by varying v_s along a direction orthogonal to v_e and v_x.

Given the explicit utility formula of Proposition 2, it is not too hard to confirm that V is decreasing (increasing) in σ for $\alpha < 0$ ($\alpha > 0$). For $\alpha = 0$, V does not depend on σ. All this is consistent with the more general discussion of Section 3, since the second derivative of f with respect to the utility argument is given by $f_{vv}(c, v) = -\alpha\beta/(1 + \alpha v)$. It follows from the proof of Proposition 2 that $1 + \alpha V_t$ is always positive, and therefore f is convex in utility for $\alpha < 0$, and concave for $\alpha > 0$.

The dependence of discount-bond prices and instantaneous variances[5] on σ is summarized in the following result. In part (c) we refer to an infinite-horizon economy, where $T = \infty$. Although we have not presented the details, the above framework extends readily to the infinite-horizon case (see Duffie and Epstein (1992b) and the proof in the Appendix).

Proposition 5. *(a) The time-zero discount bond prices, $\{P_0(\tau): \tau \in [0, T]\}$, are: (i) nondecreasing in σ, if $\alpha < 0$; (ii) nonincreasing in σ, if $\alpha > 0$; (iii) not dependent on σ, if $\alpha = 0$.*
(b) Let v stand for the instantaneous variance of any of m, r, or $\{P_t(\tau): t \leq \tau\}$, $\tau \leq T$. Then (i) v_t is nonincreasing in σ if either t, or $\max\{|v_x' v_e|, |v_x' v_s|, |v_e' v_s|\}$ *is sufficiently small; and (ii) in the infinite horizon case,* $\lim_{t\to\infty} v_t$ *exists and is nonincreasing in σ.*
(c) In the infinite-horizon case, the asymptotic yield at t is well defined by

$$\lim_{\tau \to \infty} -\frac{\log P_t(\tau)}{\tau - t},$$

and is increasing in α if $b_e v_x' v_e \geq 0$.

The proof in the Appendix also discusses weak conditions under which the terms "nonincreasing" and "nondecreasing" in Proposition 5 can be replaced by "decreasing" and "increasing," respectively.

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[5] If Y is an Ito process with $dY_t = \mu_Y(t)dt + \sigma_Y(t)dW_t$, the *instantaneous variance* of Y is the process $\{\|\sigma_Y(t)\|^2 : t \geq 0\}$.

Finally, we consider the dependence of forward rates on σ. The *time-zero (instantaneous) forward rate for maturity t* is defined by

$$F_t = -\left[\frac{d\log(P_0(\tau))}{d\tau}\right]_{\tau=t}, \quad t>0.$$

Proposition 6. *Suppose that* $(v_x'v_e)(v_e'v_s) \neq (v_e'v_e)(v_x'v_s)$ *or that* $b_e v_e' v_s \neq b_s v_e' v_e$. *Then the following strict monotonicity relationships hold:*
(*a*) *The time-zero forward rates are decreasing* (*respectively, increasing*) *in* σ *if* $\alpha<0$ (*respectively,* $\alpha>0$).
(*b*) *The time-zero forward rates become decreasingly sensitive to changes in* σ *as their maturities approach the terminal date T. That is,*

$$\left[\frac{d}{d\tau}\left|\frac{\partial F_\tau}{\partial \sigma}\right|\right]_{\tau=t} < 0,$$

for all t sufficiently close to T.
(*c*) *In the infinite-horizon case, the sensitivity of the time-zero forward rate for maturity t to changes in* σ *increases with t, if* (*i*) *t is sufficiently small, or* (*ii*) $\max\{|v_x'v_e|, |v_x'v_s|, |v_e'v_s|\}$ *is sufficiently small. That is, either* (*i*) *or* (*ii*) *imply that*

$$\left[\frac{d}{d\tau}\left|\frac{\partial F_\tau}{\partial \sigma}\right|\right]_{\tau=t} > 0.$$

