An Isomorphism Between Asset Pricing Models With and Without Linear Habit Formation

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We show an isomorphism between optimal portfolio selection or competitive equilibrium models with utilities incorporating linear habit formation, and corresponding models without habit formation. The isomorphism can be used to mechanically transform known solutions not involving habit formation to corresponding solutions with habit formation. For example, the Constantinides (1990) and Ingersoll (1992) solutions are mechanically obtained from the familiar Merton solutions for the additive utility case, without recourse to a Bellman equation or first-order conditions. More generally, recent solutions to portfolio selection problems with recursive utility and a stochastic investment opportunity set are readily transformed to novel solutions of corresponding problems with utility that combines recursivity with habit formation. The methodology also applies in the context of Hindy–Huang–Kreps (1992) preferences, where our isomorphism shows that the solution obtained by Hindy and Huang (1993) can be mechanically transformed to Dybvig's (1995) solution to the optimal consumption-investment problem with consumption ratcheting.

This article presents a general method for solving asset pricing or portfolio selection models involving linear habit formation of the type studied by Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1991), Ingersoll (1992), Chapman (1998), and others. The basic idea we pursue is that linear habit formation can be thought of as a redefinition of what constitutes consumption, to include not only the current consumption rate but also a fictitious (possibly negative) consumption rate derived from past actual consumption. By pricing out this fictitious consumption rate correctly, the economy with linear habit formation can be mechanically transformed to an equivalent economy without habit formation. This analysis simplifies and unifies existing results, but also generates novel solutions. For example, together with the results of Schroder and Skiadas (1997, 1999), our method produces optimal lifetime consumption and portfolio policies with preferences that combine recursive utility with habit formation under a stochastic

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investment opportunity set. In a related application, we will show a formal equivalence between the problem studied by Hindy and Huang (1993), involving local substitutability of consumption, and the problem studied by Dybvig (1995), involving ratcheting of consumption.

To illustrate the article's application, we now focus on a setting less general than that treated in the main part of the article. We consider a single agent with finite horizon $[0, T]$, continuous (Brownian) information, and initial wealth w, who can trade in a complete securities market. Security prices in this market are determined by a short-rate process, r, and a market-price-of-risk process, η, both of which can be stochastic and time varying. Let σ_t be the time t (possibly stochastic) volatility matrix of the instantaneous time t risky asset returns. We assume that σ_t is square and invertible. The drift term of the instantaneous excess return process is given by the process $\sigma\eta$. The agent's utility function takes the form $U(c) = \widehat{U}(\hat{c})$, where c is any state- and time-contingent consumption plan, and $\hat{c}$ is defined as

$$\hat{c} = c - bx, \quad \text{where} \quad x_t = \int_0^t e^{-a(t-s)} c_s \, ds + e^{-at} x_0,$$

for some positive constants a, b, and x_0. (In the main article we also allow for a negative b, corresponding to durability in consumption.) $\widehat{U}$ is any other utility function over consumption plans. It can, for example, take an additive form, or a more general recursive form, as defined in continuous time by Duffie and Epstein (1992). We are interested in finding the optimal consumption-portfolio plan for this agent. We will refer to the above agent and market as primal.

Next we consider a dual agent with utility function $\widehat{U}$ trading in a dual market, defined as follows. Let $P(t, s)$ be the time t price of a unit discount bond that matures at time s, and let $\theta(t, s)$ be the volatility of the instantaneous return of such a bond (all in the primal market). We define prices in the dual market in terms of the processes:

$$\rho_t = b \int_t^T e^{(b-a)(s-t)} P(t, s) \, ds,$$

and

$$\phi_t = \frac{b}{(1+\rho_t)} \int_t^T e^{(b-a)(s-t)} P(t, s) \theta(t, s) \, ds.$$

Security prices in the dual market are completely determined by the dual short-rate process $\hat{r} \equiv (r - a\rho)/(1+\rho) + b$ and the dual market-price-of-risk process $\hat{\eta} \equiv \eta - \phi$. The dual risky-asset price dynamics are defined to have the same volatility process σ, and therefore the drift term of instantaneous excess returns in the dual market is given by the process $\sigma\hat{\eta}$. The dual agent has utility function $\widehat{U}$, initial wealth $\hat{w} \equiv (w + x_0)/(1+\rho_0) - x_0$, and trades in the dual market just described. Notice that if $\widehat{U}$ is additive or recursive, the dual agent's problem involves no habit formation.

Suppose now that we have solved the dual agent's problem and we have found that the optimal consumption plan is $\hat{c}$, the associated wealth process is $\widehat{W}$, and the corresponding optimal trading strategy is $\hat{\psi}$, where $\hat{\psi}_t$ represents a vector of time t value proportions invested in the risky assets (the remainder being invested at the short rate). The corresponding quantities for the primal agent are then given as follows. The optimal consumption plan is

$$c = \hat{c} + b\hat{x}, \quad \text{where} \quad \hat{x}_t \equiv \int_0^t e^{-(a-b)(t-s)}\hat{c}_s\,ds + e^{-(a-b)t}x_0.$$

The corresponding wealth process for the primal agent is $W = (\widehat{W} + \hat{x})(1+\rho) - x$, and the consumption rule dictating the consumption rate as a proportion of wealth is given by

$$\frac{c}{W} = \frac{\hat{c}}{\widehat{W}}\left(\frac{1-\rho z}{1+\rho}\right) + bz, \quad \text{where} \quad z \equiv \frac{x}{W}.$$

Finally, the optimal trading strategy for the primal agent is

$$\psi = (1-\rho z)\hat{\psi} + (1+z)(\sigma')^{-1}\phi.$$

The above method is useful if the dual problem is easier to solve than the primal problem, or if the dual problem solution is already known. In the above setting with additive $\widehat{U}$, the dual problem is well understood from the articles of Merton (1971), Karatzas, Lehoczy, and Shreve (1987), Cox and Huang (1989), and others. If $\widehat{U}$ is stochastic differential utility (that is, continuous-time recursive utility), a general solution method to the dual problem is given by Schroder and Skiadas (1997, 1999), including essentially closed-form solutions for a homothetic recursive specification that generalizes additive HARA utility and is a continuous-time version of the CES recursive specification considered by Epstein and Zin (1989).[1] These solutions transform via the formulas of this article to solutions for a recursive utility specification that includes a linear habit term.

While we have focused on the demand problem so far, the above isomorphism readily extends to a competitive equilibrium isomorphism, provided that all agents have the same habit parameters (a, b). As an example of this interpretation, consider the constant investment opportunity set, infinite-horizon setting of Constantinides (1990). In this case, the isomorphism takes a particularly simple form because the process ρ is a deterministic constant and the process ϕ vanishes. It follows that all security prices are identical in

[1] Fisher and Gilles (1999) independently derived heuristic derivations of the solution for the homothetic CES specification.

the primal and dual economies. In other words, the economy considered by Constantinides (1990) is isomorphic to a Lucas (1978)-type economy with a modified endowment process. Moreover, the reduction to an economy with additive utilities allows us to apply familiar demand aggregation results. A related approach appears in Eichenbaum, Hansen, and Richard (1987) in the context of equilibrium with durable goods.

An analogous analysis as the one outlined above also applies with Hindy, Huang, and Kreps (1992) preferences over cumulative consumption plans. In this case, the dual agent's utility is over consumption-rate plans. The nonnegativity restriction on consumption in the Hindy and Huang (1993) formulation transforms to a dual constraint stating that the consumption rate cannot decline faster than a given constant rate, which is exactly the constraint considered by Dybvig (1995). As a consequence of this analysis, we will show that the problem solved by Dybvig (1995) is isomorphic to a version of the problem solved by Hindy and Huang (1993), and as a result the Dybvig solution can be mechanically obtained from the Hindy–Huang solution, and vice versa.[2]

Although the above applications are cast in a complete markets setting with continuous information, the underlying idea is considerably more robust, and applies with any filtration, price dynamics, and trading constraints, as well as multiple internal or external habits and/or durabilities in consumption. The main part of this article is cast in the context of a general filtration (not necessarily Brownian) and complete markets, but several other extensions are also outlined. The main limitation of the approach is that it hinges on the linearity of the habit process and hence does not apply to nonlinear versions considered by Detemple and Zapatero (1991), Haug (1998), and others.

In the following section we present the main results in the context of complete Arrow–Debreu markets with a single habit/durability process, but arbitrary information filtration, for example, allowing jumps in security prices. The specialization to the case of Brownian information is presented in Section 2. Section 3 presents another variation of the basic isomorphism that applies to the Hindy–Huang and Dybvig problems. Finally, Section 4 outlines some extensions, including multiple internal or external habits, and the case of incomplete markets. Proofs are collected in the appendix.

1. A Duality Result for Arrow–Debreu Markets

We begin with an Arrow–Debreu market setting, a context that makes the article's main conclusions transparent and general as far as price dynamics.

[2] Cuoco and Liu (2000) solve a model that includes the Hindy and Huang and Dybvig models as special cases. They do not show, however, the isomorphism of the two problems we discuss in this article. Moreover, the isomorphism we present applies quite robustly to more general (non-Brownian) price dynamics.

The underlying filtration in this section is general and is, for example, consistent with jumps in security prices. Incomplete markets are discussed in Section 4.

Our results will be presented in the form of a duality. The primal problem, which we are interested in solving, is the one that involves habit formation. We will then define a dual problem that does not involve habit formation, and is presumably easier to solve or its solution is already known. Finally, we will show how to transform a solution of the dual problem to a solution of the primal problem. Although we will adhere to the interpretational convention that the primal problem involves habit formation and the dual does not, we will also see that the resulting duality is completely symmetric and the roles of the primal and dual markets can be interchanged.

1.1 Probabilistic setting and notation

Given is a probability space $(\Omega, \mathcal{F}, P)$, a finite time horizon $[0, T]$, and a filtration $\{\mathcal{F}_t\colon t \in [0, T]\}$ (satisfying the usual technical conditions of right continuity and P-completeness, which can be safely ignored for the rest of the article). All stochastic processes introduced in this article will be assumed progressively measurable with respect to this filtration. We also assume that $\mathcal{F}_0$ is the trivial σ-algebra generated by the null events, and that $\mathcal{F}_T = \mathcal{F}$. The conditional expectation operator $E[\cdot \mid \mathcal{F}_t]$ will be abbreviated to E_t throughout (with $E = E_0$). The symbol t will always represent a time in $[0, T]$.

We let D denote the Hilbert space of progressively measurable processes of the form $x\colon \Omega \times [0, T] \to \mathbb{R}$ satisfying $E(\int_0^T x_t^2\, dt) < \infty$, with inner product

$$\langle x, y\rangle = E\left[\int_0^T x_t y_t\, dt\right], \quad x, y \in D.$$

As usual, we identify any two processes $x, x' \in D$ such that $\langle x - x', x - x'\rangle = 0$, and any two random variables whose difference has zero variance. Statements like $x_t = y_t$, where $x, y \in D$, and t is an unspecified time parameter, should be interpreted to mean that the processes x and y are equal as elements of D. D is ordered by $\leq$, where $x \leq y$ is equivalent to $\int_0^T P[x_t > y_t]\, dt = 0$. The positive cone of D is denoted D_+, while the set D_{++} consists of all $x \in D$ such that $\int_0^T P[x_t \leq 0]\, dt = 0$.

Below we introduce two Arrow–Debreu markets that we will refer to as the primal and dual markets, respectively. In terms of notation, for every object X of a certain type, we are going to define the dual object $\widehat{X}$. This hat operator will have a distinct definition for each object type, which in this section can be a consumption plan, utility function, state price density, or a parameter. To avoid possible confusion, it is important that the reader keeps such type associations in mind throughout our discussion.

1.2 Agents

An *agent* is a pair (U, e), where $U: D \to \mathbb{R} \cup \{-\infty\}$ is the agent's *utility function*, and $e \in D$ is the agent's *endowment process*. By allowing the utility function to take the value minus infinity, consumption plan constraints in an agent's optimization problem such as nonnegativity can be incorporated in the utility. We place no restrictions on the functional form of U.

