Dynamic Choice with Constant Source-Dependent Relative Risk Aversion

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September 27, 2015, forthcoming in *Economic Theory.*

Abstract

An axiomatic characterization of recursive utility with source-dependent constant relative risk aversion (CRRA), constant elasticity of intertemporal substitution, constant rate of impatience and subjective beliefs is established. The utility form is a minimal extension of Epstein-Zin-Weil utility that allows the CRRA to depend on the source of risk, a dependence that admits an ambiguity aversion interpretation. Dual representations of the proposed recursive utility are discussed and shown to be useful in tackling the central-planner problem and associated asset-pricing applications. An appendix presents the continuous-time version of the utility form, which preserves the effect of ambiguity aversion under Brownian/Poisson uncertainty, despite its smoothness.

JEL classifications: D81, D91, G12

Keywords: source-dependent risk aversion, ambiguity aversion, recursive utility, Epstein-Zin-Weil utility, small risk approximations

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1 Introduction

The traditional role of expected discounted power or logarithmic utility in Macroeconomics and Finance is increasingly being taken over by its Epstein-Zin-Weil (EZW) generalization (Epstein and Zin (1991), Weil (1989)), which retains the assumptions of a constant coefficient of relative risk aversion (CRRA) and a constant elasticity of intertemporal substitution (EIS), but does not restrict the product of the two to be one. The resulting (partial) decoupling of preferences for smoothing across time and states of the world allows a more flexible specification of time preferences and risk aversion, while retaining homotheticity with a parsimonious parameterization. This paper axiomatically develops a minimal extension of EZW utility in which preferences for smoothing are further decoupled with respect to two independent sources of risk—the single CRRA parameter of EZW is split into two parameters, one for each source of risk. The representation admits an ambiguity aversion interpretation, that is, aversion to Knightian uncertainty in the tradition of Knight (1921) and Ellsberg (1961). The ranking of the two CRRAs is characterized by essentially the same condition Gilboa and Schmeidler (1989) used to define ambiguity aversion. Thus a higher risk aversion toward risk source A than risk source B can be thought of as reflecting the decision maker's view that risk source A is more ambiguous than risk source B.

In economic modeling, source-dependent risk aversion can represent some type of specialization or expertise that leads to lower aversion to uncertainty associated with a specific risk source. The homotheticity of the proposed utility implies that a group of identical (except possibly for wealth) agents can be aggregated just as for EZW utility. One can therefore entertain tractable models with wealth effects and source-dependent risk aversion. As an example, consider a model in which all agents have identical preferences in the absence of a source of catastrophe risk, let's say hurricanes, while a subset of agents are less averse to catastrophe risk. Loosely speaking, the equilibrium price of catastrophe risk relative to the price of other risks reflects some sort of wealth-weighted average of the catastrophe-CRRA of the two types of agent, with the less catastrophe-risk-averse agents selling catastrophe insurance. At the onset of a catastrophe, agents acting as insurance writers incur an unanticipated significant wealth loss, and as a consequence their CRRA is suddenly weighed less in determining the equilibrium price of catastrophe risk, which must therefore jump upward relative to the price of other risks. Over time, catastrophe insurers restore their wealth through the collection of premia, and the price spread between catastrophe risks and other risks reverts to a lower level, not unlike the observed dynamics documented by Froot and O'Connell (1999).[1]

The purpose of this paper is not to pursue any particular application, but rather to establish a decision theoretic foundation for the proposed utility function, as well as to discuss equivalent dual representations that can be useful in tackling the type of application just outlined. The main theorem, which builds on the static theory in Skiadas (2013b), shows that a certain set of purely ordinal conditions is necessary and sufficient for an agent's preferences to be representable by a recursive utility function with constant but source-dependent CRRA, constant EIS and constant rate of impatience. These conditions do not involve probabilities—subjective beliefs are part of the representation. Moreover, all parameters, including beliefs, are shown to be unique. Besides dynamic consistency and some other structure commonly assumed of recursive preferences, the main ingredients are scale invariance (homotheticity) and conditional separability in the following sense. Preferences over deterministic plans are separable, and conditional preferences over single-period payoffs are separable when restricted to a single type of risk. Joint separability of course fails. The combination of scale invariance and separability over deterministic plans forces a constant EIS. The combination of scale invariance and conditional separability over single-type one-period payoffs forces a constant CRRA toward the given risk source. As noted earlier, an axiom that in the literature is commonly interpreted as ambiguity aversion characterizes which source of risk is associated with a higher CRRA, given that one of the risk sources is postulated to be more ambiguous.[2] An ambiguity-neutrality condition reduces the representation to EZW utility.

