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Scale-invariant uncertainty-averse preferences and source-dependent constant relative risk aversion

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Preferences are defined over payoffs that are contingent on a finite number of states representing a horse race (Knightian uncertainty) and a roulette wheel (objective risk). The class of scale-invariant (SI) ambiguity-averse preferences, in a broad sense, is uniquely characterized by a multiple-prior utility representation. Adding a weak certainty-independence axiom is shown to imply either unit coefficient of relative risk aversion (CRRA) toward roulette risk or SI maxmin expected utility. Removing the weak independence axiom but adding a separability assumption on preferences over pure horse-race bets leads to source-dependent constant-relative-risk-aversion expected utility with a higher CRRA assigned to horse-race uncertainty than to roulette risk. The multiple-prior representation in this case is shown to generalize entropic variational preferences. An appendix characterizes the functional forms associated with SI ambiguity-averse preferences in terms of suitable weak independence axioms in place of scale invariance.

KEYWORDS. Uncertainty aversion, ambiguity aversion, source-dependent risk aversion, scale invariance, homotheticity.

JEL CLASSIFICATION. D81.

1. INTRODUCTION

Assuming we agree to use the definition of Gilboa and Schmeidler (1989) for ambiguity aversion (without their certainty-independence axiom), this paper characterizes all preferences within a broad class that are ambiguity averse and scale invariant (or homothetic). Ambiguity aversion means aversion to Knightian uncertainty, as is commonly motivated by the experiments of Ellsberg (1961). Scale invariance means that the ranking of any two contingent payoffs is not reversed if all amounts are scaled by the same constant. Scale invariance is ubiquitous in models[1] of macroeconomics and finance,

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[1]Scale invariance is behind Gorman aggregation and the associated representative–agent arguments, is an essential component of balanced growth models, and generally lends numerical tractability by reducing a model's dimensionality even in models with agent heterogeneity.

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as it provides the simplest reasonably realistic way to capture wealth effects. For example, increasingly common in economic modeling is the use of Epstein–Zin–Weil utility,[2] whose certainty equivalent (CE) corresponds to expected utility with a constant coefficient of relative risk aversion (CRRA), the only possible type of homothetic expected utility. Suppose we are interested in relaxing the assumption of an expected-utility CE, while requiring ambiguity aversion, without sacrificing scale invariance. What CE parameterizations should we consider? This paper gives a parsimonious answer to this question based on a simple axiomatic foundation.

Although the paper's main results do not include probabilities among their primitives, let us temporarily focus on the Anscombe and Aumann (1963) type setup of a preference order on state-contingent objective lotteries. The states can be thought of as outcomes of a horse race and the lotteries as roulette bets, the idea being that it is easier to assign probabilities to roulette outcomes than to horse-race outcomes. Ambiguity aversion implies that the agent is less averse to roulette uncertainty than to horse-race uncertainty. A seminal contribution that quantifies this idea is Gilboa and Schmeidler (1989; henceforth GS), whose uncertainty aversion axiom A.5 forms the basis for the definition of uncertainty aversion in this paper, too (albeit without reference to objective probabilities). GS also assume what they call *certainty independence*, which is key in generating their familiar multiple-prior representation. If the assumptions of this paper's first theorem are interpreted in the GS setting, they essentially[3] amount to replacing certainty independence with scale invariance. The resulting multiple-prior utility representation associates a unique CRRA γ with roulette risk. For $\gamma = 1$, the utility form is within the variational class studied by Maccheroni et al. (2006; henceforth MMR). For $\gamma \neq 1$, the utility form is similar to (but not the same as) multiplicatively variational representations appearing in Chateauneuf and Faro (2009; henceforth CF) and Cerreia-Vioglio et al. (2011; henceforth CMMM). Whereas GS, MMR, CF, and CMMM[4] derive functional forms for aggregating horse-race uncertainty using some type of weak independence axiom, in this paper's central theorem, similar utility structures are derived as a consequence of scale invariance, without any weak independence axiom.

The paper's central representation theorem is specialized in two different directions. The first direction clarifies the role of conditions that are the analogs of GS's certainty independence (CI) and MMR's weak certainty independence (WCI), which are used to characterize the scale-invariant case of maxmin expected utility (MEU) and variational

[2]The utility of Epstein and Zin (1989) and Weil (1990) is a parameterization of homothetic Kreps and Porteus (1978) utility, which includes expected discounted power or logarithmic utility. Another widely used preference class, exemplified by expected discounted exponential utility, is characterized by translation invariance relative to a constant consumption stream. (See Chapter 6 of Skiadas 2009 for the corresponding recursive-utility formulation.) The analysis of translation-invariant preferences also reduces to the scale-invariant case by passing to log consumption.

[3]Increasing preferences and strictly positive consumption are also assumed.

[4]The reference here is to Theorem 26 of CMMM, which is not the paper's main focus. The main contribution of CMMM is the application of a general form of quasiconcave duality to establish a unified multiple-prior representation of ambiguity-averse preferences in the GS setting, without any weak independence axiom. As explained in Appendix B.3, the duality results in the present paper build on the CMMM duality, based on the implications of scale invariance.

preferences. Introducing CI naturally implies MEU with a power or logarithmic von Neumann–Morgenstern index. What is less clear is how this MEU representation relates to the main result's multiplicatively variational representation for $\gamma \neq 1$. The answer is given in terms of a concrete expression that provides an alternative functional representation of MEU. Introducing WCI naturally results in the scale-invariant case of variational preferences in the sense of MMR, but more can be said. If $\gamma = 1$, corresponding to logarithmic variational preferences, then WCI is redundant, in the sense that it is a necessary condition of scale invariance and the other assumptions of the main representation theorem. If $\gamma \neq 1$, then WCI is equivalent to CI. In other words, for nonunit CRRA toward roulette risk, the scale-invariant case of MMR-type preferences does not take us beyond MEU.

The second direction in which the paper's central theorem is specialized leaves out any form of a weak independence axiom and, instead, imposes separability of preferences over pure horse-race contingent payoffs. The result is source-dependent constant relative risk aversion, with the CRRA associated with horse-race uncertainty being potentially higher than the CRRA associated with roulette risk. The utility form in this case is within a class of source-dependent expected utility appearing in Nau (2006) and Ergin and Gul (2009) (see also Chew and Sagi 2008). It also corresponds to the earlier formulation of Schroder and Skiadas (2003) in the context of continuous-time recursive utility (as shown in Skiadas 2012a). Put in the variational form of the central representation theorem, this utility class results in a new parametric specification, which converges to the entropic variational utility of Hansen and Sargent (2001) as the CRRA toward roulette risk converges to 1. The unit-CRRA case overlaps with the analysis of Strzalecki (2011). The novel variational representations for MEU and source-dependent CRRA provide a formal link between the two.

Although unrelated to the issue of ambiguity aversion, it is worth noting that the paper's main theorem embeds a simplified axiomatic foundation for scale-invariant subjective expected utility (SEU) with a finite state space. Assuming more than two states, it is shown that if a preference order over state-contingent payoffs is continuous, is increasing, and satisfies a separability condition that allows the application of Debreu's additive representation theorem, then scale invariance is equivalent to the existence of a unique probability and a constant-relative-risk-aversion expected-utility representation relative to this probability. In contrast, an SEU axiomatization without scale invariance requires considerably more structure, whether it be that of Savage (1954), Anscombe and Aumann (1963), or any of the SEU foundations building on the theory of additive conjoint measurement, as in the contributions of Luce and Krantz (1971) and Wakker (1984, 1989). In a variant of the latter approach, Skiadas (1997, 2009, Theorem 4.12) separates the conditions leading to an additive representation and a single, but rather elaborate, state-independence condition that delivers SEU. Here the simple ordinal condition of scale invariance entirely substitutes for state independence, and further implies (globally) constant risk attitudes and smoothness. This result has essentially been noted in Theorem 3.37 of Skiadas (2009) under a minor nonordinal regularity assumption, which is dispensed with in this paper. Appendix B.1 provides details, along with further discussion of related literature.

Scale invariance plays two roles in this paper. For preferences over roulette payoffs, it helps pin down an SEU representation and associated CRRA γ. Given the latter and a preference assumption, general payoffs can be equivalently represented as horse-race contingent utility levels. The second role of scale invariance is to put structure on the function that takes these contingent utility levels as input and gives the utility of the whole payoff as output. In GS, MMR, CF, and related papers, such structure is inferred from assumed weak independence axioms. Scale invariance restricts the type of weak independence axiom that can be assumed. For $\gamma = 1$, WCI is necessary. For $\gamma \neq 1$, CI must hold if WCI is assumed. For $\gamma \neq 1$ without WCI, the functional structure implied by scale invariance does not correspond exactly to any weak independence axiom in the literature, but it can nevertheless be characterized in terms of new weak independence conditions, which are similar to Axiom 5 of CF and Axiom 10 of CMMM, as spelled out in Appendix A.

The rest of this paper proceeds as follows. Section 2 introduces the preference restrictions that are adopted throughout the main part of the paper. Section 3 presents the central representation theorem, which is related to MEU and CI in Section 4, and to variational preferences and WCI in Section 5. The theory of source-dependent constant relative risk aversion is presented in Section 6. The role of Appendix A was pointed out in the preceding paragraph, Appendix B proves the main results, and Appendix C collects proofs omitted up to that point in the paper.

2. Scale-invariant ambiguity-averse preferences

There are two sources of uncertainty, represented by the two factors of the state space

$$\{1, \dots, R\} \times \{1, \dots, S\}, \quad \text{where } R > 2.$$

Informally, we think of $1, \dots, S$ as states representing Knightian uncertainty, for example, the possible outcomes of a horse race. We think of $1, \dots, R$ as states representing better understood uncertainty, for example, the possible outcomes of a roulette spin. A generic element of the state space, or just *state*, is denoted (r, s). We refer to elements (subsets) of $\{1, \dots, R\}$ as *roulette states* (*events*) and to elements (subsets) of $\{1, \dots, S\}$ as *horse-race states* (*events*).

A *payoff* is any mapping of the form $x : \{1, \dots, R\} \times \{1, \dots, S\} \to (0, \infty)$, with $x(r, s)$ or x_r^s denoting the value of x at state (r, s). We write X for the set of all payoffs, which we identify with $(0, \infty)^{R \times S}$. A *roulette payoff* is any payoff x whose value is a function of the roulette outcome only, that is, $x(r, s) = x(r, s')$ for all $r \in \{1, \dots, R\}$ and $s, s' \in \{1, \dots, S\}$. If x is a roulette payoff, we write x_r instead of x_r^s. Analogously, a payoff x is a *horse-race payoff* if $x(r, s) = x(r', s)$ for all $r, r' \in \{1, \dots, R\}$ and $s \in \{1, \dots, S\}$, in which case we write x^s instead of x_r^s. The set of all roulette (resp. horse-race) payoffs is denoted X_R (resp. X_S) and is identified with $(0, \infty)^R$ (resp. $(0, \infty)^S$). So while X_R and X_S are subsets of X, we also think of a payoff x as an R-by-S matrix, whose columns, denoted $x_*^1, \dots, x_*^S$, are roulette payoffs. For any $x, y \in X$ and roulette event B, $x_B y$ denotes the payoff

$$(x_B y)(r, s) = \begin{cases} x_r^s & \text{if } r \in B \\ y_r^s & \text{if } r \notin B. \end{cases}$$

Note that $x, y \in X_R$ implies $x_B y \in X_R$.

The central object of study is a binary relation $\succ$ on the set of payoffs X, representing an agent's preferences: $x \succ y$ means that the agent strictly prefers x to y. As usual, the corresponding relations $\succeq$ and $\sim$ on X are defined by $[x \succeq y \iff \text{not } y \succ x]$ and $[x \sim y \iff x \succeq y \text{ and } y \succeq x]$. The restriction of $\succ$ on X_R is denoted $\succ_R$:

$$x \succ_R y \quad \iff \quad x, y \in X_R \text{ and } x \succ y.$$

The following definition lists properties of $\succ$ that are imposed in each of the main representation theorems in this paper.

Definition 1. The relation $\succ$ is

- *increasing* if for all $x, y \in X$, $x \neq y \geq x$ implies $y \succ x$
- *continuous* if for all $x \in X$, the sets $\{y : y \succ x\}$ and $\{y : x \succ y\}$ are open
- a *preference order* if $\succeq$ is complete[5] and transitive[6]
- *scale invariant* if $x \succ y$ implies $\alpha x \succ \alpha y$ for all $\alpha \in (0, \infty)$
- $\succ_R$ *-monotone* if for all $x, y \in X$,

$$x_*^s \succ_R y_*^s \text{ for all } s \in \{1, \dots, S\} \quad \Longrightarrow \quad x \succ y$$

- *ambiguity averse* if for all $x, y \in X_S$,

$$x \sim y \quad \Longrightarrow \quad x_B y \succeq x \text{ for every roulette event } B.$$

The first three conditions are commonplace, while the fourth condition is the familiar homotheticity condition; it formalizes the notion of scale invariance that along with ambiguity aversion is this paper's focal point. The last two conditions of Definition 1 are analogous to Assumptions A.4 and A.5 of GS. The $\succ_R$-monotonicity condition requires that the agent prefers payoff x to payoff y if for every horse-race outcome s, the roulette payoff x_*^s is preferred to roulette payoff y_*^s. This is not an innocuous assumption, but we follow GS and the related literature in adopting it. Ambiguity aversion requires that if the agent is indifferent between horse-race payoffs x and y, then the agent (weakly) prefers to spin the roulette wheel and select x if the ball settles in B and y otherwise. (The condition corresponds to that of "second-order risk aversion" in Ergin and Gul 2009 and Strzalecki 2011.) A commonly used illustration is as follows.

Example 2. There are only two horses ($S = 2$), about which the agent has no information. Suppose $x = (100, 1)$ and $y = (1, 100)$ are horse-race payoffs. The agent's indifference between x and y reflects the symmetry of the situation but conceals the agent's discomfort with the fact that the probability π of the first horse winning is unknown.