Appendix: Proofs

Proof of Proposition 1

(a) The conditional variance expression derived below is more general than needed in the paper. Allow x_0, s_0, and e_0 to be random variables with x_0, conditional on s_0 and e_0, to be Gaussian with variance γ_0. Then it is shown below that the conditional variance at time t is given by

$$\gamma_t = \gamma^+ \frac{1 + e^{-\delta t}\dfrac{\gamma^-}{\gamma^+}\left(\dfrac{\gamma^+ - \gamma_0}{\gamma_0 - \gamma^-}\right)}{1 + e^{-\delta t}\left(\dfrac{\gamma^+ - \gamma_0}{\gamma_0 - \gamma^-}\right)},$$

where

$$\gamma^{\pm} = -\frac{1}{2\kappa_2}(\kappa_1 \pm \delta),$$

and

$$\kappa_0 = v_x'v_x - [v_x'v_e, v_x'v_s]\Sigma^{-1}\begin{pmatrix} v_x'v_e \\ v_x'v_s \end{pmatrix},$$

$$\kappa_1 = 2b_x - 2[b_e, b_s]\Sigma^{-1}\begin{pmatrix} v_x'v_e \\ v_x'v_s \end{pmatrix},$$

$$\kappa_2 = -[b_e, b_s]\Sigma^{-1}\begin{pmatrix} b_e \\ b_s \end{pmatrix},$$

$$\delta = \sqrt{\kappa_1^2 - 4\kappa_0\kappa_2}.$$

By Theorem 12.7 in Lipster and Shiryayev (1978), the conditional variance γ satisfies the ordinary differential equation

$$\dot{\gamma}_t = 2b_x\gamma_t + v_x'v_x - [b_e\gamma_t + v_x'v_e, b_s\gamma_t + v_x'v_s]\Sigma^{-1}\begin{pmatrix} b_e\gamma_t + v_x'v_e \\ b_s\gamma_t + v_x'v_s \end{pmatrix}$$

$$= \kappa_0 + \kappa_1\gamma_t + \kappa_2\gamma_t^2.$$

The assumption that $v'v$ is positive definite ensures that $\kappa_0 > 0$ and $\kappa_2 < 0$, which imply that the roots are real and that $\gamma^- < 0 < \gamma^+$. The solution to the differential equation is well-known, but is derived below for completeness. Factoring the polynomial and rearranging:

$$1 = \frac{\dot{\gamma}_t}{\kappa_2(\gamma_t - \gamma^+)(\gamma_t - \gamma^-)} = -\frac{1}{\delta}\frac{d}{dt}\log\left(\frac{\gamma_t - \gamma^+}{\gamma_t - \gamma^-}\right).$$

The second equality it obtained by using a partial fraction expansion. Integrating and rearranging gives the expression for γ. Note that $\delta > 0$ implies that $\gamma_T \to \gamma^+$ as $T \to \infty$.

When $\gamma_0 = 0$, the formula for γ_t simplifies to

$$\gamma_t = \gamma^+ \frac{1 - e^{-\delta t}}{1 - e^{-\delta t}(\gamma^+/\gamma^-)}.$$

(b) By Theorem 12.7 in Lipster and Shiryayev (1978), W is the innovation process given by

$$dW_t = \Sigma^{-1/2}\begin{pmatrix} de_t/e_t - (a_e + b_e m_t)dt \\ ds_t - (a_s + b_s m_t)dt \end{pmatrix}.$$

(c) This result follows from Lemma 11.3 in Lipster and Shiryayev (1978). ■

Proof of Proposition 2

An application of Ito's lemma to the expression for V gives the following partial differential equation (PDE) for $V_t = J(t, m_t, e_t)$:

$$0 = J_t + J_m(a_x + b_x m) + J_e(a_e + b_e m)e$$
$$+ \tfrac{1}{2}(J_{mm}\eta_t'\Sigma^{-1}\eta_t + J_{ee}v_e'v_e e^2 + 2J_{me}(b_e\gamma_t + v_x'v_e)e) + f(e, J),$$

where,

$$\eta_t = \begin{pmatrix} b_e\gamma_t + v_x'v_e \\ b_s\gamma_t + v_x'v_s \end{pmatrix}.$$

A trial solution for J is

$$\frac{\log(1+\alpha J)}{\alpha} = q_t \log e + h_t m + k_t.$$

Note that as $\alpha \to 0$, the left hand side converges to J. The terminal condition $J(T, m, e) = 0$ implies that $q_T = h_T = k_T = 0$. Substituting the trial solution into the PDE gives

$$\begin{aligned} 0 = {} & \log(e)[\dot{q}_t + \beta(1-q_t)] + m(h_t b_x + q_t b_e + \dot{h}_t - \beta h_t) + \dot{k}_t - \beta k_t + a_x h_t \\ & + (a_e - \tfrac{1}{2}v_e' v_e)q_t + \tfrac{1}{2}\alpha[\eta_t' \Sigma^{-1}\eta_t h_t^2 + v_e' v_e q_t^2 + 2(b_e \gamma_t + v_x' v_e) q_t h_t]. \end{aligned}$$

