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Subjective Probability under Additive Aggregation of Conditional Preferences

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This paper provides an axiomatic basis for a representation of personal preferences in which the utility of an act can be expressed as an expected value of conditional utilities of the act given any set of mutually exclusive and exhaustive scenarios, under a unique subjective probability. The representation is general enough to incorporate state-dependent utilities and/or utilities with dependencies across states, as, for example, in the case of disappointment aversion. More generally, this is a model incorporating subjective probability and subjective consequences, since neither probabilities nor consequences are included among its primitives. The model reduces to subjective expected utility under the additional assumptions of separability and state-independence with respect to an objective state-contingent structure of acts. *Journal of Economic Literature* Classification Numbers: D81, D84.

1. INTRODUCTION

In the paradigm of subjective probability created by Ramsey [29] and Savage [30] the decision maker ranks acts (courses of action), each one of which is an explicit list of state-contingent consequences. In a sense, the (implied) objectivity of consequences is used to provide a decision-theoretic foundation for subjective probability. A problem is that consequences, like probabilities, are also in general subjective or ill-defined. This paper presents a decision theoretic foundation of subjective probability when consequences are also subjective.

In economic applications, acts do usually have an objective state-contingent structure in terms of money or well specified commodities. The overall subjective consequences at a given state, however, may depend on more than the corresponding objective payoff; for example, they may depend on

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the state itself, as well as on payoffs at other states. We refer to these two effects as state-dependence and nonseparability,[1] respectively. Including these effects as part of a consequence's description in Savage's setting leads to familiar problems: highly subjective and ambiguous consequences, and nonsensical acts (such as an act whose consequence in good weather is disappointment at the weather). These issues are recognized in a well developed literature, referenced below, as is the need for decision models incorporating state-dependence and nonseparability.

Another fundamental reason for considering subjective consequences is complexity. For example, someone who has to decide between living in city A and living in city B may consider the overall consequences of each alternative under a coarse set of scenarios. But the determination of the detailed consequences of each action under all economic, political, ecological, and other relevant uncertain outcomes is clearly impractical. The point is that what we call a state can always be thought of as an event on a finer state-space, and a consequence on such a state or event is typically a subjective perception of a large number of more detailed consequences that are contingent on the finer (sub)states. It is therefore not always practical to isolate the type of elementary consequences assumed by Savage.

This paper's approach is consistent with state-dependence, nonseparability, and the view of subjective consequence as representations of complex situations. The main representation result preserves a notion of additivity: conditional utility functions combine additively across events, using subjective probabilities as weights, to give an unconditional utility function. Such a representation differs from expected utility in that conditional utilities can be related to objective payoffs in a state-dependent and nonseparable way. The paper extends the theory of additive aggregation of conditional preferences developed in Skiadas [31], henceforth referred to as C&A. As explained below, the main contribution beyond C&A is the development of a sufficiently structured setting for subjective probabilities to be meaningful. As an example of an application with nonseparable preferences, we will discuss disappointment aversion, generalizing the representations of Dekel [7] and Gul [14] by incorporating subjective probabilities and possible state-dependence. Further examples that one can apply this paper's results to are discussed in C&A.

1.1. *Overview*

Unlike consequences, states of nature and acts will be regarded as objective throughout the paper. As a first illustration, we consider the

[1] Our use of the term "separable" is essentially that of Machina [25], and will be formalized later on in the paper. It is closely related to the independence axiom of von Neumann and Morgenstern, and Savage's sure-thing principle.

simplest nontrivial type of objective state space: the two outcomes of a coin toss. We also postulate a space of acts, X. The paper's main conclusions hold if the elements of X are mere names of the possible courses of action, and possess no specific structure. In order to make the notions of separability and state-independence meaningful, however, we will assume that every x in X is associated with two monetary payoffs, $x(H)$ and $x(T)$, corresponding to the outcomes of heads and tails, respectively. As in C&A, a key distinction of this paper's approach is that the overall consequences of an act on an event are not necessarily represented by the specified objective payoffs on the given event. For example, the consequences of an act on the event of heads may include a subjective sense of disappointment that the monetary payoff turned out to be lower than expected. The word "consequence" will be used informally in the widest sense possible. In particular, a consequence should include any subjective effect of the relevant state of nature, or unrealized payoffs.

Because consequences can be highly subjective (or complex), we do not wish to include them among the theory's formal primitives. Instead, their effect is reflected in the properties of the decision maker's preferences. In C&A these preferences are modeled in terms of conditional rankings of the form $x \succcurlyeq^E y$, where $x, y \in X$, and E is any nonempty event (in the current example, one of $\{H, T\}$, $\{H\}$, or $\{T\}$). The interpretation of $x \succcurlyeq^E y$ is that the decision maker considers the overall consequences of act x on event E at least as desirable as the overall consequences of act y on event E, not knowing whether E will occur or not. The conditional ranking $x \succcurlyeq^E y$ does not necessarily imply that x would be preferred to y given the knowledge that E has or will occur. For example, one can expect to be disappointed if an event resulting in a low payoff occurs, while of course there can be no (further) disappointment if the occurrence of that event is already a known fact. For any event E, $\succcurlyeq^E$ is assumed to be complete and transitive, with $\succ^E$ and $\sim^E$ representing its asymmetric and symmetric parts, respectively.

In the current context, preferences are separable if $x(H) = y(H)$ implies $x \sim^{\{H\}} y$, and $x(T) = y(T)$ implies $x \sim^{\{T\}} y$. The intuition is that the objective payoff fully captures the relevant consequences on a given event. More generally, however, we allow for the possible violation of separability. For example, if x is considered overall a better choice than y, the decision maker may rationally anticipate that on the event that x and y yield identical payoffs there will be a feeling of disappointment if act x is chosen, and a feeling of elation if act y is chosen. The ranking $y \succ^{\{H\}} x$, under the assumptions $x(H) = y(H)$ and $x \succ^{\{H, T\}} y$, can therefore be interpreted as an expression of disappointment aversion, and constitutes a violation of separability. To re-emphasize a point made earlier, notice that the decision maker would be indifferent between x and y if the coin was already tossed, and the outcome of heads was known. This formulation of disappointment aversion, introduced in C&A, will be used in this paper as a canonical example of nonseparability.

In terms of the unmodeled (subjective) consequences, the agent will be assumed to be Bayesian in the usual sense. In particular, C&A provides a set of axioms on the primitives[2] so that, in the context of the current example, $\succcurlyeq^E$ has an ordinal utility representation U^E, for any event E, and these utilities can be chosen so that

$$U^{\{H,T\}}(x) = U^{\{H\}}(x)\, P(H) + U^{\{T\}}(x)\, P(T), \qquad x \in X, \tag{1}$$

for some probability P. In the above representation, the quantity $U^E(x)$ should be interpreted as the utility of the overall consequences of x on the event E. In the separable case, one is able to write $U^{\{H\}}(x) = f(H, x(H))$ and $U^{\{T\}}(x) = f(T, x(T))$, for some function $f\colon \{H, T\} \times \mathbb{R} \to \mathbb{R}$, reducing (1) to the familiar state-dependent expected utility representation. In the case of disappointment aversion, however, conditional utilities can depend on both payoffs through the unconditional utility: $U^{\{H\}}(x) = f(H, x(H), U^{\{H,T\}}(x))$, and analogously for tails, as explained in Section 5 and in C&A.

The question that remains unanswered in C&A, and will be answered in this paper, is the meaning of the probability P in (1). In fact, without any further structure, P has no meaning. While, under the assumptions of C&A, utilities are determined up to positive affine transformations for a given P, the probability P itself is indeterminate. To see that, take any other probability, $\hat{P}$, with the same null sets as P, and define the new utilities $\hat{U}^E(x) = U^E(x)\, P(E)/\hat{P}(E)$, for any event E of positive probability. Clearly $\hat{U}^E$ is an ordinal utility representation of $\succcurlyeq^E$. Moreover, (1) remains valid with hats over U and P. Under separability and state-independence, this difficulty does not arise because $U^{\{H\}}$ and $U^{\{T\}}$ can also be thought of as ordinal utilities over acts with constant payoffs across states, and they therefore cannot be rescaled independently of each other. Under state-dependent preferences, however, a fundamental indeterminacy between utilities and probabilities arises (whether preferences are separable or not). The following subsection discusses a number of papers that propose ways of resolving this indeterminacy for the case of state-dependent but separable preferences. While the present paper's approach is new in the separable case as well, its main innovation lies in the fact that it applies to non-separable preferences. The approach will be introduced here, while a discussion of how it relates to the literature will be given in the following subsection.

[2] C&A assumes that the state space has at least three elements, and therefore, strictly speaking, does not include the current example. One can circumvent this problem by simply including an auxiliary coin toss. Alternatively, this paper's extension of C&A can be easily modified to give a theory of additive aggregation for the two-state case, as explained in Remark 2.

The novel ingredient of this paper's theory is the notion that the decision maker is able to rank the desirability of disjoint events conditionally on the choice of an act. For example, suppose that the acts under consideration are going to the office or going to the beach, and that the uncertain states of nature are good or bad weather. C&A postulates that the decision maker is able to state whether going to the office or going to the beach will yield the more desirable consequences in each state of the weather. Here we will also assume that the decision maker can state whether good weather or bad weather will yield the more desirable consequences for each choice of an act. For example, the decision maker might state that he prefers good weather when he goes to the beach, and bad weather when he goes to the office. This ranking should be thought of as representing implicit preferences over subjective consequences resulting from the corresponding combinations of contingencies and acts. While apparently new, the idea of comparing disjoint events given acts in order to separate utilities from probabilities is related to other approaches in the literature (for separable preferences), as explained in the following subsection.