[5]The relation $\succeq$ is *complete* if for all $x, y \in X$, either $x \succeq y$ or $y \succeq x$.

[6]The relation $\succeq$ is *transitive* if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Suppose also that B is a roulette event that the agent knows to have probability 1/2. Then $x_B y$ pays 100 or 1 with equal probability, for any given value of π. For this reason, $x_B y$ is preferred to either x or y. $\Diamond$

The preceding restrictions on $\succ$ imply that $\succ_R$ is an increasing, continuous, scale-invariant preference order on X_R. We will further assume that $\succ_R$ is separable.

Definition 3. The relation $\succ_R$ is *separable* if for all $x, y, z, z' \in X_R$ and $B \subseteq \{1, \dots, R\}$,

$$x_B z \succ_R y_B z \quad \iff \quad x_B z' \succ_R y_B z'.$$

In the representation theorems to follow, $\succ_R$ has a power or logarithmic expected-utility representation relative to a unique probability over roulette outcomes. Such a representation of $\succ_R$ follows from the assumption that $\succ_R$ is an increasing, continuous, scale-invariant, and separable preference order. The argument can be found as Theorem 17 in Appendix B.

3. Main representation theorem

This section presents the paper's central theorem, which characterizes all preference orders on X with the properties listed in Definitions 1 and 3. Subsequent results specialize this section's representation by imposing additional preference restrictions.

The following terminology and notation is used to state this section's theorem, as well as throughout the rest of this paper.

A *certainty equivalent* (CE) is any increasing[7] and continuous function of the form $\nu : X \to (0, \infty)$ satisfying[8] $\nu(\alpha \mathbf{1}) = \alpha$ for all $\alpha \in (0, \infty)$. The CE ν is said to *represent* $\succ$ if $\nu(x) > \nu(y)$ is equivalent to $x \succ y$.

For any positive integer n, we write

$$\Delta_n = \left\{ p \in (0,1)^n : \sum_{i=1}^{n} p_i = 1 \right\} \tag{1}$$

and $\bar{\Delta}_n$ for the closure of Δ_n, that is, $\bar{\Delta}_n = \{p \in [0,1]^n : \sum_i p_i = 1\}$. In particular, Δ_R (resp. Δ_S) is the set of all priors over roulette (resp. horse-race) states that assign a positive mass to every state.

Given any scalar γ, we write u_γ for the real-valued function on $(0, \infty)$ defined by

$$u_\gamma(z) = \begin{cases} z^{1-\gamma}/(1-\gamma) & \text{if } \gamma \neq 1 \\ \log(z) & \text{if } \gamma = 1. \end{cases} \tag{2}$$

This serves as a convenient choice of a von Neumann–Morgenstern index with constant CRRA γ, representing risk aversion toward roulette risk. Note that the image set of u_γ, denoted $u_\gamma(0, \infty)$, is equal to $(0, \infty)$ if $\gamma < 1$, to $\mathbb{R}$ if $\gamma = 1$, and to $(-\infty, 0)$ if $\gamma > 1$.

[7]Throughout this paper, we use the term *increasing* in the strict sense: $x \geq y \neq x$ implies $\nu(x) > \nu(y)$.
[8]We use the notation $\mathbf{1} = (1, 1, \dots, 1)$, the dimensionality being implied by the context.

Finally, we define a set $\mathcal{C}_\gamma$ of functions over horse-race priors, whose role is similar to those appearing in the variational utility forms of MMR and CF. The definition of $\mathcal{C}_\gamma$ is contingent on the CRRA γ in a way that reflects the image set $u_\gamma(0,\infty)$.

Definition 4. Given any $\gamma \in \mathbb{R}$, $\mathcal{C}_\gamma$ denotes the set of all functions C on Δ_S with the following properties:

- If $\gamma = 1$, then $\min C = 0$ and C is the restriction to Δ_S of a convex lower semicontinuous function $C : \bar{\Delta}_S \to \mathbb{R}_+ \cup \{\infty\}$.
- If $\gamma < 1$, then C is valued in $[1,\infty)$, $\min C = 1$, and $1/C$ is concave.
- If $\gamma > 1$, then C is valued in $(0,1]$, $\max C = 1$, and $1/C$ is convex.

Note that for every $C \in C_\gamma$ with $\gamma \neq 1$, both C and $1/C$ are finite-valued and continuous.[9] For any C, $\bar{C}$ in $\mathcal{C}_\gamma$, the notation $\bar{C} \geq C$ means $\bar{C}(q) \geq C(q)$ for all $q \in \Delta_S$.

The paper's central result follows.

Theorem 5. *Assuming $R > 2$, the following two conditions are equivalent:*

(i) The relation $\succ$ is a continuous, increasing, scale-invariant, $\succ_R$-monotone, and ambiguity-averse preference order, and $\succ_R$ is separable.

(ii) The CE $\nu : X \to (0,\infty)$ representing $\succ$ exists and takes the form

$$u_\gamma \circ \nu(x) = \min_{q \in \Delta_S} \begin{cases} \left(\sum_{s=1}^S q_s \sum_{r=1}^R p_r u_\gamma(x_r^s)\right) + C(q) & \text{if } \gamma = 1 \\ \left(\sum_{s=1}^S q_s \sum_{r=1}^R p_r u_\gamma(x_r^s)\right) C(q) & \text{if } \gamma \neq 1 \end{cases} \tag{3}$$

for some $p \in \Delta_R$, $\gamma \in \mathbb{R}$, and $C \in \mathcal{C}_\gamma$.

Assuming the two conditions are satisfied, the parameters p and γ are unique, and the function C can be uniquely selected to have the property: If representation (3) is also valid with any $\bar{C} : \Delta_S \to \mathbb{R}_+ \cup \{\infty\}$ in place of C, then $C \leq \bar{C}$ if $\gamma \leq 1$ and $\bar{C} \leq C$ if $\gamma > 1$.

We henceforth refer to the $C \in \mathcal{C}_\gamma$ of representation (3) with the preceding property as the *unique extremal* C (meaning minimal if $\gamma \leq 1$ and maximal if $\gamma > 1$).

For $\gamma = 1$, representation (3) is within the utility class characterized by MMR, whose Proposition 6 implies that the corresponding C is in fact unique[10] in $\mathcal{C}_1$ (without the requirement of minimality). We return to the relationship between the preceding theorem and MMR in Section 5. For $\gamma \neq 1$, representation (3) is closely related to but different than the representation of CF and its extension by CMMM, as further explained in Appendix A.

[9] In a finite-dimensional vector space, every convex or concave function with an open domain is continuous, a fact that can be applied here to the function $1/C$ on the open domain Δ_S.

[10] The uniqueness of C within $\mathcal{C}_1$ if $\gamma = 1$ is seen in the course of the proof of Theorem 5 (see Remark 24) to be a consequence of the Fenchel–Legendre duality that is behind the MMR representation. The multiplicative version of this duality used for $\gamma \neq 1$ does not generally imply the uniqueness of C within $\mathcal{C}_\gamma$. For instance, one can take the case $\gamma > 1$ in Example 9 and modify C by rotating upward the sloped sections of the graph of $1/C$ in Figure 1 while keeping the flat section the same. The requirement that C be extremal uniquely pins down C.

While a complete proof of Theorem 5 can be found in Appendix B, some of the underlying ideas are worth discussing here, as they lead to a better appreciation of the result and its relationship to the literature. We focus on the nontrivial implication (i) $\Longrightarrow$ (ii), and we proceed under the assumption that $\succ$ satisfies the theorem's condition (i) and ν is the CE representing $\succ$.

For $x \in X_R$, equation (3) reduces to $u_\gamma \circ \nu(x) = \sum_r p_r u_\gamma(x_r)$. The theorem's condition (i), therefore, must imply that $\succ_R$ admits an expected utility representation with prior p and a constant CRRA γ. This fact, which is proved in Appendix B.1, is the starting point of the proof of condition (ii) given condition (i). (Note also that Theorem 17 of Appendix B.1, whose relationship to the foundations of SEU was discussed in the Introduction, is the special case of Theorem 5 that is obtained by setting $S = 1$ and $R = n$.) Similarly to GS, an argument that hinges on the assumption that $\succ$ is $\succ_R$-monotone shows that there is an increasing, continuous function $f : u_\gamma(0, \infty)^S \to u_\gamma(0, \infty)$ such that $u_\gamma \circ \nu(x) = f(y_1, \dots, y_S)$, where $y_s = \sum_r p_r u_\gamma(x_r^s)$. In other words, the utility of the payoff x can be determined by first reducing x to an equivalent horse-race payoff y measured in utils and then computing $f(y)$. As in GS, ambiguity aversion is easily seen to be equivalent to the quasiconcavity of f. The central question is, what else can be said about the functional form of f?

Let us call f scale invariant (SI) if it is homogeneous of degree 1, and translation invariant (TI) if it is quasilinear with respect to the sure payoff $\mathbf{1}$. If f is quasiconcave and either SI or TI, it must also be concave. In GS, the certainty-independence axiom implies that f is both SI and TI. In this case, conjugate duality (in the sense of Section 12 of Rockafellar 1970) leads to the MEU functional form $f(y) = \min_{q \in K} q \cdot y$ for a nonempty, closed, convex $K \subseteq \Delta_S$. MMR weakened certainty independence in a way that implies that f is TI but not necessarily SI. Conjugate duality in this case leads to the functional form $f(y) = \min_{q \in \Delta_S} q \cdot y + C(q)$ for some $C \in \mathcal{C}_1$.

Certainty independence and its weaker version are discussed in the following two sections, but neither is assumed here; we rely on scale invariance instead. For $\gamma = 1$, the fact that $u_\gamma = \log$ and scale invariance of $\succ$ imply that f must be TI, resulting in an MMR-type representation in the first part of (3). For $\gamma \neq 1$, the fact that u_γ is a power function and scale invariance of $\succ$ imply that f must be SI. A corresponding duality theory in this case, developed in Appendix B, leads to the functional form $f(y) = \min_{q \in \Delta_S}(q \cdot y)C(q)$ for some $C \in \mathcal{C}_\gamma$, and hence the second part of representation (3). A subtlety in this argument involves showing that in the last representation, C and $1/C$ are finite-valued.

As this outline suggests, scale invariance restricts the type of weak independence axiom that can be assumed in a way that depends on the value of γ. If $\gamma = 1$, then scale invariance implies that f is TI, a property that is characterized by the weak certainty-independence axiom of MMR, discussed in Section 5. If $\gamma \neq 1$, then scale invariance implies that f is SI, a property that can also be characterized in terms of a certain weak independence axiom, closely related to an axiom first introduced by CF. This alternative way of characterizing the SI property of f is developed in Appendix A.

4. Scale invariance with certainty independence

The purpose of this section is to clarify the relationship between the representation of Theorem 5 and the scale-invariant case of the MEU representation characterized by GS. The essential assumption that GS make and does not appear in Theorem 5 is certainty independence. An analogous[11] condition in the current setting is the following. We write $\boldsymbol{\alpha}$ for the constant payoff that takes the value $\alpha \in (0, \infty)$ at every state.

Certainty Independence (CI). For any horse-race payoffs x, y, roulette events A, B, and $\alpha \in (0, \infty)$,

$$x_A\mathbf{1} \succ y_A\mathbf{1} \iff x_B\boldsymbol{\alpha} \succ y_B\boldsymbol{\alpha}.$$

The motivation behind certainty independence is that roulette mixing with a constant payoff cannot provide the type of hedging with respect to lack of knowledge of the prior suggested by the ambiguity-aversion condition and illustrated in Example 2.

Theorem 6 below specializes the representation of Theorem 5 by adding certainty independence, using the notation

$$\mathcal{K} = \text{set of all nonempty, closed, convex subsets of } \Delta_S$$

$$L(q) = \{x \in \mathbb{R}_+^S : q \cdot x = 1\} \quad \text{and} \quad \chi_K(q) = \begin{cases} 0 & \text{if } q \in K \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 6. *Assuming $R > 2$, the following three conditions are equivalent:*

(i) In addition to condition (i) of Theorem 5, $\succ$ satisfies Certainty Independence.

(ii) The CE $\nu : X \to (0, \infty)$ representing $\succ$ exists and takes the form

$$u_\gamma \circ \nu(x) = \min_{q \in K} \left\{ \sum_{s=1}^S q_s \sum_{r=1}^R p_r u_\gamma(x_r^s) \right\} \tag{4}$$

for some $\gamma \in \mathbb{R}$, $p \in \Delta_R$, and $K \in \mathcal{K}$.

(iii) Condition (ii) of Theorem 5 holds with $C \in \mathcal{C}_\gamma$ defined in terms of a set $K \in \mathcal{K}$ by[12]

$$C(q) = \begin{cases} \min_{\pi \in K} \max_{z \in L(q)} \pi \cdot z & \text{if } \gamma < 1 \\ \chi_K & \text{if } \gamma = 1 \\ \max_{\pi \in K} \min_{z \in L(q)} \pi \cdot z & \text{if } \gamma > 1. \end{cases} \tag{5}$$

Assuming the three conditions are satisfied, the parameters γ, p, and K are unique, and (5) defines the unique extremal $C \in \mathcal{C}_\gamma$ consistent with representation (3).

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[11] GS adopt a different formal setting of preferences over acts that are horse-race contingent objective probability distributions. Moreover, their certainty-independence condition is postulated relative to any acts x, y, not necessarily pure horse-race payoffs, as assumed in condition CI here. In the presence of our other assumptions, the latter difference is immaterial.

[12] By the minimax theorem, the order of min and max in (5) can be interchanged.

Remark 7. The equivalence of (4) and (3) with C defined in (5) remains true if the function u_γ is replaced by any function $u\colon (0,\infty)\to\mathbb{R}$ with the same image set as u_γ.

