Scale or Translation Invariant Additive Preferences

Costis Skiadas*

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Abstract

A self-contained proof is presented of the fact that any increasing, continuous preferences over contingent payoffs that admit a (possibly state-dependent) additive representation are scale (resp. translation) invariant if and only if they take the form of CRRA (resp. CARA) expected utility, with the probabilities and coefficient of risk aversion uniquely determined as part of the representation. Thus scale or translation invariance allows a significant simplification of the ordinal foundations of subjective expected utility.

*Kellogg School of Management, Northwestern University, Evanston, IL 60208.

1 Introduction

This expository paper presents a self-contained proof of a theorem characterizing all scale invariant (or homothetic) additive utilities, first shown[1] in Skiadas (2013). An isomorphic version of this result, characterizing additive translation invariant (or quasilinear) utilities, is also presented. Together, the two characterizations provide simple ordinal foundations for subjective expected utility with constant relative or absolute risk aversion.

We focus on an increasing and continuous preference order $\succ$ over a set C that is the Euclidean product of N copies of an open interval of real numbers, where N is finite. For example, the elements of C can be thought of as payoffs that are contingent on the realization of one of N states. Postponing precise definitions for now, we recall a corollary of a result by Debreu (1983): Assuming $N > 1$, the preference order $\succ$ is separable if and only if it admits an additive utility representation. The main theorem shown here states that for any $N > 0$, if $\succ$ admits an additive utility representation then it is scale invariant, meaning that $x \succ y$ implies $sx \succ sy$ for all $s \in (0, \infty)$, if and only if it admits a power-or-logarithmic expected utility representation, where the probabilities are uniquely determined as part of this representation. A transformation of the same result characterizes translation invariant additive preferences as expected exponential utility.

In the absence of scale or translation invariance, a unique subjective utility representation requires significantly more elaborate structure, as in Savage (1954), where a non-atomic probability is implied, or Anscombe and Aumann (1963), where objectively randomized payoffs are introduced, or any of the subjective expected utility foundations building on the theory of additive conjoint measurement, as in the contributions of Luce and Krantz (1971) and Wakker (1984, 1989). In a variant of the latter approach, Skiadas (1997, 2009 Theorem 4.12) separates the conditions leading to an additive representation and a single, but rather elaborate, state-independence condition that delivers a subjective expected utility representation. Here the simple ordinal condition of scale or translation invariance makes an assumption of state independence redundant.

We proceed in five sections. Section 2 reviews a well-known theorem on the additive representation of preferences. This can be thought of as providing an ordinal foundation for the utility additivity assumption, but is otherwise not part of the argument characterizing scale or translation invariant additive utilities. The uniqueness (up to positive

[1] Skiadas (2009) gives a version of the same result, but under a minor non-ordinal regularity assumption, which was removed in Skiadas (2013).

affine transformations) of additive representations on the other hand is essential for the characterization and is therefore proved in Section 3. The other key component of the argument is a consequence of a classical characterization of the so-called Cauchy equation, which is proved in Section 4. The final two sections present the main results on scale and translation invariant additive utilities, respectively.

2 Additive Utility Representation

We consider preferences on the set $C \equiv (\ell, \infty)^N$ for some positive integer N and $\ell \in \{0, -\infty\}$. The lower bound ℓ will be assumed to be zero in the scale-invariant formulation and $-\infty$ in the translation invariance formulation. It is convenient for our purposes to define utility functions on C to be monotone and continuous, although these restrictions are not standard in the literature. A function $U : C \to \mathbb{R}$ is **increasing** if $0 \neq h \geq 0$ implies $U(c+h) > U(c)$ for all $c \in C$. A **utility** (on C) is a continuous and increasing real-valued function on C.