Given agent (U, e), we now define the *dual agent*, $(\widehat{U}, \hat{e})$, in terms of the following *parameters:* a real number x_0, and a pair of bounded (and progressively measurable) processes (a, b). The dual of a consumption plan $c \in D$ is defined by

$$\hat{c}_t = c_t - b_t x_t(c), \quad \text{where} \quad x_t(c) = \int_0^t e^{-\int_s^t a_u\,du} c_s\,ds + e^{-\int_0^t a_u\,du} x_0. \tag{1}$$

Remark. *The results of Sections* 1 *and* 2 *apply with the more general specification*

$$x_t(c) = M_t + \int_0^t e^{-\int_s^t a_u\,du} c_s\,ds + e^{-\int_0^t a_u\,du} x_0,$$

where M is some (square-integrable) martingale with $M_0 = 0$. We leave the straightforward extension of the proofs to include such a term to the reader.

Introducing the dual parameters

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} a - b \\ -b \end{pmatrix} \iff \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \hat{a} - \hat{b} \\ -\hat{b} \end{pmatrix},$$

Proposition 1 shows that Equation (1) is equivalent to the symmetric expression

$$c_t = \hat{c}_t - \hat{b}_t \hat{x}_t(\hat{c}), \quad \text{where} \quad \hat{x}_t(\hat{c}) = \int_0^t e^{-\int_s^t \hat{a}_u\,du} \hat{c}_s\,ds + e^{-\int_0^t \hat{a}_u\,du} x_0.$$

Proposition 1. *For all $c \in D$, $x(c) = \hat{x}(\hat{c})$.*

In terms of economic intuition, we think of $x(c)$ as the level of a stock that, in the primal economy, depreciates at a (proportional) rate given by the process a and is replenished at a rate given by the process c. We will use the term "habit stock" for x to emphasize this interpretation. Proposition 1 states that the habit stock process is common in the primal and dual economies.

The *dual utility function* $\widehat{U}: D \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$U(c) = \widehat{U}(\hat{c}), \quad c \in D.$$

The endowment process, e, is a consumption plan, and its dual, $\hat{e}$, is also defined by Equation (1) (with $c = e$), completing the specification of the dual agent $(\widehat{U}, \hat{e})$.

In a typical application, one would take $\widehat{U}$ to have a separable expected utility form, or a recursive utility form, as defined by Duffie and Epstein (1992) in continuous time (see Example 1 in Section 2.3), in which case U can be interpreted as a utility with habit formation. A constraint of the form $c \in K$ (e.g., $c \geq 0$) in the primal agent's problem maps to the constraint $\hat{c} - \hat{b}\hat{x}(\hat{c}) \in K$ in the dual agent's problem. By letting $\widehat{U}(\hat{c}) = -\infty$ for all $\hat{c}$ violating this constraint, the results that follow apply. In practice, a positivity constraint in (primal) consumption is usually finessed by imposing an Inada-type condition on $\widehat{U}$ (infinite marginal utility near zero) with $\widehat{U}(\hat{c}) = -\infty$ for $\hat{c}$ nonpositive. This forces a positive dual optimal consumption plan $\hat{c}$, which along with a positivity assumption on b and x_0 implies the positivity of the primal consumption plan c.

While it is important to keep the above applications in mind, it should be emphasized that the duality results that follow place no restrictions whatsoever on the functional form of $\widehat{U}$. Moreover, the duality results are formally entirely symmetric, meaning that the roles of primal and dual quantities can be interchanged.

1.3 State prices and wealth processes

We assume that markets are complete; that is, every consumption plan is marketed, both in the primal and dual markets. Extensions to the incomplete market case are discussed in Section 4. A *state price density* is a strictly positive element of D. [For background on state pricing, see, e.g., Duffie (1996) for the Brownian case, and Back (1990) for more general processes.]

We fix a state price density π, and its dual $\hat{\pi}$, which will be defined below. Given any cash flow $\delta \in D$, the corresponding primal and dual wealth process are

$$W_t(\delta) = \frac{1}{\pi_t} E_t\left[\int_t^T \pi_s \delta_s \, ds\right] \quad \text{and} \quad \widehat{W}_t(\delta) = \frac{1}{\hat{\pi}_t} E_t\left[\int_t^T \hat{\pi}_s \delta_s \, ds\right].$$

It is well known that, given any consumption plan c and endowment e, we can think of $W_t(c-e)$ as the time t value of a portfolio that finances consumption plan c. In particular, $\pi_0^{-1}\langle c-e, \pi\rangle = W_0(c-e)$ is the minimal initial wealth required to finance c. Similarly, if $\delta \in D$ represents the dividend stream of a security with zero terminal price, then $W(\delta)$ is the market price process of that security. The above statements refer to the primal market, but of course the analogous results hold in the dual market.

The relationship between primal and dual state prices is given in terms of the auxiliary processes:

$$h_t = e^{-\int_0^t a_u - b_u \, du} \quad \text{and} \quad \rho_t = \frac{1}{\pi_t h_t} E_t\left[\int_t^T \pi_s b_s h_s \, ds\right].$$

The random variable $\rho_t h_t$ can be thought of as the time t market price of a consol with dividend rate process bh and zero terminal value.[3] (Equivalently, ρ is the price process of a consol with dividend rate process b and zero terminal value in a market with state price density πh.) If (a, b), and hence h, are deterministic, we can also write

$$\rho_t = \frac{1}{h_t} \int_t^T P(t, s) b_s h_s \, ds \qquad \text{(for deterministic } a, b), \tag{2}$$

where $P(t, s) = E_t[\pi_s]/\pi_t$ is the time t price of a unit discount bond with maturity $s \geq t$.

The *dual state price density*, $\hat{\pi}$, is defined as

$$\hat{\pi}_t = \pi_t(1 + \rho_t). \tag{3}$$

In order for $\hat{\pi}$ to be a state price density, the following condition will be assumed throughout:

Standing Assumption 1. $\hat{\pi} \in D_{++}$

The strict positivity of $\hat{\pi}$ is equivalent to the condition: for all t, $\rho_t > -1$ a.s., which is always satisfied if b is positive, since then ρ is also positive. The requirement that $\hat{\pi} \in D$ is implied by the condition that ρ be bounded, which is in turn implied by the existence of a bounded short rate process, as defined in Section 1.6.

Defining the dual processes

$$\hat{h}_t = e^{-\int_0^t \hat{a}_u - \hat{b}_u \, du} \quad \text{and} \quad \hat{\rho}_t = \frac{1}{\hat{\pi}_t \hat{h}_t} E_t\left[\int_t^T \hat{\pi}_s \hat{b}_s \hat{h}_s \, ds\right],$$

we have the following symmetry result:

Proposition 2. *For all* t, $(1 + \rho_t)(1 + \hat{\rho}_t) = 1$.

Therefore Equation (3) can be inverted to obtain the symmetric expression

$$\pi_t = \hat{\pi}_t(1 + \hat{\rho}_t).$$

Finally, the dual and primal wealth processes are related as follows (where, by Proposition 1, $x(c) = \hat{x}(\hat{c})$):

Proposition 3. *For all* $c \in D$, $\pi[x(c) + W(c)] = \hat{\pi}[\hat{x}(\hat{c}) + \widehat{W}(\hat{c})]$.

[3] Analogous quantities appear in other articles, for example, Detemple and Zapatero (1991) and Rallis (2000). The fundamental difference is that in all of those articles the quantity is derived in terms of first-order conditions of optimality derived from a specific utility form, which does not include, for example, recursive forms such as the one studied in Example 1. Our transformation is independent of any notion of optimality; it corresponds merely to a redefinition of consumption.

Using the habit stock level $x(c)$ as the numeraire, we can think of $x(c)+W(c)$ as the value of the habit stock plus the present value of future contributions to habit stock, and analogously for the dual economy. The relative price of habit stock in the primal and dual economies is given by $\hat{\pi}/\pi$. In those terms, Proposition 3 follows from the fact that habit stock is common to the primal and dual economies (see Proposition 1). We will use this result below to show the equivalence of the primal and dual-agent consumption problems.

1.4 Optimality

A consumption plan $c^* \in D$ is *optimal* for agent (U, e) given state price density $\pi \in D_{++}$ if

$$c^* \in \arg\max\{U(c) : \langle \pi, c-e\rangle \leq 0,\ \ c \in D\}.$$

(As noted earlier, even though we have not restricted consumption to be nonnegative in the agent's problem, such a constraint is implied if we assume that $U(c) = -\infty$ for all $c \notin D_+$.)

Theorem 1. *A consumption plan $c \in D$ is optimal for agent (U, e) given state price density π if and only if $\hat{c}$ is optimal for agent $(\widehat{U}, \hat{e})$ given $\hat{\pi}$.*

Proof. For any $c \in D$, Proposition 3 implies $\langle \pi, c\rangle + \pi_0 x_0 = \langle \hat{\pi}, \hat{c}\rangle + \hat{\pi}_0 x_0$ and $\langle \pi, e\rangle + \pi_0 x_0 = \langle \hat{\pi}, \hat{e}\rangle + \hat{\pi}_0 x_0$. Substracting the second equation from the first one, we obtain $\langle \hat{\pi}, \hat{c}-\hat{e}\rangle = \langle \pi, c-e\rangle$. Therefore c is feasible for the primal problem if and only if $\hat{c}$ is feasible for the dual problem. Since $U(c) = \widehat{U}(\hat{c})$, the theorem follows. ■

Theorem 1 shows that to compute an optimal consumption plan, c, in the primal market one can compute an optimal consumption plan, $\hat{c}$, in the dual market, and then let $c = \hat{c} - \hat{b}\hat{x}(\hat{c})$. In applications, we are often interested in the optimal consumption to wealth ratio. A simple calculation using Proposition 3 shows that, for every $c \in D$, we have

$$\frac{c_t}{W_t(c)} = \frac{\hat{c}_t}{\widehat{W}_t(\hat{c})}\left(\frac{1-\rho_t z_t(c)}{1+\rho_t}\right) + b_t z_t(c), \tag{4a}$$

where

$$z_t(c) = \frac{x_t(c)}{W_t(c)}. \tag{4b}$$

The construction of a trading strategy that finances a given optimal consumption plan follows standard hedging arguments and will not be discussed here. We will, however, indicate in Section 2 how a financing strategy for $\hat{c}$ in a (properly defined) dual securities market can be transformed directly to a financing strategy for c in the primal market. For notational simplicity, we will do that in a Brownian setting, but the arguments extend to a setting in which security prices are semimartingales (and can, for example, include jumps).

1.5 Equilibrium

So far we have expressed the basic duality result in terms of a single agent's problem. Because of the linearity of the mapping of consumption plans to their dual, the duality extends readily to equilibrium settings. We illustrate with a standard Arrow–Debreu exchange economy that can be implemented as a complete securities market.

We consider an (exchange) economy, $\mathscr{E}$, with agents (U^i, e^i), $i \in \{1, \ldots, I\}$. We let $e = \sum_i e^i$ denote the aggregate endowment process. An allocation, $(c^1, \ldots, c^I)$, is any element of D^I, and is *feasible* if $\sum_i c^i \leq e$. An *Arrow–Debreu equilibrium* of $\mathscr{E}$ is a feasible allocation $(c^1, \ldots, c^I)$, and a state price density $\pi \in D_{++}$, such that c^i is optimal for agent (U^i, e^i) given π, for all $i \in \{1, \ldots, I\}$.