The candidate solution solves the PDE if and only if

$$\begin{aligned} 0 &= \dot{q}_t + \beta(1-q_t), \\ 0 &= \dot{h}_t - (\beta - b_x)h_t + q_t b_e, \\ 0 &= \dot{k}_t - \beta k_t + a_x h_t + (a_e - \tfrac{1}{2}v_e' v_e)q_t \\ &\quad + \tfrac{1}{2}\alpha[\eta_t' \Sigma^{-1}\eta_t h_t^2 + v_e' v_e q_t^2 + 2(b_e\gamma_t + v_x' v_e)q_t h_t], \\ 0 &= q_T = h_T = k_T. \end{aligned}$$

In general, if g is a continuous function, there is a unique function f that satisfies the differential equation $\dot{f}_t - \lambda f_t + g_t = 0$, with $f_T = 0$. The solution is

$$f_t = \int_t^T e^{-\lambda(u-t)} g_u \, du.$$

The formulas for q and h follow directly by applying this result. An application to k gives

$$\begin{aligned} k_t = \int_t^T e^{-\beta(u-t)} \Big(& a_x h_u + (a_e - \tfrac{1}{2}v_e' v_e) q_u \\ & + \frac{\alpha}{2}[\eta_u' \Sigma^{-1} \eta_u h_u^2 + v_e' v_e q_u^2 + 2(b_e \gamma_u + v_x' v_e) q_u h_u] \Big) du. \end{aligned}$$

Differentiating with respect to, say, $v_s' v_s$ gives a formula that is difficult to sign. To derive a formula which makes the dependence of k on γ clearer, first rearrange the differential equation for γ as $\eta_t' \Sigma^{-1} \eta_t = -\dot{\gamma}_t + 2b_x \gamma_t + v_x' v_x$, and substitute into the formula for k:

$$\begin{aligned} k_t = \int_t^T e^{-\beta(u-t)} \Big(& a_x h_u + (a_e - \tfrac{1}{2}v_e' v_e) q_u \\ & + \frac{\alpha}{2}[(2b_x \gamma_u + v_x' v_x) h_u^2 + v_e' v_e q_u^2 + 2(b_e \gamma_u + v_x' v_e) q_u h_u] \Big) du \\ & - \frac{\alpha}{2} \int_t^T e^{-\beta(u-t)} h_u^2 \dot{\gamma}_u \, du. \end{aligned}$$

The second integral can be written as

$$\int_t^T e^{-\beta(u-t)} h_u^2 \dot{\gamma}_u du = -h_t^2\gamma_t - \int_t^T e^{-\beta(u-t)}(2h_u\dot{h}_u - \beta h_u^2)\gamma_u du$$

$$= -h_t^2\gamma_t + \int_t^T e^{-\beta(u-t)}(2(b_x-\beta)h_u^2 + 2b_e h_u q_u + \beta h_u^2)\gamma_u du,$$

where the first equality is obtained using integration by parts and the second by substituting the differential equation for h. Substituting into the formula for k and simplifying gives the result.

The uniqueness of the above solution to the partial differential equation satisfied by J can be addressed using the techniques of Duffie and Lions (1992). ■

Proof of Proposition 3

Applying Ito's lemma to the formula for π_t, we obtain

$$\frac{d\pi_t}{\pi_t} = \frac{df_c}{f_c} + f_v dt$$

$$= \frac{1}{f_c}(f_{cc}De\,dt + f_{cv}DV dt + \tfrac{1}{2}f_{ccc}(de)^2 + \tfrac{1}{2}f_{cvv}(dV)^2 + f_{ccv}de\,dV) + f_v dt$$

$$+ \frac{f_{cc}}{f_c}(de - De\,dt) + \frac{f_{cv}}{f_c}(dV - DV dt),$$

where D denotes the infinitesimal drift operator. Substituting $DV = -f$,

$$de\,dV = de\,dJ = J_m dc\,dm + J_c(de)^2$$

$$= \frac{f_c e^2}{\beta}(h_t(b_e\gamma_t + v_x'v_e) + q_t v_e' v_e),$$

and simplifying gives the formula for the state-price process. The short rate is defined as the dividend rate on a security whose price is identically equal to one. It follows that $\pi_t + \int_0^t r_s\pi_s ds$ must be a martingale, and therefore, $r_t = -D\pi_t/\pi_t$. ■