As explained in Appendix A, the continuous-time version of the proposed recursive utility appears earlier in Schroder and Skiadas (2003), but without a decision-theoretic foundation. (See also Nau (2003) for a related static notion of source-dependent risk aversion.) The certainty equivalent is smooth, in contrast to the well-known maxmin representation axiomatized by Gilboa and Schmeidler (1989). The more recent approach of Klibanoff, Marinacci, and Mukerji (2005) represents ambiguity aversion in a smooth utility function, allowing a minimal parametric extension of EZW utility that preserves homotheticity and much of the tractability of EZW utility. The quantitative effect of ambiguity aversion under this approach, however, diminishes in mainstream higher-frequency models. As argued

[1] This is an alternative to the slow-moving capital approach suggested by Duffie (2010).

[2] The same axiom can be interpreted as simply an ordinal characterization of the assumption that the agent is more afraid of one risk source than another. The model is agnostic about the reason behind this assumption.

in Skiadas (2013c), in the continuous-time limit with Brownian or Poisson information, smooth recursive utility with a certainty equivalent of the type proposed by Klibanoff, Marinacci, and Mukerji (2005) reduces to Duffie and Epstein (1992) utility, which is the continuous-time version of recursive utility with an expected-utility certainty equivalent. Appendix A shows that in the continuous-time version of this paper's preferences, the quantitative effect of ambiguity aversion is preserved.

The rest of the paper proceeds as follows. Section 2 defines the underlying finite information tree representing two sources of risk. Section 3 formulates the preference axioms used in the main utility representation theorem, which can be found in Section 4. Section 5 is about duality and its use in tackling the central-planner problem and asset pricing applications. Appendix A discusses the continuous-time case, and Appendix B collects all proofs.

2 Information Structure

The formal uncertainty structure consists of two finite information trees, representing two sources of risk over time. We will refer to these two sources of risk as "roulette uncertainty" and "horse-race uncertainty," terms that are suggestive of an ambiguity aversion condition introduced in the following section. Probabilities make no appearance in this section—they will instead be part of the main utility representation.

Fixed throughout are a finite **state space** Ω, whose subsets are called **events,** and a finite **time set** $\{0, 1, \ldots, T\}$. **Random variables** and (stochastic) **processes** are functions of the form $x : \Omega \to \mathbb{R}$ and $x : \Omega \times \{0, \ldots, T\} \to \mathbb{R}$, respectively. For any subset A of Ω or $\Omega \times \{0, \ldots, T\}$, 1_A denotes the random variable or process that takes the value one on A and vanishes outside A.

A **filtration** is any time-indexed set of algebras[3] $\mathcal{F} = \{\mathcal{F}_t : t = 0, 1, \ldots, T\}$ such that $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t \subseteq 2^{\Omega}$ for all $t > 0$. We write $\mathcal{F}_t^0$ for the partition[4] of Ω that generates[5] $\mathcal{F}_t$. A **spot** of the filtration $\mathcal{F}$ is any pair (F, t) such that $F \in \mathcal{F}_t^0$ and $t \in \{0, \ldots, T\}$. Spots can be visualized as nodes on an information tree. The **immediate successors** of spot $(F, t-1)$ are the spots $(F_1, t), \ldots, (F_n, t)$, where $F_1, \ldots, F_n$ are the elements $\mathcal{F}_t^0$ whose

[3] An *algebra* is any nonempty set of events that is closed with respect to the formation of set unions and complements.

[4] A *partition* of Ω is a set of mutually exclusive nonempty events whose union is Ω.

[5] Meaning that $\mathcal{F}_t$ is the least upper bound of $\mathcal{F}_t^0$ relative to inclusion.

union is F. Spots of the form (F, T) are **terminal** and can be identified with paths on the information tree from the time-zero spot to the terminal spots.

Let us call a filtration $\mathcal{F}$ **simple** if $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and every nonterminal spot has the same number n of immediate successor spots. This number n is the filtration's **spanning number.** In a simple filtration, a spot (F, t) is entirely specified by the event F, since there is a unique time t such that $F \in \mathcal{F}_t^0$. Since all filtrations in this paper are simple, we henceforth refer to a spot (F, t) as spot F or time-t spot F. A process x is **$\mathcal{F}$-adapted** if x_t is $\mathcal{F}_t$-measurable[6] for every time t. For any $\mathcal{F}$-adapted process x, the common value that x_t takes at spot $F \in \mathcal{F}_t^0$ is denoted x_F.