To define the dual economy, we assume that the parameters (a, b) are common to all agents, and we also fix a profile of initial habit process values $(x_0^1, \ldots, x_0^I) \in \mathbb{R}^I$. The dual economy, $\widehat{\mathscr{E}}$, consists of the I dual agents, $(\widehat{U}^i, \hat{e}^i)$, $i \in \{1, \ldots, I\}$, defined as for the single-agent case above (where the habit process $\hat{x}^i$ for agent i has initial value x_0^i). An equilibrium of the dual economy is defined analogously with the primal economy. As a consequence of Theorem 1, we have

Theorem 2. *Assuming common habit parameters, the feasible allocation and state price density $((c^1, \ldots, c^I), \pi)$ is an Arrow–Debreu equilibrium for the economy $\mathscr{E}$ if and only if $((\hat{c}^1, \ldots, \hat{c}^I), \hat{\pi})$ is an Arrow–Debreu equilibrium for the dual economy $\widehat{\mathscr{E}}$.*

The implementation of an Arrow–Debreu equilibrium as a securities market is presented by Duffie and Huang (1985) and Duffie (1996). That construction can be combined with Theorem 2 (as well as the results of Section 2) to extend this article's duality to securities market equilibria. Moreover, Theorem 2 implies that if the dual economy admits a representative agent endowed with the aggregate endowment in the economy, then so does the primal economy.

1.6 Short rate and risk premia duality

In asset pricing applications, state prices are often expressed in terms of short rates and instantaneous risk premia. In this subsection we indicate how these quantities in the primal market are related to their dual counterparts.

The process r is a *short rate process for the price density* π if the process ξ, defined as

$$\xi_t = \frac{\pi_t}{\pi_0} \exp\left(\int_0^t r_s \, ds \right), \tag{5}$$

is a martingale. In differential form, Equation (5) is equivalent to

$$\frac{d\pi_t}{\pi_{t-}} = -r_t \, dt + \frac{d\xi_t}{\xi_{t-}}.$$

(In particular, the martingale property of ξ implies that π must be a special semimartingale, and therefore r and ξ are uniquely defined.) It is well known [see Back (1990) for a general formulation] that, in a market with state price density π, if r is a short-rate process for π, then the martingale ξ determines instantaneous risk premia relative to r. We therefore refer to ξ as the *risk premia process for the price density* π.[4]

Proposition 4. *Suppose that the processes r and $\hat{r}$ satisfy*

$$\pi_t(a_t + r_t) = \hat{\pi}_t(\hat{a}_t + \hat{r}_t), \quad t \in [0, T].$$

Then r is a short rate process for the state price density π, if and only if $\hat{r}$ is a short rate process for the state price density $\hat{\pi}$.

Proposition 4 implies that a short rate process exists in the primal economy if and only if it exists in the dual economy. The exact relationship between short rates in the two economies can be understood in terms of "leasing" of the habit stock. We think of the habit stock as consisting of a durable good that can either be bought or leased. The corresponding cost-of-carry is represented by the process $a + r$, since leasing, as opposed to purchasing, saves both depreciation and financing costs. Using $x(c)$ as the numeraire, it follows that $a + r$ represents the lease rate of the habit stock in the primal economy. Applying the same argument in the dual economy and using the fact that relative habit stock prices in the two economies are given by $\hat{\pi}/\pi$, the equation of Proposition 4 becomes another expression of the fact that habit stock is common in the primal and dual economies, and therefore so are the corresponding (fictitious) leasing prices.

The following assumption will apply for the remainder of the article (and is always satisfied under a suitable choice of a numeraire).

Standing Assumption 2. *The short rate process for the price density π exists, and is denoted r. The corresponding risk premia process is denoted ξ.*

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[4] Another well-known interpretation of the martingale ξ is in terms of an equivalent martingale measure (EMM), Q, defined by the density (Radon–Nikodym derivative) $dQ/dP = \xi_T$. Since, by the martingale property, $\xi_t = E_t[dQ/dP]$, the pair (r, Q) uniquely specifies the state price density, π, and for every $\delta \in D$, $W_t(\delta) = E_t^Q[\int_t^T e^{-\int_t^s r_u \, du} \delta_s \, ds]$.

A simple calculation, using Proposition 4, shows that $\hat{r}$ and $\hat{\xi}$, defined by

$$\hat{r}_t = \frac{r_t - a_t \rho_t}{1+\rho_t} + b_t, \quad \hat{\xi}_t = \xi_t \frac{1+\rho_t}{1+\rho_0} \exp\left(\int_0^t \hat{r}_u - r_u \, du \right), \tag{6}$$

are, respectively, a short rate and risk premia process for $\hat{\pi}$. Because of symmetry, one can also write by inspection the dual expressions, giving r and ξ in terms of $\hat{r}$ and $\hat{\xi}$.

The following result summarizes two important simplifications of short rate and risk premia duality:

Proposition 5. *(a) If (a, b) and r are deterministic processes, then $\hat{\xi} = \xi$.*
(b) Suppose that $T = \infty$, that (a, b) and r are constants (deterministic and constant over time), and that $r > b - a$. Then $\hat{r} = r$, $\hat{\xi} = \xi$, and $\widehat{W}(\delta) = W(\delta)$ for all $\delta \in D$.

Part (*b*) of Proposition 5 includes as a special case the setting of Constantinides (1990), showing that all security prices in that model are the same in the primal and dual economies. The implication is that if one wants to test the validity of the Constantinides model, one can instead test a Lucas-type model without habit formation after appropriately transforming the consumption data to correspond to the dual problem's definition of consumption.

2. The Case of Continuous Information

This section discusses the important special case of continuous information, modeled by a Brownian filtration. The following assumption will be adopted throughout this section:

Standing Assumption 3. *The underlying filtration, $\{\mathcal{F}_t\colon t \in [0, T]\}$, is the (augmented) filtration generated by a n-dimensional standard Brownian motion, $B = (B^1, \ldots, B^n)'$.*

2.1 Market price of risk

A process, η, is a market price of risk process for the state price density π if

$$\frac{d\xi_t}{\xi_t} = -\eta_t' \, dB_t.$$

The martingale representation theorem implies the existence of a (unique) market price of risk for π under weak integrability restrictions [see Karatzas and Shreve (1988, 1998)]. For example, it is sufficient that r be bounded, since in that case $\xi \in D$. From now on we will directly assume:

Standing Assumption 4. *η is the (unique) market price of risk process for π.*

Proposition 6. *There exists a (progressively measurable) process* ϕ *such that*

$$\frac{d\rho_t}{1+\rho_t} = (r_t - \hat{r}_t + \eta_t'\phi_t)\,dt + \phi_t'\,dB_t. \tag{7}$$

Moreover, $\hat{\eta} = \eta - \phi$ *is the market price of risk process for* $\hat{\pi}$*, that is,*

$$\frac{d\hat{\xi}_t}{\hat{\xi}_t} = -\hat{\eta}_t'\,dB_t.$$

We henceforth assume that ϕ is defined by Equation (7). We will see below that the process ϕ plays an important role in relating trading strategies in the primal and dual markets. For deterministic (a, b), an expression for ϕ can be obtained in terms of the discount bond prices $P(t, s) = \pi_t^{-1}E_t[\pi_s]$. We first recall that Ito's lemma implies

$$\frac{dP(t,s)}{P(t,s)} = (r_t + \eta_t'\theta(t,s))\,dt + \theta(t,s)'\,dB_t,$$

for some process $\{\theta(t, s) : t \le s\}$.

Proposition 7. *Suppose that* $E(\int_0^T \int_0^T P^2(t,s)\theta(t,s)'\theta(t,s)\,ds\,dt) < \infty$, *and that* (a, b) *are deterministic. Then*

$$\eta_t - \hat{\eta}_t = \phi_t = \frac{1}{h_t(1+\rho_t)}\int_t^T \theta(t,s)P(t,s)b_s h_s\,ds.$$

2.2 Trading strategies

In order to discuss trading strategies, we now introduce securities markets that implement the primal and dual Arrow–Debreu markets we have considered so far.

The *primal securities market* consists of short-term default-free borrowing and lending, at a rate given by the process r, and trading in n risky securities, one for each component of the Brownian motion B. The risky security instantaneous excess returns (relative to r) are represented by the n-dimensional Ito process $R_t = [R_t^1, \ldots, R_t^n]'$, with Ito decomposition $dR_t = \mu_t\,dt + \sigma_t\,dB_t$, where μ and σ are progressively measurable processes valued in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively, and satisfy $\int_0^T (|\mu_t| + \sigma_t(\sigma_t)')\,dt < \infty$ a.s. We assume that σ_t is invertible almost everywhere, and therefore the market price of risk satisfies $\eta_t = \sigma_t^{-1}\mu_t$.[5]

A *trading strategy* is any progressively measurable process, ψ, valued in $\mathbb{R}^n$, such that $\int_0^t (|\psi_s'\mu_s| + \psi_s'\sigma_s(\sigma_s)'\psi_s)\,ds < \infty$ a.s. for all t. We interpret

[5] Sufficient conditions on the primitives μ, σ, and r in order for η to define a martingale risk-premia process, ξ, are discussed in detail by Karatzas and Shreve (1998). It is sufficient, for example, that η satisfies the Novikov condition.

ψ_t^i as the time t proportion of wealth invested in security $i \in \{1, \dots, n\}$, the remaining wealth being invested at the short rate r. Given a consumption plan c, we say that the trading strategy ψ finances c in the primal market if

$$dW_t(c) = W_t(c)(\psi_t' dR_t + r_t\, dt) - c_t\, dt.$$

Note that, for every given pair (ψ, c) and initial wealth level, the above equation determines a wealth process that must equal $W(c)$ as defined in Section 1.3. The above definition therefore bypasses some standard arguments equating the wealth process generated by the budget equation to the wealth process obtained as a present value of future consumption. [For general expositions see Duffie (1996) and Karatzas and Shreve (1998). For a discussion of models not covered in the above definition, see Loewenstein and Willard (1999) and the references therein.]

The *dual securities market* is defined as the primal market, except that the short rate process is $\hat{r}$ and the risky asset instantaneous excess returns (relative to $\hat{r}$), $\widehat{R}$, have the Ito decomposition

$$d\widehat{R}_t = \sigma_t \hat{\eta}_t\, dt + \sigma_t\, dB_t. \tag{8}$$

This ensures that the market price of risk in the dual market is $\hat{\eta}$. The trading strategy $\hat{\psi}$ finances $\hat{c}$ in the dual market if

$$d\widehat{W}_t(\hat{c}) = \widehat{W}_t(\hat{c})\big(\hat{\psi}_t' d\widehat{R}_t + \hat{r}_t\, dt\big) - \hat{c}_t\, dt.$$

The following result relates trading strategies in the primal and dual markets. For its statement we recall the definitions of the processes $\phi = \eta - \hat{\eta}$ (see Propositions 6 and 7) and $z(c)$ [see Equation (4b)].

Proposition 8. *Suppose that $c \in D$ and the trading strategies ψ and $\hat{\psi}$ are related by*

$$\psi_t = (1 - \rho_t z_t(c))\hat{\psi}_t + (1 + z_t(c))(\sigma_t')^{-1}\phi_t. \tag{9}$$

Then ψ finances c in the primal market if and only if $\hat{\psi}$ finances $\hat{c}$ in the dual market. If (a, b) and r are deterministic, then $\phi = 0$.

It is worth noting that the portfolio $(\sigma_t')^{-1}\phi_t$ appearing in the second term of Equation (9) can be written as the difference of the portfolio $(\sigma_t\sigma_t')^{-1}\mu_t$, which is instantaneously mean-variance efficient in the primal market, and $(\sigma_t\sigma_t')^{-1}\hat{\mu}_t$ which is instantaneously mean-variance efficient in the dual market. Both of the last two portfolios are optimal for an agent with time-additive logarithmic utility in their respective market.