The argument leading to representation (4) is closely related to that of GS, as outlined in the discussion following Theorem 5. The third condition transforms the familiar representation of GS to one that fits within the unifying representation of Theorem 5. The proof of Theorem 6 in Appendix B shows that expression (5) is an easy consequence of the duality behind the proof of Theorem 5. It is worth noting, however, a direct proof of the equivalence of conditions (ii) and (iii) of Theorem 6, as well as of Remark 7, based on the following proposition, whose simple proof can be found in Appendix C.

Proposition 8. *Given any $K\in\mathcal{K}$ and $y\in(0,\infty)^S$,*

$$\min_{q\in K} q\cdot y = \min_{q\in\Delta_S}\left\{(q\cdot y)\min_{\pi\in K}\max_{z\in L(q)}\pi\cdot z\right\}. \tag{6}$$

The result remains true if the roles of min *and* max *are interchanged.*

The equivalence of representations (4) and (3) with C defined in (5) is a corollary of this proposition. If $\gamma<1$, then the image set $u_\gamma(0,\infty)$ is $(0,\infty)$ and the claim follows by applying identity (6) with $y_s=\sum_r p_r u_\gamma(x_r^s)$. If $\gamma>1$, then $u_\gamma(0,\infty)=(-\infty,0)$ and the claim follows by applying the dual identity to (6) obtained by interchanging max and min. The case $\gamma=1$ is obvious. This argument depends on u_γ only through its image set, thus verifying Remark 7.

We close the section with an example illustrating the membership of the function C defined in (5) to the set $\mathcal{C}_\gamma$ defined in the last section.

Example 9. Suppose $S=2$ and the CE ν is given by the following special case of (4):

$$u_\gamma\circ\nu(x)=\min\left\{w\sum_{r=1}^{R}p_r u_\gamma(x_r^1)+(1-w)\sum_{r=1}^{R}p_r u_\gamma(x_r^2) : w\in\left[\tfrac{1}{3},\tfrac{2}{3}\right]\right\}$$

for some $p\in\Delta_R$ and CRRA γ. An equivalent representation (3) also holds with C computed in (5). For $\gamma=1$, $C\in\mathcal{C}_1$ is defined by

$$C(w,1-w)=\begin{cases}0 & \text{if } w\in[\frac{1}{3},\frac{2}{3}]\\ \infty & \text{otherwise.}\end{cases}$$

For $\gamma\neq 1$, a simple calculation shows that (5) implies

$$\frac{1}{C(w,1-w)}=\begin{cases}\min\{3w,1,3(1-w)\} & \text{if } \gamma<1\\ \max\{(\frac{3}{2})(1-w),1,(\frac{3}{2})w\} & \text{if } \gamma>1.\end{cases}$$

The above function is graphed as the piecewise linear dashed lines in Figure 1, with the case $\gamma<1$ corresponding to the bottom, thin line and the case $\gamma>1$ corresponding to the top, thick line. The diagram clearly illustrates the fact that $C\in\mathcal{C}_\gamma$. The superimposed curves of Figure 1 are analogous examples of $1/C$ for a specification with source-dependent CRRA that is developed in Section 6. ◊

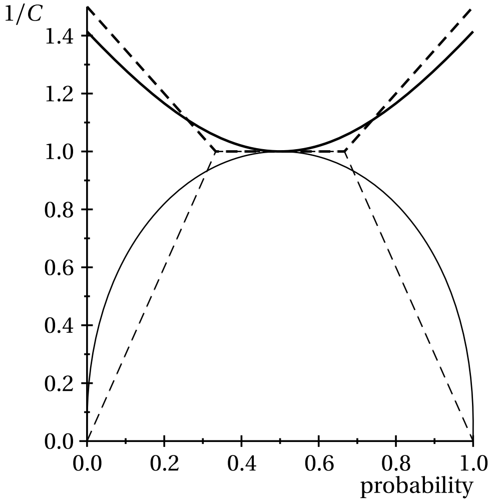

FIGURE 1. Graph of $1/C(w, 1-w)$ as a function of the probability weight w in Examples 9 and 14. The dashed lines correspond to Example 9: the thick line for $\gamma > 1$ and the thin line for $\gamma < 1$. The solid curves correspond to Example 14: the thick curve for $(\gamma, \hat{\gamma}) = (2, 3)$ and the thin curve for $(\gamma, \hat{\gamma}) = (0.1, 1.1)$.

5. Scale invariance with weak certainty independence

The GS characterization of MEU was extended by MMR to a class of variational preferences by weakening the certainty-independence axiom of GS. The analogous weakening of CI in our setting (similar to A2′ in Strzalecki 2011) is as follows.

WEAK CERTAINTY INDEPENDENCE (WCI). For any horse-race payoffs x, y, *roulette event* B, and $\alpha \in (0, \infty)$,

$$x_B \mathbf{1} \succ y_B \mathbf{1} \quad \Longleftrightarrow \quad x_B \boldsymbol{\alpha} \succ y_B \boldsymbol{\alpha}.$$

CI implies WCI, corresponding to the fact that the GS preference class is a subset of the MMR preference class. In contrast to CI, WCI allows the possibility that $x_A \mathbf{1} \succ y_A \mathbf{1}$ but not $x_B \mathbf{1} \succ y_B \mathbf{1}$ for some roulette events A, B. For example, suppose A is highly unlikely and B is certain, while x is more ambiguous but also more promising than y. It is conceivable that the agent is able to tolerate the ambiguity of x if there is only a tiny probability attached to it, but not if it is the whole bet.

Assuming roulette probabilities $p \in \Delta_R$ and a constant CRRA γ toward roulette risk, the MMR representation of a CE ν, adapted to the current setting, takes the form

$$u_\gamma \circ \nu(x) = \min_{q \in \Delta_S} \left\{ \left(\sum_{s=1}^{S} q_s \sum_{r=1}^{R} p_r u_\gamma(x_r^s) \right) + \mathbf{C}(q) \right\} \tag{7}$$

for some $\mathbf{C} \in \mathcal{C}_1$. Clearly, if $\succ$ has a CE representation ν of the form (7), then WCI must hold. The MEU form (4) is obtained by setting $\mathbf{C} = \chi_K$. Note that for $\gamma \neq 1$, the $\mathbf{C}$ of representation (7) is not the same type of object as the C of representation (3).

MMR use weak certainty independence to characterize representation (7) with a general von Neumann–Morgenstern index u in place of u_γ. The conditions of the main Theorem 5 not only imply that $u = u_\gamma$ for some γ, but also restrict the role of WCI and the associated representation (7) as follows.

THEOREM 10. *Suppose the two equivalent conditions of Theorem 5 are satisfied.*

(i) If $\gamma = 1$, then Weak Certainty Independence is necessarily satisfied. If $\gamma \neq 1$, then Weak Certainty Independence is satisfied if and only if Certainty Independence is satisfied (in which case Theorem 6 applies).

(ii) For any $\mathbf{C} \in \mathcal{C}_1$, the CE ν admits the representation (7) if and only if either $\mathbf{C} = \chi_K$ for some $K \in \mathcal{K}$ or $\gamma = 1$ and $\mathbf{C}$ equals the C of representation (3).

Part (i) says that given the conditions of Theorem 5, WCI is redundant if $\gamma = 1$ and equivalent to CI if $\gamma \neq 1$. Theorem 10 restricted to the case $\gamma = 1$ is a corollary of Theorem 5, since representations (3) and (7) coincide if $\gamma = 1$. To see the simple idea behind Theorem 10 for $\gamma \neq 1$, recall the discussion following Theorem 5, where it was pointed out that the variational representation (7) results from a convex dual representation of the function f under the assumption that f is TI, a property that is equivalent to WCI (see Lemma 20(iii)). If $\gamma \neq 1$, scale invariance implies that f must also be SI, but the case in which f is both TI and SI corresponds to MEU, characterized by CI. For this reason, if $\gamma \neq 1$, WCI implies CI and representation (7) reduces to MEU.

6. Source-dependent CRRA

In the last two sections, the scale-invariant and ambiguity-averse preference class of Theorem 5 was specialized by imposing conditions CI or WCI. In this section, the same class of preferences is specialized in another direction by imposing separability on preferences over (pure) horse-race payoffs, without any weak independence axiom. The result is a utility representation in which both preferences on horse-race payoffs and preferences on roulette payoffs admit a constant-relative-risk-aversion expected-utility representation with a unique prior. The key feature of the representation is that the CRRA assigned to horse-race payoffs can be higher than the CRRA assigned to roulette payoffs, as a reflection of ambiguity aversion. Of course, the dual, multiple-prior representation (3) remains valid in this case for a function C that is specified below. For $\gamma = 1$, C takes the form of relative entropy, corresponding to a well studied class of entropic variational preferences. For $\gamma \neq 1$, the function C takes a new form that generalizes the entropic specification, which can be obtained by letting γ approach 1.

We use the following extension of earlier notation and terminology. For any $x, y \in X_S$ and $A \subseteq \{1, \dots, S\}$, $x^A y$ denotes the horse-race payoff defined by

$$(x^A y)^s = \begin{cases} x^s & \text{if } s \in A \\ y^s & \text{if } s \notin A. \end{cases}$$

We let $\succ_S$ denote the restriction of $\succ$ to the set of horse-race payoffs:

$$x \succ_S y \quad \Longleftrightarrow \quad x, y \in X_S \text{ and } x \succ y.$$

We say that $\succ_S$ is *separable* if for every $x, y, z, z' \in X_S$ and $A \subseteq \{1, \ldots, S\}$,

$$x^A z \succ_S y^A z \quad \Longleftrightarrow \quad x^A z' \succ_S y^A z'.$$

THEOREM 11. *Assuming $R, S > 2$, the following three conditions are equivalent:*

(i) Condition (i) of Theorem 5 is satisfied and $\succ_S$ is separable.

(ii) There exist $(p, \gamma) \in \Delta_R \times \mathbb{R}$ and $(\hat{p}, \hat{\gamma}) \in \Delta_S \times \mathbb{R}$ with $\hat{\gamma} \geq \gamma$ such that the CE ν representing $\succ$ is given by

$$u_{\hat{\gamma}} \circ \nu(x) = \sum_{s=1}^{S} \hat{p}_s u_{\hat{\gamma}} \circ u_{\gamma}^{-1}\left(\sum_{r=1}^{R} p_r u_{\gamma}(x_r^s)\right), \quad x \in X. \tag{8}$$

(iii) Condition (ii) of Theorem 5 is satisfied with C given in terms of $\hat{p} \in \Delta_S$ as follows.

- *If $\gamma = 1$, then for some $\theta \in (0, \infty]$,*

$$C(q) = \theta \sum_{s=1}^{S} q_s \log\left(\frac{q_s}{\hat{p}_s}\right), \quad q \in \Delta_S \quad (\textit{with } \infty \cdot 0 = 0). \tag{9}$$

- *If $\gamma \neq 1$, then for some $\eta \in [-\infty, 1) \cup (1, \infty]$,*

$$\frac{1}{C(q)} = \left(\sum_{s=1}^{S} \hat{p}_s \left(\frac{q_s}{\hat{p}_s}\right)^{\eta}\right)^{1/\eta}, \tag{10}$$

where the cases $\eta \in \{0, \pm\infty\}$ are computed by taking a corresponding limit:

$$\frac{1}{C(q)} = \begin{cases} \exp(\sum_s \hat{p}_s \log(q_s/\hat{p}_s)) & \textit{if } \eta = 0 \\ \max_s\{q_s/\hat{p}_s\} & \textit{if } \eta = \infty \\ \min_s\{q_s/\hat{p}_s\} & \textit{if } \eta = -\infty. \end{cases} \tag{11}$$

Assuming these equivalent conditions are satisfied, all parameters in conditions (ii) and (iii) are unique, the parameters p, $\hat{p}$, and γ are common between the two conditions, and the remaining parameters are related by

$$\theta = \frac{1}{\hat{\gamma} - 1} \quad \textit{and} \quad \eta = \frac{\hat{\gamma} - 1}{\hat{\gamma} - \gamma}. \tag{12}$$

REMARK 12. The following extensions of Theorem 11 are also shown in its proof.

(i) Conditions (i) and (ii) remain equivalent if ambiguity aversion is omitted in condition (i) and the requirement $\hat{\gamma} \geq \gamma$ is omitted in condition (ii). In this case, $\succ$ is ambiguity averse if and only if $\hat{\gamma} \geq \gamma$.

(ii) The equivalence of conditions (ii) and (iii) is valid without the assumption $R, S > 2$.

Representation (8) is within a class of source-dependent expected utilities studied by Ergin and Gul (2009) and Nau (2006) (which should not be confused with the second-order expected utility studied by Klibanoff et al. 2005). Here scale invariance provides more structure and simplifies the axiomatic foundations. As explained in Skiadas (2012a), a continuous-time version of representation (8) already appears in Schroder and Skiadas (2003), albeit without a decision-theoretic foundation. The dual representation for $\gamma = 1$ is an instance of the entropic variational preferences of Hansen and Sargent (2001), and the preceding characterization is consistent with Strzalecki (2011). For $\gamma \neq 1$, the dual representation is (to my knowledge) new and extends the case $\gamma = 1$, which can be obtained as a limiting version by letting $\gamma \to 1$, as outlined in the following remark.

