Infinite Horizon Stochastic Differential Utility

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Abstract: Existence and uniqueness of an infinite horizon stochastic differential utility function is proved. The issue of existence is analogous to that of stability of a non-linear feedback system. Stability is guaranteed by imposing a "uniform sector condition" on the "feedback function." The basic properties of the finite horizon recursive utility presented by Duffie and Epstein generalize directly to the infinite horizon case.

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1. Introduction

Duffie and Epstein (1989) introduced the notion of a stochastic differential utility over a finite horizon $[0, T]$ and applied it to asset pricing problems. They showed that for an appropriately integrable consumption process, c, and under regularity conditions on f, there is a unique integrable semimartingale, V^T, the recursive utility process corresponding to c, defined by

$$V_t^T = E\left[\int_t^T f(c_s, V_s^T)\ ds \,\middle|\, \mathcal{F}_t\right] \quad a.s., \quad t \in [0, T]. \tag{1}$$

A utility function, U^T, is then defined by letting $U^T(c) = V_0^T$. In this paper it is shown that, under regularity conditions on f and c, the finite horizon recursive utility process corresponding to c converges to an integrable semimartingale as the horizon length goes to infinity:

$$V_t = \lim_{T \to \infty} V_t^T \quad a.s. \tag{2}$$

Furthermore, V_t satisfies, for all t and $T \geq t$,

$$V_t = E\left[\int_t^T f(c_s, V_s)\ ds + V_T \,\middle|\, \mathcal{F}_t\right] \quad a.s. \tag{3}$$

It is also shown that V is the unique integrable semimartingale satisfying (3) and a transversality condition of the form:

$$\lim_{t \to \infty} e^{-\nu t} E\left(|V_t|\right) = 0, \tag{4}$$

for a suitable constant ν. We call V the infinite horizon recursive utility process corresponding to c. An infinite horizon stochastic differential utility function, U, is then defined by letting $U(c) = V_0$, and is shown to posess all of the elementary properies of U^T discussed by Duffie and Epstein (1989). Duffie and Lions (1990) show existence of infinite-horizon stochastic differential utility by partial differential equation techniques in a Markov diffusion setting, admitting some weakening of the conditions below on f.

We now proceed with the formal details of the model. For simplicity we will omit the a.s. (almost sure) qualification whenever it obviously applies. The basic primitive of the model is a filtered probability space $(\Omega, \mathcal{F}, I\!F, P)$, where the filtration $I\!F = \{\mathcal{F}_t : t \geq 0\}$ is assumed to satisfy the usual conditions. It is also assumed that $\mathcal{F}_0$ is trivial, that is, it only contains events of probability one or zero. Consumption processes are valued in a closed convex subset, $\mathcal{C}$, of some separable Banach lattice. The reader may choose to think of $\mathcal{C}$ as a finite dimensional Euclidean space or its positive cone. We let $\|c\|$ denote the norm of $c \in \mathcal{C}$. For fixed $\nu \in \mathbb{R}$, $\mathcal{D}_\nu$ is defined to be the space of all optional, $\mathcal{C}$-valued processes, $c = (c_t\ ;\ t \geq 0)$, such that $E\left(\int_0^\infty e^{-\nu t}\|c_t\|^2\, dt\right) < \infty$, and is equipped with the norm $\|c\|_{\mathcal{D}_\nu} = \left[E\left(\int_0^\infty e^{-\nu t}\|c_t\|^2\, dt\right)\right]^{\frac{1}{2}}$. Finally, for any horizon length $T < \infty$, $\mathcal{D}[0,T]$ is defined to be the space of all optional, $\mathcal{C}$-valued processes, c, such that $E\left(\int_0^T \|c_t\|^2\, dt\right) < \infty$ and $c_t = 0$ for all $t \geq T$. Notice that $\mathcal{D}[0,T] \subset \mathcal{D}_\nu$ for all choices of $T < \infty$ and ν. Duffie and Epstein defined a recursive utility U^T over $\mathcal{D}[0,T]$. In this paper the definition is extended to all of $\mathcal{D}_\nu$ for an appropriate ν.

2. Assumptions

The issue of existence of the infinite horizon recursive utility has the flavor of that of stability of non-linear feedback systems. We can view f as a non-linear feedback function. In control theory literature, f is often required to satisfy a "sector condition." (See for example Vidyasagar (1978).) Here we will employ a more stringent "uniform sector condition," which can also be viewed as a generalized Lipschitz condition.