2.3 Optimal trading strategies

In our context, markets are complete; that is, every consumption plan can be financed by some trading strategy [see Duffie (1996) or Karatzas and Shreve (1998)]. Given this fact, we can define a pair of a consumption plan and a trading strategy (c, ψ) to be optimal for agent (U, e) in the primal market if c is optimal for (U, e) and ψ finances c in the primal market.[6] Optimality in the dual market is defined analogously. Putting together our earlier results, we can now summarize this section's main conclusion:

Theorem 3. *Suppose that $c \in D$ and the trading strategies ψ and $\hat{\psi}$ are related by Equation* (9) (*where $\phi = 0$ for deterministic (a, b) and r*). *Then (c, ψ) is optimal for agent (U, e) in the primal market if and only if $(\hat{c}, \hat{\psi})$ is optimal for agent $(\widehat{U}, \hat{e})$ in the dual market. The optimal consumption-to-wealth ratios in the dual and primal markets are related by Equation* (4).

While we have focused on the problem of a single agent, Theorem 2 suggests an equilibrium version of Theorem 3, which is left to the reader [see Duffie and Huang (1985) or Duffie (1996) for the implementation of Arrow–Debreu equilibria as securities markets].

Example 1. Recursive utility with habit formation

Using Theorem 3 and the results in Schroder and Skiadas (1997, 1999), hereafter S-S, we are now in a position to give a complete solution to the optimal consumption-portfolio problem for a utility specification that combines a recursive homothetic specification with linear habit formation. The parametric recursive form we consider is a continuous-time version of the CES specification considered by Epstein and Zin (1989), and nests the utility specification used by Constantinides (1990). The utility functional form was derived by Duffie and Epstein (1992), while existence and basic properties are proved in S-S. The solution given below applies for a general (not necessarily Markovian) stochastic investment opportunity set.

The utility function is defined as $U(c) = \widehat{U}(\hat{c}) = \widehat{V}_0(\hat{c})$, where

$$\widehat{V}_t(\hat{c}) = \begin{cases} E_t\left[\displaystyle\int_t^T e^{-\beta(s-t)}\left((\hat{c}_s^\gamma/\gamma)\,ds + (\alpha/2)\widehat{V}(\hat{c})_s^{-1}d[\widehat{V}(\hat{c})]_s\right)\right], & \text{if } \gamma \neq 0; \\ E_t\left[\displaystyle\int_t^T e^{-\beta(s-t)}\left(\log(\hat{c}_s)\,ds + (\alpha/2)d[\widehat{V}(\hat{c})]_s\right)\right], & \text{if } \gamma = 0, \end{cases}$$

[6] This characterization of optimality forms the basis of the approach to solving the lifetime consumption/investment problem introduced by Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989) [as opposed to Merton's (1971) dynamic programming approach] for the case of additive preferences. The method was extended to settings with nonadditive preferences by Detemple and Zapatero (1991) for the case of habit formation, and by Schroder and Skiadas (1997, 1999) for the case of recursive preferences [see also Duffie and Skiadas (1994)].

with parameter restrictions:

$$\beta \geq 0 \quad \text{and} \quad \begin{cases} \alpha > -1, \text{ and } \gamma < \min\{1, (1+\alpha)^{-1}\}, & \text{if } \gamma \neq 0; \\ \alpha \leq \beta, & \text{if } \gamma = 0. \end{cases}$$

The notation $[\widehat{V}(\hat{c})]$ denotes the quadratic variation of $\widehat{V}(\hat{c})$. Note that the familiar additive power utility specification is obtained for $\alpha = 0$.

For technical reasons, we assume that the agent considers only consumption plans, c, such that $\hat{c}$ is strictly positive and $E(\int_0^T \hat{c}_t^l \, dt) < \infty$ for every real number l. For such consumption plans we show in S-S that the above utility is well defined (in a sense, stated rigorously in S-S). To embed this restriction into this article's setting, we simply assume that the utility takes the value minus infinity for all consumption plans violating the above constraints (which are not binding at the optimum). We also assume that $\hat{r}$ and $\hat{\eta}$ are bounded, although possibly stochastic and time varying.

The optimal solution will be stated in terms of the processes

$$k_t = \begin{cases} (1-\gamma(1+\alpha))^{-1}, & \text{if } \gamma \neq 0; \\ \beta[\beta - \alpha(1-e^{-\beta(T-t)})]^{-1}, & \text{if } \gamma = 0, \end{cases}$$

$$p_t = \hat{r}_t + \frac{k_t}{2}\hat{\eta}_t'\hat{\eta}_t, \quad q_t = \frac{\beta - \gamma p_t}{1-\gamma},$$

and

$$\widetilde{B}_t = B_t + \int_0^t (1-k_s)\hat{\eta}_s \, ds.$$

By Girsanov's theorem, $\widetilde{B}$ is n-dimensional standard Brownian motion under a properly defined probability $\widetilde{P}$ that is equivalent to P [see Equation (12) of S-S]. We let $\widetilde{E}$ denote the expectation operator with respect to $\widetilde{P}$.

Finally, we introduce the auxiliary process pair (Y, Z) as an adapted solution to the backward SDE:[7]

$$dY_t = \begin{cases} -[(1+\alpha)^{\gamma/(1-\gamma)} - q_t Y_t + (\alpha k_t Z_t' Z_t)/(2Y_t)]\, dt + Z_t' \, d\widetilde{B}_t, & \text{if } \gamma \neq 0; \\ -[(1-k_t)(\beta - p_t) + k_t(\alpha - \beta)Y_t + Z_t' Z_t/2]\, dt + Z_t' \, d\widetilde{B}_t, & \text{if } \gamma = 0, \end{cases}$$

with terminal condition $Y_T = Y_{T-} = 0$. As shown in S-S, (Y, Z) is uniquely defined by the above recursion (in a properly defined space), and if $\gamma \neq 0$, Y is strictly positive. Alternative characterizations of the process Y are obtained

[7] The process (Y, Z) corresponds to the process $\gamma(J, Z)$ in S-S if $\gamma \neq 0$, and to the process (J, Z) if $\gamma = 0$.

by applying Theorems A1 and A2 of S-S. If $\gamma \neq 0$, then $k_t = k$ is a constant, $1 + \alpha k > 0$, and

$$\frac{Y_t^{1+\alpha k}}{1+\alpha k} = (1+\alpha)^{\gamma/(1-\gamma)} \widetilde{E}_t\left[\int_t^T e^{-\int_t^s (1+\alpha k) q_u du} Y_s^{\alpha k}\, ds\right] \quad (\gamma \neq 0).$$

The above expression simplifies significantly for $\alpha = 0$, corresponding to a time-additive power dual utility function:

$$Y_t = \widetilde{E}_t\left[\int_t^T e^{-\int_t^s q_u\, du}\, ds\right], \quad (\alpha = 0,\ \gamma \neq 0).$$

For $\gamma = 0$ (and possibly nonzero α), Y solves

$$e^{Y_t} = \widetilde{E}_t\left[\int_t^T \exp\left(\int_t^s [(1-k_u)(\beta - p_u) + k_u(\alpha - \beta) Y_u]\, du\right) ds\right] \quad (\gamma = 0).$$

In Markovian settings the pair (Y, Z) is characterized in terms of a partial differential equation, as explained in S-S.

Theorems 3 and 4 of S-S, together with Theorem 1 and Proposition 3, imply that the optimal consumption-to-wealth ratio for the primal agent is given by Equation (4), where the dual optimal consumption-to-wealth ratio is

$$\frac{\hat{c}_t}{\widehat{W}_t(\hat{c})} = \begin{cases} (1+\alpha)^{\gamma/(1-\gamma)} Y_t^{-1}, & \text{if } \gamma \neq 0; \\ \beta(1 - e^{-\beta(T-t)})^{-1}, & \text{if } \gamma = 0. \end{cases}$$

The dynamics for $\hat{c}$ are given in closed form [in terms of (Y, Z)] by Theorems 3 and 4 of S-S applied to the dual agent and market.

Finally, combining the solutions in S-S with Propositions 6 and 8, we obtain that the optimal trading strategy for the primal agent is

$$\psi_t = (\sigma_t')^{-1}\big[(1 - \rho_t z_t(c))(k_t \eta_t + K_t Z_t) + (1 - k_t + (1 + \rho_t k_t) z_t(c)) \phi_t\big],$$

where

$$K_t = \begin{cases} (1 + \alpha k_t)/Y_t, & \text{if } \gamma \neq 0; \\ 1, & \text{if } \gamma = 0. \end{cases}$$

The above solutions simplify if r and (a, b) are deterministic, in which case ϕ vanishes. Moreover, if η is also deterministic, then Z vanishes as well.

This example's solutions are extended to the case of general homothetic recursive preferences and incomplete markets in Schroder and Skiadas (2002).

3. Hindy–Huang–Kreps Preferences

This section establishes an equivalence between the model of Dybvig (1995), in which an agent maximizes utility over consumption rate processes, but under a constraint on the rate of decline of consumption,[8] and the model of Hindy and Huang (1993), whose utility specification over cumulative consumption plans captures a notion of local substitutability of consumption. This equivalence is shown as a consequence of a variant of the isomorphism we have developed so far that applies considerably more generally than the Hindy–Huang and Dybvig articles.

While Dybvig considers an infinite horizon, we choose to present the finite-horizon case, because the finite-horizon version of Dybvig's problem that preservers the isomorphism with the finite-horizon Hindy–Huang problem may not be immediately obvious. The infinite-horizon extension amounts to a straightforward exercise of letting the terminal date stretch to infinity, and simplifies both the problem statement and its solution.

3.1 Consumption spaces and agents

We consider the probabilistic setting of Section 1.1. In particular, the underlying filtration is general, and the assumptions of Section 2 are not required until the discussion of trading strategies in Section 3.3 below.

Fixing a bounded nonnegative process a throughout, we let D^a be the space containing any progressively measurable and right-continuous process x satisfying $x_0 \geq 0$, $x_t \geq e^{-\int_s^t a_u\,du} x_s$ for all $t > s$, and $E(x_T^2) < \infty$. Similarly, D^0 denotes the space consisting of any progressively measurable, right-continuous, and nondecreasing process x satisfying $x_0 \geq 0$ and $E(x_T^2) < \infty$. As a matter of interpretation, we think of elements of D^0 as cumulative consumption plans, as in the Hindy–Huang model, and elements of D^a as consumption-rate plans whose decline rate is bounded by a, as in Dybvig's model. Given any times $s \leq t$, we adopt the notational conventions: $\int_{s-}^t = \int_{[s,t]}$ and $\int_s^t = \int_{(s,t]}$.

Given any $c \in D^0$, we define the process $\tilde{c}$ as

$$\tilde{c}_t = \int_{0-}^t e^{-\int_s^t a_u\,du} dc_s + x_{0-} e^{-\int_0^t a_u\,du},$$

where $x_{0-} \in [0, \infty)$ is a constant that is fixed throughout the section. For every $c \in D^0$ we adopt the notational conventions $c_{0-} = 0$ and $\tilde{c}_{0-} = x_{0-}$, implying $\tilde{c}_0 = c_0 + x_{0-}$.

Hindy and Huang discuss a durable goods interpretation of their model that provides useful insight to our results below. We think of increments in c as additions to a capital stock which at time t has level $\tilde{c}_t$ and depreciates at the rate a_t. In those terms, it should be clear that the capital stock rate of change will not fall below the depreciation rate if and only if capital

[8] As discussed in Problem 2 of Dybvig (1995), this is equivalent, after a change of variables, to a problem in which the agent is intolerant of any decline in the consumption rate.

stock is only added and not removed. This intuition supports the following result:

Proposition 9. *$c \in D^0$ if and only if $\tilde{c} \in D^a$.*

We take as primitive an *agent*, formally a pair (U, e) where U: $D^0 \to \mathbb{R} \cup \{-\infty\}$ is the agent's *utility function* and $e \in D^0$ is the agent's *endowment*. We assume that the utility function $\widetilde{U}$: $D^a \to \mathbb{R} \cup \{-\infty\}$ is such that

$$U(c) = \widetilde{U}(\tilde{c}), \quad c \in D^0,$$

and we refer to the pair $(\widetilde{U}, \tilde{e})$ as the *dual agent*.