We fix throughout a binary relation $\succ$ on C (that is, a subset of $C \times C$) with the interpretation that $a \succ b$ (meaning, $(a,b) \in \succ$) indicates strict preference of a over b. We assume that $\succ$ is irreflexive (there is no c such that $c \succ c$), monotone (if $0 \neq h \geq 0$, then $b = c$ or $b \succ c$ implies $b + h \succ c$), and continuous (if $a \succ b$, then $x \succ y$ for all x in a sufficiently small open ball centered at a and all y in a sufficiently small open ball centered at b). We denote the complement of $\succ$ by $\succeq$ ($x \succeq y$ if and only if not $y \succ x$) and we assume that $\succeq$ is complete and transitive. It is well-known and simple to show (see, for example, Chapter 3 of Skiadas, 2009), that there exists a utility U that **represents** $\succ$, meaning that $a \succ b \iff U(a) > U(b)$. Conversely, if the relation $\succ$ admits such a representation, then it must satisfy all the preceding conditions assumed of $\succ$.

We are interested in additive utility representations in the following sense.

Definition 1 *The utility U is* ***additive*** *if there exist functions $U_n : (\ell, \infty) \to \mathbb{R}$ such that*

$$U(x) = \sum_{n=1}^{N} U_n(x_n), \quad x \in (\ell, \infty)^N . \tag{1}$$

The existence of an additive utility representation is characterized by a separability condition on $\succ$ that we now define. Given any $x, y \in (\ell, \infty)^N$ and $A \subseteq \{1, \ldots, N\}$, we write $x_A y$ to denote the element of C defined by

$$(x_A y)_n = \begin{cases} x_n, & \text{if } n \in A; \\ y_n, & \text{if } n \notin A. \end{cases}$$

Definition 2 *The preference $\succ$ is* ***separable*** *if*

$$x_A z \succ y_A z \quad \Longleftrightarrow \quad x_A \tilde{z} \succ y_A \tilde{z}, \tag{2}$$

for all $x, y, z, \tilde{z} \in (\ell, \infty)^N$ *and* $A \subseteq \{1, \ldots, N\}$.

A preference that admits an additive utility representation is clearly separable. The following theorem gives a converse. The result is a corollary of a theorem by Debreu (1983) (based on a 1952 working paper), but also captures the essential aspects of that theorem. Debreu's theorem is part of a broader theory of measurement, which is reviewed in the monographs of Krantz, Luce, Suppes, and Tversky (1971) and Narens (1985). These authors present an algebraic theory that generalizes Debreu's topological results (see also Wakker (1988)). A detailed proof of Debreu's additive representation theorem can be found in Wakker (1989).

Theorem 3 (Existence of Additive Representations) *Suppose $N > 2$ and the relation $\succ$ on C admits some utility representation. Then $\succ$ admits an additive utility representation if and only if it is separable.*

It is worth noting that for $N = 2$, separability is not sufficient for the existence of an additive representation; a more elaborate ordinal condition is required.

3 Uniqueness of Additive Representations

The following theorem on the uniqueness of additive representations is also part of the additive representation theory cited earlier. It is an important component of the proof of our later characterization of scale/translation-invariant additive preferences.

Two utilities are said to be **ordinally equivalent** if they represent the same preference. The additive utilities U and $\tilde{U}$ on $(\ell, \infty)^N$ are said to be **related by a positive affine transformation** if there exist $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}^N$ such that $\tilde{U}_n = aU_n + b_n$ for every $n \in \{1, \ldots, N\}$.

Theorem 4 (Uniqueness of Additive Representations) *For any integer $N \geq 2$, two additive utilities on $(\ell, \infty)^N$ are ordinally equivalent if and only if they are related by a positive affine transformation.*

Proof. We assume throughout that $\ell = -\infty$. This is without loss of generality, since for $\ell = 0$, we can apply the result for $\ell = -\infty$ to the utility functions

$$U^*(z) = \sum_n U_n(e^{z_n}) \quad \text{and} \quad \tilde{U}^*(z) = \sum_n \tilde{U}_n(e^{z_n}).$$