Proof of Proposition 4

The stochastic differential equation for the short rate can be written as

$$dr_t = b_x r_t + \xi_t + \gamma_t b_e(\alpha\beta h_t - b_e) + b_e\eta_t'\Sigma^{-1/2}d\tilde{W}_t,$$

where η is defined in the proof of Proposition 2, and the process $\tilde{W}_t$ is standard Brownian motion under the equivalent martingale measure Q (see footnote 4).

The equation has the explicit solution

$$r_\tau = r_t e^{b_x(\tau - t)} + \int_t^\tau e^{b_x(\tau - u)}(\xi_u + \gamma_u b_e(\alpha\beta h_u - b_e))du$$

$$+ b_e \int_t^\tau e^{b_x(\tau - u)} \eta_u' \Sigma^{-1/2} d\tilde{W}_u.$$

The integral of the interest rate is given by

$$\int_t^\tau r_u du = K(\tau - t)r_t + \int_t^\tau K(\tau - u)(\xi_u + \gamma_u b_e(\alpha\beta h_u - b_e))du$$

$$+ b_e \int_t^\tau K(\tau - u)\eta_u' \Sigma^{-1/2} d\tilde{W}_u.$$

Using standard results on normal random variables, the price of a discount bond can be expressed as

$$P_t(\tau) = E_t^Q \exp\left(- \int_t^\tau r_u du \right)$$

$$= \exp\left[-E_t^Q\left(\int_t^\tau r_u du \right) + \frac{1}{2} Var_t^Q\left(\int_t^\tau r_u du \right) \right].$$

The variance term can be written as

$$Var_t^Q\left(\int_t^\tau r_u du \right) = b_e^2 \int_t^\tau K^2(\tau - u)\eta_u' \Sigma^{-1} \eta_u du$$

$$= b_e^2 \int_t^\tau K^2(\tau - u)(2b_x \gamma_u + v_x' v_x - \dot{\gamma}_u)du,$$

where the second equality is obtained by substituting for the differential equation satisfied by γ. The term involving $\dot{\gamma}$ can be simplified using integration by parts:

$$\int_t^\tau K^2(\tau - u)\dot{\gamma}_u du = -\gamma_t K^2(\tau - t) + 2\int_t^\tau K(\tau - u)[1 + b_x K(\tau - u)]\gamma_u du.$$

Substituting into the variance expression yields

$$Var_t^Q\left(\int_t^\tau r_u du \right) = b_e^2 K^2(\tau - t)\gamma_t + b_e^2 \int_t^\tau K(\tau - u)(K(\tau - u)v_x' v_x - 2\gamma_u)du.$$

Substituting into the discount bond price expression gives the result. ■

Proof of Proposition 5

(a) The comparative statics for discount bond prices follow directly from the explicit formulas, the fact that $b_e h_t > 0$ for any $t < T$, and the property that γ_t is nondecreasing in σ for any $t > 0$. The latter can be shown by an application of Jensen's inequality, or by the proof in part (b) below.

Remark: If $(v_x'v_e)(v_e'v_s) \neq (v_e'v_e)(v_x'v_s)$, or $b_e v_e' v_s \neq b_s v_e' v_e$, then time-zero discount prices are increasing (decreasing) in σ for $\alpha < 0$ $(\alpha > 0)$. This follows from the proof in part (b), which shows that under either condition, γ_t is increasing in σ for all $t > 0$, except possibly at one point in time.

(b) Since $(dr_t)^2 = b_e^2(dm_t)^2$, and $(dP_t(\tau))^2 = \mathrm{P}_t(\tau)^2(dr_t)^2$, it suffices to consider the case in which $v_t = (dm_t)^2 = \eta_t'\Sigma^{-1}\eta_t$, where η is defined in the proof of Proposition 2.