We consider an agent whose preferences will be the central object of interest. The agent's information over time is represented by the two simple filtrations

$$\mathcal{R} = \{\mathcal{R}_t : t = 0, 1, \ldots, T\} \quad \text{and} \quad \mathcal{S} = \{\mathcal{S}_t : t = 0, 1, \ldots, T\},$$

with respective spanning numbers R and S. We refer to $\mathcal{R}$ as the **roulette filtration** and to $\mathcal{S}$ as the **horse-race filtration.** The agent's complete information is specified by the filtration $\mathcal{F}$, where[7]

$$\mathcal{F}_t = \mathcal{R}_t \vee \mathcal{S}_t, \quad t = 0, 1, \ldots, T.$$

Without loss of generality, we assume throughout that $\mathcal{F}_T = 2^\Omega$, meaning that each state is identified with a terminal spot of $\mathcal{F}$. The set of all $\mathcal{F}$-adapted processes is denoted $\mathcal{L}$. Preferences will be defined over the set $\mathcal{L}_{++}$ of strictly positive $\mathcal{F}$-adapted process. (A process x is **strictly positive** if $x(\omega, t) > 0$ for all $\omega \in \Omega$ and $t \in \{0, \ldots, T\}$.)

The unqualified term **spot** will always refer to the filtration $\mathcal{F}$, while **roulette spots** and **horse-race spots** refer to spots of $\mathcal{R}$ and $\mathcal{S}$, respectively. We follow the convention of using the letters F, G and H to denote elements of $\mathcal{F}_t^0$, $\mathcal{R}_t^0$, and $\mathcal{S}_t^0$, respectively. Every spot F can be uniquely expressed as $F = G \cap H$ where G is a roulette spot and H is a horse-race spot. (This use of G and H can be remembered as "Games and Horses.") Assuming that F is nonterminal, its immediate successors will be denoted

$$F_{rs} = G_r \cap H_s, \quad G_r \in \mathcal{R}_{t+1}^0, \quad H_s \in \mathcal{S}_{t+1}^0, \quad (r, s) \in \{1, \ldots, R\} \times \{1, \ldots, S\}. \tag{1}$$

Note that the spanning number of the filtration $\mathcal{F}$ is $R \times S$ and the single period uncertainty

[6] Given any algebra $\mathcal{A} \subseteq 2^\Omega$, a random variable is $\mathcal{A}$-*measurable* if it can be expressed as $\sum_i \alpha_i 1_{A_i}$ for $\alpha_i \in \mathbb{R}$ and $A_i \in \mathcal{A}$.

[7] Here $\vee$ denotes a least upper bound relative to inclusion.

following spot F can be identified with the single-period state space

$$\{1,\ldots,R\}\times\{1,\ldots,S\}\,. \tag{2}$$

We use the notation (r,s) for a generic element of the set (2), with r referred to as a **roulette outcome** and s as a **horse-race outcome.**

A (single-period) **payoff** is any mapping of the form

$$x:\{1,\ldots,R\}\times\{1,\ldots,S\}\rightarrow(0,\infty)\,.$$

The set of all payoffs is denoted $X_{R\times S}$ and can be identified with $(0,\infty)^{R\times S}$. We equivalently think of a payoff x as an R-by-S matrix with entry $x_{rs}=x\,(r,s)$ in the rth row and sth column. A **roulette payoff** is any payoff x whose value is a function of the roulette outcome only, that is, $x\,(r,s)=x\,(r,s')$ for all $r\in\{1,\ldots,R\}$ and $s,s'\in\{1,\ldots,S\}$. If x is a roulette payoff, we write x_r instead of x_{rs}. Analogously, a payoff x is a **horse-race payoff** if $x\,(r,s)=x\,(r',s)$ for all $r,r'\in\{1,\ldots,R\}$ and $s\in\{1,\ldots,S\}$, in which case we write x_s instead of x_{rs}. The set of all roulette (resp. horse-race) payoffs is denoted X_R (resp. X_S) and is identified with $(0,\infty)^R$ (resp. $(0,\infty)^S$). So while X_R and X_S are subsets of $X_{R\times S}$, we also think of the payoff x as a matrix whose columns, denoted $x_{*1},\ldots,x_{*S}$, are roulette payoffs.