Clearly, two additive utilities related by a positive affine transformation are ordinally equivalent. To show the converse, consider any ordinally equivalent additive utilities U and $\tilde{U}$ on $\mathbb{R}^N$ that satisfy

$$U_n(0) = \tilde{U}_n(0) = 0, \quad n = 1, 2, \ldots, N, \quad \text{and} \quad U_1(1) = \tilde{U}_1(1) = 1. \tag{3}$$

We will show that $\tilde{U}_n = U_n$ for all $n \in \{1, \ldots, N\}$. This proves the claim, since any additive utility on $\mathbb{R}^N$ can be made to satisfy normalization (3) after a positive affine transformation.

Let us fix arbitrary $n \in \{2, \ldots, N\}$, $L \in (-\infty, 0)$ and scalar Δ such that $L + \Delta > 1$. We will show that

$$U_1(x) = \tilde{U}_1(x) \quad \text{and} \quad U_n(x) = \tilde{U}_n(x) \quad \text{for all } x \in [L, L+\Delta]. \tag{4}$$

Since every $x \in \mathbb{R}$ is in an interval of the form $[L, L+\Delta] \supset [0,1]$, this argument proves $\tilde{U}_n = U_n$.

Define the functions $f, g : [0,1] \to \mathbb{R}$ by

$$f(z) = \frac{U_1(L + z\Delta) - U_1(L)}{U_1(L+\Delta) - U_1(L)}, \quad g(z) = \frac{U_n(L + z\Delta) - U_n(L)}{U_1(L+\Delta) - U_1(L)}. \tag{5}$$

Define also $\tilde{f}$ and $\tilde{g}$ by putting a tilde over f, g and every instance of U in the above display. Applying Lemma 5 bellow, we conclude that $(f, g) = (\tilde{f}, \tilde{g})$, from which (4) follows easily, using the normalizations (3). ■

Lemma 5 *Suppose the functions $f, g, \tilde{f}, \tilde{g} : [0,1] \to \mathbb{R}$ are increasing and continuous, and they satisfy*

$$f(0) = g(0) = \tilde{f}(0) = \tilde{g}(0) = 0 \quad \textit{and} \quad f(1) = \tilde{f}(1) = 1, \tag{6}$$

Suppose also that for all $x, y, z, w \in [0,1]$,

$$f(x) + g(y) = f(z) + g(w) \iff \tilde{f}(x) + \tilde{g}(y) = \tilde{f}(z) + \tilde{g}(w). \tag{7}$$

Then $(f, g) = (\tilde{f}, \tilde{g})$.

Proof. Let N be any positive integer such that $2^{-N} < g(1)$. Given any $n \in \{N, N+1, \dots\}$, define the points $x_k^n \in [0,1]$ and $y^n \in (0, g(1))$ by

$$f(x_k^n) = k2^{-n}, \quad k = 0, 1, \dots, 2^n, \quad \text{and} \quad g(y^n) = 2^{-n}. \tag{8}$$

Note that $x_0^n = 0$ and $x_{2^n}^n = 1$. Since $g(0) = 0$, we have

$$f(x_k^n) + g(0) = f\left(x_{k-1}^n\right) + g(y^n), \quad k = 1, \dots, 2^n.$$

By assumption (7), it is also true that

$$\tilde{f}(x_k^n) + \tilde{g}(0) = \tilde{f}\left(x_{k-1}^n\right) + \tilde{g}(y^n), \quad k = 1, \dots, 2^n. \tag{9}$$

Since $\tilde{g}(0) = \tilde{f}(0) = 0$, it follows that

$$1 = \tilde{f}(1) = \sum\nolimits_{k=1}^{2^n} \tilde{f}(x_k^n) - \tilde{f}\left(x_{k-1}^n\right) = 2^n \tilde{g}(y^n).$$

This proves that $\tilde{g}(y^n) = 2^{-n}$, which together with (9) shows that $\tilde{f}(x_k^n) = k2^{-n}$ for $k > 0$. Comparing this conclusion to (8), we have proved that the functions f^{-1} and $\tilde{f}^{-1}$ are equal on the set $D_n = \{k2^{-n} : k = 0, \dots, 2^n\}$, for every $n \geq N$. Since the set $\bigcup_{n \geq N} D_n$ is dense in $[0,1]$ and the functions f^{-1} and $\tilde{f}^{-1}$ are continuous, it follows that $f^{-1} = \tilde{f}^{-1}$, and therefore $f = \tilde{f}$.