(i) From the proof of Proposition 1, the conditional variance γ satisfies the ordinary differential equation $\dot{\gamma}_t = 2b_x\gamma_t + v_x'v_x - \eta_t'\Sigma^{-1}\eta_t$, with $\gamma_0 = 0$. Differentiating both sides with respect to σ implies $\dot{f}_t = \lambda_t f_t + \mu_t$, with $f_0 = 0$, where

$$f_t = \frac{\partial \gamma_t}{\partial \sigma},$$

$$\lambda_t = 2(b_x - [b_e, b_s]\Sigma^{-1}\eta_t),$$

$$\mu_t = \frac{1}{\det(\Sigma)}\eta_t'\left[\Sigma^{-1} - \begin{pmatrix} (v_e'v_e)^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right]\eta_t$$

$$= \frac{1}{v_e'v_e \det(\Sigma)^2}((v_x'v_e)(v_e'v_s) - (v_e'v_e)(v_x'v_s) + \gamma_t(b_e v_e'v_s - b_s v_e'v_e))^2.$$

Since $\mu_t \geq 0$, for all $t \geq 0$, it follows that $f_t \geq 0$, for all $t \geq 0$, and $\dot{f}_t \geq 0$ at least for small t.

Rearranging the differential equation for γ and differentiating, we have

$$\frac{\partial}{\partial \sigma}(\eta_t'\Sigma^{-1}\eta_t) = 2b_x f_t - \dot{f}_t.$$

Since $b_x < 0$, the above derivative must be nonpositive for small $t \geq 0$.

If $v_x'v_e = v_x'v_s = v_e'v_s = 0$, the instantaneous variance of m_t is given by:

$$(dm_t)^2 = \frac{v_x'v_x(\delta^2 - 4b_x^2)(1 - e^{-\delta t})^2}{(2b_x - \delta - e^{-\delta t}(2b_x + \delta))^2}dt$$

where,

$$\delta = 2\sqrt{v_x'v_x(b_x^2/v_x'v_x + b_e^2/v_e'v_e + b_s^2/\sigma)}$$

Some tedious computations show that the partial derivative of $(dm_t)^2$ with respect to δ is strictly positive for $t > 0$. By continuity, the same holds for sufficiently small $|v_x'v_e|$, $|v_x'v_s|$, and $|v_e'v_s|$. Since δ is decreasing in σ, the result is shown.

Remark: If $(v_x'v_e)(v_e'v_s) \neq (v_e'v_e)(v_x'v_s)$, or $\mathrm{b}_e v_e'v_s \neq b_s v_e'v_e$, the variance of m_t is decreasing in σ for small $t > 0$. This follows because under either condition, $f_t > 0$ and $\dot{f}_t > 0$, at least for small $t > 0$. If both conditions are violated, it is easy to contruct an example in which γ_t does not depend on σ, for any t, even though $v'v$ is positive definite.

(ii) The stationary variance, γ^+ satisfies $2b_x\gamma^+ + v_x'v_x - (\eta^+)'\Sigma^{-1}\eta^+ = 0$, where

$$\eta^+ = \begin{pmatrix} b_e\gamma^+ + v_x'v_e \\ b_s\gamma^+ + v_x'v_s \end{pmatrix}.$$

Rearranging and differentiating the limiting instantaneous variance of m_t with respect to σ, we obtain

$$\frac{\partial}{\partial\sigma}((\eta^+)'\Sigma^{-1}\eta^+) = 2b_x\frac{\partial\gamma^+}{\partial\sigma}.$$

The assumption that $b_x < 0$ ensures that the right hand side is nonpositive.

Remark: Under the assumption that

$$\gamma_+(b_e v_e' v_s - b_s v_e' v_e) \neq -(v_x'v_e)(v_e'v_s) + (v_e'v_e)(v_x'v_s),$$

the limiting variance of m_t is decreasing in σ. Under this assumption, μ_t converges to some positive constant as $t \to \infty$. Furthermore, the convergence of γ_t to γ^+ implies that λ_t converges to some finite (negative) constant. If follows that $\partial\gamma^+/\partial\sigma > 0$. Note that the above condition is both necessary and sufficient for $\partial\gamma^+/\partial\sigma > 0$.