All existing decision-theoretic foundations of subjective probability share the feature that the decision maker is required to rank a very large class of objects, typically involving either a set of acts yielding arbitrary combinations of consequences (as in Savage [30]), or a set of objective probabilities allowing arbitrary mixing (as in Anscombe and Aumann [1]). The present paper faces the same limitation by relying on structural "solvability" assumptions, which effectively imply that acts can yield arbitrary combinations of utility levels on any finite set of scenarios. As a result, given our structural assumptions, the conditional ranking of disjoint events given acts is uniquely determined by a set of indifference relations of the form $\sim^x$, where x is any act. For disjoint events, E and F, the statement $E \sim^x F$ has the interpretation that the overall consequences of x on E are equally desirable as the overall consequences of x on F. Non-disjoint events need not be comparable under any $\sim^x$. The formal theory of this paper adds the relations $\sim^x$ to the primitives of C&A, as well as a list of related axioms. In the additive representation (1), it will also be required that $E \sim^x F$ implies $U^E(x) = U^F(x)$. This additional restriction will imply that conditional utilities can no longer be arbitrarily rescaled on disjoint events, a fact that will be utilized to calibrate the underlying probabilities.

The structural restrictions of the paper will also imply that there exists a unique preference order, $\succcurlyeq$, on the space of all pairs of the form (x, E), where x is an act and E is an event, that is compatible with the primitives, in the sense that $x \succcurlyeq^E y \Leftrightarrow (x, E) \succcurlyeq (y, E)$, and $E \sim^x F \Rightarrow (x, E) \sim (x, F)$. Moreover, $\succcurlyeq$ has a utility representation, U, that satisfies (1) (with $U(x, E) = U^E(x)$). An arbitrary act-event pair, (x, E), which we will formally call a "situation," should be thought of as an indirect representation of all consequences of act x on event E. The existence and uniqueness of the preference order $\succcurlyeq$ shows that our primitives and axioms indirectly specify a consistent complete ranking of all subjective consequences of any possible situation. Clearly, one could rewrite the whole paper by replacing at the outset all conditional preferences with a single complete preference order over all situations, as in Luce and Krantz [22] and Fishburn [11]. This would afford us minor simplifications of the axioms (although it would necessitate the addition of a new axiom, as explained in Section 4). However, direct comparisons of arbitrary situations can be harder to interpret than conditional comparisons. In our earlier example, the decision maker's conditional preferences seem more natural than a statement of the form: going to the office under good weather is less (or more) desirable than going to the beach, without specifying the weather condition while at the beach.

1.2. *Related Literature*

The literature of nonseparable preferences is extensive, and well motivated by an abundance of empirical findings. Relevant discussions and surveys include: Fishburn [12], Machina [24, 25], Karni and Schmeidler [18], and Epstein [10]. On the other hand, there are only few papers on the meaning of subjective probability in the context of nonseparable preferences. If we restrict attention to transitive preferences and additive probabilities, as we do throughout this paper,[3] the theories of Machina and Schmeidler [26, 27] (with Grant's [13] extension) seem to be the only alternatives. The two papers by Machina and Schmeidler, which generalize those by Savage [30] and Anscombe and Aumann [1], respectively, differ from the present paper significantly. First, they assume a state-contingent structure of acts, thereby implicitly assumming objective consequences. Second, they assume state-independence. Neither of the two are assumed in this paper. On the other hand, within the class of state-independent preferences with objective consequences, the present paper's primitives and representation is more structured (and therefore less general) than those in the Machina–Schmeidler theories, in that conditional utilities are meaningful, and can be aggregated additively.

The well known reasons for considering state-dependent preferences are reviewed by Karni [15, 16, 17], while Aumann [2], in a letter to Savage, articulates the conceptual difficulties that arise due to subjective and state-dependent consequences. Under the additional assumption of preference

[3] Subjective probability has also been axiomatized in the context of a certain type of nontransitive preferences, surveyed by Fishburn [12] (also see Sugden [32]). These so-called SSB (skew-symmetric bilinear) representations are based on an interpretation of "expected regret." In Skiadas [31] it is shown that regret can also be modeled in the context of the present paper, with transitive preferences.

separability, this paper provides a new foundation for state-dependent subjective expected utility, complementing a number of related papers, briefly discussed below. When this paper's setting is further restricted to separable and state-independent preferences, it reduces to essentially Wakker's [36] theory of subjective expected utility.[4] The mathematical structure of the main result's proof is a direct extension of Wakker's arguments.[5] The remainder of this section briefly reviews, and relates to this paper, the existing literature of state-dependent separable preferences.

One of the earliest axiomatic formulations of subjective probability under state-dependent preferences was provided by Fishburn [11] (and extended by Balch and Fishburn [3]). Fishburn's approach relies on a mixture-set structure of acts in terms of objective mixing probabilities, in the tradition of Anscombe and Aumann [1]. Since linearity of conditional utilities in the probabilities is a crucial ingredient of this approach, it does not seem suited for applications in which utilities are nonlinear in the probabilities, as in the case of disappointment aversion. Like Luce and Krantz [22], Fishburn postulates preferences over pairs of acts and events. But while Luce and Krantz rely on an objective state-contingent consequence structure of acts, Fishburn never introduces a space of consequences among his primitives. In this sense, Fishburn's theory can be interpreted as a subjective-consequence theory, but in a more limited sense than in the present paper, since subjective consequences cannot depend on unrealized payoffs.

Another approach to subjective probability under state-dependent preferences is suggested by Karni, Schmeidler, and Vind [19], and is also followed by Wakker [34]. The basic idea is to assume that the primitives include the decision maker's hypothetical preferences in an imaginary world in which all probabilities are given objectively in some arbitrarily prescribed way. Under this assumption, equation (1) would have to hold with P being the prescribed hypothetical probability, and $U^{\{H, T\}}$ representing the hypothetical unconditional ranking of acts. Since the probability P is fixed, utilities are now calibrated (up to positive affine transformations). Passing from probability P to the decision maker's actual subjective beliefs should not affect the

[4] Strictly speaking, this paper's theory does not reduce to Wakker's under state-independence and separability, but the differences are not that significant. Remark 2 of Section 4 outlines a variant of the paper's main result which is a strict generalization of Wakker's theorem.

[5] Closely related to Wakker's formulation, and therefore to the present paper, are the theories of subjective expected utility of Luce and Krantz [22] and Nakamura [28], who use the (formally more general) "algebraic approach" to additive conjoint measurement, as opposed to the (simpler) "topological approach" adopted by Wakker and this paper (see also Krantz, *et al.* [20] and Wakker [35]). Whether in a topological or algebraic form, the type of generalization discussed in this paper could also be applied to Nakamura's theory, or to its variations discussed by Chew and Karni [5].

conditional utilities $U^{\{H\}}$ and $U^{\{T\}}$, and thus there can only be one set of subjective beliefs consistent with additivity of the already calibrated utilities. Whatever the merits or drawbacks of this approach in the separable case, it does not appear to be a viable solution under nonseparable preferences. Considering again the example of disappointment aversion, one can see that the assumption that the conditional utilities given heads or tails remain unaltered when changing the assumed underlying probabilities is no longer valid. Changing the probabilities also changes unconditional expectations, and therefore the degree of disappointment or elation felt in each state. The hypothetical-probabilities approach is therefore not feasible without being explicit about the structure of potential non-separabilities.

A third general approach for determining probabilities under state-dependence can be loosely characterized by the idea that there exists some way of calibrating utilities that does not make use of any objective or hypothetical probabilities. Dréze [8, 9], for example, postulates the existence of two "omnipotent" acts ("games" in his terminology) whose conditional utilities are directly normalized to be constant across states. Karni [17] avoids axioms involving utilities (which should be endogenously derived) by introducing the idea of "constant valuation" acts. In terms of our earlier discussion, constant valuation acts can be thought of as acts that have equally desirable consequences in every state of the world, where again (unlike Karni) we use the word "consequence" in the wide sense, to include the state-dependent subjective impact of objective payoffs. Constant valuation acts in Karni's theory play the role of constant acts in Savage's [30] theory, and are used to calibrate utilities and probabilities. (Karni [16] applies an analogous approach in the Anscombe–Aumann [1] setting.) The present paper's theory is essentially a variation of this third approach, since its structural assumptions imply the existence of acts with equally desirable consequences on all states, analogous to Dréze's omnipotent acts, or Karni's constant valuation acts. Unlike the above papers, however, this paper derives such acts endogenously, in terms of properties of conditional comparisons of disjoint events, given acts. Of course, as indicated earlier, the more substantial innovation of this paper is that, unlike any of the above references on subjective probability under state-dependence, it accommodates nonseparable preferences.

The rest of the paper is organized in four sections and two appendices. Section 2 reviews from C&A the idea of additive aggregation of conditional preferences. Section 3 provides the additional primitives and assumptions required to make subjective probability meaningful. Section 4 states and discusses the central representation theorem. Section 5 concludes with examples. Appendix A extends the main Theorem to the case of countably additive representations, and Appendix B contains mathematical proofs.

2. ADDITIVE AGGREGATION OF CONDITIONAL PREFERENCES

This section reviews the relevant theory of conditional preferences and additive aggregation from C&A [31], which can be consulted for further discussion and examples. The new primitives and axioms that are required for this paper's main result, and that cannot be found in C&A, are presented in the following section.