DEFINITION. *A function $g : \mathbb{R} \to \mathbb{R}$ is said to satisfy* a sector condition *if, for some constants α and β and all $x \in \mathbb{R}\backslash\{0\}$,*

$$\alpha \leq \frac{g(x) - g(0)}{x} \leq \beta.$$

We denote this by $g \in sector[\alpha, \beta]$, and say that g satisfies a uniform sector condition *if for all $x \in \mathbb{R}$, $g(\cdot - x) \in sector[\alpha, \beta]$. We denote this by $g \in usector[\alpha, \beta]$.*

Intuitively, $g \in usector[\alpha, \beta]$ if and only if the slope of the line connecting any two points of the graph of g lies between α and β.

In the sequel $f : \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ will be taken to be a measurable function satisfying the following regularity conditions:

ASSUMPTION 1. *f satisfies a growth condition in consumption. That is, for some constants k_1 and k_2, and all $c \in \mathcal{C}$, $|f(c, 0)| \leq k_1 + k_2\|c\|$.*

ASSUMPTION 2. *f satisfies a uniform sector condition in utility. That is, for some constants ν and k with $\nu < k$, and all $c \in \mathcal{C}$, $f(c, \cdot\,) \in usector[-k, -\nu]$.*

Remarks:

1. For $\nu = -k < 0$, Assumption 2 is equivalent to a Lipschitz condition, justifying our claim that a uniform sector condition can be viewed as a generalized Lipschitz condition.

2. The notation in Assumption 2 has been chosen having in mind that, in general, $\nu \geq 0$. However, this is not a condition required by our theory.

3. If $f(c, \cdot\,)$ is concave, then Assumption 2 is equivalent to $f(c, \cdot\,) \in sector[-k, -\nu]$.

3. Existence and Uniqueness

Duffie and Epstein (1989) show that if f satisfies Assumptions 1 and 2, with $\nu = -k$, then for any $c \in \mathcal{D}[0, T]$ there is a unique integrable semimartingale[1] V^T that satisfies the recursive relation (1) for $t \in [0, T]$ and such that $V_t^T = 0$ for $t \geq T$. The finite horizon recursive utility, $U^T : \mathcal{D}[0, T] \to I\!\!R$, is then defined by letting $U^T(c) = V_0^T$. The following theorem allows us to extend this definition.

THEOREM 1. *Under Assumptions 1 and 2 and for all $c \in \mathcal{D}_\nu$, the limit in (2) exists and satisfies (3). Furthermore, (2) defines the unique integrable semimartingale V satisfying (3) and the transversality condition (4)*

Assumption 2 and the requirement that $c \in \mathcal{D}_\nu$ play antagonistic roles. We want to find the largest possible ν for which Assumption 2 holds, in order to allow the largest possible class of consumption processes implied by the theorem.

The infinite horizon recursive utility, $U : \mathcal{D}_\nu \to I\!\!R$, can now be defined by setting $U(c) = V_0$. Notice that, for $0 \in \mathcal{C}$ and $f(0, 0) = 0$, U^T is the restriction of U to $\mathcal{D}[0, T]$. Before proving the theorem we give two lemmas, proved in Appendix I, which will be useful in subsequent proofs. Until the end of the proof of Theorem 1, we fix some $c \in \mathcal{D}_\nu$.

LEMMA 1. *Suppose V is an integrable semimartingale satisfying (3), for some $T > 0$. Then, for all stopping times τ_1, τ_2 bounded by T,*

$$V_{\tau_1} = E\left[\int_{\tau_1}^{\tau_2} f(c_s, V_s)\, ds + V_{\tau_2} \,\middle|\, \mathcal{F}_{\tau_1}\right], \quad \text{on } \{\tau_1 < \tau_2\}.$$

PROOF. See Appendix I.

LEMMA 2. *There exist constants K_1 and K_2 such that for all $t \leq T$,*

$$|V_t^T| \leq K_1 + K_2 e^{\nu t} E\left[\int_t^T \|c_s\| e^{-\nu s}\, ds \,\middle|\, \mathcal{F}_t\right].$$

[1] We identify semimartingales that are versions of each other.

PROOF. See Appendix I.