Remark 13. Let ν be the CE of condition (ii) of Theorem 11. The restriction of ν to the set of horse-race payoffs is the expected-utility CE $\nu_S(x) = u_{\hat{\gamma}}^{-1}(\sum_s \hat{p}_s u_{\hat{\gamma}}(x^s))$, $x \in X_S$. For $\gamma = 1$, the fact that (8) is equivalent to (3) with C given in (9) is equivalent to the well known identity

$$\log \nu_S(x) = \min_{q \in \Delta_S} \left\{ \sum_{s=1}^{S} q_s \log(x_s) + \theta \sum_{s=1}^{S} q_s \log\left(\frac{q_s}{\hat{p}_s}\right) \right\}, \quad x \in (0, \infty)^S, \tag{13}$$

with the convention $\infty \cdot 0 = 0$. For $\gamma \neq 1$, the fact that (8) is equivalent to (3) with C given in (10) is equivalent to the new identity

$$u_\gamma \circ \nu_S(x) = \min_{q \in \Delta_S} \frac{\sum_{s=1}^{S} q_s u_\gamma(x_s)}{(\sum_{s=1}^{S} \hat{p}_s^{1-\eta} q_s^{\eta})^{1/\eta}}, \quad x \in (0, \infty)^S, \tag{14}$$

with the limit conventions in (11). Given (12), identity (13) is the limiting version of identity (14) as $\gamma \to 1$. To verify this claim, multiply (14) by $(1-\gamma)$, take logs, divide by $(1-\gamma)$, and use (12) to find

$$\log \nu_S(x) = \min_{q \in \Delta_S} \left\{ \frac{1}{1-\gamma} \log\left(\sum_{s=1}^{S} q_s x_s^{1-\gamma} \right) + \frac{\theta}{\eta - 1} \log\left(\sum_{s=1}^{S} q_s \left(\frac{q_s}{\hat{p}_s}\right)^{\eta-1} \right) \right\}.$$

As $\gamma \to 1$, and therefore $\eta \to 1$, the first term inside the curly brackets converges to $\sum_s q_s \log(x_s)$ and the second term converges to $\theta \sum_s q_s \log(q_s/\hat{p}_s)$, thus reducing the above expression to (13).

The following is an example of the dual representation of condition (iii) of Theorem 11 that is a smooth analog to the MEU Example 9.

Example 14. Suppose that $S = 2$ (see Remark 12(ii)) and the CE ν is given by (8) with $\hat{p} = (\frac{1}{2}, \frac{1}{2})$ and $1 \neq \hat{\gamma} > \gamma \neq 1$. Letting $\eta = (\hat{\gamma} - 1)/(\hat{\gamma} - \gamma)$, the function C of the dual representation (3) is given by (10), which in this context becomes

$$\frac{1}{C(w,1-w)}=\left(\frac{1}{2}\right)^{(1-\eta)/\eta}(w^{\eta}+(1-w)^{\eta})^{1/\eta},\quad w\in(0,1).$$

Two examples of this function are graphed in Figure 1 on top of analogous examples in the MEU specification of Example 9. The top, thick curve corresponds to $\hat{\gamma}=3$ and $\gamma=2$, while the bottom, thin curve corresponds to $\hat{\gamma}=1.1$ and $\gamma=0.1$. ◊

We close with some remarks on the order of aggregation in expression (8), that is, the fact that roulette payoffs are collapsed to their certainty equivalent first and the resulting horse-race payoff is collapsed to a certainty equivalent second. As noted in Remark 12(i), ambiguity aversion is not relevant to this issue. Without ambiguity aversion, the roles of roulette and horse-race uncertainty are symmetric, except for $\succ_R$-monotonicity, which dictates the order of aggregation in (8). The situation is analogous to the partial separation of time preferences and risk aversion in Epstein–Zin–Weil utility (see footnote 2), which over a single period is achieved by aggregating over states first and then over time. Similarly here, to achieve a partial separation of risk attitudes toward two risk sources, one source of risk is aggregated prior to the other. This paper follows the tradition of GS in assuming that risk is aggregated prior to uncertainty (although the reverse order seems worthy of future research). The topic is further explored in Skiadas (2012a), where a minimal extension of Epstein–Zin–Weil utility to reflect CE (8) is axiomatically established. It is also shown there that for small incremental risks, corresponding to Brownian or Poisson uncertainty, the order of aggregation becomes approximately irrelevant, a symmetry that becomes exact in the continuous-time version[13] of the utility in Schroder and Skiadas (2003).

Appendix A: Other weak independence axioms

Theorem 5 established the utility functional form (3) for evaluating horse-race uncertainty. For $\gamma=1$, this functional form corresponds to the specification of MMR, which is characterized by Weak Certainty Independence (WCI). Moreover, we saw in Theorem 10 that if $\gamma\neq 1$, the multiplicative variational form in (3) is consistent with WCI if and only if Certainty Independence (CI) is satisfied, corresponding to an MEU representation. In the case in which $\gamma\neq 1$ and CI is not satisfied, the functional structure of (3) is related to formulations by CF and CMMM, but lacks an exact foundation based on weak independence conditions, rather than scale invariance. The purpose of this appendix is to close this gap, formulating weak independence axioms that characterize all functional forms for aggregating horse-race uncertainty in (3), without assuming scale invariance.

The relevant weak independence conditions, in addition to WCI (see Section 5), are listed below. Recall that $\boldsymbol{\alpha}$ denotes the constant payoff taking the value α at all states.

[13]The effect of ambiguity aversion in this model survives in the continuous-time Brownian/Poisson limit, in contrast to the argument made in Skiadas (2012b) with regard to the smooth second-order expected utility of Klibanoff et al. (2005).

Low-Constant Independence (LCI). For any $x, y \in X_S$ and roulette event B, there exists $\varepsilon > 0$ such that

$$x \succ y \iff \text{for all } \alpha \in (0, \varepsilon), x_B\boldsymbol{\alpha} \succ y_B\boldsymbol{\alpha}.$$

High-Constant Independence (HCI). For any $x, y \in X_S$ and roulette event B, there exists $M > 0$ such that

$$x \succ y \iff \text{for all } \alpha \in (M, \infty), x_B\boldsymbol{\alpha} \succ y_B\boldsymbol{\alpha}.$$

Suppose the CE representation ν of $\succ$ admits the representation (3) for some $p \in \Delta_S$, $\gamma \in \mathbb{R}$, and $C \in \mathcal{C}_\gamma$. Then the following implications are easily seen to be true.

$$\gamma = 1 \implies \text{WCI}, \qquad \gamma < 1 \implies \text{LCI}, \qquad \gamma > 1 \implies \text{HCI}.$$

LCI and HCI are variants of Axiom 5 of CF and Axiom A.10 of CMMM. The latter formulates weak independence relative to a fixed reference outcome, while the former further assumes that the outcome is the worst possible. In our setting, there is no worst (or best) outcome—LCI is weak certainty independence relative to all sufficiently bad constant payoffs, and HCI is weak certainty independence relative to all sufficiently good constant payoffs.

We show that in the absence of scale invariance, WCI, LCI, and HCI entirely characterize the functional structure (3) toward horse-race uncertainty. For technical reasons, we do so in a modified model in which roulette outcomes are uniformly distributed on $[0, 1]$, essentially embedding our earlier treatment in a model with objective roulette probabilities, just as in the related literature of GS, MMR, CF, and CMMM.

For the remainder of this appendix, the roulette state space $\{1, \ldots, R\}$ is replaced with the unit interval $[0, 1]$. An objective distribution over roulette outcomes is given as Lebesgue measure λ on $[0, 1]$. A *roulette event* is now any Borel subset of $[0, 1]$. A *roulette payoff* is any Borel-measurable simple random variable of the form $z : [0, 1] \to (0, \infty)$, meaning that there exist finitely many disjoint roulette events $B_1, \ldots, B_n$ and corresponding $z_1, \ldots, z_n \in (0, \infty)$ such that $z = \sum_{i=1}^n z_i 1_{B_i}$. The corresponding expectation is $\mathbb{E}z = \sum_{i=1}^n z_i \lambda(B_i)$. A *payoff* is any mapping of the form $x : [0, 1] \times \{1, \ldots, S\} \to \mathbb{R}$ such that for every horse-race state s, the section $x^s : [0, 1] \to \mathbb{R}$, defined by $x^s(r) = x(r, s)$, is a roulette payoff. As before, we identify a roulette payoff with a payoff that does not depend on the horse-race state, while a *horse-race payoff* can be viewed as either a payoff that does not depend on the roulette state or an element of $(0, \infty)^S$.

As in Section 2, we take as given a relation $\succ$ on the set of payoffs X, whose restriction on the set of roulette payoffs X_R (resp. horse-race payoffs X_S) is denoted $\succ_R$ (resp. $\succ_S$). We further assume that $\succ_R$ has a von Neumann–Morgenstern (vNM) representation. For the purpose of this discussion, a *vNM index* is any increasing continuous function of the form $u : (0, \infty) \to \mathbb{R}$ and is said to *represent* $\succ_R$ if $x \succ_R y$ is equivalent to $\mathbb{E}u(x) > \mathbb{E}u(y)$ for all $x, y \in X_R$. We focus on the case in which $\succ_R$ has an unbounded vNM representation u. Since we are free to choose any positive affine transformation of u, we assume, without loss of generality, that the image set of u is $\mathbb{R}$ or $\pm(0, \infty)$ (meaning $(0, \infty)$ or $(-\infty, 0)$).

The representation theorem that follows essentially modifies Theorem 5 by replacing scale invariance with a weak independence axiom, which one depending on the image of u. The theorem refers to the set $\mathcal{C}_\gamma$ of Definition 4. Note that $\mathcal{C}_\gamma$ depends on γ only through the image set $u_\gamma(0,\infty)$, which can be $\mathbb{R}$ or $\pm(0,\infty)$.

THEOREM 15. *Suppose the vNM index u is such that $u(0,\infty)=\mathbb{R}$ or $u(0,\infty)=\pm(0,\infty)$, and select any $\gamma\in\mathbb{R}$ such that $u(0,\infty)=u_\gamma(0,\infty)$. Then the following conditions are equivalent:*

(i) The relation $\succ$ is an increasing, $\succ_R$-monotone, and ambiguity-averse preference order, and $\succ_S$ is continuous. Moreover, $\succ_R$ has the vNM representation u, and $\succ$ satisfies WCI if $u(0,\infty)=\mathbb{R}$, LCI if $u(0,\infty)=(0,\infty)$, and HCI if $u(0,\infty)=(-\infty,0)$.

(ii) The CE $\nu:X\to(0,\infty)$ representing $\succ$ exists and takes the form

$$u\circ\nu(x)=\min_{q\in\Delta_S}\begin{cases}\left(\sum_{s=1}^S q_s\mathbb{E}u(x^s)\right)+C(q) & \text{if } u(0,\infty)=\mathbb{R}\\ \left(\sum_{s=1}^S q_s\mathbb{E}u(x^s)\right)C(q) & \text{if } u(0,\infty)=\pm(0,\infty)\end{cases}\tag{15}$$

for some $C\in\mathcal{C}_\gamma$.

Suppose these two conditions are satisfied. The function C can be uniquely selected so that if representation (15) is also valid with any $\bar{C}:\Delta_S\to\mathbb{R}_+\cup\{\infty\}$ in place of C, then $C\le\bar{C}$ if $\gamma\le 1$ and $\bar{C}\le C$ if $\gamma>1$. Moreover, the following statements are true:

- *CI is satisfied if and only if C takes the form of condition (iii) of Theorem 6. For $u(0,\infty)=\pm(0,\infty)$, CI is satisfied if and only if WCI is satisfied.*
- *Assuming $S>2$, the preference order $\succ_S$ on X_S is separable if and only if C takes the form of condition (iii) of Theorem 11.*

REMARK 16. Representation (15) corresponds to one form of the function f that maps the vector $(\mathbb{E}u(x^1),\dots,\mathbb{E}u(x^S))$ to $u\circ\nu(x)$. If *CI* is satisfied, then f can alternatively be expressed as $f(z)=\min_{q\in K}q\cdot z$ for some (unique) $K\in\mathcal{K}$, by the same argument used in Theorem 6. (This fact is of course known from GS.) If $S>2$ and $\succ_S$ is separable, then f is alternatively given by Example 26 if $u(0,\infty)=\mathbb{R}$ and by Example 27 if $u(0,\infty)=\pm(0,\infty)$, just as in the proof of Theorem 11.

In the case $u(0,\infty)=\mathbb{R}$, the utility representation (15) and the characterization of the entropic form (9) are familiar thanks to MMR and Strzalecki (2011), respectively. In this case, the function C of (15) is unique in $\mathcal{C}_1$ (see Remark 24), just as in the context of Theorem 5 for $\gamma=1$.

The case $u(0,\infty)=\pm(0,\infty)$ is related to CF and Theorem 26 of CMMM, but is different in terms of the restrictions placed on C, reflecting the difference between LCI or HCI and the corresponding weak independence assumptions of CF and CMMM. The characterization in the case of separable $\succ_S$ with $u(0,\infty)=\pm(0,\infty)$ and C given by (10) is new.

Appendix B: Proof of representation theorems

This appendix proves the theorems of the main part of the paper and Appendix A, and explains their underlying structure. The first section presents a key representation theorem for scale-invariant separable preferences that is of interest in its own right. The second section relates preference properties to primal CE representations. The third section develops convex duality results that, in conjunction with the primal representations, lead to the multiple-prior representations of the main results, whose proofs are concluded in the last five sections. Omitted lemma proofs can be found in Appendix C.

B.1 *Scale-invariant separable preferences*

In preparation for the main analysis, this section states and proves Theorem 17, providing a characterization of scale-invariant separable preferences. The result, which is of interest in its own right, improves Theorem 3.37 of Skiadas (2009) by removing the nonordinal assumption that the utility is continuously differentiable in some arbitrarily small neighborhood. As discussed in the Introduction, the remarkable aspect of Theorem 17 is that separability together with scale invariance substitute for a more elaborate SEU theory on a finite state space, delivering the power or logarithmic expected utility structure under a unique probability. Related insights are provided by Hens (1992) and Werner (2005). Hens notes that if a continuously differentiable additive utility has a constant marginal rate of substitution along the certainty line, then it must take the form of expected utility. Werner shows that an additive utility that is more risk averse than risk neutral relative to an exogenously given probability must be expected utility relative to this probability. These arguments are not special to scale-invariant preferences, but rely on nonordinal assumptions. Theorem 17 makes only ordinal assumptions—utility smoothness and the existence of the unique probability p are all consequences of these ordinal assumptions, as is the fact that the utility is either (globally) risk averse or risk seeking.