Let us now apply the same argument with the functions $(F, G, \tilde{F}, \tilde{G})$ in place of $(f, g, \tilde{f}, \tilde{g})$, where

$$F(z) = \frac{g(z)}{g(1)}, \quad \tilde{F}(z) = \frac{\tilde{g}(z)}{\tilde{g}(1)}, \quad G(z) = \frac{f(z)}{g(1)}, \quad \tilde{G}(z) = \frac{\tilde{f}(z)}{\tilde{g}(1)}.$$

The conclusion $F = \tilde{F}$ implies that $g = a\tilde{g}$ for some positive scalar a. Choose any $\varepsilon, \delta > 0$ such that $f(\varepsilon) = g(\delta)$, and therefore $f(\varepsilon) + g(0) = f(0) + g(\delta)$ (by 6). By (7), it must also be the case that $\tilde{f}(\varepsilon) = \tilde{g}(\delta)$. Since $f = \tilde{f}$ and $\tilde{g} = ag$, this shows that $a = 1$ and therefore $g = \tilde{g}$, completing the proof. ■

4 Cauchy's Functional Equation

This section is based on Aczél (2006), who provides further historical context. The section's objective is to prove Lemma 7, which is key for the characterization of scale/translation-invariant additive preferences in the following sections. The lemma solves a functional equation by reducing it to the so-called Cauchy functional equation, stated as equation (10) below.

Lemma 6 *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}, \tag{10}$$

and there exists some nonempty open interval on which f is bounded. Then $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

Proof. We are to prove that the function $\delta(x) = f(x) - f(1)x$, $x \in \mathbb{R}$, vanishes on the entire real line. Note that $\delta(1) = 0$ and $\delta(x+y) = \delta(x) + \delta(y)$ for every $x, y \in \mathbb{R}$. Iterating the last equation, we have $0 = \delta(1) = \sum_{i=1}^{n} \delta(1/n)$, and therefore $\delta(1/n) = 0$ for every positive integer n. Similarly, for positive integers m and n, we have $\delta(m/n) = \sum_{i=1}^{m} \delta(1/n) = 0$. Therefore, δ vanishes on the set of positive rational numbers. Since $\delta(0+0) = \delta(0) + \delta(0)$, it also vanishes at zero, and since $\delta(0) = \delta(r) + \delta(-r)$, it also vanishes on the set of negative rationals. In short, $\delta(r) = 0$ for every rational r. The function δ inherits from f the property that it is bounded on some nonempty open interval (a, b). Given any $x \in \mathbb{R}$, we can find a rational r such that $x + r \in (a, b)$, and since $\delta(x) = \delta(x + r)$, it follows that δ is bounded on the entire real line. Finally, for any real x, the set of all $\delta(nx) = n\delta(x)$ as n ranges over the positives integers remains bounded only if $\delta(x) = 0$. ■

Lemma 7 *Suppose that the functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy*

$$f(x+y) = f(x) g(y) + h(y) \quad \text{for all } x, y \in \mathbb{R}, \tag{11}$$

and f is increasing. Then there exist constants $\alpha \in (0, \infty)$ and $\beta, \gamma \in \mathbb{R}$ such that either

$$f(x) = \alpha x + \beta \quad \text{and} \quad g(x) = 1, \tag{12}$$

or

$$f(x) = \alpha \frac{e^{(1-\gamma)x}}{1-\gamma} + \beta \quad \text{and} \quad g(x) = e^{(1-\gamma)x}, \quad \text{with } \gamma \neq 1.$$