Uncertainty is modeled by a state-space Ω, whose elements are called *states*, and an algebra[6] $\mathscr{F}$ of subsets of Ω. We let L denote the set of all *random variables* (that is, $\mathscr{F}$-measurable functions of the form $V: \Omega \to \mathbb{R}$). We also take as primitive a subset $\mathscr{N}$ of $\mathscr{F}$, whose elements are the *null sets*, and should be thought of as contingencies that the decision maker considers so unlikely so that any corresponding consequences can be ignored ex ante. The reader who wishes to only consider the case of a finite Ω can safely assume that the only null set is the empty set. More generally, $\mathscr{N}$ will be assumed throughout the paper to have the following properties:[7]

(a) $\Omega \notin \mathscr{N}$.

(b) For all $F \in \mathscr{F}$ and $N \in \mathscr{N}$, $F \subseteq N$ implies $F \in \mathscr{N}$.

(c) For all disjoint $N_1, N_2 \in \mathscr{N}$, $N_1 \cup N_2 \in \mathscr{N}$.

An *event* is any *non-null* element of $\mathscr{F}$. The set of all events is denoted $\mathscr{E} = \mathscr{F} \backslash \mathscr{N}$.

Preferences will be defined over a set X, whose elements we call *acts*. An act is to be thought of as a label of a course of action, and not necessarily as a specification of state-contingent consequences. As explained in the Introduction, consequences will not be part of our primitives, because of their possible subjective or vague nature. We will, however, use the term "consequence" informally in discussing our assumptions.

In order to obtain an additive representation, the following structural topological assumption will be made:

A1. X is a connected compact topological space.

As discussed in Remark 3 of Section 4, the compactness assumption is made mainly for simplicity of exposition.

[6] An *algebra* in this context is a nonempty set of sets that is closed with respect to disjoint unions, and complementation.

[7] Alternatively, null sets could be derived in terms of properties of preferences. Since this aspect of the theory is of secondary importance here, we find it simpler to simply postulate the existence of the set $\mathscr{N}$. In the language of Boolean Algebra, $\mathscr{N}$ is assumed to be any subset of $\mathscr{F}$ that is a proper ideal.

Given any event E, the decision maker is able to rank any two acts on the basis of their consequences on E. We model that with a set of *preference orders*, that is, complete and transitive (binary) relations,[8] on X:

A2. For every $E \in \mathscr{E}$, $\succcurlyeq^E$ is a continuous[9] preference order on X. Moreover, $E \triangle F \in \mathscr{N}$ implies $\succcurlyeq^E = \succcurlyeq^F$, for all $E, F \in \mathscr{E}$.

Here $E \triangle F$ denotes the symmetric difference between E and F. The symmetric and asymmetric parts[10] of $\succcurlyeq^E$ are denoted $\sim^E$ and $\succ^E$, respectively.

The statement $x \succ^E y$ represents the decision maker's ex-ante judgment that, on event E, the overall consequences of x are preferred to those of y. The implicit consequences of x on E can be subjective, they may depend on E, and they may depend on x's payoffs at states not belonging to E. As indicated in the Introduction, $x \succ^E y$ does *not* necessarily imply that the consequences of x are preferred to those of y given the knowledge that E has in fact occurred. We clarify these interpretations by reviewing two examples from C&A:

Example 1 (Separable Preferences). We fix a space M, whose elements represent objective payoffs. For example, M could be the real line, representing all possible monetary payoffs. We also assume that every act is a function from Ω to M. The payoff $x(\omega)$ should not, however, be confused with the overall consequences of act x at ω, which could be subjective. We say that "$x = y$ on E" if $x(\omega) = y(\omega)$ for all $\omega \in E$. Preferences are defined to be *separable* if

$$x = y \text{ on } E \Rightarrow x \sim^E y, \qquad x, y \in X, \quad E \in \mathscr{E}. \tag{2}$$

The interpretation of separability is that the objective payoff structure of an act on any given event is sufficient for evaluating the act's overall consequences on that event. Example 1 will be further discussed in Section 5.

Example 2 (Disappointment Aversion). Suppose that acts have the same state-contingent payoff structure as in Example 1. Instead of separability, however, we now only require that

$$(x = y \text{ on } E \text{ and } y \succcurlyeq^{\Omega} x) \Rightarrow x \succcurlyeq^E y, \qquad x, y \in X, \quad E \in \mathscr{E}. \tag{3}$$

[8] A binary relation, R, on X is *complete* if, for any $x, y \in X$, xRy or yRx, and *transitive* if, for all $x, y, z \in X$, xRy and yRz implies xRz.

[9] A preference order, $\succcurlyeq$, on X is *continuous* if the sets $\{y : y \succcurlyeq x\}$ and $\{y : x \succcurlyeq y\}$ are closed for all $x \in X$.

[10] Given preference order $\succcurlyeq$, its *symmetric part* is the relation $\sim$ on X defined by $(x \sim y) \Leftrightarrow (x \succcurlyeq y \text{ and } y \succcurlyeq x)$, and its *asymmetric part* is the relation $\succ$ on X defined by $(x \succ y) \Leftrightarrow (x \succcurlyeq y$ and not $y \succcurlyeq x)$.

This represents (weak) disappointment aversion: Suppose that acts x and y yield identical payoffs on event E, but y is expected to yield overall no less desirable consequences than x. Then, on event E, y will yield no more desirable consequences than x, since it may involve a feeling of disappointment, while x may involve a feeling of elation. A strict version of disappointment aversion is obtained by replacing all weak preferences with strong preferences in (3). This example illustrates a point made earlier: the consequences of an act on an event need not coincide with the consequences of the act given the knowledge that the event has occurred. In particular, one cannot expect to be disappointed about an event that is already known. Example 2 is further developed in C&A and in Section 5 of this paper.

The agent's conditional preferences will be required to satisfy the following monotonicity condition, called "strict coherence" in C&A:

A3. For any disjoint events E, F, and any acts x, y, $x \succcurlyeq^E y$ and $x \succcurlyeq^F y$ implies $x \succcurlyeq^{E \cup F} y$; and $x \succ^E y$ and $x \succcurlyeq^F y$ implies $x \succ^{E \cup F} y$.

The interpretation of strict coherence is essentially the informal "sure-thing principle" of Savage [30]: If the consequences of x are preferred to those of y under each of two scenarios, then the same is true under the combined scenarios. The two implications in A3 formalize a weak and a strict version of this basic idea, respectively. Coherence differs, however, from the sure-thing principle as manifested in Savage's axioms in that the latter incorporate separability in the sense of Example 1, while coherence is compatible with nonseparable preferences.

Assumptions A2 and A3 are necessary for the representation we are seeking. We will use two more assumptions to obtain an additive structure, both of which are structural (not necessary).

We will assume that, given any finite number of scenarios and corresponding set of acts, there is a single act that exactly compensates for not following act i on scenario i, for every i. Formally, this is expressed by the following "solvability" condition:

A4. Given any finite number of pairwise disjoint events $E_1, ..., E_n$, and any acts $x_1, ..., x_n$ there exists an act x such that $x \sim^{E_i} x_i$ for all $i \in \{1, ..., n\}$.

Under state-contingent acts and separable preferences (as in Example 1), the act x in A4 can be chosen so that $x = x_i$ on E_i for every i. For nonseparable preferences, such a construction is clearly not sufficient in general. Nevertheless, one can still imagine that appropriate state-contingent payments can be adjusted in a way that makes the overall subjective consequence on event E_i just as desirable as the prescribed act x_i, for every i. For example, C&A and Example 3 of Section 5 provide formulations of disappointment aversion (in the sense of Example 2) in which solvability holds.

Our next assumption imposes a certain degree of non-degeneracy, and is of secondary importance.

A5. For any given event E, there exist acts x, y such that $x \succ^E y$. Moreover, there exist at least three pairwise disjoint events.

A modification of the theory that works with only two states is outlined in Remark 2 of Section 4.

Suppose for now that $\mathscr{F}$ is finite. It is shown in C&A (Theorem 2), using a version of Debreu's [6] theorem, that A1 through A5 imply the existence of a probability P and a function $U: X \to L$ such that

$$x \succcurlyeq^E y \Leftrightarrow \int_E U(x)(\omega)\, dP(\omega) \geqslant \int_E U(y)(\omega)\, dP(\omega), \qquad E \in \mathscr{E}, \tag{4}$$

and $P(F) = 0 \Leftrightarrow F \in \mathscr{N}$, for all $F \in \mathscr{F}$. This is what we mean by "additive aggregation of conditional preferences," which should not be confused with a state-dependent expected utility representation. Under additive aggregation, acts are not assumed to have a state-contingent structure, and if they do, it need not be the case that we can write $U(x)(\omega) = f(\omega, x(\omega))$ for almost every $\omega \in \Omega$, for some function f. This point is further discussed in Section 5.

Another important point to notice is that the probability P in the above representation plays a primarily notational role. Indeed, let Q be an arbitrary probability with the same null sets as P, and let dP/dQ denote the Radon–Nikodym derivative of P with respect to Q. Then (4) remains valid with $U\, dP/dQ$ in place of U, and Q in place of P. In this sense, the underlying measure is arbitrary (up to null sets) and not necessarily one that corresponds to the decision maker's subjective beliefs. Our objective in the following section is to introduce sufficient additional structure that will allow us to select a unique probability that is compatible with an interpretation of subjective beliefs. A by-product of this additional structure is an additive aggregation theorem for an infinite number of events (that differs from the one given in C&A).