Besides the single-period payoff spaces just defined, we are going to use two-period roulette payoffs, which can be thought of as payoffs that are contingent on the consecutive outcomes of two spins of the roulette. Formally, the set of **two-period roulette payoffs,** denoted $X_{R\times R}$, is defined exactly as $X_{R\times S}$ but with $\{1,\ldots R\}$ in place of $\{1,\ldots,S\}$. The set of **two-period horse race payoffs,** which can be thought of as as payoffs that are contingent on two consecutive horse races, is defined analogously and is denoted $X_{S\times S}$.

3 Preference Restrictions

Prior to stating the conditions used in the main representation theorem, we review some more or less standard terminology relating to any binary relation $\succ$ on $(0,\infty)^N$, where N is any positive integer. As usual, $x\succ y$ means $(x,y)\in\succ$. Associated with $\succ$ are the relation $\succeq$ on $(0,\infty)^N$, defined by $x\succeq y\iff$ not $y\succ x$, and the **indifference relation** $\sim$ on $(0,\infty)^N$, defined by $x\sim y\iff x\succeq y$ and $y\succeq x$.

The relation $\succ$ is a **preference order** if $\succeq$ is[8] complete and transitive; it is **increasing**

[8] The relation $\succeq$ is *complete* if for all $x,y\in(0,\infty)^N$, either $x\succeq y$ or $y\succeq x$; and it is *transitive* if $x\succeq y$ and $y\succeq z$ implies $x\succeq z$.

if for all $x, y \in X$, $x \neq y \geq x$ implies $y \succ x$; and **continuous** if for all $x \in X$, the sets $\{y : y \succ x\}$ and $\{y : x \succ y\}$ are open.

The relation $\succ$ is said to be **scale invariant (SI)** if

$$x \succ y \quad \text{implies} \quad sx \succ sy \text{ for all } s \in (0, \infty).$$

For any $A \subseteq \{1, \ldots, N\}$ and $x, y \in (0, \infty)^N$, the notation $x_A y$ stands for the element of $(0, \infty)^N$ defined by

$$(x_A y)_n = \begin{cases} x_n & \text{if } n \in A, \\ y_n & \text{if } n \notin A. \end{cases} \tag{3}$$

The relation $\succ$ is said to be **separable** if for all $A \subseteq \{1, \ldots, N\}$ and $x, y, z, z' \in (0, \infty)^N$, $x_A z \succ y_A z$ implies $x_A z' \succ y_A z'$.

We henceforth focus on a specific binary relation $\succ$ on the set $\mathcal{L}_{++}$ of strictly positive $\mathcal{F}$-adapted processes, with $a \succ b$ having the interpretation that the decision maker strictly prefers consumption plan a to consumption plan b from the perspective of the time-zero spot. The set $\mathcal{L}_{++}$ can be identified with $(0, \infty)^N$, where N is the total number of spots of the filtration $\mathcal{F}$, thus making the preceding notation and terminology applicable. We will impose a number of restrictions on $\succ$ that together characterize this paper's utility class in Theorem 11.

The first restriction on $\succ$ includes the central assumption of scale invariance, which will lead to a constant EIS and a constant CRRA toward each type of risk.

Condition 1 (scale-invariant preferences) $\succ$ *is a scale-invariant, continuous and increasing preference order.*

The relation $\succ$ represents preferences from the perspective of time zero. We now define the conditional preferences induced by $\succ$, given any other spot. For any set $A \subseteq \Omega \times \{0, 1, \ldots, T\}$ and any processes a, b, we write $a_A b$ to denote the process

$$(a_A b)(\omega, t) = \begin{cases} a(\omega, t) & \text{if } (\omega, t) \in A, \\ b(\omega, t) & \text{if } (\omega, t) \notin A. \end{cases}$$

For each $F \in \mathcal{F}_t^0$, the relation $\succ^F$ on $\mathcal{L}_{++}$ is defined by letting

$$a \succ^F b \quad \Longleftrightarrow \quad a_{F \times \{t, \ldots, T\}} \mathbf{1} \succ b_{F \times \{t, \ldots, T\}} \mathbf{1}. \tag{4}$$

In the current context, $\mathbf{1}$ denotes the process identically equal to one. We will assume that the preceding definition does not depend on the choice of 1 as a common payoff outside $F \times \{t, \ldots T\}$, reflecting the irrelevance of past or unrealized consumption at spot F.