PROOF OF THEOREM 1.

Existence: We show that the limit in (1) exists by proving that, for each time $t \geq 0$, the sequence $\{V_t^n : n \geq 0\}$ is Cauchy, almost everywhere on Ω. Suppose $n, m \geq N$, fix t, and define the stopping times: $\tau = \inf\{s : s \geq t,\ V_s^n \leq V_s^m\}$ and $\tau_N = \tau \wedge N$. Lemma 1 and Assumption 2 imply, for all $u \in [t, T]$,

$$\begin{aligned}
(V_u^n - V_u^m)1_{\{\tau>u\}} &= E\left[\int_u^{\tau_N} \left(f(c_s, V_s^n) - f(c_s, V_s^m)\right) 1_{\{\tau>u\}}\, ds \,\middle|\, \mathcal{F}_u\right] \\
&\quad + E\left[(V_{\tau_N}^n - V_{\tau_N}^m)1_{\{\tau>u\}} \,\middle|\, \mathcal{F}_u\right] \\
&\leq E\left[\int_u^{\tau_N} -\nu(V_s^n - V_s^m)1_{\{\tau>u\}}\, ds + (V_N^n - V_N^m)1_{\{\tau>N\}} \,\middle|\, \mathcal{F}_u\right] \\
&= E\left[\int_u^{N} -\nu(V_s^n - V_s^m)1_{\{\tau>s\}}\, ds + (V_N^n - V_N^m)1_{\{\tau>N\}} \,\middle|\, \mathcal{F}_u\right].
\end{aligned}$$

By the stochastic Gronwall-Bellman inequality,

$$\begin{aligned}
(V_t^n - V_t^m)^+ &\leq e^{-\nu(N-t)} E\left[(V_N^n - V_N^m)1_{\{\tau>N\}} \,\middle|\, \mathcal{F}_t\right] \\
&\leq e^{-\nu(N-t)} E\left[|V_N^n| + |V_N^m| \,\middle|\, \mathcal{F}_t\right].
\end{aligned}$$

Because of symmetry, $|V_t^n - V_t^m| \leq 2e^{-\nu(N-t)} E\left[|V_N^n| + |V_N^m| \,\middle|\, \mathcal{F}_t\right]$. Combining this result with Lemma 2 proves our assertion.

To show that V satisfies (3), we start with $V_t^n = E\left[\int_t^T f(c_s, V_s^n)\, ds + V_T^n \,\middle|\, \mathcal{F}_t\right]$. Noting that $|f(c_s, V_s^n)| \leq |f(c_s, V_s^n) - f(c_s, 0)| + |f(c_s, 0)| \leq \max(k, -\nu)|V_s^n| + k_1 + k_2\|c_s\|$ and using dominated convergence and Lemma 2, we can let $n \to \infty$ to derive (2).

Uniqueness: Suppose $\hat{V}$ and V both satisfy (3) and (4). Arguing exactly as above, we find $|\hat{V}_t - V_t| \leq 2e^{-\nu(N-t)} E\left[|\hat{V}_N| + |V_N| \,\middle|\, \mathcal{F}_t\right]$. Taking expectations on both sides and letting $N \to \infty$, we find $\hat{V}_t = V_t$.

QED.

We conclude this section by discussing the representation:

$$V_t = E\left[\int_t^\infty f(c_s, V_s)\, ds \,\middle|\, \mathcal{F}_t\right]. \tag{5}$$

Tempting as it is, such a representation is not always valid.

EXAMPLE. Let $f(c,v) \equiv 1 - v$. Then, clearly, Assumptions 1 and 2 are satisfied. For any c, the corresponding utilities are, $V_t^T = 1 - e^{(t-T)}$ and $V_t = 1$. While (2) and (3) are satisfied, (5) fails. One might suggest that (5) be modified to

$$V_t = E\left[\int_t^\infty f(c_s, V_s)\, ds \,\middle|\, \mathcal{F}_t\right] + V_\infty,$$

for some random variable V_∞ (identically equal to 1 in our case). But then consider $\overline{f}(c,v) \equiv c - (1 + \frac{1}{c})v$ and take $c_t = 1 + t$. Again Assumptions 1 and 2 are satisfied while the corresponding utilities are $\overline{V}_t^T = (1+t)V_t^T$ and $\overline{V}_t = (1+t)V_t$, with V^T and V as above. Now (5) fails even more seriously. However, we have the following positive result.