In the Hindy–Huang and Dybvig applications, we interpret (U, e) as the agent in the Hindy–Huang model (e representing a cumulative endowment process), and $(\widetilde{U}, \tilde{e})$ as the agent in the Dybvig model ($\tilde{e}$ representing an endowment rate process). In this case, $\widetilde{U}$ is taken to be a standard additive utility function. In a more general example, $\widetilde{U}$ can be assumed to have a recursive form [as in Duffie and Epstein (1992)] that is not necessarily time additive.

3.2 Prices, wealth, and optimality

The primal agent can trade in a complete market with an Arrow–Debreu state price density π and corresponding short rate process r (see Section 1.6). We assume throughout that r is bounded and that $E(\sup_t \pi_t^2) < \infty$.[9] We also assume that $a + r$ is strictly positive, and we define the dual state price density process

$$\tilde{\pi} = \pi(a + r).$$

In terms of the durable good interpretation, we think of π as being a state price density process pricing increments to capital stock, while $\tilde{\pi}$ is a state price density process pricing leasing of capital stock. Leasing saves both the financing cost of purchasing and depreciation of the capital stock. The lease rate is therefore the product of this cost of carry (short rate plus depreciation rate) and the purchase price.

The wealth process associated with a consumption plan is defined differently for the primal and dual markets. In the primal market, $c \in D^0$ represents a cumulative consumption plan. We therefore define

$$W_t(c) = \frac{1}{\pi_t} E_t\left[\int_t^T \pi_s \, dc_s\right], \quad W_{0-}(c) = c_0 + W_0(c), \quad c \in D^0.$$

[9] Since r is bounded, the integrability restriction on π is equivalent to ξ being a square-integrable martingale. In the Brownian case, a coarse sufficient condition is that η be bounded. Also note that while we introduced π as an element of D, we can no longer identify any two state price densities π and π' such that $\langle \pi - \pi', \pi - \pi' \rangle = 0$. Instead, π and π' must have almost surely identical paths.

Note that $W_t(c)$ represents wealth right after time t consumption. In the dual market on the other hand, $\tilde{c} \in D^a$ represents a consumption-rate process. We define the dual wealth processes as

$$\widetilde{W}_t(\tilde{c}) = \frac{1}{\tilde{\pi}_t} E_t\left[\int_t^T \tilde{\pi}_s \tilde{c}_s \, ds + \tilde{\pi}_T \frac{\tilde{c}_T}{a_T + r_T}\right], \quad \tilde{c} \in D^a.$$

The last term in the above wealth expression is unconventional, and stems from the problem formulation of the dual agent. In particular, we assume that the dual agent is constrained to choose a consumption rate process whose decline rate is bounded by a, and moreover the terminal wealth is at least equal to $\tilde{c}_T/(a_T + r_T)$. The latter quantity is the present value of a perpetual consumption-rate process with initial value $\tilde{c}_T$ and declining at the constant rate a_T, using r_T as a discount rate. (In an infinite-horizon version of this formulation, we would postulate that the last term of the wealth expression vanishes as T approaches infinity for every admissible consumption process c.)

Returning to the durable good interpretation, and taking time t capital stock as the numeraire, $W_t(c)$ represents the time t present value of future increments to capital stock. The expression $(\tilde{\pi}_t/\pi_t)\widetilde{W}_t(\tilde{c})$, on the other hand, represents the time t present value of leasing from t to T, then buying the amount $\tilde{c}_T$ at T (hence the unconventional last term in the dual wealth expression). Proposition 10 proves the intuitive result that the present value of purchasing, given by $W_t(c) + \tilde{c}_t$, equals the present value of leasing up to T and then buying:

Proposition 10. *For all* $c \in D^0$, $W_t(c) + \tilde{c}_t = \widetilde{W}_t(\tilde{c})(a_t + r_t)$, $t \geq 0$.

A cumulative-consumption plan $c \in D^0$ is optimal for the agent (U, e) in the primal market if $W_{0-}(c) \leq W_{0-}(e)$, and there exists no $c' \in D^0$ such that $W_{0-}(c') \leq W_{0-}(e)$ and $U(c') > U(c)$. Similarly, a consumption-rate plan $\tilde{c} \in D^a$ is optimal for agent $(\widetilde{U}, \tilde{e})$ in the dual market if $\widetilde{W}_0(\tilde{c}) \leq \widetilde{W}_0(\tilde{e})$, and there exists no $c' \in D^a$ such that $\widetilde{W}_0(c') \leq \widetilde{W}_0(e)$ and $\widetilde{U}(c') > \widetilde{U}(\tilde{c})$.

This section's central conclusion follows.

Theorem 4. *For every* $c \in D^0$, *c is optimal for agent (U, e) in the primal market if and only if $\tilde{c}$ is optimal for agent $(\widetilde{U}, \tilde{e})$ in the dual market.*

Proof. By definition, $U(c) = \widetilde{U}(\tilde{c})$, and by Proposition 9, $c \in D^0 \Leftrightarrow \tilde{c} \in D^a$. It therefore suffices to show the equivalence of the budget constraints. By Proposition 10, we have

$$W_{0-}(c) = W_0(c) + c_0 = \widetilde{W}_0(\tilde{c})(a_0 + r_0) - \tilde{c}_0 + c_0 = \widetilde{W}_0(\tilde{c})(a_0 + r_0) - x_{0-}.$$

Subtracting the analogous expression with e in place of c, we obtain the desired result: $W_{0-}(c) \leq W_{0-}(e) \Leftrightarrow \widetilde{W}_0(\tilde{c}) \leq \widetilde{W}_0(\tilde{e})$. ■

An equilibrium version of the above theorem is left to the reader.

3.3 Trading strategies

To extend this section's isomorphism to include trading strategies, we now consider (for simplicity) the Brownian information setting of Section 2. The primal securities market in which agent (U, e) trades is as in Section 2.2, except that the budget equation must now be modified to reflect the fact that c represents a cumulative-consumption plan rather than a consumption-rate plan. We say that trading strategy ψ finances $c \in D^0$ in the primal market if

$$dW_t(c) = W_t(c)(\psi_t' dR_t + r_t\, dt) - dc_t.$$

As in Section 2.2, this definition bypasses standard arguments equating wealth processes in securities and corresponding Arrow–Debreu markets.

We assume that $a + r$ is an Ito process, and we define the dual securities market as in Section 2, with the short rate $\tilde{r}$ and the market price of risk $\tilde{\eta}$ being obtained from the Ito decomposition:

$$\frac{d\tilde{\pi}_t}{\tilde{\pi}_t} = -\tilde{r}_t\, dt - \tilde{\eta}_t\, dB_t.$$

The excess return dynamics in the dual market are therefore

$$d\widetilde{R}_t = \sigma_t \tilde{\eta}_t\, dt + \sigma_t\, dB_t.$$

A trading strategy $\tilde{\psi}$ finances $\tilde{c}$ in the dual market if

$$d\widetilde{W}_t(\tilde{c}) = \widetilde{W}_t(\tilde{c})(\tilde{\psi}_t' d\widetilde{R}_t + \tilde{r}_t\, dt) - \tilde{c}_t\, dt.$$

Proposition 11. *Suppose $c \in D^0$ and the trading strategies ψ and $\tilde{\psi}$ are related by*

$$\psi_t = \left(1 + \frac{\tilde{c}_t}{W_t(c)}\right)\big(\tilde{\psi}_t + (\sigma_t')^{-1}(\eta_t - \tilde{\eta}_t)\big). \tag{10}$$

Then ψ finances c in the primal market if and only if $\tilde{\psi}$ finances $\tilde{c}$ in the dual market.

Equation (10) simplifies if a and r are deterministic constants, in which case the short rate and risk premia processes are identical in the primal and dual economies, and the last term of Equation (10) is therefore eliminated.

As in Section 2, market completeness allows us to define a consumption plan and trading strategy pair (c, ψ) to be optimal for agent (U, e) in the primal market if c is optimal for (U, e) and ψ finances c in the primal market. Optimality in the dual market is defined analogously.

As a corollary of Theorem 4 and Proposition 11, we have

Theorem 5. *Suppose $c \in D^0$ and the trading strategies ψ and $\tilde{\psi}$ are related by Equation* (10). *Then (c, ψ) is optimal for agent (U, e) in the primal market if and only if $(\tilde{c}, \tilde{\psi})$ is optimal for agent $(\widetilde{U}, \tilde{e})$ in the dual market.*

Example 2. The Hindy–Huang–Dybvig application
Dybvig (1995) considers the portfolio/consumption problem which amounts to an infinite-horizon version of the dual agent's problem of this section, with the following simplifications: $\widetilde{U}$ is a discounted time-additive expected utility of the HARA type, a and r are deterministic constants, and therefore $\tilde{r} = r$ and $\tilde{\eta} = \eta$. The primal problem in this case corresponds to a problem solved by Hindy and Huang (1993). The closed-form solutions for the infinite-horizon problems with a constant investment opportunity set derived by Hindy and Huang (1993) and Dybvig (1995) can therefore be mechanically transformed to each other, as we now outline.

The optimal consumption policy derived by Hindy and Huang is to consume only when the wealth to weighted average consumption ratio, $W(c)/\tilde{c}$, hits some upper barrier u, and to consume only as much as is needed to prevent the ratio from exceeding the barrier. The optimal consumption policy derived by Dybvig is the same type of reflecting barrier policy imposed on the wealth to consumption ratio, $\widetilde{W}(\tilde{c})/\tilde{c}$, but with upper barrier $\tilde{u} = (a+r)^{-1}(u+1)$. The relationship between u and $\tilde{u}$ is an immediate consequence of (the infinite-horizon version of) Proposition 10.

The dollar investments in the risky assets in the Dybvig model are given by the portfolio weights in the Hindy–Huang model times $\widetilde{W} - \tilde{c}/(a+r)$, which has the interpretation as the wealth in excess of the declining (at rate a) perpetuity value of current consumption.[10] This is an immediate consequence of (the infinite-horizon version of) Theorem 5 (with $\eta = \hat{\eta}$).

New solutions to the Dybvig problem in a stochastic investment opportunity setting can be obtained applying our isomorphism to the results of Bank and Riedel (1999). They extend the Hindy–Huang analysis and show, in a complete-markets, stochastic investment opportunity setting that the optimal consumption policy is obtained by reflecting the weighted-average past consumption process, $\tilde{c}$, on a stochastic lower bound. The policy is solved in closed form in a homogeneous setting (infinite horizon and a state price density characterized by the exponential of a Lévy process) with time-additive power utility. We leave the details of this exercise to the interested reader.

4. Extensions

4.1 Multiple habits
In a straightforward extension of Sections 1 and 2 we can allow the process x to be k-dimensional, for any positive integer k. (This framework includes the case of a habit process x that follows a higher-order linear differential

[10] It is easy to verify Equation (10) after observing that Dybvig's parameter R^* (after modifying the other parameters according to his Problem 2 in Section 1) is identical to one minus the parameter α^* of Hindy and Huang. To reconcile the optimal consumption policies, we use the above relationship between u and $\tilde{u}$. Note that $(r^*)^{-1}$ is Dybvig's notation for our u, and k^* is the Hindy–Huang notation for our $\tilde{u}/a$.

equation, by using the standard trick of reducing ODEs to first-order ODEs of higher dimension.)