Condition 2 (irrelevance of past or unrealized consumption) *For every* $F \in \mathcal{F}_t^0$ *and* $a, b \in \mathcal{L}_{++},\ a \succ^F b$ *if and only if* $a_{F\times\{t,\dots,T\}}c \succ b_{F\times\{t,\dots,T\}}c$ *for every* $c \in \mathcal{L}_{++}$.

An essential assumption will be that of dynamic consistency, requiring that if at spot F the agent prefers plan a to plan b, and a and b are identical outside the subtree rooted at spot F, then at time zero the agent also prefers a to b.

Condition 3 (dynamic consistency) *For any* $F \in \mathcal{F}_t^0$ *and* $a, b \in \mathcal{L}_{++}$, *suppose that* $a(\omega, s) = b(\omega, s)$ *for all* $(\omega, s) \notin F \times \{t, \dots, T\}$. *Then* $a \succ^F b$ *implies* $a \succ b$.

In order to obtain the simplest parametric utility representation with the desired features, we will assume that at every spot the agent has the same preferences for substituting present consumption for a constant stream of future consumption, a condition stated more precisely below.

Condition 4 (constant time preferences) *For any nonterminal times* t_1, t_2 *and corresponding spots* $F_i \in \mathcal{F}_{t_i}^0$, *and any* $x, y, z \in (0, \infty)$,

$$x1_{F_1\times\{t_1\}} + y1_{F_1\times\{t_1+1,\dots,T\}} \succ^{F_1} z1_{F_1\times\{t_1,\dots,T\}} \iff x1_{F_2\times\{t_2\}} + y1_{F_2\times\{t_2+1,\dots,T\}} \succ^{F_2} z1_{F_2\times\{t_2,\dots,T\}}.$$

Given the finite horizon of the model, the preceding condition may at first seem unwarranted, since the annuity $y1_{F_1\times\{t_1,\dots,T\}}$ from the perspective of spot F_1 need not have the same duration as the annuity $y1_{F_2\times\{t_2,\dots,T\}}$ from the perspective of spot F_2, which brings up the interpretation of preferences for terminal consumption (or bequest). At each terminal spot, consumption at level y should be thought of as being equivalent to consumption of a perpetuity that begins at the terminal date paying y in every period. Therefore, from the perspective of any spot F, the consumption plan $y1_{F\times\{t,\dots,T\}}$ can be thought of as being equivalent to a consumption perpetuity at a constant rate y per period, which explains why the plans $y1_{F_1\times\{t_1+1,\dots,T\}}$ and $y1_{F_2\times\{t_2+1,\dots,T\}}$ in Condition 4 are viewed as equivalent. This device will allow us to derive a stationary representation without dealing with the additional complications of an infinite-horizon model. That said, the theory extends in a straightforward manner if Condition 4 is modified to require $t_1 = t_2$, resulting in possible time dependence of the recursive utility representation.

A process x is **deterministic** if $x(\omega, t) = x(\omega', t)$ for all $\omega, \omega' \in \Omega$ and $t \in \{0, \dots, T\}$. We will impose preference separability in the absence of all risk.

Condition 5 (time separability) *The restriction*[9] *of* $\succ$ *to the set of deterministic plans is separable.*

Note that the definition of separability given earlier applies in this context by identifying the set of deterministic plans with $(0,\infty)^T$.

For any nonterminal spot F, the conditional preference $\succ^F$ induces a preference order $\succ^F_{R\times S}$ on the set $X_{R\times S}$ of (single-period) payoffs: For any $x, y \in X_{R\times S}$,

$$x \succ^F_{R\times S} y \quad \Longleftrightarrow \quad 1_{F\times\{t\}} + \sum_{r,s} x_{rs} 1_{F_{rs}\times\{t+1,\ldots,T\}} \succ^F 1_{F\times\{t\}} + \sum_{r,s} y_{rs} 1_{F_{rs}\times\{t+1,\ldots,T\}},$$

where we have used the successor spot notation (1). The remaining conditions relate to these induced preferences for single-period payoffs.

The following condition expresses the assumption that spot-F consumption is irrelevant to the agent's preferences over payoffs from the perspective of spot F.