PROPOSITION 1. *Suppose $\nu \leq 0$, Assumptions 1 and 2 are satisfied, and $c \in \mathcal{D}_\nu$. Then (3) and (5) are equivalent.*

PROOF. Clearly, (3) implies (5). For the converse, we wish to let $T \to \infty$ and then apply the dominated convergence theorem as in the proof of the existence part of the above theorem. The reader can verify that Lemma 2 and Fubini's theorem yield the required integrability condition. It is here that the condition $\nu \leq 0$ is crucial.

QED.

4. Properties

We conclude this paper by briefly reviewing some properties of the infinite horizon stochastic differential utility. For this section, f is taken to satisfy Assumptions 1 and 2. Most of the basic properties are direct consequences of their finite horizon counterparts and equation (2). Thus

- U is (strictly) increasing whenever f is (strictly) increasing in consumption.
- U is concave whenever f is concave.
- U is time consistent in the sense explained in Duffie and Epstein (1989).

The discussion on risk aversion and homotheticity in Duffie and Epstein (1989) also extends in an obvious manner to the infinite horizon case. A less obvious result is continuity relative to the norm $\|\cdot\|_{\mathcal{D}_\nu}$ on $\mathcal{D}_\nu$.

THEOREM 2 (CONTINUITY). *The utility function $U : \mathcal{D}_\nu \to I\!R$ is continuous provided f is continuous.*

PROOF. See Appendix I.

Finally, the discussion of the Bellman equation by Duffie and Epstein (1989) also generalizes to an infinite horizon. The only change is the terminal value condition for the value function which is now replaced by a transversality condition as in equation (4). The reader who has read the proofs of this paper will have no trouble in applying the same approach in modifying the finite horizon argument.

An alternative approach to Bellman's equation for solving optimization problems in the presence of convexity is generalized Kuhn-Tucker theory. This requires the computation of the gradient of the utility function. Work in this direction is reported in Duffie and Skiadas (1990).

APPENDIX I

This appendix contains the proofs omitted in the main text.

LEMMA 1. *Suppose V is an integrable semimartingale satisfying (3), for some $T > 0$. Then, for all stopping times τ_1, τ_2 bounded by T,*

$$V_{\tau_1} = E\left[\int_{\tau_1}^{\tau_2} f(c_s, V_s)\, ds + V_{\tau_2} \,\middle|\, \mathcal{F}_{\tau_1}\right], \quad \text{on } \{\tau_1 < \tau_2\}.$$

PROOF. Equation (3) implies

$$\int_0^t f(c_s, V_s)\, ds + V_t = E\left[\int_0^T f(c_s, V_s)\, ds + V_T \,\middle|\, \mathcal{F}_t\right],$$

which is a martingale. Doob's Optional Stopping Theorem then allows us to replace t by τ_1 or τ_2 in the above equation and hence in (3). Therefore,

$$V_{\tau_1} = E\left[\int_{\tau_1}^{T} f(c_s, V_s)\, ds + V_T \,\middle|\, \mathcal{F}_{\tau_1}\right],$$

and using the fact that $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$,

$$E\left[V_{\tau_2} \,\middle|\, \mathcal{F}_{\tau_1}\right] = E\left[\int_{\tau_2}^{T} f(c_s, V_s)\, ds + V_T \,\middle|\, \mathcal{F}_{\tau_1}\right].$$

Subtracting the last equation from the second to last, the result follows.

QED.

LEMMA 2. *There exist constants K_1 and K_2 such that for all $t \leq T$,*

$$|V_t^T| \leq K_1 + K_2 e^{\nu t} E\left[\int_t^T \|c_s\| e^{-\nu s}\, ds \,\middle|\, \mathcal{F}_t\right].$$