3. COMPARISON OF CONSEQUENCES ACROSS EVENTS

This section presents the primitives and assumptions that this paper adds to the theory of additive aggregation of C&A, outlined in the last section, in order to make subjective probabilities meaningful. Specifically, we are going to postulate that, for any given act, the decision maker is able to identify any two disjoint events on which the act's consequences are believed to be equally desirable. The purpose of the section's assumptions is twofold. First, to make the formalization of this new type of comparisons given acts consistent with their informal interpretation in terms of consequences, and second to provide a sufficiently rich set of acts, so that there is a unique underlying probability that is consistent with additive aggregation and the agent's subjective beliefs.

Formally, we start with the following relations:

A6. For every act x, $\sim^x$ is a (binary) relation on $\mathscr{E}$ satisfying, for all pairwise disjoint $E, F, G \in \mathscr{F}$,

(a) $E \sim^x F \Leftrightarrow F \sim^x E$; and

(b) $(E \sim^x F \text{ and } F \sim^x G) \Rightarrow E \sim^x G$.

The interpretation of the statement $E \sim^x F$, where E and F are disjoint events, is that act x has equally desirable consequences on event E as it does on event F. This is not the same as saying that the decision maker would prefer the consequences of x under the knowledge that E will occur to the consequences of y under the knowledge that F will occur. As in the analogous discussion of the last section, the case of disappointment aversion should make the distinction clear. In general, the subjective consequences implicit in all relations of the form $\succcurlyeq^E$ or $\sim^x$ (where x is an act and E is an event) should be interpreted as consequences perceived ex ante, without any knowledge about the "true" state of nature that is not already reflected in the decision maker's beliefs.

Assumption A6 does not include any statement of completeness. In fact, the formal structure so far does not preclude the possibility that each $\sim^x$ is empty. The degree of completeness of $\sim^x$ will be dictated by our next two assumptions.

A7. Given any pairwise disjoint events E_1, E_2, F, and any act x such that $E_1 \sim^x E_2$, we have $E_1 \cup E_2 \sim^x F \Leftrightarrow E_1 \sim^x F$.

The idea behind A7 is that if x has equally desirable consequences on E_1 and E_2, then to say that the consequences of x on $E_1 \cup E_2$ are equally desirable to the consequences of x on F should be the same as saying that the latter are equally desirable to the consequences of x on either E_1 or E_2.

A more restrictive structural assumption is the following "solvability" condition, which should be thought of as complementing A4:

A8. Given any act x and disjoint events E and F, there exists an act y such that $y \sim^E x$ and $E \sim^y F$.

Like A4, this condition is automatically satisfied in Savage's [30] setting, but it is nevertheless a demanding assumption.[11] The role of the assumption is to provide a sufficiently rich space of acts, so that utility levels of (implied) consequences can be compared meaningfully across disjoint events. Such comparisons will then be used to calibrate probabilities, and to interpret these probabilities as representations of subjective beliefs. A consequence of A8 is that any two disjoint events are comparable under some $\sim^x$. In general, we are not going to require that non-disjoint events are comparable under any $\sim^x$ (but see Section 4).

So far, the assumptions of this section have not involved conditional preferences given events. Clearly, relations of the form $\sim^x$ and $\succcurlyeq^E$ must be consistent, in the following sense:

A9. For all disjoint events, E, F, and acts, x, y, we have

(a) $(E \sim^x F$ and $E \sim^y F)$ implies $(x \succcurlyeq^E y \Leftrightarrow x \succcurlyeq^F y)$.

(b) $(x \sim^E y$ and $x \sim^F y)$ implies $(E \sim^x F \Leftrightarrow E \sim^y F)$.

Condition A9(a) says that if x has equally desirable consequences on E and F, and likewise for y, then the ranking of the consequences of x and y on the two events has to be consistent. Condition A9(b) has a similar interpretation.

Our final assumption concerns the compatibility of statements about compensating trade-offs across events. Although somewhat complicated, this is an intuitive and necessary condition for the representation of the main theorem. In the separable case it reduces to a variant of Wakker's [36] "no-contradictory-trade-offs" condition. Some preliminary notation and discussion will help simplify its statement and meaning. First, given any disjoint events E, F, and acts x, y, we will write $x_E y_F$ to denote any choice of an act z that satisfies $z \sim^E x$ and $z \sim^F y$. For example, a statement of the form "there exists $x_E y_F$ such that..." should be interpreted as "there exists $z \in X$ satisfying $z \sim^E x$ and $z \sim^F y$ such that..." Second, we define a notion of "equal trade-offs on events," an idea that builds on earlier formulations by Luce and Krantz [22] and Wakker [36].

Before any formal statements, it is instructive to consider Fig. 1, with the following interpretations: The symbols E and F represent two disjoint events, while v, w, x, y, z are all acts. The position of these acts on the leftmost vertical line should be thought of as representing the subjective degree of desirability of the consequences of each act on event E, and analogously for the other vertical line. The higher the position on the line,

[11] It is of some interest to notice that, under the topological assumptions A1 and A2, A8 is implied by the following condition: for any act x and disjoint events E and F, the set $\{y: E \sim^y F\}$ is connected, and it contains acts $\bar{y}$ and $\underline{y}$ such that $\bar{y} \succcurlyeq^E x \succcurlyeq^E \underline{y}$.

the more desirable the consequences. The line connecting act x on E to act v on F represents some act $x_E v_F$, which, according to our earlier notational convention, has equally desirable consequences with x on E, and with v on F. The remaining lines connecting the two vertical lines also represent acts with the analogous properties. We assume that all the acts represented by the lines of Fig. 1 exist. Suppose now, that $x_E v_F \sim^{E \cup F} y_E w_F$ and $y_E v_F \sim^{E \cup F} z_E w_F$. The last two indifferences reveal that the reduction in desirability of consequences on event F from the level represented by v to that represented by w is exactly compensated for by an increase in desirability of consequences on event E either from the level of x to that of y, or from the level of y to that of z. In this sense, the two indifferences reveal that the pairs of acts (x, y) and (y, z) represent equal trade-offs on event E as measured by compensating consequences on event F. We denote this fact by $(x, y) =^E_F (y, z)$.

Definition 1 below formally summarizes the discussion of the last paragraph, and also introduces the relation $\neq^E_F$ on the set of act pairs. The interpretation of $(x, y) \neq^E_F (y, z)$ is that the agent's conditional preferences reveal that the act pairs (x, y) and (y, z) represent unequal trade-offs on event E as measured by compensating consequences on event F.

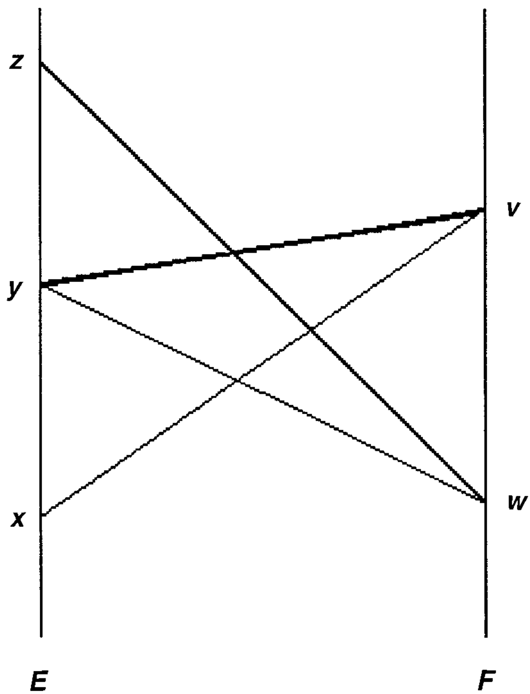

FIGURE 1

DEFINITION 1. For any disjoint events E, F, the relations $=_F^E$ and $\neq_F^E$ on X^2 are defined by:

(a) $(x, y) =_F^E (y, z)$ if there exist acts $v, w, x_E v_F, y_E w_F, z_E w_F$, and $y_E v_F$ such that $x_E v_F \sim^{E \cup F} y_E w_F$ and $y_E v_F \sim^{E \cup F} z_E w_F$.

(b) $(x, y) \neq_F^E (y, z)$ if there exist acts $v, w, x_E v_F, y_E w_F, z_E w_F$, and $y_E v_F$ such that $x_E v_F \sim^{E \cup F} y_E w_F$ and *not* $y_E v_F \sim^{E \cup F} z_E w_F$.

Given our assumptions and definitions so far, $(x, y) \neq_F^E (y, z)$ is not formally equivalent to not $(x, y) =_F^E (y, z)$. Nevertheless, under our interpretation of the primitives, $\neq_F^E$ should be thought of as the complement of $=_F^E$, a fact that will become formally true as well, once we introduce our last assumption, A10, below.

Consider now acts x_1, x_2, x_3, and disjoint events E, F. Loosely speaking, our last assumption states that, in deciding that $(x_1, x_2) =_F^E (x_2, x_3)$, the decision maker disregards the likelihood of E. In other words, the equality of trade-offs is a statement about desirability of consequences only, and not about the relative likelihood of E and F. To express this in the language of our primitives, consider a third event G that is disjoint of $E \cup F$, and suppose that each x_i has equally desirable consequences on E as on G, that is, $E \sim^{x_i} G$, for all $i \in \{1, 2, 3\}$. While the event G need not have the same probability of occurrence as E, the equality of trade-offs on E, $(x_1, x_2) =_F^E (x_2, x_3)$, should imply a corresponding equality of trade-offs on G, $(x_1, x_2) =_F^G (x_2, x_3)$, which in turn should be contradicted by the statement $(x_1, x_2) \neq_F^G (x_2, x_3)$. This is precisely the content of our last assumption:

A10. Given any pairwise disjoint events E, F, G, and any acts x_i, $i \in \{1, 2, 3\}$, such that $E \sim^{x_i} G$ for all i, it is not the case that $(x_1, x_2) =_F^E (x_2, x_3)$ and $(x_1, x_2) \neq_F^G (x_2, x_3)$.