Let a be a bounded progressively measurable process valued in $\mathbb{R}^{k\times k}$, the space of all $k\times k$ matrices. Also let b be a bounded progressively measurable process valued in $\mathbb{R}^k$, and let g denote a k-dimensional constant vector. Given $x_0\in\mathbb{R}^k$, we define the dual consumption plan

$$\hat{c}_t = c_t - b_t' x_t(c), \quad \text{where} \quad dx_t(c) + a_t x_t(c)\,dt = g c_t\,dt, \quad x_0(c) = x_0. \tag{11a}$$

For example, Ingersoll (1992) obtains a closed-form solution for a linear two-dimensional habit formation problem with diagonal a and a time-additive HARA utility function. The solution for $x(c)$ is $x_t(c) = \hat{h}_t x_0 + \hat{h}_t \int_0^t \hat{h}_s^{-1} g c_s\,ds$, where $\hat{h}_t$ is the $\mathbb{R}^{k\times k}$-valued process satisfying $d\hat{h}_t = -a_t \hat{h}_t \times dt$, $\hat{h}_0 = I$ (where I denotes the identity matrix). For example, if a is either diagonal or a constant, then $\hat{h}_t = \exp(-\int_0^t a_s\,ds)$. Defining the dual parameters $\hat{a} = a - gb'$ and $\hat{b} = -b$, we get the symmetric expression

$$c_t = \hat{c}_t - \hat{b}_t' \hat{x}_t(\hat{c}), \quad \text{where} \quad d\hat{x}_t(\hat{c}) + \hat{a}_t \hat{x}_t(\hat{c})\,dt = g\hat{c}_t\,dt, \quad \hat{x}_0(\hat{c}) = x_0, \tag{11b}$$

with a solution $\hat{x}_t(\hat{c}) = h_t x_0 + h_t \int_0^t h_s^{-1} g\hat{c}_s\,ds$, where h is the $\mathbb{R}^{k\times k}$-valued process satisfying $dh_t = -\hat{a}_t h_t\,dt$, $h_0 = I$.

Dual state prices are defined by

$$\hat{\pi}_t = \pi_t(1 + \rho_t' g),$$

where the $\mathbb{R}^k$-valued process ρ is given by

$$\rho_t = \frac{1}{\pi_t}(h_t')^{-1} E_t\left[\int_t^T \pi_s h_s' b_s\,ds\right]. \tag{12}$$

The extensions of the main equations of Sections 1 and 2 follow. Note that $z(c)$ is $\mathbb{R}^k$ valued and ϕ is $\mathbb{R}^{n\times k}$ valued. The expression for ϕ is based of the assumptions of Proposition 7:

$$\begin{aligned}
W_t(\hat{c})\hat{\pi}_t &= [W_t(c) - \rho_t' x_t(c)]\pi_t\\
\frac{c_t}{W_t(c)} &= \frac{\hat{c}_t}{W_t(\hat{c})}\left(\frac{1-\rho_t' z_t(c)}{1+\rho_t' g}\right) + b_t' z_t(c),\\
\hat{r}_t &= \frac{r_t - \rho_t' a_t g}{1+\rho_t' g} + b_t' g, \quad \hat{\xi}_t = \xi_t \frac{1+\rho_t' g}{1+\rho_0' g}\exp\left(\int_0^t \hat{r}_u - r_u\,du\right),\\
\phi_t &= \frac{1}{(1+\rho_t' g)}\left(\int_t^T \theta(t,s)P(t,s)b_s' h_s\,ds\right)h_t^{-1}, \quad \eta_t - \hat{\eta}_t = \phi_t g,\\
\psi_t &= (\sigma_t')^{-1}\phi_t(g + z_t(c) + [z_t(c)g' - g z_t(c)']\rho_t)\\
&\quad + (1-\rho_t' z_t(c))\hat{\psi}_t.
\end{aligned}$$

Theorems 1 and 2 continue to hold [for Theorem 2, the habit parameters (a, b, g) must also be common to all agents], as does Proposition 5 after replacing the part (b) assumption that $r+a-b>0$ with the assumption that all the eigenvalues of $r+a-gb'$ have strictly positive real parts.

4.2 Multiple internal and external habits

In this section we allow the k-dimensional habit process to depend not only on the agent's own past consumption, but on the past consumption of other agents. We use an isomorphism to eliminate both the internal and external habits in the dual problem. With some simplifying assumptions, we can essentially decouple the dual optimization problems. Once we compute dual initial wealth, which is a simple affine function of all the agents' initial wealth, we eliminate all interagent dependence in the dual problem.

Let the primal and dual consumption of the I agents be given by the $\mathbb{R}^I$-valued processes $c, \hat{c} \in D^I$. They are again related by Equations (11a) and (11b), but now g and b_t are valued in $\mathbb{R}^{k\times I}$ (as before, $x_t, \hat{x}_t \in \mathbb{R}^k$ and $a_t \in \mathbb{R}^{k\times k}$). Agent i's utility function in the primal economy is defined in terms of some function $\widehat{U}^i$: $D^I \to \mathbb{R} \cup \{-\infty\}$. The dual problems will decouple under the assumption that $\widehat{U}^i$ depends only on $\hat{c}^i$, the ith element of $\hat{c}$. The primal problem for agent i, with endowment process $e^i \in D$, is to choose $c^i \in D$ to maximize $\widehat{U}^i(\hat{c})$ subject to $\langle \pi, c^i - e^i \rangle \leq 0$, where $\hat{c}$ is defined in Equation (11a).

For the remainder of the section, assume an infinite horizon ($T=\infty$), constant habit parameter matrices a and b, and monotonicity of $\widehat{U}^i$ in $\hat{c}^i$, for all i. We also assume that $\mathbf{I}+\rho' g$ (where $\mathbf{I}$ denotes the identity matrix) is invertible and has strictly positive diagonal elements, and, to ensure that ρ is finite, we assume that all the eigenvalues of $r+a-gb'$ have strictly positive real parts. Under our assumptions, ρ is constant, prices in the primal and dual economies are the same (see Proposition 5b), and, as seen below, dual budget constraints depend only on each agent's own consumption process and other agents' wealth levels.

For any cash flow vector $\delta \in D^I$, define the I-dimensional wealth process as

$$W_t(\delta) = \frac{1}{\pi_t} E_t\left[\int_t^T \pi_s \delta_s \, ds\right].$$

A proof analogous to that of Proposition 3 shows that wealth can be expressed in terms of dual consumption as

$$W_t(c) = (\mathbf{I}+\rho' g) W_t(\hat{c}) + \rho' x_t(c), \tag{13}$$

where $\rho \in \mathbb{R}^{k\times I}$ is again given by Equation (12).

The dual problem for agent i is to choose $\hat{c}^i \in D$ to maximize $\widehat{U}^i(\hat{c})$ subject to the dual budget constraint $\langle \pi, \hat{c}^i - \hat{e}^i \rangle \leq 0$. Equation (13) and the assumption of monotonicity (which implies $\langle \pi, \hat{c}^j - \hat{e}^j \rangle = 0$ for all $j \neq i$)

together imply that Theorem 1 holds for each agent. That is, $c^i \in D$ is optimal for agent (U^i, e^i) if and only if $\hat{c}^i$ is optimal for agent $(\widehat{U}^i, \hat{e}^i)$.

Once each dual problem has been solved, we can recover the optimal primal consumption : wealth ratios and primal portfolio plans. Letting superscript i indicate the ith component of a vector, and superscript $i\cdot$ the ith row of a matrix, the primal and dual consumption to wealth ratios are related by

$$\frac{c_t^i}{W_t^i(c)} = \frac{\hat{c}_t^i}{W_t^i(\hat{c})}[(\mathbf{I}+\rho' g)^{-1}]^{i\cdot}\left(\frac{W_t(c)-\rho' x_t(c)}{W_t^i(c)}\right) + \frac{(b')^{i\cdot} x_t(c)}{W_t^i(c)}.$$

Finally, letting ψ_t^i and $\hat{\psi}_t^i$ denote agent i's ($\mathbb{R}^n$-valued) portfolio weight vectors in the primal and dual economies, respectively, and using superscript ij to indicate the matrix element in the ith row and jth column, we have

$$W_t^i(c)\psi_t^i = \sum_{j=1}^{I} \hat{\psi}_t^j (\mathbf{I}+\rho' g)^{ij}[(\mathbf{I}+\rho' g)^{-1}]^{j\cdot}(W_t(c)-\rho' x_t(c)).$$

The expression simplifies when the dual portfolio weights are identical for all agents (for example, time-additive power dual utility with identical coefficients of relative risk aversion across agents), in which case

$$\psi_t^i = \hat{\psi}[1-(\rho')^{i\cdot} x_t(c)/W_t^i(c)].$$

Theorem 2 applies if agents have common internal and external habit parameters in the following sense. Suppose each agent has K internal habits, with agent i's habit vector process $x^i(c)$ satisfying $dx_t^i(c) + Ax_t^i(c)\,dt = Gc_t^i$, where $A \in \mathbb{R}^{K\times K}$ and $G \in \mathbb{R}^{K\times 1}$. Note that agents share the same depreciation matrix A and consumption weighting vector G. Let agent i's dual consumption be given by $\hat{c}_t^i = c_t^i - F' x_t^i(c) - H' \sum_{i=1}^{I} x_t^i(c)$, where $F, H \in \mathbb{R}^{K\times 1}$. For every agent, the weight vector F is applied to the agent's own habit process, while the weight vector H is applied to other agents' habit processes as well.[11] Given these common habit parameters (but possibly different utility functions), $((c^1, \ldots, c^I), \pi)$ is an Arrow–Debreu equilibrium for the economy $\mathscr{E}$ if and only if $((\hat{c}^1, \ldots, \hat{c}^I), \pi)$ is an Arrow–Debreu equilibrium for the dual economy $\widehat{\mathscr{E}}$.

4.3 Trading restrictions

Theorem 1 extends to include incomplete markets or other trading constraints. The abstract formulation proceeds as follows.

A *market* is a pair (M, π), where $M \subseteq D$ is the *market space*, representing a set of *marketed cash flows*, and $\pi \in D_{++}$ is a *state price density*. We assume

[11] This model can be embedded into our general setting as follows. We stack the habit processes by letting $x(c) = [x^1(c)', \ldots, x^I(c)']'$. Let $\mathbf{1}$ denote I-dimensional vector of ones and $\otimes$ denote the Kronecker product. The habit parameters are then defined as $a = \mathbf{I}\otimes A$, $b = \mathbf{I}\otimes F + \mathbf{1}\mathbf{1}'\otimes H$, and $g = \mathbf{I}\otimes G$.

throughout that $0 \in M$, allowing the agent to not trade. The shape of M depends on market restrictions. For example, in frictionless complete markets $M = D$, while under incomplete markets M would be a linear subspace of D. A consumption plan $c^* \in D$ is *optimal* for agent (U, e) given market (M, π) if

$$c^* \in \arg\max\{U(c) : c = e + m,\ \langle \pi, m \rangle \leq 0,\ m \in M,\ c \in D\}. \tag{14}$$

For every marketed cash flow $m \in M$, the dual cash flow $\hat{m}$ is defined by

$$\hat{m}_t = m_t - b_t \int_0^t e^{-\int_s^t a_u\, du} m_s\, ds.$$

The *dual market space* is the set of dual marketed cash flows: $\widehat{M} = \{\hat{m} : m \in M\}$. By Proposition 1 (applied with $x_0 = 0$ and $c = m$), Equation (14) is equivalent to

$$m_t = \hat{m}_t - \hat{b}_t \int_0^t e^{-\int_s^t \hat{a}_u\, du} \hat{m}_s\, ds.$$

An argument similar to that used for Theorem 1 shows the following extension: *For any consumption plan* $c \in D$, c is *optimal for agent* (U, e) *given market* (M, π) *if and only if* $\hat{c}$ *is optimal for agent* $(\widehat{U}, \hat{e})$ *given market* $(\widehat{M}, \hat{\pi})$.