Theorem 17 is stated in terms of a binary relation $\succ$ on $(0,\infty)^n$ for some positive integer n. (This is not the same $\succ$ as in the main part of the paper; the result is applied to $\succ_R$ with $n=R$ and to $\succ_S$ with $n=S$.) We refer to Definition 1 for the meaning of the terms increasing, continuous, scale-invariant, and preference order. We also use Definition 3: $\succ$ is *separable* if $x_A z \succ y_A z$ implies $x_A z' \succ y_A z'$ for all $x, y, z, z' \in (0,\infty)^n$, and $A \subseteq \{1,\dots,n\}$, where

$$(x_A z)_i = \begin{cases} x_i & \text{if } i \in A \\ z_i & \text{if } i \notin A. \end{cases}$$

We refer to (1) and (2) for the definition of the notation Δ_n and u_γ.

Theorem 17. *Suppose $\succ$ is a binary relation on $(0,\infty)^n$ for an integer $n > 2$. Then the following two conditions are equivalent:*

(i) The relation $\succ$ is an increasing, continuous, separable, and scale-invariant preference order.

(ii) *There exist unique* $p \in \Delta_n$ *and* $\gamma \in \mathbb{R}$ *such that*

$$x \succ y \iff \sum_{i=1}^{n} p_i u_\gamma(x_i) > \sum_{i=1}^{n} p_i u_\gamma(y_i) \quad \text{for all } x, y \in (0,\infty)^n. \tag{16}$$

Proof. Clearly, the second condition implies the first (even without uniqueness of p and γ). Conversely, suppose that $\succ$ satisfies the first condition. By Debreu's additive representation theorem (see Debreu 1983, Krantz et al. 1971, and Wakker 1988), there exist increasing and continuous functions $U_i : (0,\infty) \to \mathbb{R}$ such that

$$x \succ y \iff \sum_{i=1}^{n} U_i(x_i) > \sum_{i=1}^{n} U_i(y_i) \quad \text{for all } x, y \in (0,\infty)^n. \tag{17}$$

Moreover, the representation is unique up to a positive affine transformation: If (17) holds for functions $\tilde{U}_i : (0,\infty) \to \mathbb{R}$ in place of the U_i, there exist $a \in (0,\infty)$ and $b \in \mathbb{R}^n$ such that $\tilde{U}_i = aU_i + b_i$ for all i. Given any $s \in (0,\infty)$, scale invariance states that $x \succ y \iff sx \succ sy$ and, therefore, the functions $\tilde{U}_i(z) = U_i(sz)$ define another additive representation of $\succ$. There exist, therefore, functions $a : (0,\infty) \to (0,\infty)$ and $b : (0,\infty) \to \mathbb{R}^n$ such that

$$U_i(sz) = U_i(z)a(s) + b_i(s), \quad s, z \in (0,\infty), i = 1, \ldots, n. \tag{18}$$

Let us also define the functions $f_i, h, k_i : \mathbb{R} \to \mathbb{R}$ by

$$f_i(x) = U_i(e^x), \qquad h(x) = a(e^x), \quad \text{and} \quad k_i(x) = b_i(e^x).$$

We can then restate restriction (18) as

$$f_i(x+y) = f_i(x)h(y) + k_i(y), \quad x, y \in \mathbb{R}, i = 1, \ldots, n. \tag{19}$$

(Note that the h are common for all i.) Given that each f_i is strictly monotone, all solutions to functional equations (19) are fully characterized in[14] Corollary 1 in Section 3.1.3 of Aczél (2006) (whose argument reduces (19) to the classical Cauchy functional equation). This characterization implies that if the f_i are increasing and equations (19) are satisfied, then there exist constants $\alpha_i \in (0,\infty)$ and $\beta_i, \gamma \in \mathbb{R}$ such that either

$$f_i(x) = \alpha_i x + \beta_i \quad \text{and} \quad h(x) = 1$$

or

$$f_i(x) = \alpha_i \frac{e^{(1-\gamma)x}}{1-\gamma} + \beta_i \quad \text{and} \quad h(x) = e^{(1-\gamma)x}, \quad \text{with } \gamma \neq 1.$$

(The fact that h does not depend on i implies that γ also does not depend on i.) The above conditions on the f_i can be summarized as

$$U_i(z) = \alpha_i u_\gamma(z) + \beta_i.$$

[14] I thank an anonymous referee for this reference, which improves on my original bare-hands proof.

After a positive affine transformation of the U_i, we can set $\alpha_i = p_i$ for some $p \in \Delta_n$ and $\beta_i = 0$. Uniqueness of the additive representation (17), up to a positive affine transformation, implies there is a unique choice of $p \in \Delta_n$ and $\gamma \in \mathbb{R}$ that is consistent with representation (16). □

B.2 *Primal CE representation*

Up to the final Section B.8, where Theorem 15 is proved, we assume the finite state space setting of Section 2. This section's focus is on primal CE representations. Corresponding multiple-prior representations are derived in the next section by convex duality arguments.

Definition 18. Suppose $D \subseteq \mathbb{R}$ is an interval. A *certainty equivalent* (CE) on D^S is any increasing and continuous function $f : D^S \to D$ with the property $f(\alpha \mathbf{1}) = \alpha$ for all $\alpha \in D$. The CE f is defined to be

- *scale invariant* (SI) if $f(\alpha z) = \alpha f(z)$ for all $\alpha \in (0, \infty)$ and $z \in D^S$ such that $\alpha z \in D^S$
- *translation invariant* (TI) if $f(z + \alpha \mathbf{1}) = f(z) + \alpha$ for all $\alpha \in \mathbb{R}$ and $z \in D^S$ such that $z + \alpha \mathbf{1} \in D^S$.

From Section 2, recall that $\succ$ is a relation on the set of payoffs X and that $\succ_R$ is its restriction on the set of roulette payoffs X_R.

Lemma 19. *The following two conditions are equivalent:*

(i) The relation $\succ$ is a continuous, increasing, and $\succ_R$-monotone preference order, and $\succ_R$ is separable and scale invariant.

(ii) The CE $\nu : X \to (0, \infty)$ representing $\succ$ exists and takes the form

$$u_\gamma \circ \nu(x) = f\left(\sum_{r=1}^{R} p_r u_\gamma(x_r^1), \ldots, \sum_{r=1}^{R} p_r u_\gamma(x_r^S)\right), \quad x \in X, \tag{20}$$

for unique $p \in \Delta_R$, $\gamma \in \mathbb{R}$, and CE f on $u_\gamma(0, \infty)^S$.

Proof. That (ii) $\Longrightarrow$ (i) is clear (even without uniqueness). Conversely, suppose condition (i) is satisfied. Since $\succ$ is a continuous, increasing preference order,

$$\nu(x) = \inf\{\alpha \in (0, \infty) : \alpha \mathbf{1} \succ x\} \tag{21}$$

defines a CE ν that represents $\succ$. Applying Theorem 17 to $\succ_R$, it follows that there exist unique $p \in \Delta_R$ and $\gamma \in \mathbb{R}$ such that $x \succ_R y$ is equivalent to $\sum_r p_r u_\gamma(x_r) > \sum_r p_r u_\gamma(y_r)$ for all $x, y \in X_R$. The fact that $\succ$ is increasing and $\succ_R$-monotone implies that the function $f : D^S \to D$ is well defined by (20). Indeed, suppose $x, y \in X$ are such that $\sum_r p_r u_\gamma(x_r^s) = \sum_r p_r u_\gamma(y_r^s)$ and, therefore, $x_*^s \sim_R y_*^s$ for every s. For any $\varepsilon > 0$, $(x + \varepsilon \mathbf{1})_*^s \succ y_*^s$ and, therefore, $\nu(x + \varepsilon \mathbf{1}) > \nu(y)$ by $\succ_R$-monotonicity. Letting $\varepsilon \downarrow 0$, we have $\nu(x) \geq \nu(y)$. By symmetry, $\nu(x) = \nu(y)$ and, therefore, f takes the same value whether it is defined in terms of x or y in (20). The proof that f is a unique CE is also straightforward and is left to the reader. □

Given the representation of the last lemma, the following lemma maps properties of $\succ$ to corresponding properties of the CE $f: u_\gamma(0,\infty)^S \to u_\gamma(0,\infty)$.

Lemma 20. *Assume that $R > 1$ and the two equivalent conditions of Lemma 19 are satisfied. Then the following statements are true.*

(i) The relation $\succ$ is ambiguity averse if and only if f is quasiconcave.

(ii) The relation $\succ$ is scale invariant if and only if

$$f \text{ is } \begin{cases} TI & \text{if } \gamma = 1 \\ SI & \text{if } \gamma \neq 1. \end{cases}$$

(iii) Weak Certainty Independence (see Section 5) is satisfied if and only if f is TI.

Proof. (i) The "if" part is immediate. Conversely, suppose $\succ$ is ambiguity averse. Fix any $B \subseteq \{1, \ldots, R\}$ such that $\pi \equiv \sum_{r \in B} p_r \in (0,1)$. Then for any horse-race payoffs $x, y \in X_S$, we can write

$$\sum_{r=1}^{R} p_r u_\gamma((x_B y)_r^S) = \pi u_\gamma(x^s) + (1-\pi) u_\gamma(y^s).$$

Given representation (20), ambiguity aversion requires that for all $x, y \in X_S$,

$$L \equiv f(u_\gamma(x^1), \ldots, u_\gamma(x^S)) = f(u_\gamma(y^1), \ldots, u_\gamma(y^S))$$
$$\implies \quad f(\pi u_\gamma(x^1) + (1-\pi)u_\gamma(y^1), \ldots, \pi u_\gamma(x^S) + (1-\pi) u_\gamma(y^S)) \geq L.$$

The same condition can be stated more simply as

$$f(x) = f(y) \quad \implies \quad f(\pi x + (1-\pi)y) \geq f(y) \quad \text{for all } x, y \in u_\gamma(0,\infty)^S. \tag{22}$$

Because f is increasing and continuous, condition (22) is equivalent to

$$f(x) \geq f(y) \quad \implies \quad f(\pi x + (1-\pi)y) \geq f(y) \quad \text{for all } x, y \in u_\gamma(0,\infty)^S. \tag{23}$$

To see why, suppose $x, y \in u_\gamma(0,\infty)^S$ satisfy $f(x) > f(y)$. Pick any $z \in u_\gamma(0,\infty)^S$ such that $z \leq x, y$ and let $\delta = x - z \geq 0$. Since $f(x) > f(y) \geq f(z)$, δ is nonzero. The decreasing continuous function $h: [0,1] \to \mathbb{R}$ defined by $h(\alpha) = f(x - \alpha\delta)$ satisfies $h(0) > f(y) > h(1)$. Let $\alpha \in (0,1)$ be such that $h(\alpha) = f(y)$. By monotonicity and (22), we conclude that $f(\pi x + (1-\pi)y) \geq f(\pi(x - \alpha\delta) + (1-\pi)y) \geq f(y)$. This proves (23). Applying the same conclusion with the complement of B in place of B, and the notation for x and y interchanged, we also have

$$f(y) \geq f(x) \quad \implies \quad f(\pi x + (1-\pi)y) \geq f(x) \quad \text{for all } x, y \in u_\gamma(0,\infty)^S. \tag{24}$$

Using (23) and (24) together, we show that f is quasiconcave. For any given $z \in u_\gamma(0,\infty)^S$, we are to prove that the set $C \equiv \{x : f(x) \geq f(z)\}$ is convex. Suppose $x, y \in C$. Using condition (23) if $f(x) \geq f(y)$ and condition (24) if $f(y) \geq f(x)$, it follows that

$$x, y \in C \quad \Longrightarrow \quad \pi x + (1-\pi)y \in C. \tag{25}$$

This is not quite the definition of convexity of C, since π is fixed, but it implies convexity of C given the continuity of f. To show this claim, let $J_0 = \{0, 1\}$ and $J_{n+1} = \{\pi\alpha + (1-\pi)\beta : \alpha, \beta \in J_n\}$, $n = 1, 2, \ldots$. The set $J = \bigcup_{n=1}^{\infty} J_n$ is dense in $[0, 1]$. Fix any x and y in C, and consider the set $K = \{\phi \in [0,1] : \phi x + (1-\phi)y \in C\}$. An induction using (25) shows that $J \subseteq K$. Since K is closed and contains a dense subset of $[0, 1]$, it contains all of $[0, 1]$. Therefore C is convex.

(ii) Expression (21) implies that $\succ$ is scale invariant if and only if ν is SI (that is, homogeneous of degree 1). Given this observation, the claim is immediate from the definitions.