Proof. Letting $x = 0$ in equation (11), we have $f(y) = f(0) g(y) + h(y)$ for every $y \in \mathbb{R}$. Subtracting the last equation from equation (11) results in

$$f(x+y) - f(y) = (f(x) - f(0)) g(y), \quad x, y \in \mathbb{R}. \tag{13}$$

Let the function $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = f(x) - f(0)$, $x \in \mathbb{R}$. Then $\phi(0) = 0$ and

$$\phi(x+y) - \phi(y) = \phi(x) g(y), \quad x, y \in \mathbb{R}. \tag{14}$$

The same equation with the variable names interchanged is

$$\phi(x+y) - \phi(x) = \phi(y)\, g(x), \quad x, y \in \mathbb{R}.$$

Subtracting the second last equation from the last one and rearranging, we obtain

$$\phi(x)(g(y) - 1) = \phi(y)(g(x) - 1), \quad x, y \in \mathbb{R}.$$

If $g(x) = 1$ for all $x \in \mathbb{R}$, then equation (14) and Lemma 6 imply (12). Suppose instead that $g(z) \neq 1$ for some z. The last displayed equation implies

$$\phi(x) = A(g(x) - 1), \quad x \in \mathbb{R}, \tag{15}$$

where $A = \phi(z) / (g(z) - 1)$. Since f is assumed increasing, it must be that $A \neq 0$. Substituting expression (15) for ϕ into equation (14) and simplifying, we find that A cancels out and

$$g(x+y) = g(x)\, g(y), \quad x, y \in \mathbb{R}.$$

Note that $\log g$ satisfies the Cauchy equation of Lemma 6. Equation (13) implies that

$$g(y) = \frac{f(1+y) - f(y)}{f(1) - f(0)}, \quad y \in \mathbb{R}.$$

Therefore g is the difference of two increasing functions, which is necessarily bounded on some open interval. By Lemma 6, we can write $\log g(x) = (1-\gamma)\, x$ for some scalar $\gamma \neq 1$ (where the last inequality follows from the fact that we are analyzing the case in which g is not identically equal to one). Equation (15) therefore becomes

$$f(x) - f(0) = Ae^{(1-\gamma)x} - A, \quad x \in \mathbb{R}.$$

Letting $\alpha = (1-\gamma)\, A$ and $\beta = f(0) - A$, we obtain

$$f(x) = \alpha \frac{e^{(1-\gamma)x}}{1-\gamma} + \beta.$$

Note that $f'(x) = \alpha e^{(1-\gamma)x} > 0$, since f is increasing. Therefore, $\alpha > 0$. ■

5 Scale-Invariant Additive Utility

We are now ready to show that additivity coupled with scale invariance implies a unique power-or-logarithmic expected utility functional form. We assume that $\ell = 0$ and define the relation $\succ$ to be **scale invariant** if $x \succ y$ is equivalent to $sx \succ sy$ for all $x, y \in (0, \infty)^N$ and $s \in (0, \infty)$.

Theorem 8 *Suppose the relation* $\succ$ *on* $(0,\infty)^N$ *admits an additive utility representation (Definition 1). Then* $\succ$ *is scale invariant if and only if it admits a utility representation of the form*

$$U(x) = \sum\nolimits_{n=1}^{N} w_n \frac{x_n^{1-\gamma} - 1}{1-\gamma}, \quad x \in (0,\infty)^N, \tag{16}$$

for unique parameters $\gamma \in \mathbb{R}$ *and* $w_1, \ldots, w_N \in (0,1)$ *such that* $\sum_n w_n = 1$*. The convention for* $\gamma = 1$ *is that* $\left(x^{1-\gamma} - 1\right)/(1-\gamma)$ *is equal to* $\log x$*, which is the limit as* $\gamma \to 1$*.*