Condition 6 (irrelevance of current consumption for attitude toward uncertainty) *For every spot* $F \in \mathcal{F}^0_t$ *with immediate successors* (1) *and any* $\alpha \in (0,\infty)$,

$$x \succ^F_{R\times S} y \quad \Longleftrightarrow \quad \alpha 1_{F\times\{t\}} + \sum_{r,s} x_{rs} 1_{F_{rs}\times\{t+1,\ldots,T\}} \succ^F \alpha 1_{F\times\{t\}} + \sum_{r,s} y_{rs} 1_{F_{rs}\times\{t+1,\ldots,T\}}.$$

We will assume that the decision maker considers roulette uncertainty and horse-race uncertainty entirely unrelated and will therefore not modify preferences over roulette (resp. horse-race) payoffs based on past horse-race (resp. roulette) outcomes.

Condition 7 (independence of roulettes and horses) *For any time* $t < T$ *and nonterminal spots* $G, G' \in \mathcal{R}^0_t$ *and* $H, H' \in \mathcal{S}^0_t$, *the following are true.*

- *For all* $x, y \in X_R$, $x \succ^{G\cap H}_{R\times S} y$ *if and only if* $x \succ^{G\cap H'}_{R\times S} y$.
- *For all* $x, y \in X_S$, $x \succ^{G\cap H}_{R\times S} y$ *if and only if* $x \succ^{G'\cap H}_{R\times S} y$.

Let $\succ^F_R$ (resp. $\succ^F_S$) denote the restriction of $\succ^F_{R\times S}$ on the set X_R of roulette payoffs (resp. the set X_S of horse-race payoffs):

$$x \succ^F_R y \quad \Longleftrightarrow \quad x, y \in X_R \text{ and } x \succ^F_{R\times S} y,$$
$$x \succ^F_S y \quad \Longleftrightarrow \quad x, y \in X_S \text{ and } x \succ^F_{R\times S} y.$$

[9] The **restriction** of a binary relation to a set S is the intersection of the relation with $S \times S$.

The corresponding indifference relations are denoted $\sim_R^F$ and $\sim_S^F$.

The following condition corresponds to the monotonicity axiom A.4 of Gilboa and Schmeidler (1989).

Condition 8 ($\succ_R$-monotonicity) *For any nonterminal spot F and any $x, y \in X_{R\times S}$, if $x_{*s} \succ_R^F y_{*s}$ for all $s \in \{1, \ldots, S\}$, then $x \succ_{R\times S}^F y$.*

In order to discuss conditions that will help us compare risk aversion at different spots, we define preferences induced by $\succ$ over the set $X_{R\times R}$ of two-period roulette payoffs and over the set $X_{S\times S}$ of two-period horse-race payoffs. Each $x \in X_{R\times R}$ is represented by an $R \times R$ matrix whose rows $x_{1*}, \ldots, x_{R*}$ are identified with corresponding elements of X_R. Analogously, each $x \in X_{S\times S}$ is represented by an $S \times S$ matrix whose rows $x_{1*}, \ldots, x_{S*}$ are identified with corresponding elements of X_S. Given any time $t < T-1$, consider any time-t roulette spot G and horse race spot H, and let $G_1, \ldots, G_R$ and $H_1, \ldots, H_S$ denote their immediate successors on the roulette and horse-race filtrations, respectively. We define corresponding binary relations $\succ_{R\times R}^G$ on $X_{R\times R}$ and $\succ_{S\times S}^H$ on $X_{S\times S}$ as follows, where $\mathbf{1}$ denotes a vector of ones whose dimensionality is implied by the context.

- $x \succ_{R\times R}^G y$ if and only if $x, y \in X_{R\times R}$ and there exist $\bar{x}, \bar{y} \in X_R$ such that $\bar{x} \succ_R^G \bar{y}$ and for every $r \in \{1, \ldots, R\}$, $\bar{x}_r \mathbf{1} \sim_R^{G_r} x_{r*}$ and $\bar{y}_r \mathbf{1} \sim_R^{G_r} y_{r*}$.
- $x \succ_{S\times S}^H y$ if and only if $x, y \in X_{S\times S}$ and there exist $\bar{x}, \bar{y} \in X_S$ such that $\bar{x} \succ_S^H \bar{y}$ and for every $s \in \{1, \ldots, S\}$, $\bar{x}_s \mathbf{1} \sim_S^{H_s} x_{s*}$ and $\bar{y}_s \mathbf{1} \sim_S^{H_s} y_{s*}$.

Our final condition stipulates that the two relations just introduced are separable.