(c) As in Duffie and Epstein (1992), the infinite horizon utility is defined as the limit as $T \to \infty$ of the finite horizon utility. It is easy to show that $q(t) \to 1$, $h(t) \to b_e/(\beta - b_x)$, and $\xi(t) \to \xi^*$, for all t, where

$$\xi^* = b_e a_x - b_x a_e - \beta b_x + v_e'v_e b_x(1-\alpha) + v_x'v_x\alpha\frac{b_e^2}{\beta - b_x} + v_x'v_e b_e\frac{\beta(\alpha-1)+b_x}{\beta - b_x}.$$

The asymptotic yield is given by

$$\lim_{\tau\uparrow\infty} -\frac{\log P_t(\tau)}{\tau - t} = -b_e\frac{a_x}{b_x} + a_x + \beta - b_e v_x'v_e - \frac{1}{2}v_x'v_x\frac{b_e^2}{b_x^2} - v_e'v_e$$
$$+ \alpha v_e'v_e - \alpha\frac{b_e^2}{b_x(\beta - b_x)}\left(v_x'v_x + \beta\gamma^+ - v_x'v_e\beta\frac{b_x}{b_e}\right).$$

Noting again that $b_x < 0$, a sufficient (but not necessary) condition for the yield to be increasing in α is $b_e v_x'v_e \geq 0$. ■

Proof of Proposition 6

(a) From the discount bond price formula of Proposition 4, we obtain:

$$\left[\frac{\partial F_\tau}{\partial\sigma}\right]_{\tau=t} = \alpha\beta b_e\int_0^t e^{b_x(t-u)}h(u)\frac{\partial\gamma(u)}{\partial\sigma}\,du.$$

Given our assumption, the proof of part (b) of Proposition 5 shows that γ_t is (strictly) increasing in σ for all $t > 0$, and part (a) follows.

(b) Differentiating the above expression with respect to the maturity date, we obtain

$$\left[\frac{\partial^2 F_\tau}{\partial\tau\partial\sigma}\right]_{\tau=t} = \alpha\beta b_e\left(h(t)\frac{\partial\gamma(t)}{\partial\sigma} + b_x\int_0^t e^{b_x(t-u)}h(u)\frac{\partial\gamma(u)}{\partial\sigma}du\right).$$

Noting that $b_e h(t)\downarrow 0$ as $t\uparrow T$, and recalling that $b_x < 0$, the result follows.

(c) Given an infinite horizon, we substitute $h(t) = b_e/(\beta - b_x)$ in the last expression, and we integrate by parts, using the fact that $\partial\gamma(0)/\partial\sigma = 0$, to obtain

$$\left[\frac{\partial^2 F_\tau}{\partial\tau\partial\sigma}\right]_{\tau=t} = \frac{\alpha\beta b_e^2}{\beta - b_x}\int_0^t e^{b_x(t-u)}\frac{\partial^2\gamma(u)}{\partial u\partial\sigma}du.$$

The proof of part (b) of Proposition 5 shows that

$$\left[\frac{\partial^2\gamma(u)}{\partial u\partial\sigma}\right]_{u=t} > 0,$$

for all sufficiently small t. This proves the result under assumption (i). To show the result under assumption (ii) one could confirm the above inequality for all t by differentiating the explicit formula for γ_t. This avenue is exceedingly tedious, however. Instead, we define $f_t = \partial\gamma_t/\partial\sigma$, which satisfies $\dot{f}_t = \lambda_t f_t + \mu_t$, with $f_0 = 0$. The function f is given explicitly by $f_t = \int_0^t \exp(\int_u^t \lambda_s ds)\mu_u du$, and therefore

$$\dot{f}_t = \mu_t + \lambda_t\int_0^t e^{\int_u^t \mu_s ds}\mu_u du.$$

Integration by parts implies that

$$\dot{f}_t = \lambda_t e^{\int_0^t \lambda_s ds}(\mu_0/\lambda_0) + \lambda_t\int_0^t e^{\int_u^t \lambda_s ds}\frac{d}{du}\left(\frac{\mu_u}{\lambda_u}\right)du.$$

When the covariances are sufficiently small, $\lambda_t < 0$ for $t \geq 0$, and a straightforward calculation shows that

$$\frac{d}{du}\left(\frac{\mu_u}{\lambda_u}\right) < 0,$$

everywhere on $(0, \infty)$. Combining these facts with $\mu_0 \geq 0$ implies that $\dot{f}_t > 0$ for all $t > 0$, proving the desired result.

Remark: If $(v_x' v_e)(v_e' v_s) = (v_e' v_e)(v_x' v_s)$ and $b_e v_e' v_s = b_s v_e' v_e$, then

$$\left[\frac{\partial F_\tau}{\partial\sigma}\right]_{\tau=t} = 0, \quad t \geq 0. \quad \blacksquare$$

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