PROOF. Fix t and define the stopping times:

$$\tau^+ = \inf\{s : s \geq t, V_s^T \leq 0\} \quad \text{and} \quad \tau^- = \inf\{s : s \geq t, V_s^T \geq 0\},$$

with the convention that $\inf \emptyset = \infty$. Let τ represent either τ^+ or τ^-. Since $\tau \leq T$, Lemma 1 implies that, for all $u \in [t, T]$,

$$V_u^T 1_{\{\tau > u\}} = E\left[\left(\int_u^\tau f(c_s, V_s^T)\, ds + V_\tau^T \right) 1_{\{\tau > u\}} \,\middle|\, \mathcal{F}_u \right],$$

and since $V_{\tau^+}^T 1_{\{\tau^+ > u\}} \leq 0$,

$$V_u^T 1_{\{\tau^+ > u\}} \leq E\left[\int_u^T f(c_s, V_s^T) 1_{\{\tau^+ > s\}}\, ds \,\middle|\, \mathcal{F}_u \right].$$

The sector condition on f then implies that, for all $u \in [t, T]$,

$$V_u^T 1_{\{\tau^+ > u\}} \leq E\left[\int_u^T \left(|f(c_s, 0)| - \nu V_s^T \right) 1_{\{\tau^+ > s\}}\, ds \,\middle|\, \mathcal{F}_u \right].$$

Similarly, for all $u \in [t, T]$,

$$-V_u^T 1_{\{\tau^- > u\}} \leq E\left[\int_u^T \left(|f(c_s, 0)| - \nu(-V_s^T) \right) 1_{\{\tau^- > s\}}\, ds \,\middle|\, \mathcal{F}_u \right].$$

The stochastic version of the Gronwall-Bellman inequality (stated in the appendix) gives

$$V_t^T 1_{\{\tau^+ > t\}} \leq E\left[\int_t^T e^{-\nu(s-t)} |f(c_s, 0)| 1_{\{\tau^+ > s\}}\, ds \,\middle|\, \mathcal{F}_t \right],$$

and

$$-V_t^T 1_{\{\tau^- > t\}} \leq E\left[\int_t^T e^{-\nu(s-t)} |f(c_s, 0)| 1_{\{\tau^- > s\}}\, ds \,\middle|\, \mathcal{F}_t \right].$$

Adding the last two inequalities and noting that $\tau^+ = \tau^- = 0 \Rightarrow V_t^T = 0$, we conclude that

$$|V_t^T| \leq E\left[\int_t^T e^{-\nu(s-t)} |f(c_s, 0)|\, ds \,\middle|\, \mathcal{F}_t \right].$$

The result follows by the growth condition on f.

QED.

THEOREM 2 (CONTINUITY). *The utility function $U : \mathcal{D}_\nu \to I\!R$ is continuous provided f is continuous.*

PROOF. Suppose $c, \hat{c} \in \mathcal{D}_\nu$ and let V and $\hat{V}$ be the respective associated utility processes. Fix t and define the stopping times: $\tau = \inf\{s : s \geq t,\ V_s \leq \hat{V}_s\}$ and $\tau_N = \tau \wedge N$. Lemma 1 and Assumption 2 imply that, for all $u \in [t, T]$,

$$\begin{aligned}
(V_u - \hat{V}_u)1_{\{\tau > u\}} =& E\left[\int_u^{\tau_N} \left(f(c_s, V_s) - f(\hat{c}_s, \hat{V}_s)\right) 1_{\{\tau > u\}}\, ds + (V_{\tau_N} - \hat{V}_{\tau_N})1_{\{\tau > u\}} \,\middle|\, \mathcal{F}_u\right] \\
\leq& E\left[\int_u^{N} \left(f(c_s, V_s) - f(\hat{c}_s, V_s) + f(\hat{c}_s, V_s) - f(\hat{c}_s, \hat{V}_s)\right) 1_{\{\tau > s\}}\, ds \,\middle|\, \mathcal{F}_u\right] \\
&+ E\left[(V_N - \hat{V}_N)1_{\{\tau > N\}} \,\middle|\, \mathcal{F}_u\right] \\
\leq& E\left[\int_u^{N} \left(|f(c_s, V_s) - f(\hat{c}_s, V_s)| - \nu(V_s - \hat{V}_s)1_{\{\tau > s\}}\right) ds \,\middle|\, \mathcal{F}_u\right] \\
&+ E\left[(V_N - \hat{V}_N)1_{\{\tau > N\}} \,\middle|\, \mathcal{F}_u\right].
\end{aligned}$$