(iii) We prove the "only if" part, the converse being straightforward. Suppose WCI is satisfied and fix any roulette event B such that $\pi \equiv \sum_{r\in B} p_r \in (0, 1)$. Suppose we are given any $a, b \in u_\gamma(0,\infty)^S$, and $t \in \mathbb{R}$ is such that $a + t\mathbf{1}, b + t\mathbf{1} \in u_\gamma(0,\infty)^S$. Suppose further that t is restricted so that $t(1-\gamma) > 0$. It is then not hard to show that there exist $x, y \in X_S$ and $\alpha, \beta \in (0, \infty)$ such that

$$\begin{aligned} a_s &= \pi u_\gamma(x^s) + (1-\pi)u_\gamma(\alpha) \\ b_s &= \pi u_\gamma(y^s) + (1-\pi)u_\gamma(\alpha) \\ t &= (1-\pi)(u_\gamma(\beta) - u_\gamma(\alpha)). \end{aligned}$$

(The idea is to pick α so that $u_\gamma(\alpha)$ is close to zero: if $\gamma = 1$, choose $\alpha = 1$; if $\gamma < 1$, choose α very small; if $\gamma > 1$, choose α very large. Given sufficiently small $u_\gamma(\alpha)$, clearly x and y can be selected to satisfy the stated condition, while β can be chosen to be positive thanks to the assumed restriction $t(1-\gamma) > 0$.) The assumed structure of the CE ν implies that $f(a) = u \circ \nu(x_B\boldsymbol{\alpha})$, $f(a + t\mathbf{1}) = u \circ \nu(x_B\boldsymbol{\beta})$, $f(b) = u \circ \nu(y_B\boldsymbol{\alpha})$, and $f(b + t\mathbf{1}) = u \circ \nu(y_B\boldsymbol{\beta})$. Using WCI, it follows that

$$f(a) > f(b) \quad \Longleftrightarrow \quad f(a + t\mathbf{1}) > f(b + t\mathbf{1}). \tag{26}$$

Consider now any $z \in u_\gamma(0,\infty)^S$ such that $z + t\mathbf{1} \in u_\gamma(0,\infty)^S$. Definition 18 implies that $f(z) \in u_\gamma(0,\infty)$. Consider any $\varepsilon > 0$ sufficiently small so that $f(z) + \varepsilon \in u_\gamma(0,\infty)$. Applying (26) with $a = (f(z) + \varepsilon)\mathbf{1}$ and $b = z$ shows that

$$f(z) + \varepsilon + t = f(a + t\mathbf{1}) > f(b + t\mathbf{1}) = f(z + t\mathbf{1}).$$

Since this is true for every sufficiently small $\varepsilon > 0$, we have $f(z) + t \geq f(z + t\mathbf{1})$. Similarly, applying (26) with $a = z$ and $b = (f(a) - \varepsilon)\mathbf{1}$ for sufficiently small $\varepsilon > 0$, we conclude that $f(z) + t \leq f(z + t\mathbf{1})$. We have therefore proved that $f(z + t\mathbf{1}) = f(z) + t$ for any $z \in u_\gamma(0,\infty)^S$ and $t \in \mathbb{R}$ such that $z + t\mathbf{1} \in u_\gamma(0,\infty)^S$ and $t(1-\gamma) > 0$. The last inequality entails no loss of generality, as we can always relabel $z + t\mathbf{1}$ as z and flip the sign of t. This completes the proof that f is TI. □

B.3 *Dual CE representation*

The dual, multiple-prior version of the representation of Lemma 19 is based on the following result, which can be understood in terms of basic demand theory, but whose application to uncertainty aversion is an insight of CMMM. A proof is given in Appendix C.

Lemma 21. *Suppose $D \subseteq \mathbb{R}$ is an open connected interval and the function $f : D^S \to D$ is increasing, continuous, and quasiconcave. Let the function $G : D \times (0,\infty)^S \to \mathbb{R} \cup \{\infty\}$ be defined by*

$$G(z, q) \equiv \sup\{f(x) : q \cdot x \leq z, x \in D^S\}. \tag{27}$$

Then for every $x \in D^S$,

$$f(x) = \min_{q \in \Delta_S} G(q \cdot x, q). \tag{28}$$

The following technical lemma, which is proved in Appendix C, is key in showing that C is strictly positive if $\gamma > 1$. (The lemma's nontrivial part shows that F stays bounded away from zero on the boundary of $(0,\infty)^S \cap \{x : q \cdot x = 1\}$.)

Lemma 22. *Suppose that $F : (0,\infty)^S \to (0,\infty)$ is increasing, convex, and homogeneous of degree 1, and $F(\mathbf{1}) = 1$. Then $\inf\{F(x) : q \cdot x = 1\} > 0$ for every $q \in (0,\infty)^S$.*

Recall the notation u_γ for the function defined by (2) and $\mathcal{C}_\gamma$ for the set of Definition 4. Note that $\mathcal{C}_\gamma$ depends on γ only through the image set $u_\gamma(0,\infty)$. The duality of Lemma 21 is specialized in this paper as follows.

Lemma 23. *Suppose that $f : D^S \to D$ is a CE, and either $D = \mathbb{R}$ and f is TI or $D = \pm(0,\infty)$ and f is SI. Then the function $G : D \times (0,\infty)^S \to \mathbb{R} \cup \{\infty\}$ defined by (27) takes the form*

$$G(z, q) = \begin{cases} z + C(q) & \text{if } f \text{ is TI on } \mathbb{R}^S \\ zC(q) & \text{if } f \text{ is SI on } \pm(0,\infty)^S, \end{cases} \tag{29}$$

where $C : (0,\infty)^S \to \mathbb{R}_+ \cup \{\infty\}$ is defined by

$$C(q) = \begin{cases} \sup\{f(x) : q \cdot x \leq 0, x \in \mathbb{R}^S\} & \text{if } f \text{ is TI on } \mathbb{R}^S \\ \sup\{f(x) : q \cdot x \leq 1, x \in (0,\infty)^S\} & \text{if } f \text{ is SI on } (0,\infty)^S \\ -\sup\{f(x) : q \cdot x \leq -1, x \in (-\infty,0)^S\} & \text{if } f \text{ is SI on } (-\infty,0)^S. \end{cases} \tag{30}$$

The restriction of C to Δ_S is an element of $\mathcal{C}_\gamma$ for any γ such that $D = u_\gamma(0,\infty)$.

Proof. The validity of (29) and (30) follows easily from the definitions. Fix any $\gamma \in \mathbb{R}$ such that $D = u_\gamma(0,\infty)$. We verify that the restriction of C on Δ_S is in $\mathcal{C}_\gamma$ by considering cases.

Case of $D = \mathbb{R}$ and f TI (and therefore $\gamma = 1$). Setting $x = \mathbf{1}$ in (28) and using (29) shows that $\min C = 0$. We extend C by letting $C(q) = \sup_{x \in \mathbb{R}^S}\{f(x) : q \cdot x = 0\}$ for all $q \in \bar{\Delta}_S$.

We show that $C:\bar{\Delta}_S \to \mathbb{R}_+ \cup \{\infty\}$ is lower semicontinuous by showing that its epigraph is a closed set. Suppose we have a sequence (q^n, α^n) in the epigraph C that converges to $(q, \alpha) \in \bar{\Delta}_S \times \mathbb{R}$. We are to show that (q, α) is also in the epigraph of C, which is to say that $q \cdot x = 0$ implies $f(x) \leq \alpha$. Consider any $x \in \mathbb{R}^S$ such that $q \cdot x$ and let $x^n = x + \mathbf{1}(q - q^n) \cdot x$, which converges to x as $n \to \infty$. By construction, $q^n \cdot x^n = 0$ and, therefore, $f(x^n) \leq \alpha^n$ (since (q^n, α^n) is in the epigraph of C). Letting $n \to \infty$ and using the continuity of f, it follows that $f(x) \leq \alpha$, completing the proof of the lower semicontinuity of C. We show the convexity of C by confirming that for any given $q^1, q^2 \in \bar{\Delta}_S$,

$$\tfrac{1}{2}(C(q^1) + C(q^2)) \geq C\Big(\tfrac{1}{2}(q^1 + q^2)\Big).$$

This inequality is equivalent to

$$\sup\{f(x^1) : q^1 \cdot x^1 = 0\} + \sup\{f(x^2) : q^2 \cdot x^2 = 0\} \geq \sup\{2f(x) : (q^1 + q^2) \cdot x = 0\}.$$

Consider any $x \in \mathbb{R}^S$ such that $(q^1 + q^2) \cdot x = 0$. Then the preceding inequality follows if we can produce x^1 and x^2 such that

$$q^1 \cdot x^1 = q^2 \cdot x^2 = 0 \quad \text{and} \quad f(x^1) + f(x^2) = 2f(x). \tag{31}$$

This is achieved by setting $x^1 = x + (q^2 \cdot x)\mathbf{1}$ and $x^2 = x + (q^1 \cdot x)\mathbf{1}$. The fact that $\mathbf{1} \cdot q^1 = \mathbf{1} \cdot q^2 = 1$ and $(q^1 + q^2) \cdot x = 0$ implies the first two equalities in (31). The fact that f is TI implies that $f(x^1) = f(x) + q^2 \cdot x$ and $f(x^2) = f(x) + q^1 \cdot x$. Adding the last two equations and using $(q^1 + q^2) \cdot x = 0$ gives the last equality of (31). This completes that proof that C is convex on Δ_S.

Case of $D = (0, \infty)$ *and* f *SI* (and therefore $\gamma < 1$). Given any $q \in (0, \infty)^S$, let $m = \max\{q_1^{-1}, \ldots, q_S^{-1}\}$. Then $x \in (0, \infty)^S$ and $q \cdot x \leq 1$ implies $x \leq m\mathbf{1}$. Since f is increasing, the definition of $C(q)$ in (30) implies that $C(q) \leq f(m\mathbf{1}) = m$ and, therefore, C is finite-valued. Setting $x = \mathbf{1}$ in (28) and using (29) shows that $\min\{C(q) : q \in \Delta_S\} = 1$. Basic demand theory tells us that C is quasiconvex and, therefore, $1/C$ is quasiconcave. Since f is SI, it can easily be confirmed that $1/C$ is homogeneous of degree 1 on $(0, \infty)^S$ and therefore a concave function.

Case of $D = (-\infty, 0)$ *and* f *SI* (and therefore $\gamma > 1$). Consider any $q \in (0, \infty)^S$. Since f is increasing, the constraint $q \cdot x \leq -1$ in the definition of C in (30) is binding. We use Lemma 22 with $F(x) = -f(-x)$ to conclude that $C(q) > 0$ for every $q \in (0, \infty)^S$. Arguing as in the last case, C is quasiconcave and, therefore, $1/C$ is quasiconvex. Moreover, $1/C$ is homogeneous of degree 1 and, therefore, convex on Δ_S. Setting $x = -\mathbf{1}$ in (28) and using (29) shows that $\max\{C(q) : q \in \Delta_S\} = 1$. □

Expression (29) means that (28) can be restated as

$$f(x) = \min_{q \in \Delta_S} \begin{cases} q \cdot x + C(q) & \text{if } f \text{ is TI on } \mathbb{R}^S \\ (q \cdot x)C(q) & \text{if } f \text{ is SI on } \pm(0, \infty)^S. \end{cases} \tag{32}$$

Remark 24. Suppose f is TI on $\mathbb{R}^S$ and $C \in \mathcal{C}_1$ satisfies (32). By Definition 4, C is the restriction to Δ_S of a convex, lower semicontinuous function C on $\bar{\Delta}_S$. We extend C to all of $\mathbb{R}^S$ by letting $C(q) = \infty$ for $q \notin \bar{\Delta}_S$. It follows that $F(x) \equiv -f(-x) = \max_q\{q \cdot x - C(q)\}$, meaning that F is the Fenchel–Legendre conjugate of C. As shown in Section 12 of Rockafellar (1970), we can invert this relationship to write $C(q) = \sup_x\{q \cdot x - F(x)\}$ (which is the Lagrangian dual of $C(q) = \sup_x\{f(x) : q \cdot x \leq 0\}$ given that f is TI). In particular, C is uniquely determined in $\mathcal{C}_1$ given f.

The uniqueness of an extremal C consistent with representation (32) is spelled out in the following result.

Lemma 25. *Under the same assumptions as Lemma 23, suppose that C is defined by (30), γ is such that $D = u_\gamma(0, \infty)$, and (32) is satisfied with $\bar{C} : \Delta_S \to \mathbb{R} \cup \{\infty\}$ in place of C. Then $C \leq \bar{C}$ if $\gamma \leq 1$ and $\bar{C} \leq C$ if $\gamma > 1$.*

Proof. Fixing any $q \in \Delta_S$, we show that $C(q) \leq \bar{C}(q)$ if $\gamma \leq 1$ and $\bar{C}(q) \leq C(q)$ if $\gamma > 1$, by considering cases.

Case of $D = \mathbb{R}$ and f TI. Let $L(q) = \{x \in \mathbb{R}^S : q \cdot x = 0\}$. For any $x \in L(q)$, setting $p = q$ implies $p \cdot x + \bar{C}(p) = \bar{C}(q)$. Therefore,

$$f(x) = \min_{p \in \Delta_S}\{p \cdot x + \bar{C}(p)\} \leq \bar{C}(q), \quad x \in L(q).$$

This in turn implies

$$C(q) = \sup\{f(x) : x \in L(q)\} \leq \bar{C}(q). \tag{33}$$

Case of $D = (0, \infty)$ and f SI. Let $L(q) = \{x \in \mathbb{R}^S : q \cdot x = 1\}$. For any $x \in L(q)$, setting $p = q$ implies $(p \cdot x)\bar{C}(p) = \bar{C}(q)$. Therefore,

$$f(x) = \min_{p \in \Delta_S}\{(p \cdot x)\bar{C}(p)\} \leq \bar{C}(q), \quad x \in L(x).$$

Again (33) must hold.

Case of $D = (-\infty, 0)$ and f SI. Let $L(q) = \{x \in \mathbb{R}^S : q \cdot x = -1\}$. For any $x \in L(q)$, setting $p = q$ implies $(p \cdot x)\bar{C}(p) = -\bar{C}(q)$. Therefore,

$$f(x) = \min_{p \in \Delta_S}\{(p \cdot x)\bar{C}(p)\} \leq -\bar{C}(q), \quad x \in L(q),$$

which implies

$$-C(q) = \sup\{f(x) : x \in L(q)\} \leq -\bar{C}(q).$$

□

The following two examples, which are essential in proving Theorems 11 and 15, establish the form of duality (32) when f is a TI or SI CE representing a separable and convex preference order. It follows from Theorem 17 that if f is TI (which is to say $\exp \circ f \circ \log$ is SI), then it must take the form of Example 26, and if f is SI, it must take the form of Example 27.