Of course, the practical use of this result depends on the tractability of the dual problem. In the Brownian case with time-additive $\widehat{U}$, the results in Cvitanić and Karatzas (1992) can be utilized. Assume that $1 - \rho_t z_t(c) > 0$, $t \in [0, T]$ (this is implied if optimal dual consumption is strictly positive, which, in turn, is implied if $\widehat{U}(\hat{c}) = -\infty$ for all $\hat{c} \notin D_{++}$). Constraints on the primal trading strategy of the form $\psi_t \in K_t$ are then equivalent to the dual trading strategy constraints $\hat{\psi}_t \in \widehat{K}_t$, where

$$\widehat{K}_t = \{x \in \mathbb{R}^n : (1 - \rho_t z_t(c))x + (1 + z_t(c))(\sigma_t')^{-1}(\eta - \hat{\eta}) \in K_t\}.$$

(Propositions 1 and 3 and be used to express $\widehat{K}_t$ in terms of dual quantities only.) Moreover, $\widehat{K}_t$ is closed, convex, and nonempty valued if and only if K_t has the same properties. When r and (a, b) are deterministic (which implies $\eta = \hat{\eta}$), and K_t takes values that are cones containing the origin, then $\widehat{K}_t = K_t$ and the conditions discussed in Section 16 of Cvitanić and Karatzas (1992) all hold. Some technical difficulties arise when the assumption of deterministic r and (a, b) is removed. For example, there is generally no uniform lower bound on the family of support functions described in Sections 4 and 16 of Cvitanić and Karatzas. But their Remark 4.2 states that the duality and existence results in Sections 12 and 13, which rely on this assumption, may be established directly in certain cases.

Explicit solutions under incomplete markets, extending Example 1, are presented in Schroder and Skiadas (2001).

5. Conclusion

The main contribution of this article is to show the equivalence, or duality, of certain classes of models that have previously been treated separately in the literature. In some cases this duality relates known solutions, and in other cases it produces novel solutions. The classes of models considered have the central property that consumption enters utility through some linear transformation. The key idea is to redefine consumption, to what we call dual consumption, through such a linear transformation, and then properly define dual Arrow–Debreu prices to correctly price dual consumption. In a linear habit-formation model, the dual consumption is the excess of (primal) consumption over the habit stock, and the dual Arrow–Debreu prices are easily computed in terms of the primal Arrow–Debreu prices and the price of a fictitious consol. Dual expressions are also derived for interest rates, risk premia, and optimal trading strategies. Through this duality, we were able to transform available optimal consumption/portfolio strategies with recursive preferences without habit formation to corresponding solutions with linear habit formation. In the Hindy–Huang model, the dual consumption is the habit stock and the dual problem is that analyzed by Dybvig (1995). The solution of either model mechanically produces a solution to the other. The basic idea behind this article can be applied with arbitrary information structure, price dynamics, and trading constraints (such as incomplete markets). As with any duality result, in some cases the dual problem is easier to solve; in some others it is simply interesting to know the range of models that an available solution applies to.

Appendix: Proofs

Proof of Proposition 1. Substituting $c = \hat{c} + bx(c)$ into Equation (1) implies

$$dx_t(c) + (a_t - b_t)x_t(c)\,dt = \hat{c}_t\,dt, \quad x_0(c) = x_0,$$

which is the same as the ordinary differential equation satisfied by $\hat{x}(\hat{c})$. The uniqueness of the solution implies that $x(c) = \hat{x}(\hat{c})$. ■

Proof of Proposition 2. We define $\mathcal{A}_t = \exp\left(\int_0^t a_u\,du\right)$, $\mathcal{B}_t = \exp\left(\int_0^t b_u\,du\right)$, $y_t = \pi_t/\mathcal{A}_t$, and analogously with hats over all processes. Note that $\widehat{\mathcal{A}}_t = \mathcal{A}_t/\mathcal{B}_t$, $\widehat{\mathcal{B}}_t = \mathcal{B}_t^{-1}$, and $\pi_t h_t = y_t \mathcal{B}_t$. The definition of ρ implies that

$$y_t \mathcal{B}_t \rho_t = -\int_0^t y_s\,d\mathcal{B}_s + E_t\left[\int_0^T y_s\,d\mathcal{B}_s\right].$$

Using integration by parts,[12] we obtain the identity

$$y_t \mathscr{B}_t(1+\rho_t) = y_0 + \int_0^t \mathscr{B}_s\, dy_s + E_t\left[\int_0^T y_s\, d\mathscr{B}_s\right].$$

Since $\hat{\pi}_t = \pi_t(1+\rho_t) \Leftrightarrow \hat{y}_t = y_t\mathscr{B}_t(1+\rho_t)$, it follows that $\hat{\pi}_t = \pi_t(1+\rho_t)$ if and only if

$$d\hat{y}_t = \mathscr{B}_t dy_t + dN_t, \quad \hat{y}_T = y_T\mathscr{B}_T, \tag{15}$$

for some martingale N. By symmetry, the equation $\pi_t = \hat{\pi}_t(1+\hat{\rho}_t)$ is equivalent to

$$dy_t = \widehat{\mathscr{B}}_t\, d\hat{y}_t + d\widehat{N}_t, \quad y_T = \hat{y}_T\widehat{\mathscr{B}}_T, \tag{16}$$

for some martingale $\widehat{N}$. Since $\widehat{\mathscr{B}}_t = \mathscr{B}_t^{-1}$, if we set $d\widehat{N}_t = -\widehat{\mathscr{B}}_t\, dN_t$, Equation (15) holds if and only if Equation (16) does. This proves $\hat{\pi}_t = \pi_t(1+\rho_t) \Leftrightarrow \pi_t = \hat{\pi}_t(1+\hat{\rho}_t)$. ■

Proof of Proposition 3. We substitute

$$x_s(c) = \hat{x}_s(\hat{c}) = h_t^{-1}h_s\hat{x}_t(\hat{c}) + \int_t^s h_u^{-1}h_s\hat{c}_u\, du$$

into

$$\pi_t W_t(c) = E_t\left[\int_t^T \pi_s(\hat{c}_s + b_s x_s(c))\, ds\right],$$

and we use Fubini's theorem, the law of iterated expectations, and the definitions of ρ and $\hat{\pi}$, to obtain

$$\begin{aligned}
\pi_t(x_t(c)+W_t(c)) &= \left(\pi_t + h_t^{-1}E_t\left[\int_t^T \pi_s b_s h_s\, ds\right]\right)\hat{x}_t(\hat{c}) \\
&\quad + E_t\left[\int_t^T \left(\pi_s + h_s^{-1}\left(\int_s^T \pi_u b_u h_u\, du\right)\right)\hat{c}_s\, ds\right] \\
&= \pi_t(1+\rho_t)\hat{x}_t(\hat{c}) + E_t\left[\int_t^T \pi_s(1+\rho_s)\hat{c}_s\, ds\right] \\
&= \hat{\pi}_t(\hat{x}_t(\hat{c}) + \widehat{W}_t(\hat{c})).
\end{aligned}$$

The proof is completed by using the definition of ρ. ■

Proof of Proposition 4. Suppose that r is a short rate process for the primal market, and let LM be short for "some local martingale" (which can be different in each occurrence of the abbreviation). Since $d\pi_t = -r_t\pi_t\, dt + d(LM)_t$, we have

$$d(h_t\pi_t) = h_t\pi_t(b_t - a_t - r_t)\, dt + d(LM)_t.$$

On the other hand, Proposition 2 implies

$$d(h_t\hat{\pi}_t) = d(h_t\pi_t) - b_t h_t \pi_t\, dt + d(LM)_t.$$

[12] A minor technicality here is the meaning of integration with respect to y (and $\hat{y}$), since we have not assumed that y is a semimartingale. That is not an issue, however, since $\mathscr{B}_t\, dy_t$ can be defined through integration by parts. Alternatively, one can easily write down a version of this proof in which the semimartingale $\hat{\pi} - \pi$ is treated as the variable that a backward SDE must be solved for. In applications, we typically assume that $d\pi_t/\pi_{t-} = -r_t\, dt + dN_t$, where r is the short rate process and N is a martingale, implying that π is a (special) semimartingale.

Combining the last two expressions with the proposition's assumption results in

$$d(h_t\hat{\pi}_t) = h_t\hat{\pi}_t(b_t - a_t - \hat{r}_t)\,dt + d(LM)_t.$$

Reversing the first step above, this implies that $\hat{r}$ is a short rate process for the dual market. The converse follows by symmetry. ■

Proof of Proposition 5. (a) Given that (a, b) are deterministic, Equation (2) for ρ applies. Also, if r is deterministic, the discount bond prices $P(t, s)$ are also deterministic. It follows that ρ is absolutely continuous. Applying integration by parts for semimartingales [see Protter (1990)] to Equation (3), it follows that $d\pi_t/\pi_{t-}$ and $d\hat{\pi}_t/\hat{\pi}_{t-}$ have the same martingale part, and therefore $\xi = \hat{\xi}$.

(b) Under the given assumptions, $\rho_t = b(r + a - b)^{-1}$. Equation (3) then implies that $\hat{\pi}$ is proportional to π, and therefore the two state price densities generate the same wealth processes for any given cash flow. The equality of the short rate processes can be inferred by noticing that the bounded variation terms of $d\pi_t/\pi_{t-}$ and $d\hat{\pi}_t/\hat{\pi}_{t-}$ must coincide. ■

Proof of Proposition 6. Recall that $\rho_t h_t$ is the time t market price of a consol with dividend rate process bh. By Ito's lemma

$$d(h_t\rho_t) = (r_t h_t\rho_t - b_t h_t + \eta_t'\nu_t)\,dt + \nu_t'\,dB_t$$

for some process ν. Integration by parts implies

$$d\rho_t = \frac{d(h_t\rho_t)}{h_t} + (a_t - b_t)\rho_t\,dt.$$

Combining these equations and defining

$$\phi_t = \frac{\nu_t}{(1+\rho_t)h_t},$$

we obtain

$$d\rho_t = (\rho_t(a_t + r_t) + (1+\rho_t)(\eta_t'\phi_t - b_t))\,dt + (1+\rho_t)\phi_t'\,dB_t.$$

Equation (7) then follows by using the expression for $\hat{r}$ in Equation (6). Finally, the relationship between η and $\hat{\eta}$ follows by applying integration by parts to the identity $\hat{\pi}_t = (1+\rho_t)\pi_t$. ■

Proof of Proposition 7. Let BV be short for "some bounded variation process" (which can be different in each occurrence of the abbreviation). By Equation (7), $(1+\rho_t)\phi$ is the diffusion coefficient in the Ito expansion for ρ. The result then follows from the following Ito expansion:

$$d\rho_t = h_t^{-1}\left(\int_t^T P(t,s)\theta(t,s)'b_s h_s\,ds\right)dB_t + d(BV)_t.$$

To prove the last expression, we start with Equation (2) and we apply Fubini's theorem [in the form of Theorem 46 in Protter (1990)]:

$$\begin{aligned}
h_t\rho_t &= \int_t^T P(t,s)b_s h_s\,ds \\
&= \int_t^T\left(\int_0^t P(u,s)\theta(u,s)'dB_u\right)b_s h_s\,ds + (BV)_t \\
&= \int_0^t\left(\int_t^T P(u,s)\theta(u,s)'b_s h_s\,ds\right)dB_u + (BV)_t \\
&= \int_0^t\left(\int_u^T P(u,s)\theta(u,s)'b_s h_s\,ds - \int_u^t P(u,s)\theta(u,s)'b_s h_s\,ds\right)dB_u + (BV)_t
\end{aligned}$$

$$= \int_0^t \left(\int_u^T P(u,s)\theta(u,s)' b_s h_s \, ds \right) dB_u$$
$$- \int_0^t \left(\int_0^s P(u,s)\theta(u,s)' dB_u \right) b_s h_s \, ds + (BV)_t.$$

The claimed expression for the martingale part of ρ follows because h and the second expression on the right side of the last equality are of bounded variation. ■

Proof of Proposition 8. Let BV be short for "some bounded variation process" (which can be different in each occurrence of the abbreviation). We also let $W = W(c)$, $\widehat{W} = \widehat{W}(c)$, and $x = x(c) = \hat{x}(\hat{c})$ (see Proposition 1). By Proposition 3, we have

$$\pi_t(W_t + x_t) = \hat{\pi}(\widehat{W}_t + x_t). \tag{17}$$

Using integration by parts, we obtain:

$$\pi_t \, dW_t - \pi_t(W_t + x_t)\eta_t' \, dB_t = \hat{\pi}_t \, d\widehat{W}_t - \hat{\pi}(\widehat{W}_t + x_t)\hat{\eta}_t' \, dB_t + d(BV)_t.$$