By the stochastic Gronwall-Bellman inequality, applied on $[t, T]$,

$$\begin{aligned}
(V_t - \hat{V}_t)^+ \leq& e^{-\nu(N-t)} E\left[|V_N| + |\hat{V}_N| \,\middle|\, \mathcal{F}_t\right] \\
&+ E\left[\int_t^N e^{-\nu(N-s)} |f(c_s, V_s) - f(\hat{c}_s, V_s)|\, ds \,\middle|\, \mathcal{F}_t\right].
\end{aligned}$$

Therefore, by symmetry,

$$\begin{aligned}
|V_t - \hat{V}_t| \leq& 2e^{-\nu(N-t)} E\left[|V_N| + |\hat{V}_N| \,\middle|\, \mathcal{F}_t\right] \\
&+ 2E\left[\int_t^N e^{-\nu(N-s)} |f(c_s, V_s) - f(\hat{c}_s, V_s)|\, ds \,\middle|\, \mathcal{F}_t\right].
\end{aligned} \tag{6}$$

Suppose now that $c^n \to c$ in $\mathcal{D}_\nu$. By Jensen's inequality, this implies that $\|c^n - c\|^2$ converges[2] to zero in $L^1(\Omega \times [0, \infty), \mathcal{F} \otimes \mathcal{B}[0, \infty), \mu)$, where μ is the product measure defined by $\mu(A \times B) = E\left(\int_B e^{-\nu t} 1_A\, dt\right)$. It follows that, given any subsequence $(c^{n'_k}; k = 1, 2, \ldots)$, there is a further subsequence $(c^{n''_k}; k = 1, 2, \ldots)$

[2] Recall that an optional process is necessarily progressively measurable.

that converges to c, μ-$a.e.$ as $k \to \infty$. By continuity of f, $|f(c_t, V_t) - f(c_t^{n''_k}, V_t)| \to 0$ as $k \to \infty$, μ-$a.e.$ Noting that

$$\begin{aligned}|f(c_t, V_t) - f(c_t^{n''_k}, V_t)| &\leq |f(c_t, V_t) - f(c_t, 0)| + |f(c_t, 0)| + |f(c_t^{n''_k}, 0)| \\ &\quad + |f(c_t^{n''_k}, V_t) - f(c_t^{n''_k}, 0)| \\ &\leq 2\max(k, -\nu)|V_t| + 2k_1 + k_2(\|c_t\| + \|c_t^{n''_k}\|),\end{aligned}$$

and using Lemma 2, the reader can verify that the dominated convergence theorem allows us to conclude that for every $N > 0$,

$$E\left[\int_0^N e^{-\nu(N-s)}|f(c_s, V_s) - f(c_s^{n''_k}, V_s)|\, ds\right] \to 0 \quad as \quad k \to \infty.$$

Let $V^{n''_k}$ be the utility process corresponding to $c^{n''_k}$. Then given $\varepsilon > 0$, the above result and (6), applied at $t = 0$, imply that there exists K such that for all $k > K$,

$$|V_0 - V_0^{n''_k}| \leq 2e^{-\nu N} E\left[|V_N| + |V_N^{n''_k}|\right] + \varepsilon.$$

Letting $N \to \infty$ and using (4), it follows that $|V_0 - V_0^{n''_k}| \leq \varepsilon$ for all $k > K$.

We have shown that every subsequence of $\{U(c^n) - U(c)\}$ has a further subsequence converging to zero. Therefore, the original sequence also converges to zero.

QED.

APPENDIX II

We state the Stochastic Gronwall-Bellman Inequality. A proof can be found in the appendix of Duffie and Epstein (1989).

THE STOCHASTIC GRONWALL-BELLMAN INEQUALITY. *Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space whose filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ satisfies the usual conditions. Suppose $\{X_s\}$ and $\{Y_s\}$ are optional integrable processes and α is a constant. Suppose, for all t, that $s \mapsto E\left[Y_s \mid \mathcal{F}_t\right]$ is continuous almost surely. If, for all t, $Y_t \leq E\left[\int_t^T (X_s + \alpha Y_s)\, ds \mid \mathcal{F}_t\right] + Y_T$, then, for all t,*

$$Y_t \leq e^{\alpha(T-t)} E\left[Y_T \mid \mathcal{F}_t\right] + E\left[\int_t^T e^{\alpha(s-t)} X_s\, ds \,\middle|\, \mathcal{F}_t\right] \quad \text{a.s.}$$

The same result holds if the sense of the above inequalities is reversed.

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