This completes the presentation of the assumptions used in the paper's main result, which is the topic of the following section.

4. REPRESENTATION THEOREM

This section discusses the main theorem of the paper. Intuitively, the result states that the overall (unmodeled) consequences of acts on events can be assigned utilities that on the one hand are compatible with the comparisons expressed by the relations of the form $\succcurlyeq^E$ and $\sim^x$, and on the other hand they can be aggregated additively under a unique probability. After the statement of the main result, we will argue that this unique probability is consistent with the decision maker's subjective beliefs.

The formal statement of the paper's main result follows. Its proof can be found in Appendix B.

THEOREM 1. *Suppose that assumptions* A1 *through* A10 *hold. Then there exist a function* $U: X \times \mathscr{E} \to \mathbb{R}$ *and a finitely additive probability* P *on* $\mathscr{F}$ *such that*

(a) *For any disjoint* $E, F \in \mathscr{E}$, *and any* $x, y \in X$,

$$x \succcurlyeq^E y \Leftrightarrow U(x, E) \geqslant U(y, E) \tag{5}$$

$$E \sim^x F \Leftrightarrow U(x, E) = U(x, F); \tag{6}$$

$$U(x, E \cup F)\, P(E \cup F) = U(x, E)\, P(E) + U(x, F)\, P(F). \tag{7}$$

Moreover, for all $F \in \mathscr{F}$,

$$P(F) = 0 \Leftrightarrow F \in \mathscr{N}. \tag{8}$$

(b) *Any other pair* $(\tilde{U}, \tilde{P})$ *with the above properties of* (U, P) *satisfies* $\tilde{P} = P$ *and* $P[\tilde{U} = \alpha U + \beta] = 1$ *for some* $\alpha \in (0, \infty)$ *and* $\beta \in \mathbb{R}$.

(c) *For every* $E \in \mathscr{E}$, *the function* $U(\cdot, E)$ *is continuous, and maps* X *onto a (closed and bounded) interval that is independent of the choice of* E.

Given the representation (U, P) of Theorem 1, two natural questions arise. First, in what sense is P compatible with a notion of subjective probability? Second, what meaning, if any, should we attribute to general comparisons of the form $U(x, E) \geqslant U(y, F)$? Both questions have simple answers. To show that, let us assume, for the remainder of this section, that A1 through A10 hold, and (U, P) is as in Theorem 1.

To address the first question, we formulate a notion of revealed probability comparisons in the tradition of Ramsey [29] and Savage [30]. We adopt the same notational conventions used in Definition 1.

DEFINITION 2. The binary relation $\geqslant$ on $\mathscr{F}$ is defined as follows:

(a) For all disjoint $E, F \in \mathscr{E}$, we let $E \geqslant F$ if $x_E y_F \succcurlyeq^{E \cup F} y_E x_F$ for any choice of x, y, $x_E y_F$, $y_E x_F$ such that $E \sim^x F$, $E \sim^y F$, and $x \succ^E y$.

(b) For all $N \in \mathscr{N}$ and $F \in \mathscr{F}$, we have: (i) $F \geqslant N$, and (ii) $N \geqslant F \Rightarrow F \in \mathscr{N}$.

(c) Given (a) and (b), for arbitrary $E, F \in \mathscr{F}$, we have $E \geqslant F \Leftrightarrow (E \backslash F) \geqslant (F \backslash E)$.

The interpretation of $E \geqslant F$ in part (a) is that the decision maker believes E to be more likely than F, and therefore would rather have the more desirable consequences occur on E, and the less desirable consequences on F, than the other way around. Once again, this interpretation is consistent with state-dependent valuation of objective payoffs, because the effect of such state-dependence is part of the subjective consequences implied by the conditional rankings of acts. Part (b) of Definition 2 states that a null set is less likely than any other set in $\mathscr{F}$, and an element of $\mathscr{F}$ that is less likely than a null set is also null. Part (c) compares the likelihood of overlapping events by removing from both their intersection and using parts (a) and (b) to compare the remaining disjoint parts.

The reader can easily confirm that

$$P(E) \geqslant P(F) \Leftrightarrow E \geqslant F, \qquad E, F \in \mathscr{F}.$$

In this sense, the probability P is compatible with the decision maker's subjective beliefs. It should be emphasized that P need not be the only probability that represents the relation $\geqslant$ on $\mathscr{F}$. This is to be expected, because, unlike Savage [30], we do not require nonatomicity (for example, the state space can be finite). Our assumptions only imply that among all probabilities consistent with the decision maker's revealed probabilistic rankings, one and only one is compatible with additive aggregation.

To answer the second question posed above, regarding the interpretation of utilities, we think of a pair $(x, E) \in X \times \mathscr{E}$ as an indirect objective representation of the (possibly subjective) consequences of act x on event E. For instance, recalling an example from the Introduction, if x represents the act of living in city A, and E represents the event of politician X becoming mayor of A, then the pair (x, E) objectively represents the overall consequences of living in city A with X as mayor. The decision maker can objectively describe the *situation* leading to these consequences, but the exact nature of the consequences themselves remains subjective and complex. With this motivation, we formally call an element of $X \times \mathscr{E}$ a *situation*. We now wish to interpret the ranking of situations implied by the utility function U.

A preference order, $\succcurlyeq$, on $X \times \mathscr{E}$ will be called *compatible* (with the decision maker's conditional preferences), if for all $x, y \in X$ and $E, F \in \mathscr{E}$,

$$(x, E) \succcurlyeq (y, E) \Leftrightarrow x \succcurlyeq^E y \qquad \text{and} \qquad (x, E) \succ (x, F) \Rightarrow \text{not } E \sim^x F,$$

where $\succ$ is the asymmetric part of $\succcurlyeq$.

PROPOSITION 1. *If $\succcurlyeq$ is a compatible preference order on $X \times \mathscr{E}$, then for every $x, y \in X$, and any disjoint $E, F \in \mathscr{E}$, we have*

$$(x, E) \succcurlyeq (y, F) \Leftrightarrow U(x, E) \geqslant U(y, F). \tag{9}$$

Proof. Suppose E and F are disjoint events, and $x, y \in X$. Choose any $z \in X$ such that $z \sim^E x$ and $E \sim^z F$ (there is one by A8). By the compatibility of $\succcurlyeq$, we then have $(x, E) \succcurlyeq (y, F) \Leftrightarrow z \succcurlyeq^F y$, and $U(x, E) = U(z, F)$, from which (9) follows. ∎

Proposition 1 shows that there is only one compatible ranking of any two situations, (x, E) and (y, F), with E and F disjoint, and this ranking is the one defined by U. There remains the question, however, of the interpretation of (9) when E and F are not disjoint. Our assumptions so far do not rule out the possibility that non-disjoint events are not comparable under any $\sim^x$. As a result, compatible preference orders on $X \times \mathscr{E}$ can rank situations involving events that are not disjoint in a non-unique way. To remedy this situation, we introduce a new assumption, that has no effect on any of our earlier results:

A11. For every $x \in X$, and any disjoint $E, F \in \mathscr{F}$, $E \sim^x F$ implies $E \sim^x E \cup F$.

The following is proved in Appendix B, and completes our interpretation of U as a utility function of preferences over situations:

PROPOSITION 2. *Under the additional assumption A*11, (9) *defines the only possible compatible preference order*, $\succcurlyeq$, *on* $X \times \mathscr{E}$.

The above discussion shows that Theorem 1 and its assumptions could be formulated directly in terms of a preference order on the space of situations, simplifying somewhat its formal structure. A complete ranking of arbitrary situations is, however, more difficult to interpret as a primitive than the rankings expressed by conditional preferences with respect to events and acts. The more elements two situations have in common, the easier it is for a decision maker to visualize them and compare them. Conditional preferences involve the comparison of situations having an underlying event or act in common. A preference order over situations, on the other hand, involves the comparison of potentially completely unrelated situations. What Proposition 2 shows is that, under our assumptions, the consistent ranking of related situations induces a unique ranking of all situations.

Finally, given the preference order $\succcurlyeq$ on situations of Proposition 2, we can return to our first question, and formulate another sense in which P is consistent with the decision maker's subjective beliefs, which directly generalizes Savage's [30] formulation: for any $x, y \in X$ and $E, F \in \mathscr{E}$ such that $(x, E) \sim (y, F) \succ (x, E^c) \sim (y, F^c)$, we have $x \succcurlyeq^{\Omega} y \Leftrightarrow P(E) \geqslant P(F)$. This is an easy consequence of the fact that U is additive under P, and represents $\succcurlyeq$.

The following section provides examples of the application of Theorem 1 with separable and nonseparable preferences. We close this section with brief remarks on some variations of Theorem 1, which should be omitted on a first reading.

Remark 1. If $\mathscr{F}$ is finite, and (U, P) is as in the conclusion of Theorem 1, we can also view U as a function of the form $U: X \to L$ satisfying

$$U(x, E) = \frac{1}{P(E)} \int_E U(x)(\omega)\, dP(\omega), \qquad x \in X, \quad E \in \mathscr{E}. \tag{10}$$

Appendix A provides a sufficient and necessary condition for such a representation to be valid (in the context of Theorem 1) with an infinite number of events and P countably additive.