Proof. The "if" part is immediate. Conversely, suppose that $\succ$ is scale invariant and let U be an additive utility representing $\succ$. For any $s \in (0,\infty)$, the scale invariance of $\succ$ implies that $U(sz)$ as a function of $z \in (0,\infty)^N$ defines another additive utility representation of $\succ$. By Theorem 4, there exist functions $a : (0,\infty) \to (0,\infty)$ and $b : (0,\infty) \to \mathbb{R}^N$ such that

$$U_n(sz) = U_n(z)\, a(s) + b_n(s), \quad s, z \in (0,\infty), \quad n = 1, \ldots, N. \tag{17}$$

Let us also define the functions $f_n, g, h_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = U_n(e^x), \quad g(x) = a(e^x) \quad \text{and} \quad k_n(x) = b_n(e^x).$$

We can then restate restriction (17) as

$$f_n(x+y) = f_n(x)\, g(y) + h_n(y), \quad x, y \in \mathbb{R}, \quad n = 1, \ldots, N. \tag{18}$$

Note that each f_n is increasing, and g does not depend on the index n. By Lemma 7, there exist constants $\alpha_n \in (0,\infty)$ and $\beta_n, \gamma \in \mathbb{R}$ such that either

$$f_n(x) = \alpha_n x + \beta_n \quad \text{and} \quad g(x) = 1,$$

or

$$f_n(x) = \alpha_n \frac{e^{(1-\gamma)x}}{1-\gamma} + \beta_n \quad \text{and} \quad g(x) = e^{(1-\gamma)x}, \quad \text{with } \gamma \neq 1.$$

(The fact that g does not depend on n implies that γ also does not depend on n.) The preceding conditions on the f_n can be summarized as

$$U_n(z) = \alpha_n \frac{x_n^{1-\gamma} - 1}{1-\gamma} + \beta_n, \quad (= \alpha_n \log x_n + \beta_n \text{ if } \gamma = 1).$$

After a positive affine transformation of the U_n, we can set $\alpha_n = w_n$, where $\sum_n w_n = 1$, and $\beta_n = 0$. Since an additive representation is unique up to a positive affine transformation, it follows that the w_n and $\gamma \in \mathbb{R}$ of the preceding argument are also unique. ■

6 Translation-Invariant Additive Utility

In what is essentially a transformed version of last section's result, we show that additivity coupled with translation invariance results in a unique exponential functional form. Let $\mathbf{1} = (1, 1, \ldots, 1)$. We assume that $\ell = -\infty$ and define the relation $\succ$ to be **translation invariant** if $x \succ y$ is equivalent to $x + t\mathbf{1} \succ y + t\mathbf{1}$ for all $x, y \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

Theorem 9 *Suppose the relation $\succ$ on $\mathbb{R}^N$ admits an additive utility representation (Definition 1). Then $\succ$ is translation invariant if and only if it admits a utility representation of the form*

$$U(x) = \sum\nolimits_{n=1}^{N} w_n \frac{1 - \exp(-\alpha x_n)}{\alpha}, \quad x \in \mathbb{R}^N, \tag{19}$$

for unique parameters $\alpha \in \mathbb{R}$ and $w_1, \ldots, w_N \in (0, 1)$ such that $\sum_n w_n = 1$. The convention for $\alpha = 0$ is that $(1 - \exp(-\alpha x))/\alpha$ is equal to x, which is the limit as $\alpha \to 0$.

Proof. Using the notation $\log x = (\log x_1, \ldots, \log x_N)$, define the preference $\succ^{\log}$ on $(0, \infty)^N$ by letting

$$x \succ^{\log} y \quad \iff \quad \log x \succ \log y.$$

Note that $\succ$ is translation invariant if and only if $\succ^{\log}$ is scale invariant. The theorem's proof is now easily completed by applying Theorem 8 to $\succ^{\log}$. ∎

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