Let $z_t = x_t/W_t$. Applying Equation (17) on the right-hand side, dividing by $W_t \pi_t$, and using the fact (from Proposition 3) that $(1+\rho_t)(\widehat{W}_t/W_t) = 1 - \rho_t z_t$, we obtain

$$\frac{dW_t}{W_t} = (1 - \rho_t z_t)\frac{d\widehat{W}_t}{\widehat{W}_t} + (1 + z_t)(\eta_t' - \hat{\eta}_t') \, dB_t + d(BV)_t.$$

On the other hand, the budget equations in the primal and dual markets imply

$$\frac{dW_t}{W_t} = \psi_t'\sigma_t \, dB_t + d(BV)_t \quad \text{and} \quad \frac{d\widehat{W}_t}{\widehat{W}_t} = \hat{\psi}_t'\sigma_t \, dB_t + d(BV)_t.$$

Combining the last three equations, and matching the martingale parts, gives the result. ■

Proof of Proposition 9. Let $\mathcal{A}_t = \exp(\int_0^t a_u \, du)$ and $y_t = \mathcal{A}_t \tilde{c}_t$. From the definition of $\tilde{c}$, we have $dy_t = \mathcal{A}_t \, dc_t$, or equivalently $dc_t = \mathcal{A}_t^{-1} \, dy_t$. Therefore y is nondecreasing if and only if c is nondecreasing. The result is an immediate consequence of this observation and the assumption that a is bounded. ■

Proof of Proposition 10. Integration by parts [see Protter (1992)] and the fact that $\tilde{c}$ is a bounded variation process imply

$$d(\pi_t \tilde{c}_t) = \pi_t \, d\tilde{c}_t + \tilde{c}_{t-} \, d\pi_t = \pi_t \, d\tilde{c}_t - r_t \pi_t \tilde{c}_t \, dt - dM_t,$$

for some local martingale M. Fix any time t, and let $\{\tau_n\}$ be a sequence of stopping times valued in $[t, T)$ and monotonically almost surely converging to T, such that $\{M_{s \wedge \tau_n} : t \le s \le T\}$ is a martingale for every n. Using the definition of $W(c)$, the equation $dc_t = a_t \tilde{c}_t \, dt + d\tilde{c}_t$, and the above observation, we obtain

$$\pi_t(W_t(c) + \tilde{c}_t) = E_t\left[\int_t^{\tau_n} \pi_s a_s \tilde{c}_s \, ds + \int_t^{\tau_n} \pi_s d\tilde{c}_s + \pi_{\tau_n} W_{\tau_n}(c) + \pi_t \tilde{c}_t\right]$$
$$= E_t\left[\int_t^{\tau_n} (a_s + r_s)\pi_s \tilde{c}_s \, ds + \pi_{\tau_n}(W_{\tau_n}(c) + \tilde{c}_{\tau_n})\right]. \tag{18}$$

Letting n approach infinity, and using $\tilde{\pi} = \pi(a + r)$ and the definition of $\widetilde{W}(\tilde{c})$, we claim that

$$\pi_t(W_t(c) + \tilde{c}_t) = E_t\left[\int_t^T (a_s + r_s)\pi_s \tilde{c}_s \, ds + \pi_T \tilde{c}_T\right] = \tilde{\pi}_t \widetilde{W}_t(\tilde{c}),$$

which simplifies to $W_t(c)+\tilde{c}_t = \widetilde{W}_t(c)(a_t+r_t)$. There remains to justify the above limit. The first term of Equation (18) converges to the obvious limit by monotone convergence. Let us consider now the second term of Equation (18). First, we observe that by the martingale convergence theorem (applied to ξ, recalling that r is bounded and $\sup_t \pi_t$ is square-integrable), $\lim_{n\to\infty} \pi_{\tau_n} = \pi_T$ a.s. Using this fact, we have

$$E_t[\pi_{\tau_n} W_{\tau_n}(c)] = E_t\left[\int_{\tau_n}^T \pi_s\, dc_s\right] \longrightarrow E_t[\pi_T(c_T - c_{T-})] \quad \text{as} \quad n \longrightarrow \infty,$$

justified by the dominated convergence theorem (since $\int_{\tau_n}^T \pi_s\, ds \le \int_t^T \pi_s\, dc_s$ a.s.). We also have the limit $E_t[\pi_{\tau_n}\tilde{c}_{\tau_n}] \to E_t[\pi_T \tilde{c}_{T-}]$ as $n \to \infty$, justified again by dominated convergence, since

$$0 \le \pi_{\tau_n}\tilde{c}_{\tau_n} \le \pi_{\tau_n}\tilde{c}_T \exp\left(\int_{\tau_n}^T a_u\, du\right) \le \left(\sup_t \pi_t\right)\tilde{c}_T \exp\left(T \sup_u |a_u|\right).$$

Combining the last two limits, we have shown

$$E_t[\pi_{\tau_n}(W_{\tau_n}(c)+\tilde{c}_t)] \;\longrightarrow\; E_t[\pi_T(c_T - c_{T-} + \tilde{c}_{T-})] = E_t[\pi_T \tilde{c}_T] \quad \text{as} \quad n \longrightarrow \infty,$$

justifying the limit for the second term of Equation (18). ■

Proof of Proposition 11. Let BV stand for "a process of bounded variation," and let $W = W(c)$ and $\widetilde{W} = \widetilde{W}(\tilde{c})$. By Proposition 3, we have

$$\pi_t(W_t + \tilde{c}_t) = \widetilde{W}_t \tilde{\pi}_t. \tag{19}$$

Integration by parts gives

$$\pi_t\, dW_t - (W_t + \tilde{c}_t)\pi_t \eta_t\, dB_t = \tilde{\pi}_t\, d\widetilde{W}_t - \widetilde{W}_t \tilde{\pi}_t \tilde{\eta}_t\, dB_t + d(BV)_t.$$

Dividing by $\pi_t W_t$ and using Equation (19), we obtain

$$\frac{dW_t}{W_t} = \left(1 + \frac{\tilde{c}_t}{W_t}\right)\left(\frac{d\widetilde{W}_t}{\widetilde{W}_t} + (\eta_t - \tilde{\eta}_t)\, dB_t\right) + d(BV)_t.$$

On the other hand, the budget equations in the primal and dual markets imply

$$\frac{dW_t}{W_t} = \psi_t' \sigma_t\, dB_t + d(BV)_t \quad \text{and} \quad \frac{d\widetilde{W}_t}{\widetilde{W}_t} = \tilde{\psi}_t' \sigma_t\, dB_t + d(BV)_t.$$

Combining the last three equations, and matching the martingale parts, gives the result. ■

References

Back, K., 1990, "Asset Pricing for General Processes," *Journal of Mathematical Economics*, 20, 371–396.

Bank, P., and F. Riedel, 1999, "Optimal Consumption Choice Under Uncertainty with Intertemporal Substitution," working paper, Institut für Mathematik, Humboldt-Universität zu Berlin.

Chapman, D., 1998, "Habit Formation and Aggregate Consumption," *Econometrica*, 66, 1223–1230.

Constantinides, G. M., 1990, "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy*, 98, 519–543.

Cox, J., and C.-F. Huang, 1989, "Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, 33–83.

Cuoco, D., and H. Liu, 2000, "Optimal Consumption of a Divisible Durable Good," *Journal of Economic Dynamics and Control*, 24, 561–613.

Cvitanić, J., and I. Karatzas, 1992, "Convex Duality in Constrained Portfolio Optimization," *Annals of Applied Probability*, 2, 767–818.

Detemple, J., and F. Zapatero, 1991, "Asset Prices in an Exchange Economy With Habit Formation," *Econometrica*, 59, 1633–1658.

Duffie, D., 1996, *Dynamic Asset Pricing Theory*, 2nd ed., Princeton University Press, Princeton, NJ.

Duffie, D., and L. Epstein (appendix with C. Skiadas), 1992, "Stochastic Differential Utility," *Econometrica*, 60, 353–394.

Duffie, D., and C.-F. Huang, 1985, "Implementing Arrow–Debreu Equilibria by Continuous Trading of Few Long-Lived Securities," *Econometrica*, 53, 1337–1356.

Duffie, D., and C. Skiadas, 1994, "Continuous-Time Security Pricing: A Utility Gradient Approach," *Journal of Mathematical Economics*, 23, 107–131.

Dybvig, P., 1995, "Dusenberry's Ratcheting of Consumption: Optimal Dynamic Consumption and Investment Given Intolerance for any Decline in Standard of Living," *Review of Economic Studies*, 62, 287–313.

Eichenbaum, M., L. Hansen, and S. Richard, 1987, "Aggregation, Durable Goods, and Nonseparable Preferences in an Equilibrium Asset Pricing Model," unpublished working paper.

Epstein, L., and S. Zin, 1989, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969.

Fisher, M., and C. Gilles, 1999, "Consumption and Asset Prices with Recursive Preferences," Working Paper FEDS 98-40, Federal Reserve Board.

Haug, J., 1998, "Explicity Characterization of Financial Prices Under Habit Formation," working paper, Norwegian School of Economics and Business Administration.

Hindy, A., and C.-F. Huang, 1992, "Intertemporal Preferences for Uncertain Consumption: A Continuous Time Approach," *Econometrica*, 60, 781–802.

Hindy, A., and C.-F. Huang, 1993, "Optimal Consumption and Portfolio Rules with Durability and Local Substitution," *Econometrica*, 61, 85–121.

Hindy, A., C.-F. Huang, and D. Kreps, 1992, "On Intertemporal Preferences With a Continuous Time Dimension: The Case of Certainty," *Journal of Mathematical Economics*, 21, 401–440.

Ingersoll, J. E., Jr., 1992, "Optimal Consumption and Portfolio Rules With Intertemporally Dependent Utility of Consumption," *Journal of Economic Dynamics and Control*, 16, 681–712.

Karatzas, I., J. Lehoczky, and S. Shreve, 1987, "Optimal Portfolio and Consumption Decisions for a 'Small Investor' on a Finite Horizon," *SIAM Journal of Control and Optimization*, 25, 1557–1586.

Karatzas, I., and S. Shreve, 1988, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.

Karatzas, I., and S. Shreve, 1998, *Methods of Mathematical Finance*, Springer-Verlag, New York.

Loewenstein, M., and G. Willard, 1999, "Local Martingales, Arbitrage, and Viability: Free Snacks and Cheap Thrills," forthcoming in *Economic Theory*.

Lucas, R., 1978, "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445.

Merton, R., 1971, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, 3, 373–413.

Protter, P., 1990, *Stochastic Integration and Differential Equations*, New York, Springer-Verlag.

Rallis, N., 2000, "Term Structure Based Capital Asset Pricing in a General Multi-Agent Dynamically Incomplete Setting with Intertemporally Dependent Preferences," working paper, London Business School.

Schroder, M., and C. Skiadas, 1997, "Optimal Portfolio Selection With Stochastic Differential Utility," Finance Working Paper no. 226, Kellogg School of Management, Northwestern University.

Schroder, M., and C. Skiadas, 1999, "Optimal Portfolio Selection With Stochastic Differential Utility," *Journal of Economic Theory*, 89, 68–126.

Schroder, M., and C. Skiadas, 2002, "Optimal Lifetime Consumption-Portfolio Strategies in Incomplete Markets Under Recursive Homothetic Preferences," Working Paper no. 285, Kellogg School of Management, Northwestern University.

Sundaresan, S., 1989, "Intertemporally Dependent Preferences and the Volatility of Consumption and Wealth," *Review of Financial Studies,* 2, 73–89.