Remark 2. Another variation of Theorem 1 applies with only two states of uncertainty. This is achieved by eliminating the assumption in A5 that there are more than two disjoint events, while replacing the last phrase in A10 with "it is not the case that $(x_1, x_2) =^{E}_{\Omega\setminus E} (x_2, x_3)$ and $(x_1, x_2) \neq^{G}_{\Omega\setminus G} (x_2, x_3)$," thus eliminating the need for the third event F. A proof of such a version of Theorem 1 is a generalization of Wakker's [36] proof of his subjective expected utility theorem, using similar arguments as those used in the current proof of Theorem 1. An underlying additivity theorem that works with only two components can be used, based on the "hexagon condition" (see Wakker [36] and Krantz *et al.* [20]), which can be verified using the modified condition A10. The details are left to the interested reader.

Remark 3. In some applications the assumption that X is compact may appear overly restrictive. Using Theorem 2 of C&A, however, we can modify the assumptions of Theorem 1 by relaxing compactness, while strengthening A4 (solvability) to a condition of "continuous solvability" stated in C&A. The main results remain valid under these assumptions, without any substantial change in their proof. Moreover, Example 3 below can easily be modified so that M is unbounded (and therefore X is not compact), and continuous solvability holds. Intuitively, continuous solvability states that, in A4, small perturbations of the acts x_i should correspond to small changes in the compensating act x.

5. EXAMPLES: SEPARABILITY AND DISAPPOINTMENT AVERSION

We conclude with examples that illustrate the application of Theorem 1 under separable or nonseparable preferences. For simplicity of exposition, and to avoid purely technical issues, we assume the following throughout this section: Ω is finite, $\mathcal{N} = \{\varnothing\}$, $\mathcal{F}$ consists of all subsets of Ω, and X consists of all functions of the form $x: \Omega \to M$. As in Examples 1 and 2, M represents a space of objective payoffs.[12] We also assume throughout that

[12] The interested reader will have no difficulty providing an extension of this discussion to the case of an infinite number of events, using Theorem 2 of Appendix A in conjunction with Theorem A4 of C&A. (For the separable case, also see the treatments in Vind and Grodal [33, Chapter VI], and Wakker [36, Chapter V].)

(U, P) is as in Theorem 1, and by taking an appropriate affine transformation of U, we assume, without loss of generality, that $U(\cdot, E)$ maps X onto the unit interval, $[0, 1]$, for every $E \in \mathscr{E}$.

In general, $U(x, \{\omega\})$ represents the utility of the subjective consequences of x at state ω. Under the separability condition (2) (applied with $E = \{\omega\}$), it follows immediately that $U(x, \{\omega\}) = f(x(\omega), \omega)$, for some function $f: M \times \Omega \to [0, 1]$. In this case, the representation of Theorem 1 reduces to state-dependent expected utility. State-independent expected utility is obtained if, in addition to separability, one assumes that

$$(x = m \text{ on } E \cup F \text{ and } E \cap F = \emptyset) \Rightarrow E \sim^x F, \quad E, F \in \mathscr{E}, \quad x \in X, \quad m \in M. \tag{11}$$

This condition simply states that a payoff m is valued the same on any two disjoint events. (The condition is formulated in a form that applies even when all singleton events are null. In the current special context, it suffices to assume that (11) holds for singleton E and F.) Given separability, (2), and state independence, (11), it follows easily that $U(x, \omega) = f(x(\omega))$ for some function $f: M \to [0, 1]$, and the representation of Theorem 1 reduces to a state-independent expected utility representation.

While Theorem 1 reduces to familiar situations under separability, the main innovation of the result is that it can also accommodate nonseparable preferences. We demonstrate with the case of disappointment aversion, introduced in Example 2, where the utility of a consequence of an act at a state depends not only on the act's payoff at the given state, but also on the unconditional utility of the act. The following result formalizes such a representation, and also summarizes the discussion of the last paragraph. Part (a) of Proposition 3 characterizes the separable case, as discussed above. Part (b) characterizes state-independent preferences, but more broadly than discussed above, since we now allow for the possibility of nonseparable state-independent preferences exhibiting disappointment aversion. Finally, part (c) gives a fixed-point characterization of an act's unconditional utility under disappointment aversion:

Proposition 3. *In the context of this section, disappointment aversion in the sense of* (3) *is equivalent to the existence of a function* $f: \Omega \times M \times [0, 1] \to [0, 1]$ *that is nonincreasing in its last argument, and satisfies*

$$U(x, E) = \int_E f(\omega, x(\omega), U(x, \Omega))\, dP(\omega), \qquad x \in X, \quad E \in \mathscr{E}. \tag{12}$$

Moreover, the following hold:

(a) *The function f can be chosen not to depend on its third (utility) argument if and only if preferences are separable (that is, (2) holds).*

(b) *The function f can be chosen not to depend on its first (state) argument if and only if preferences are state-independent (that is, (11) holds).*

(c) *$U(x, \Omega)$ uniquely solves the equation*

$$U(x, \Omega) = \int_{\Omega} f(\omega, x(\omega), U(x, \Omega))\, dP(\omega). \tag{13}$$

The proof is a straightforward extension of the proof of Proposition 2 of C&A, and is therefore omitted. It can also be easily confirmed that strict monotonicity of f in its third (utility) argument corresponds to strict disappointment aversion in the sense of Example 2.

Under state-independence, the implicit utility representation (13) is of the type modeled by Dekel [7], who considered preferences over objective probability distributions. In Dekel's representation f is monotone in its second (payoff) argument, rather than its third (utility) argument, and consequently his representation is not linked to disappointment aversion. Dekel's formulation is easily embedded in our setting, however, by replacing (3) with the assumption

$$(x \geqslant y \text{ on } E \text{ and } x \sim^{\Omega} y) \Rightarrow x \succcurlyeq^{E} y, \qquad x, y \in X, \quad E \in \mathscr{E},$$

assuming that M is ordered by $\geqslant$. (The latter condition can also be combined with the disappointment-aversion condition, (3), obtaining a function f in Proposition 3 that is nondecreasing in its payoff argument, and nonincreasing in its utility argument.) In more special parametric models using objective probabilities, disappointment has been formalized by Bell [4], Loomes and Sugden [23], and within Dekel's class of preferences, by Gul [14].

Finally, we use the current context to provide a concrete instance of the primitives of this paper for which all the assumptions of Theorem 1 are satisfied, without assuming separability or state-independence. For simplicity, we will make this example more restrictive than necessary. (See Appendix A of C&A for the infinite state-space case.)

EXAMPLE 3. With $(\Omega, \mathscr{F}, \mathscr{N}, X)$ as fixed earlier in this section, and with $M = [0, 1]$, we will define conditional preferences on X in terms of a strictly positive probability P on $\mathscr{F}$, and a function $f\colon \Omega \times M \times [0, 1] \to [0, 1]$, assumed to satisfy the following conditions:

(a) f is nonincreasing in its last argument.

(b) f is jointly continuous in its last two arguments at every state.

(c) $f(\omega, 0, v) = 1 - f(\omega, 1, v) = 0$ for all $\omega \in \Omega$ and $v \in [0, 1]$.

Given such an f and P, (13) has a unique solution in $U(x, \Omega)$. Uniqueness follows from (a), just as in Proposition 3. For an existence proof, let $g: [0, 1] \to [0, 1]$ be the continuous function defined by $g(v) = \int_\Omega f(\omega, x(\omega), v)\, dP(\omega) - v$, and notice that $g(0) \geqslant 0$ while $g(1) \leqslant 0$, implying that g vanishes somewhere on $[0, 1]$. Having defined $U(x, \Omega)$ implicitly by (13), we use (12) to define $U(x, E)$ for all $x \in X$ and $E \in \mathscr{E}$, and (5) and (6) to define all relations of the form $\succcurlyeq^E$ and $\sim^x$. The reader can now confirm that A1 through A10 hold. The preferences just defined satisfy strict disappointment aversion, and are therefore nonseparable.

C&A discusses settings in which acts are not assumed to have a state-contingent payoff structure in the sense of the above examples, but Theorem 1 still applies. For example, X could be (or have a component that is) a set of opportunity sets, a formalism useful for modeling "regret" as in Example 2 of C&A, or "preferences for flexibility" as in Kreps [21]. Or X could be a set of algebras, representing possible information sets. Such a setting is useful in modeling the subjective value of information (Example 3 of C&A).

APPENDIX A: COUNTABLE ADDITIVITY

In this appendix we show that adding a (necessary) continuity requirement in the list of assumptions of Theorem 1 allows us to conclude that the probability P is countably additive, and that a representation of the form given by (10) exists.

Throughout this appendix we assume that $\mathscr{F}$ is a σ-algebra and $\mathscr{N}$ is a σ-ideal. (That is, in addition to their previously assumed properties, $\mathscr{F}$ and $\mathscr{N}$ are closed under countable disjoint unions.) The additional assumption we will use is as follows:

A12. Suppose that $\{x_n : n = 1, 2, ...\}$ is a sequence of acts converging to some act x, and that $\{E_n : n = 1, 2, ...\}$ is an increasing ($E_{n+1} \supseteq E_n$ for all n) sequence of events such that $\bigcup_{n=1}^{\infty} E_n = \Omega$. If there is some $y \in X$ such that $x_n \sim^{E_n} y$ for all n, then $x \sim^{\Omega} y$.