Condition 9 (conditional separability) *For every time $t < T - 1$ and any time-t spot F, the relations $\succ_{R\times R}^F$ and $\succ_{S\times S}^F$ are separable.*

We have stated all the conditions characterizing this paper's utility representation. To relate this representation to ambiguity aversion, we formally define what it means for $\succ$ to be ambiguity averse, analogously to the uncertainty-aversion axiom A.5 of Gilboa and Schmeidler (1989). The underlying intuitive idea behind this notion of ambiguity aversion is that the agent is less averse to betting on roulettes than horses, since it is easier to assign probabilities on roulette outcomes.

Definition 10 (ambiguity aversion) *The relation* $\succ$ *is* ambiguity averse *if for every nonterminal spot* F *and any* $B \subseteq \{1, \ldots, R\}$,

$$x \sim_S^F y \quad \Longrightarrow \quad x_B y \succeq_{R\times S}^F y.$$

Note that in this definition, $x, y \in X_S$ and $z = x_B y$ is the element of $X_{R\times S}$ defined by letting $z(r,s) = x(s)$ if $r \in B$ and $z(r,s) = y(s)$ if $r \notin B$.

4 Representation Theorem

Last section's conditions uniquely characterize a recursive utility representation of $\succ$ with a constant elasticity of intertemporal substitution, a constant rate of impatience and source-dependent constant relative risk aversion, which is higher for horse-race uncertainty than roulette uncertainty if and only if preferences are ambiguity averse. This claim is stated precisely in this section, following the introduction of some requisite additional notation and terminology.

Let Δ denote the set of every probability $P : 2^{\Omega} \to [0,1]$ such that $P(A) > 0$ for every nonempty event A. Given a reference probability, we write $\mathbb{E}$ for the corresponding expectation operator and we use the following abbreviations for conditional expectations:

$$\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_t] \quad \text{and} \quad \mathbb{E}_t[\cdot \mid \mathcal{A}] = \mathbb{E}[\cdot \mid \mathcal{A}, \mathcal{F}_t].$$

$\Delta_{\mathcal{R}\perp\mathcal{S}}$ denotes the set of every probability $P \in \Delta$ relative to which the algebras $\mathcal{R}_T$ and $\mathcal{S}_T$ are stochastically independent.

For any parameter $\gamma \in \mathbb{R}$, the function $\phi_\gamma : (0,\infty) \to \mathbb{R}$ is defined by

$$\phi_\gamma(z) = \begin{cases} z^{1-\gamma}, & \text{if } \gamma \neq 1, \\ \log(z), & \text{if } \gamma = 1. \end{cases} \tag{5}$$

(Note that $\phi_1(z) = \lim_{\gamma\to 1}(\phi_\gamma(z) - 1)/(1-\gamma) \neq \lim_{\gamma\to 1}\phi_\gamma(z)$.)

A **certainty equivalent (CE)** on $\mathcal{L}_{++}$ is any continuous function $U_0 : \mathcal{L}_{++} \to (0,\infty)$ that is increasing ($x \geq y \neq x$ implies $U(x) > U(y)$) and satisfies $U_0(\alpha\mathbf{1}) = \alpha$ for all $\alpha \in (0,1)$. The CE $\succ$ **represents** $\succ$ on $\mathcal{L}_{++}$ if $a \succ b$ is equivalent to $U_0(a) > U_0(b)$.

The main representation theorem is stated below, with the utility function specified in terms of a **prior** $P \in \Delta_{\mathcal{R}\perp\mathcal{S}}$, an **impatience** parameter β, an **inverse EIS** parameter δ, a **roulette CRRA** γ_R, and a **horse-race CRRA** γ_S.

Theorem 11 *Suppose $R, S > 1$ and $T > 2$. For any binary relation $\succ$ on $\mathcal{L}_{++}$, the following two conditions are equivalent:*

1. *$\succ$ satisfies Conditions 1 through 9.*

2. *There exist $P \in \Delta_{\mathcal{R}\perp\mathcal{S}}$, $\beta \in (0,1)$ and $\gamma_R, \gamma_S, \delta \in \mathbb{R}$ such that $\succ$ admits the CE representation U_0, where for each $c \in \mathcal{L}_{++}$, $U_0(c)$ is the initial value U_0^c of the process $U^c \in \mathcal{L}_{++}$ that is determined by the backward recursion*

$$U_t^c = \phi_\delta^{-1}\left((1-\beta)\,\phi_\delta(c_t) + \beta\phi_\delta \circ \phi_{\gamma_S}^{-1}\left(\mathbb{E}_t\left[\phi_{\gamma_S} \circ \phi_{\gamma_R}^{-1}\left(\mathbb{E}_t\left[\phi_{\gamma_R}\left(U_{t+1}^c\right) \mid \mathcal{S}_{t+1}\right]\right)\right]\right)\right),$$

with terminal condition $U_T^c = c_T$, and $\mathbb{E}$ denoting expectation under P.