Example 26. Given parameter $\theta \in (0, \infty]$, define the TI CE $\psi_\theta : \mathbb{R}^S \to \mathbb{R}$ by

$$\psi_\theta(x) = -\theta \log\left(\sum_{s=1}^{S} \hat{p}_s \exp\left(-\frac{x_s}{\theta}\right)\right) \quad \text{with } \psi_\infty(x) = \hat{p} \cdot x.$$

For $f = \psi_\theta$, duality (32) takes the form

$$\psi_\theta(x) = \min_{q \in \Delta_S}\{q \cdot x + C(q)\}, \quad C(q) = \theta \sum_{s=1}^{S} q_s \log\left(\frac{q_s}{\hat{p}_s}\right), \text{ where } \infty \cdot 0 = 0.$$

This is a well known identity that already appears in Donsker and Varadhan (1975). It can also be easily verified by computing C using (30). ◊

Example 27. Given parameter $\eta \in [-\infty, 1) \cup (1, \infty]$, define the SI CE ϕ_η as

$$\phi_\eta(x) = \begin{cases} (\sum_{s=1}^{S} \hat{p}_s x_s^{\eta/(\eta-1)})^{(\eta-1)/\eta} & \text{for } x \in (0, \infty)^S \text{ and } \eta \in [-\infty, 1) \\ -(\sum_{s=1}^{S} \hat{p}_s (-x_s)^{\eta/(\eta-1)})^{(\eta-1)/\eta} & \text{for } x \in (-\infty, 0)^S \text{ and } \eta \in (1, \infty]. \end{cases}$$

For $\eta = 0$ or $\pm\infty$, the above expressions are interpreted by taking a limit:

$$\phi_0(x) = \exp\left(\sum_{s=1}^{S} \hat{p}_s \log x_s\right) \quad \text{and} \quad \phi_{\pm\infty}(x) = \hat{p} \cdot x.$$

For $f = \phi_\eta$, duality (32) takes the form

$$\phi_\eta(x) = \min_{q \in \Delta_S}\{(q \cdot x)C(q)\}, \quad \text{where } \frac{1}{C(q)} = \left(\sum_{s=1}^{S} \hat{p}_s^{1-\eta} q_s^{\eta}\right)^{1/\eta},$$

with the convention (11) for $\eta \in \{0, \pm\infty\}$. The expression for C is easily verified using (30). ◊

B.4 *Proof of Theorem 5*

The proof of Theorem 5 amounts to compiling earlier lemmas. That (ii) $\Longrightarrow$ (i) is straightforward to confirm. Conversely, suppose condition (i) is satisfied. Applying Lemma 19, we obtain the primal representation (20) for unique $p \in \Delta_R$, $\gamma \in [0, \infty)$, and CE f on $u_\gamma(0, \infty)^S$. By Lemma 20, f is quasiconcave, and TI if $\gamma = 1$, and SI if $\gamma \neq 1$. (Therefore, f is in fact concave.) Applying Lemmas 21 and 23, we can express f as in (32) for some $C \in \mathcal{C}_\gamma$, which in combination with representation (20) gives the main representation (3). An application of Lemma 25 proves that C in representation (20) can be uniquely selected to be minimal if $\gamma \leq 1$ and maximal if $\gamma > 1$. (For $\gamma = 1$, the argument of Remark 24 shows that the C of representation (3) is unique in $\mathcal{C}_1$.)

B.5 *Proof of Theorem 6*

We show the equivalence of the three conditions; uniqueness is a corollary of Theorem 5.

(ii) $\Longrightarrow$ (i). This implication is straightforward and is left to the reader.

(i) $\Longrightarrow$ (ii). Assuming the conditions of Theorem 5, we use CI to establish the MEU representation (4) of the second condition. As in the last proof, Lemma 19 implies representation (20) for unique $p \in \Delta_R$, $\gamma \in [0, \infty)$, and CE f on $u_\gamma(0,\infty)^S$. We treat the cases $\gamma = 1$ and $\gamma \neq 1$ separately.

Case of $\gamma = 1$. In this case, we know from the preceding analysis that f is TI and admits the representation (32) for a unique (see Remark 24) $C \in \mathcal{C}_1$, given by Lemma 23 as

$$C(q) = \sup\{f(z) : q \cdot z \leq 0, z \in \mathbb{R}^S\}. \tag{34}$$

We use CI to establish that there exists a positive scalar $s \neq 1$ such that

$$f(sz) = sf(z), \quad z \in \mathbb{R}^S. \tag{35}$$

Given the assumption $R > 2$, there exist distinct roulette events A and B, with respective probabilities $\pi_A = \sum_{r \in A} p_r$ and $\pi_B = \sum_{r \in B} p_r$. We verify (35) for $s = \pi_A/\pi_B$. CI implies

$$x_A \mathbf{1} \succ \boldsymbol{\alpha}_A \mathbf{1} \quad \Longleftrightarrow \quad x_B \mathbf{1} \succ \boldsymbol{\alpha}_B \mathbf{1} \quad \text{for all } x \in X_S \text{ and } \alpha \in (0, \infty). \tag{36}$$

By identity (20) defining f (with $u_\gamma = \log$), condition (36) is equivalent to

$$f(\pi_A \log x) > \pi_A \log \alpha \quad \Longleftrightarrow \quad f(\pi_B \log x) > \pi_B \log \alpha, \quad x \in X_S, \alpha \in (0, \infty), \tag{37}$$

where $\log x = (\log x^1, \ldots, \log x^S)$. Making the change of variables $s = \pi_A/\pi_B$, $z = \pi_B \log x$, and $l = \pi_B \log \alpha$, condition (37) can be restated as $f(sz) > sl \iff f(z) > l$ for all $z \in \mathbb{R}^S$ and $l \in \mathbb{R}$, a condition that is clearly equivalent to the claimed condition (35).

Combining (34) and (35), we have

$$C(q) = \sup\{f(sz) : q \cdot (sz) \leq 0\} = s \sup\{f(z) : q \cdot z \leq 0\} = sC(q).$$

Since $s \neq 1$, it follows that $C(q) \in \{0, \infty\}$. We have shown that C can only take the values 0 or ∞ on the open set Δ_S and, therefore, $C = \chi_K$, where $K = \{q \in \Delta_S : C(q) = 0\}$, corresponding to the MEU representation (4). Clearly, K is nonempty and convex. There remains to show that K is closed. Since C is lower semicontinuous on Δ_S, it is enough to show that for any $\bar{q}$ on the boundary of Δ_S, $C(q) \to \infty$ as $q \to \bar{q}$. Suppose instead that there is a sequence $(q_1, q_2, \ldots)$ in Δ_S that convergences to a point $\bar{q}$ on the boundary of Δ_S, while $C(q_n) = 0$ for all n. Let $s \in \{1, \ldots, S\}$ be such that $\bar{q}^s = 0$. Relabeling horse-race states if necessary, we assume $s = 1$. The corresponding sequence of first components, $(q_1^1, q_2^1, \ldots)$, converges to zero. Let $\varepsilon_n = q_n^1/(q_n^1 - 1)$, which converges to zero as $n \to \infty$. Since $C(q_n) = 0$ and $q_n^1 + (1 - q_n^1)\varepsilon_n = 0$, it follows from (34) that $f(1, \varepsilon_n, \ldots, \varepsilon_n) \leq 0$ for all n. Letting $n \to \infty$, this implies $f(1, 0, \ldots, 0) \leq 0$, since f is continuous. Since f is a CE in the sense of Definition 18, $f(1, 0, \ldots, 0) > f(\mathbf{0}) = 0$, a contradiction.

Case of $\gamma \neq 1$. In this case, we know from Lemma 20 that f is quasiconcave and both SI (since $\succ$ is SI) and TI (by WCI, which is implied by CI). The domain of f is $u_\gamma(0,\infty)^S$, where $u_\gamma(0,\infty) = \pm(0,\infty)$. The function f has a unique TI extension to all of $\mathbb{R}^S$. To verify this claim, for any $x \in \mathbb{R}^S$, define $f(x) = f(x + t\mathbf{1}) - t$ for any $t \in \mathbb{R}$ such that $x + t\mathbf{1} \in u_\gamma(0,\infty)^S$. Since f is TI on $u_\gamma(0,\infty)^S$, any such choice of t gives the same value $f(x)$. Given the TI property, the extension just given is clearly unique. It is now straightforward to check that the unique TI extension of f to $\mathbb{R}^S$ preserves scale invariance, quasiconcavity, and the CE property of f. For instance, for any $x \in \mathbb{R}^S$, $s \in (0,\infty)$, and $t \in \mathbb{R}$ such that $x + t\mathbf{1} \in u_\gamma(0,\infty)^S$, it is also the case that $sx + st\mathbf{1} \in u_\gamma(0,\infty)^S$ and

$$f(sx) = f(sx + st\mathbf{1}) - st = s(f(x + t\mathbf{1}) - t) = sf(x).$$

This shows that f is SI on $\mathbb{R}^S$. Other claimed properties of f can be shown similarly by translating the property to be proved to $u_\gamma(0,\infty)^S$. Applying Lemma 23, we can now complete the proof just as for $\gamma = 1$, except here (35) need not be proved, as we already know f is SI.

(ii) $\iff$ (iii). This follows directly from Proposition 8, but it is worth noting that the claimed expression (5) for C follows easily from (30) with $f(z) = \min_{\pi \in K} \pi \cdot z$. For example, suppose that $\gamma < 1$ and, therefore, $C(q) = \max\{f(x) : x \in L(q)\}$, where we use the fact that f is increasing and q is strictly positive, which implies that $L(q)$ is compact. Since K is also compact, we can apply the minimax theorem to conclude that

$$C(q) = \max_{x \in L(q)} \min_{\pi \in K} \pi \cdot z = \min_{\pi \in K} \max_{x \in L(q)} \pi \cdot z.$$

The case $\gamma > 1$ is analogous, while the case $\gamma = 1$ is trivial.

B.6 *Proof of Theorem 10*

We assume the validity of the two equivalent conditions of Theorem 5 and we show parts (i) and (ii) together. The case $\gamma = 1$ is straightforward: representation (3) coincides with (7) for a unique $\mathbf{C} = C$, and WCI is implied by this representation. Suppose now that $\gamma \neq 1$. As in the last two proofs, Lemma 19 implies representation (20) for unique $p \in \Delta_R$, $\gamma \in \mathbb{R}$, and a CE f on $u_\gamma(0,\infty)^S$, which is quasiconcave and SI by Lemma 20. For the case $\gamma \neq 1$, the preceding proof of the part (i) $\Longrightarrow$ (ii) of Theorem 6 only used WCI (rather than the stronger condition CI). Therefore, assuming $\gamma \neq 1$, WCI implies the validity of condition (ii) of Theorem 6 and, therefore, the validity of CI. Of course, representation (7) implies WCI and, therefore, the same condition. Moreover, the function $\mathbf{C}$ in (7) is unique by Remark 24, which is to say that $\mathbf{C} = \chi_K$.

B.7 *Proof of Theorem 11 and Remark 12*

Suppose that condition (i) of Theorem 5 is satisfied, except $\succ$ need not be ambiguity averse, for now. Let also ν, f, p, and γ be defined by the second condition of Lemma 19. Adding the assumption that $\succ_S$ is separable, we can apply Theorem 17 to $\succ_S$ to conclude that there exist unique $\hat{p} \in \Delta_S$ and $\hat{\gamma} \in \mathbb{R}$ such that the restriction of the CE ν on the set of horse-race lotteries is given by $u_{\hat{\gamma}} \circ \nu(x) = \sum_s \hat{p}_s u_{\hat{\gamma}}(x^s)$, $x \in X_S$. Representation (20) implies that

$$u_\gamma \circ \nu(x) = f(u_\gamma(x^1), \ldots, u_\gamma(x^S)), \quad x \in X_S.$$

Combining the last two equations, we have

$$u_{\hat{\gamma}} \circ u_\gamma^{-1} \circ f(y_1, \ldots, y_S) = \sum_{s=1}^{S} \hat{p}_s u_{\hat{\gamma}} \circ u_\gamma^{-1}(y_s), \quad y \in u_\gamma(0, \infty)^S. \tag{38}$$

Using this f back in the CE expression (20) gives the claimed representation (8). By Lemma 20, $\succ$ is ambiguity averse if and only if f is quasiconcave, a condition that is clearly equivalent to $\hat{\gamma} \geq \gamma$, given expression (38). This proves that (i) $\Longrightarrow$ (ii) and the analogous statement of Remark 12. The converse is straightforward and left to the reader.

Assuming ambiguity aversion, the equivalence (ii) $\Longleftrightarrow$ (iii) corresponds to the duality of Lemmas 21 and 23 for the specific function f defined in (38). With θ and η defined in (12), condition (38) can be restated as

$$f(y) = \begin{cases} (\sum_s \hat{p}_s y_s^{\eta/(\eta-1)})^{(\eta-1)/\eta} & \text{if } 1 > \gamma < \hat{\gamma} \neq 1 \\ -\theta \log(\sum_s \hat{p}_s \exp(-y_s/\theta)) & \text{if } 1 = \gamma < \hat{\gamma} \\ -(\sum_s \hat{p}_s (-y_s)^{\eta/(\eta-1)})^{(\eta-1)/\eta} & \text{if } 1 < \gamma < \hat{\gamma} \\ \exp(\sum_s \hat{p}_s \log y_s) & \text{if } \gamma < \hat{\gamma} = 1 \\ \sum_s \hat{p}_s y_s & \text{if } \gamma = \hat{\gamma}. \end{cases} \tag{39}$$

Examples 26 and 27 apply, resulting in the claimed dual representation with C given by expressions (9) and (10), with the limiting interpretations (11).

B.8 *Proof of Theorem 15*

We assume the setting, notation, and terminology of Appendix A, in which the roulette state space is the unit interval. Lemmas 19 and 20 in this context are modified as follows.

Lemma 28. *Suppose the relation $\succ_S$ on X_S is a continuous, increasing preference order, the relation $\succ_R$ on X_R has the vNM representation u, and $\succ$ is $\succ_R$-monotone. Then the CE ν on X representing $\succ$ exists and takes the form*

$$u \circ \nu(x) = f(\mathbb{E}u(x^1), \ldots, \mathbb{E}u(x^S)), \quad x \in X, \tag{40}$$

for a unique CE f on $u(0, \infty)^S$. Moreover, the following statements are true:

(i) If $u(0, \infty) = \mathbb{R}$, then f is TI if and only WCI holds.