The following result extends Theorem 1, by deriving countable additivity and a density representation as in (10). We call two functions of the form $U: X \to L$ and $\tilde{U}: X \to L$ versions of each other if $\{U(x) \neq \tilde{U}(x)\} \in \mathscr{N}$ for every $x \in X$.

THEOREM 2. *Suppose that A1 through A10, and A12 are all satisfied. Then there exist a function $U: X \times \mathscr{E} \to \mathbb{R}$ and a countably additive probability P satisfying (a), (b), and (c) of Theorem 1. Given U and P, (10) holds for some, unique up to versions, $U: X \to L$. Finally, A12 is necessary for this extension of Theorem 1.*

Proof. Let U and P be as in Theorem 1. We will first show that P is countably additive. By parts (b) and (c) of Theorem 1, we can and do assume that $U(\cdot, E)$ maps onto the interval $[0, 1]$ for every $E \in \mathscr{E}$. Let $\{E_n: n = 1, 2, ...\}$ be a decreasing sequences of events such that $\bigcap_{n=1}^{\infty} E_n = \varnothing$. To show countable additivity of P, it suffices to prove that the sequence $\{P(E_n)\}$ converges to zero. For any given n, let $\mathscr{F}_n$ be the (finite) algebra generated by the events $\{E_1, ..., E_n\}$. Using A4 (solvability), we can then choose $y_n \in X$ such that $U(y_n, F) = 0$ for every non-null $F \in \mathscr{F}_n$. Similarly, there is an $x_n \in X$ such that $U(x_n, E_n) = 1$ and $U(x_n, \Omega \backslash E_n) = 0$. Given that X is compact, we can assume (after passing to a subsequence) that the sequences $\{x_n\}$ and $\{y_n\}$ converge to some x and y, respectively. Clearly, $U(y, \Omega) = 0$. Also, for every given n, we have $x_n \sim^{\Omega \backslash E_n} y_m$ for all $m \geqslant n$, and therefore $x_n \sim^{\Omega \backslash E_n} y$ for all n. Using A12, we conclude that $x \sim^{\Omega} y$. Finally, (7) implies that

$$P(E_n) = U(x_n, \Omega) \to U(x, \Omega) = U(y, \Omega) = 0 \qquad \text{as} \quad n \to \infty.$$

This proves that P is countably additive.

For any given $x \in X$, we can now define the finite measure μ_x on $\mathscr{F}$, by letting $\mu_x(E) = U(x, E)\, P(E)$ for all $E \in \mathscr{E}$ and $\mu_x(N) = 0$ for all $N \in \mathscr{N}$. The measure μ_x is countably additive, since U is valued in $[0, 1]$, and is of course dominated by P. Applying the Radon–Nikodym theorem, the unique (up to versions) representation in (10) follows.

The proof of necessity is straightforward, and is left to the reader. ∎

APPENDIX B: PROOFS

This appendix contains the proofs omitted from the main text.

Proof of Theorem 1

The following proof generalizes an argument due to Wakker [36, Theorem IV.2.7], whose setting assumes state-contingent acts, separability, and state-independence. (In fact, our setting differs from Wakker's even under these assumptions, but not in a significant way). We assume throughout that A1–A10 hold.

Step 0: Preliminaries. We begin by assuming that $\mathscr{F}$ is finite and that $\mathscr{N} = \{\varnothing\}$. This assumption will be in effect until Step 4 below. We also collect here some terminology, notation, and facts that will be of use throughout the proof. A pair (U, P) is an *additive representation* if

(a) U is a function of the form $U: X \to L$, P is a probability on $\mathscr{F}$, and (4) holds.

(b) Any other function $\tilde{U}$ such that (a) holds with $\tilde{U}$ in place of U satisfies: $P[\tilde{U}=\alpha U+\beta]=1$ for some $\alpha \in (0, \infty)$ and $\beta \in L$.

(c) $\int_E U(\cdot)(\omega)\, dP(\omega)$ is continuous for every $E \in \mathscr{E}$.

Theorem 2 of C&A implies that, given our standing assumptions, an additive representation exists. Given an additive representation (U, P), we also regard U as a function of the form $U: X \times \mathscr{E} \to \mathbb{R}$ satisfying (10). Moreover, the following are true: (i) For any $\alpha \in (0, \infty)$ and $\beta \in L$, $(\alpha U+\beta, P)$ is also an additive representation; and (ii) If Q is another probability with the same null sets as P, then $(U\, dP/dQ, Q)$ is also an additive representation (where dP/dQ denotes a Radon–Nikodym derivative). These facts will be used without further explanation.

Next, we introduce some convenient notation. We denote by $\{F_1, ..., F_n\}$ the partition of Ω that generates $\mathscr{F}$. Our standing assumptions imply that each F_i is non-null, and that $n \geqslant 3$. We simplify the notation by using the index i to denote the set F_i, for every i. Therefore, $\succcurlyeq^i = \succcurlyeq^{F_i}$, and we write $i \succcurlyeq^x j$ instead of $F_i \succcurlyeq^x F_j$. The statements $x \sim^i y$ and $i \sim^x j$ are analogously defined. In the same vein, we write $=^i_j$ instead of $=^{F_i}_{F_j}$, $x_i y_i$ instead of $x_{F_i} y_{F_j}$, and so on. (Consequently, $x_i y_j$ represents any choice of an act v that satisfies $v \sim^i x$ and $v \sim^j y$.) By A4, an $x_i y_j$ always exists.

We now fix a reference probability Q, defined by $Q(F_i)=1/n$ for all $i \in \{1, ..., n\}$, and a function $V: X \to L$ such that (V, Q) is an additive representation. For every i and x, we define $V_i(x)=V(x)(\omega)$ for all $\omega \in F_i$. Notice that the function V_i is then a utility representation of $\succcurlyeq^i$. The pair (U, P) of Theorem 1 will be constructed from (V, Q) by appropriate transformations of the type (i) and (ii) discussed above.

Step 1: Construction of U and P. Given any $i, j \in \{1, ..., n\}$, and any act x, A8 implies that there exists an act y such that $y \sim^i x$ and $i \sim^y j$. For any given x, we let x_{ij} denote a choice of such an act y. From A9, it follows that no matter what the particular choice of x_{ij} and y_{ij} is, we have

$$x \succcurlyeq^i y \Leftrightarrow x_{ij} \succcurlyeq^i y_{ij} \Leftrightarrow x_{ij} \succcurlyeq^j y_{ij}. \tag{14}$$

We now define the functions $U_i: X \to \mathbb{R}$, by letting $U_1=V_1$, and for $i \in \{2, ..., n\}$, $U_i(x)=V_1(x_{i1})$. Because of (14), the definition of $U_i(x)$ is independent of the particular choice of x_{i1}, and U_i is a utility representation of $\succcurlyeq^i$.

For every i, let $I_i=\{V_i(x): x \in X\}$, an interval since X is assumed connected and V_i is continuous. By A5, each interval I_i has nonzero length. Fix any $i \in \{1, ..., n\}$. Since U_i and V_i both represent the same preference order, there exists a strictly increasing function $\phi_i: I_i \to I_1$ such that $U_i(x) = \phi_i(V_i(x))$ for all $x \in X$. Moreover, ϕ_i is surjective. (This is because, by A8 and A9, an x_{1i} exists and $U_1(x)=V_i(x_{1i})$, for every x.) Therefore, ϕ_i is also continuous.

LEMMA. *ϕ_i is affine.*

Proof. Let I_k^0 denote the interior of I_k for every k. Following Wakker [36], we will show that, for any $\alpha \in I_i^0$, there is a positive ε such that

$$\phi_i(\alpha+\delta)-\phi_i(\alpha)=\phi_i(\alpha)-\phi_i(\alpha-\delta), \qquad \text{for all} \quad \delta \in (0, \varepsilon). \tag{15}$$

A standard exercise in real analysis shows that this property of ϕ_i, together with continuity, implies that ϕ_i is affine.

Fix any $\alpha \in I_i^0$ and any $j \notin \{1, i\}$. By A8, there exists an act y such that

$$\alpha = V_i(y) \qquad \text{and} \qquad 1 \sim^{y} i. \tag{16}$$

We fix such a y, and we notice, using A8 and A9, that $V_1(y) \in I_1^0$. There exists, therefore, a sufficiently small $\varepsilon > 0$ with the following property: Given any $\delta \in (0, \varepsilon)$, there exist $x, z \in X$ such that

$$V_i(z)-V_i(y)=V_i(y)-V_i(x)=\delta, \tag{17}$$

$$V_i(x)+V_j(v)=V_i(y)+V_j(w), \tag{18}$$

$$V_1(x)+V_j(v')=V_1(y)+V_j(w'), \tag{19}$$

for some acts v, v', w, w'. Furthermore, by A8 we can and do assume that the x, z above are always chosen so that

$$1 \sim^{x} i \qquad \text{and} \qquad 1 \sim^{z} i. \tag{20}$$

Using the shorthand notation $\sim^{ij} = \sim^{F_i \cup F_j}$, one can easily check, using solvability (A4) and coherence (A3), that

$$x \sim^{ij} y \Leftrightarrow V_i(x)+V_j(x)=V_i(y)+V_j(y), \qquad x, y \in X.$$

Combining this fact with (17) and (18), we obtain

$$x_i v_j \sim^{ij} y_i w_j \qquad \text{and} \qquad y_i v_j \sim^{ij} z_i w_j.$$

Therefore $(x, y) =_j^i (y, z)$, and by A10 it is not the case that $(x, y) \neq_j^1 (y, z)$. But (19) gives $x_1 v'_j \sim^{1j} y_1 w'_j$, and therefore $y_1 v'_j \sim^{1j} z_1 w'_j$. The last two indifferences together give

$$V_1(z)-V_1(y)=V_1(y)-V_1(x)=V_j(v')-V_j(w').$$

Using (16) and (20), and the definition of U_i, we therefore have $U_i(z)-U_i(y) = U_i(y)-U_i(x)$. Finally, using the definition of ϕ_i, and (16) and (17), we see that (15) holds, and the proof that ϕ_i is affine is complete. ▮

Given the Lemma, suppose now that $U_i = \alpha_i V_i + \beta_i$ for some $\alpha_i \in (0, \infty)$ and $\beta_i \in \mathbb{R}$, for all $i \in \{1, ..., n\}$. We define the probability P on F by letting $P(F_i)/Q(F_i) = \alpha/\alpha_i$, where α is the unique constant that makes P a probability. The function $U: X \to L$ is then defined by letting $U(x)(\omega) = U_i(x)$ whenever $\omega \in F_i$.

Step 2: (U, P) is an additive representation satisfying (6) for all disjoint $E, F \in \mathscr{E}$. Define $\beta \in L$ by letting, for every i, $\beta = \beta_i P(F_i)/Q(F_i)$ on F_i. It follows from the definitions that $U(dP/dQ) = \alpha V + \beta$, and therefore $(U(dP/dQ), Q)$ is also an additive representation, and therefore so is (U, P). Defining $U: X \times \mathscr{E} \to \mathbb{R}$ through (10), it follows that (5), (7), and (8) hold for all $x \in X$ and disjoint events E, F.

To show (6), we use induction on the *size* of $E \cup F$, defined as the number of events of the partition $\{F_1, ..., F_n\}$ whose union is $E \cup F$. For size two, we have $E, F \in \{F_1, ..., F_n\}$, and the claim is immediate from the definition of U. Suppose now that $s \in \{3, 4, ...\}$, and (6) holds for all $x \in X$ and disjoint events E, F such that $E \cup F$ has size less than s. Fixing any disjoint events E and F such that $E \cup F$ has size s, we now prove (6) for any $x \in X$.

Let $E = E_1 \cup \cdots \cup E_m$, where $E_i \in \{F_1, ..., F_n\}$ for all i. Since both $U(x, E)$ and $U(x, F)$ are in I_1, the range of each V_k, there exists, for any $i \in \{1, ..., m\}$ an act y_i such that $U(y_i, E_i) = U(x, E)$. By A4, we can then choose a single act y such that $y \sim^F x$ and $y \sim^{E_i} y_i$ for all i. Fixing such a y, we have $U(y, F) = U(x, F)$ and $U(y, E_i) = U(y, E) = U(x, E)$ for all i. By the induction hypothesis, we then have $E_1 \sim^y E \backslash E_1$. On the other hand, A9 implies that $E \sim^x F \Leftrightarrow E \sim^y F$. Given these facts, (6) now follows directly from A7 and the induction hypothesis.

Step 3: Part (b) of the Theorem holds. Let $\tilde{U}: X \times \mathscr{F} \to \mathbb{R}$ and the probability $\tilde{P}$ also satisfy part (a) of the Theorem, as (U, P) has been shown to do. Through equation (10), we regard U and $\tilde{U}$ as functions from X to L, as well. Since $(U(dP/d\tilde{P}), \tilde{P})$ is an additive representation, there exist $\alpha \in (0, \infty)$ and $\beta \in L$ such that

$$\tilde{U} = \alpha U \frac{dP}{d\tilde{P}} + \beta. \tag{21}$$

We will show that in fact both $dP/d\tilde{P}$ and β must be constant, thus proving the result. For every $x \in X$ and $i \in \{1, ..., n\}$, let $U_i(x) = U(x)(\omega)$ for all $\omega \in F_i$, $P_i = P(F_i)$, and define $\tilde{U}_i$ and $\tilde{P}_i$ analogously. From (21) we then obtain

$$(\tilde{U}_k(x) - \tilde{U}_k(y)) = \alpha(U_k(x) - U_k(y)) \frac{P_k}{\tilde{P}_k}, \qquad x, y \in X, \quad i \in \{1, ..., n\}. \tag{22}$$

Fix any given $i, j \in \{1, ..., n\}$, divide the equation obtained from (22) by setting $k = i$ with the equation obtained from (22) by setting $k = j$. Moreover, assume that $x = v_{ij}$ and $y = w_{ik}$ for some $v, w \in X$ such that $v \succ^i w$, and that $i \neq j$. The result is the equation $P_i/\tilde{P}_i = P_j/\tilde{P}_j$. This proves that $dP/d\tilde{P}$ is constant, and therefore $P = \tilde{P}$, and (21) reduces to $\tilde{U} = \alpha U + \beta$. Given any distinct i, j, we can then derive the equation

$$\tilde{U}_i(x) - \tilde{U}_j(x) = \alpha(U_i(y) - U_j(y)) + (\beta_i - \beta_j), \qquad x, y \in X.$$

By choosing $x, y \in X$ such that $i \sim^x j$ and $i \sim^y j$, it follows that $\beta_i = \beta_j$, and therefore β is constant.

This proves part (b). Part (c) is immediate from the definition of U, and we have therefore proved the Theorem for a finite number of events and no nonempty null sets.

Step 4: The general case. Suppose first that $\mathscr{F}$ is finite, but there exist nonempty null sets. Then there is a unique decomposition $\Omega = \Omega_0 \cup \Omega_1$, where $\mathscr{N} = \{F \in \mathscr{F} : F \subseteq \Omega_0\}$ and $\Omega_0 \cap \Omega_1 = \emptyset$. The result follows by applying Steps 0 through 4 on the restricted state-space Ω_1, and by letting $P(N) = 0$ and $U(x, E \cup N) = U(x, E)$ for all $N \in \mathscr{N}$, $x \in X$, and $E \in \mathscr{E}$ such that $E \subseteq \Omega_1$.

Consider now the general case, with no restrictions on $\mathscr{F}$ or $\mathscr{N}$. Let an algebra of events be *nice* if it is finite and it contains three given disjoint events (fixed arbitrarily). Given any nice algebra $\mathscr{G}$, one can apply Theorem 1 for the finite case, to obtain a representation $(U_{\mathscr{G}}, P_{\mathscr{G}})$ satisfying all the conditions that (U, P) satisfies in the theorem, but with $\mathscr{G}$ in place of $\mathscr{F}$. If further $U_{\mathscr{G}}(\cdot, E)$ maps X onto $[0, 1]$ for all $E \in \mathscr{E}$, we call the pair $(U_{\mathscr{G}}, P_{\mathscr{G}})$ *a nice* $\mathscr{G}$*-representation*. There are two important facts to notice: First, given any nice algebra $\mathscr{G}$, any two nice $\mathscr{G}$-representations are identical. Second, if $\mathscr{H} \subseteq \mathscr{G}$ are nested nice algebras, then the nice $\mathscr{H}$-representation is the restriction of the nice $\mathscr{G}$-representation on $\mathscr{H}$. Given these observations, we can now consistently define (U, P) as follows: Given any $G \in \mathscr{F}$, pick any nice algebra $\mathscr{G}$ containing G, with corresponding nice $\mathscr{G}$-representation $(U_{\mathscr{G}}, P_{\mathscr{G}})$, and let $U(\cdot, G) = U_{\mathscr{G}}(\cdot, G)$ and $P(G) = P_{\mathscr{G}}(G)$. It is not hard to see that the definition of $(U(\cdot, G), P(G))$ does not depend on the specific choice of $\mathscr{G}$. Using this fact, one can then easily confirm that P is a finitely additive probability, and that all nice representations are restrictions of (U, P). The remaining conclusions of Theorem 1 follow from this observation and the finite case. ∎

Proof of Proposition 2

Clearly, U is the utility representation of a compatible preference order on situations. Conversely, suppose that $\succcurlyeq$ is a compatible preference order on $X \times \mathscr{E}$. We will show that (9) must hold. Fix arbitrary situations (x, E)

and (y, F). If $E=F$, then (9) clearly holds. We will therefore assume that $F\backslash E \neq \varnothing$ (the case in which $E\backslash F \neq \varnothing$ is analogous). Using A8, we also fix a $z \in X$ such that $z \sim^E x$ and $E \sim^z F\backslash E$. By compatibility of $\succcurlyeq$ and the properties of U, we have

$$U(x, E) = U(z, E) = U(z, F\backslash E) = U(z, E \cup F).$$

Using compatibility of $\succcurlyeq$, A11, the assumed properties of U, and the last set of equalities, we now confirm (9) in each of the following two cases:

Case A. $E \subseteq F$. In this case the following is true:

$$(x, E) \succcurlyeq (y, F) \Leftrightarrow (z, E) \succcurlyeq (y, F) \Leftrightarrow (z, F) \succcurlyeq (y, F)$$
$$\Leftrightarrow z \succcurlyeq^F y \Leftrightarrow U(z, F) \geqslant U(y, F) \Leftrightarrow U(x, E) \geqslant U(y, F).$$

Case B. $E \nsubseteq F$. In this case, we use, in addition to the earlier stated properties, the fact that Case A is already proved:

$$(x, E) \succcurlyeq (y, F) \Leftrightarrow (z, E) \succcurlyeq (y, F) \Leftrightarrow (z, F\backslash E) \succcurlyeq (y, F)$$
$$\Leftrightarrow U(z, F\backslash E) \geqslant U(y, F) \Leftrightarrow U(x, E) \geqslant U(y, F).$$

This completes the proof of Proposition 2. ∎

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