(ii) If $u(0, \infty) = (0, \infty)$, then f is SI if and only if LCI holds.

(iii) If $u(0, \infty) = (-\infty, 0)$, then f is SI if and only if HCI holds.

(iv) The relation $\succ$ is ambiguity averse if and only if f is quasiconcave.

Proof. A CE $\nu_S : X_S \to (0,\infty)$ representing the preference order $\succ_S$ on horse-race lotteries is well defined by $\nu_S(x) = \inf\{\alpha \in (0,\infty) : \alpha\mathbf{1} \succ x\}$. We extend this CE to the whole of X by letting

$$\nu(x) = \nu_S(u^{-1}\mathbb{E}u(x^1), \dots, u^{-1}\mathbb{E}u(x^S)), \quad x \in X. \tag{41}$$

It is straightforward to confirm that $\nu : X \to (0,\infty)$ is a CE. Let us now show that ν represents $\succ$. Given any $x \in X$, let $\bar{x}^s = u^{-1}\mathbb{E}u(x^s)$, $s = 1, \dots, S$, which defines a horse-race payoff $\bar{x}$. By construction, $\bar{x}^s\mathbf{1} \sim_R x^s$ for every s. Arguing as in the proof of Lemma 19, using the $\succ_R$-monotonicity of $\succ$, we have $x \succ y$ if and only if $\bar{x} \succ \bar{y}$, which is in turn equivalent to $\nu_S(\bar{x}) > \nu_S(\bar{y})$. By construction, $\nu(x) = \nu_S(\bar{x})$ for every $x \in X$. This completes the proof that ν represents $\succ$. Equation (40) follows from (41) with $f(y) = u \circ \nu_S(u^{-1}(y^1), \dots, u^{-1}(y^S))$.

Finally, we prove claims (i)–(iv).

(i) The proof of part (i) is the same as for Lemma 20(iii), with $\pi = \lambda(B)$ being the probability of the given roulette event B. (In fact, the argument can be simplified somewhat in the current context, since the availability of every Borel subset of $[0,1]$ as a roulette event means that π can take any value in $[0,1]$.)

(ii) Suppose $u(0,\infty) = (0,\infty)$. We identify X_S with $(0,\infty)^S$ and write

$$u(x) = (u(x^1), \dots, u(x^S)), \quad x \in X_S.$$

Making the change of variables $a = u(x)$ and $b = u(y)$, LCI can be restated as the requirement that for any given roulette event B and corresponding probability $\pi = \lambda(B)$, and for any $a, b \in (0,\infty)^S$, there exists small enough $\varepsilon > 0$ such that $f(a) > f(b)$ if and only if for all $\alpha \in (0,\varepsilon)$, $f(\pi a + (1-\pi)u(\alpha)) > f(\pi b + (1-\pi)u(\alpha))$. Since $u(\alpha) \to 0$ as $\alpha \to 0$ and f is continuous, the last part of this condition is equivalent to

$$\text{for all } a, b \in (0,\infty)^S \text{ and } \pi \in (0,1), \quad f(a) > f(b) \iff f(\pi a) > f(\pi b).$$

Applying the last equivalence with $\pi^{-1}a$ and $\pi^{-1}b$ in place of a and b, it is easy to see that the preceding condition is equivalent to

$$\text{for all } a, b \in (0,\infty)^S \text{ and } s \in (0,\infty), \quad f(a) > f(b) \iff f(sa) > f(sb).$$

Since f is a CE, the last condition is equivalent to f being SI (as shown, for example, in Section 3.5.1 of Skiadas 2009).

(iii) This proof is similar to the proof of part (ii).

(iv) The equivalence of ambiguity aversion and quasiconcavity of f follows by the same argument as for Lemma 20(i). (As with part (i), the argument can be simplified in this context, since ambiguity aversion implies that (22) holds for every $\pi \in (0,1)$.) □

The equivalence (i) $\iff$ (ii) of Theorem 15 follows analogously to the corresponding equivalence of Theorem 5 in the last section, with Lemma 28 in place of Lemmas 19 and 20. That (ii) $\implies$ (i) is immediate. Conversely, Lemma 28 gives representation (40) for a quasiconcave CE f that is TI or SI according to (i)–(iii) (and therefore also concave). Lemmas 21 and 23 imply that f can be expressed, for some $C \in \mathcal{C}_\gamma$, as in (32), which, in combination with representation (40), gives the main representation (15). An application of Lemma 25 shows the existence of a unique extremal $C \in \mathcal{C}_\gamma$ consistent with (15).

Suppose now the theorem's two equivalent conditions are satisfied and, therefore,

$$u \circ \nu(x) = f(u(x^1), \dots, u(x^S)), \quad x \in X_S,$$

for a CE f that is either TI on $\mathbb{R}^S$ or SI on $\pm(0,\infty)^S$. We show the theorem's two bullet points. In both cases, the "if" part is immediate, so we focus on the "only if" part.

For the first bullet point, suppose CI is satisfied. If $u(0,\infty) = \mathbb{R}$, we assume, without loss of generality, that $u(1) = 1$, in which case we can apply the same argument used in the proof of Theorem 6 for the special case $u = \log$. Similarly, if $u(0,\infty) = (0,\infty)$ (resp. $(-\infty, 0)$), we can apply the argument used to prove Theorem 6 for the case $\gamma < 1$ (resp. $\gamma > 1$), while the claim regarding WCI follows just as for Theorem 10.

For the second bullet point, define the SI CE μ on $(0,\infty)^S$ by

$$\mu(x) = \begin{cases} \exp f(\log x^1, \dots, \log x^S) & \text{if } u(0,\infty) = \mathbb{R} \text{ and } f \text{ is TI} \\ f(x^1, \dots, x^S) & \text{if } u(0,\infty) = (0,\infty) \text{ and } f \text{ is SI} \\ -f(-x^1, \dots, -x^S) & \text{if } u(0,\infty) = (-\infty,0) \text{ and } f \text{ is SI.} \end{cases}$$

Suppose that $\succ_S$ is separable. Then the preference order on $(0,\infty)^S$ represented by μ satisfies the conditions of Theorem 17. There exist, therefore, unique $\delta \geq 0$ and $\hat{p} \in \Delta_S$ such that

$$\mu(x) = u_\delta^{-1}\left(\sum_{s=1}^S \hat{p}_s u_\delta(x^s)\right).$$

Solving for f, we find that it takes the forms (39) for any γ such that $u(0,\infty) = u_\gamma(0,\infty)$, with $\hat{\gamma} = \delta$ if $u(0,\infty) = \mathbb{R}$ and with $\hat{\gamma}$ such that $1 - \delta = (1-\hat{\gamma})/(1-\gamma)$ if $u(0,\infty) = \pm(0,\infty)$. The proof is completed just as for Theorem 11.

Appendix C: Remaining proofs

This appendix collects all proofs omitted so far.

C.1 *Proof of Proposition 8*

We first show that

$$\min_{q \in \bar{\Delta}_S} \left\{ (q \cdot y) \min_{\pi \in K} \max_{z \in L(q)} \pi \cdot z \right\} \geq \min_{\pi \in K} \pi \cdot y. \tag{42}$$

The inequality holds because the left-hand side cannot increase if the operator $\max_{z \in L(q)}$ is replaced with $\max_{z \in \{\bar{y}\}}$, where $\bar{y} = (q \cdot y)^{-1} y \in L(q)$. After this replacement, the factor $q \cdot y$ cancels out, giving the right-hand side of (42). Similarly, the left-hand side of (42) does not decrease if $\min_{q \in \bar{\Delta}_S}$ is replaced with $\min_{q \in K}$, and $\min_{\pi \in K}$ is replaced with $\min_{\pi \in \{q\}}$, resulting in

$$\min_{q \in \bar{\Delta}_S} \left\{ (q \cdot y) \min_{\pi \in K} \max_{z \in L(q)} \pi \cdot z \right\} \leq \min_{q \in K} \left\{ (q \cdot y) \max_{z \in L(q)} q \cdot z \right\} = \min_{q \in K} q \cdot y. \tag{43}$$

(The last equality is true because $q \cdot z = 1$ for every $z \in L(q)$.) Inequalities (42) and (43) imply identity (6), albeit with $\min_{q \in \bar{\Delta}_S}$ instead of $\min_{q \in \Delta_S}$ on the right-hand side. But if the minimizing q were on the boundary of Δ_S, then we would have $\max_{z \in L(q)} \pi \cdot z = \infty$ for all $\pi \in K$, contradicting (43) (whose right-hand side is positive and finite).

Finally, interchanging the roles of min and max in (6) simply reverses inequalities (42) and (43), with analogous reasoning.

C.2 *Proof of Lemma 21*

Fix any $x \in D^S$. By the definition of G, it is clear that $f(x) \leq G(q \cdot x, q)$ for all $q \in \Delta_S$. We must, therefore, demonstrate the existence of one $q \in \Delta_S$ such that $f(x) = G(x \cdot q, q)$. By the supporting hyperplane theorem, there is a nonzero q in $\mathbb{R}^S$ that supports the convex set $\{y \in D^S : f(y) > f(x)\}$ at x, meaning that

$$f(y) > f(x) \quad \Longrightarrow \quad q \cdot y \geq q \cdot x. \tag{44}$$

Since f is increasing, if $\delta \in \mathbb{R}^S_+$ is nonzero and small enough so that $x + \delta \in D^S$, we have $f(x + \delta) > f(x)$ and, therefore, $q \cdot \delta \geq 0$. Therefore, $0 \neq q \geq 0$. Next we show that

$$f(y) > f(x) \quad \Longrightarrow \quad q \cdot y > q \cdot x. \tag{45}$$

Suppose $q \cdot y \leq q \cdot x$. For every $\varepsilon > 0$ small enough so that $y - \varepsilon q \in D^S$, we have $q \cdot (y - \varepsilon q) < q \cdot x$ and, therefore, $f(y - \varepsilon q) \leq f(x)$ by (44). Letting ε go to zero, it follows that $f(y) \leq f(x)$, confirming (45). Applying the latter with $y = x + \delta$ for all nonzero sufficiently small $\delta \in \mathbb{R}^S_+$ shows that q is in fact strictly positive. After positive scaling, we can further assume that $q \in \Delta_S$. The proof is now complete, since (45) is a restatement of the condition $f(x) = G(q \cdot x, q)$.

C.3 *Proof of Lemma 22*

Fix any $q \in (0, \infty)^S$ and let $K = \{x \in (0, \infty)^S : q \cdot x = 1\}$. Since $\bar{K}$ is compact, we can select a sequence $\{x^{(n)}\}$ in K that converges to some $\bar{x} \in \bar{K}$ such that $F(\bar{x}) \equiv \lim_{n \to \infty} F(x^{(n)}) = \inf F$. If $\bar{x} \in K$, then $\inf F = F(\bar{x}) > 0$. Suppose now that $\bar{x}$ is on the boundary of K, meaning that at least one of the coordinates of $\bar{x}$ vanishes. Relabeling the coordinates if necessary, we assume that

$$L \equiv \min\{\bar{x}_1, \ldots, \bar{x}_m\} > 0 \quad \text{and} \quad \bar{x}_{m+1} = \cdots = \bar{x}_S = 0.$$

Let us also define the function $H : (0, \infty)^2 \to (0, \infty)$ by letting

$$H(a, b) = F(x), \quad \text{where } x_s = \begin{cases} a & \text{for } s = 1, \ldots, m \\ b & \text{for } s = m + 1, \ldots, S. \end{cases}$$

Consider the sequence $\{y^{(n)}\}$ in $(0, \infty)^S$, defined by

$$y_s^{(n)} = \begin{cases} a^{(n)} \equiv \min\{x_s^{(n)}, L\} & \text{for } s = 1, \ldots, m \\ b^{(n)} \equiv \min\{x_{m+1}^{(n)}, \ldots, x_S^{(n)}\} & \text{for } s = m + 1, \ldots, S. \end{cases}$$

Since

$$\lim_{n\to\infty}(a^{(n)}, b^{(n)}) = (L, 0) \tag{46}$$

and $y^{(n)} \leq x^{(n)}$, we have

$$H(L, 0) \equiv \lim_{n\to\infty} H(a^{(n)}, b^{(n)}) = \lim_{n\to\infty} F(y^{(n)}) \leq \lim_{n\to\infty} F(x^{(n)}) = \inf F. \tag{47}$$

We complete the proof by showing that $H(L, 0) > 0$.

Note that H inherits the assumed properties of F, that is, it is increasing, convex, and homogeneous of degree 1 and satisfies $H(1, 1) = 1$. Let the function $G: (0, \infty) \to \mathbb{R}$ be the increasing, convex function defined by $G(r) = H(1, r)$. Since H is homogeneous of degree 1,

$$H(a, b) = aG\left(\frac{b}{a}\right), \quad a, b \in (0, \infty).$$

Let $r^{(n)} = b^{(n)}/a^{(n)}$. The limit (46) implies that $\lim_{n\to\infty} r^{(n)} = 0$ and

$$\frac{H(L, 0)}{L} = \lim_{n\to\infty} \frac{a^{(n)}}{L} G\left(\frac{b^{(n)}}{a^{(n)}}\right) = \lim_{r\downarrow 0} G(r) \equiv G(0). \tag{48}$$

By the convexity of G, for all n large enough so that $r^{(n)} < 1$, we have the slope inequality

$$G(2) - 1 = \frac{G(2) - G(1)}{1} \geq \frac{G(1) - G(r^{(n)})}{1 - r^{(n)}} \to 1 - G(0) \quad \text{as } n \to \infty.$$

This proves that

$$G(0) \geq 2 - G(2). \tag{49}$$

By the monotonicity of H, we have

$$\tfrac{1}{2}G(2) = H\left(\tfrac{1}{2}, 1\right) < H(1, 1) = 1.$$

Therefore, $2 - G(2) > 0$, which combined with (49) and (48) proves that $H(L, 0) > 0$ and, therefore, $\inf F > 0$ by (47).

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