Assuming the last condition is true, the parameters $P, \beta, \gamma_R, \gamma_S, \delta$ are all unique, and $\succ$ is ambiguity averse if and only if $\gamma_S \geq \gamma_R$.

The Theorem's utility recursion can be broken down into three steps. Given time-t information, the roulette uncertainty associated with U_{t+1}^c is first aggregated for every single-period horse-race scenario using a constant-CRRA expected-utility conditional CE:

$$\bar{U}_{t+1}^c = \phi_{\gamma_R}^{-1}\left(\mathbb{E}_t\left[\phi_{\gamma_R}\left(U_{t+1}^c\right) \mid \mathcal{S}_{t+1}\right]\right). \tag{6}$$

The horse-race uncertainty associated with $\bar{U}_{t+1}^c$ is aggregated through another constant-CRRA expected-utility conditional CE:

$$\upsilon_t\left(\bar{U}_{t+1}^c\right) = \phi_{\gamma_S}^{-1}\left(\mathbb{E}_t\left[\phi_{\gamma_S}\left(\bar{U}_{t+1}^c\right)\right]\right). \tag{7}$$

The CRRAs γ_R and γ_S can differ. Finally, the conditional CE value of $\bar{U}_{t+1}^c$ is aggregated with time-t consumption just as for EZW utility:

$$U_t^c = \phi_\delta^{-1}\left((1-\beta)\,\phi_\delta(c_t) + \beta\phi_\delta\left(\upsilon_t\left(\bar{U}_{t+1}^c\right)\right)\right). \tag{8}$$

The specification reduces to EZW utility if $\gamma_R = \gamma_S$, or if U_0 is restricted to either the set of $\mathcal{R}$-adapted processes in $\mathcal{L}_{++}$ or the set of $\mathcal{S}$-adapted processes in $\mathcal{L}_{++}$.

5 Dual Representations

Well-known constructions from convex analysis can be used to formulate equivalent dual expressions for expected utility, which can then be combined to obtain equivalent dual representations of the utility function of Theorem 11. One reason to be aware of this type of duality is in order to recognize that some functional forms appearing in the literature, often under the rubric of robust preferences, are in fact equivalent dual representations of expected utility. More significantly, duality can be helpful in asset pricing applications, as proposed by Dumas, Uppal, and Wang (2000) in a different setting (with Duffie-Epstein utility) and illustrated in a simple example below in the current context. We refer to Skiadas (2013a) for a detailed development of duality for single-period source-dependent CRRA. Rolling back the single-period expressions in a recursive utility setting can result in tediously elaborate functional forms. Rather than seeking general forms, therefore, we focus on a particularly tractable case corresponding to unit CRRA toward roulette uncertainty and a simple application of Legendre-Fenchel duality (see Lemma 18). The dual functional form we discuss overlaps with forms appearing in Hansen and Sargent (2001). A continuous-time version for the case of EZW utility was given in Skiadas (2003).

We focus on the recursive utility form of Theorem 11 with the parameter restrictions $\gamma_S > \gamma_R = 1$ and $\delta \geq 1$. As part of the utility form, we also fix a probability $P \in \Delta_{\mathcal{R}\perp\mathcal{S}}$ with expectation operator $\mathbb{E}$, and an impatience parameter $\beta \in (0,1)$. To state the dual representation, we let $\mathcal{M}^{\mathcal{S}}$ denote the set of all P-martingales relative to the filtration $\mathcal{S}$,[10] and we define the set of corresponding unit-mean strictly positive martingales:

$$\Xi = \left\{\xi \in \mathcal{M}^{\mathcal{S}}_{++} : \xi_0 = 1\right\}.$$

A process ρ is said to be **predictable** if ρ_t is $\mathcal{F}_{t-1}$-measurable for every time $t > 0$. We write $\mathcal{D}$ for the set of all strictly positive predictable processes ρ satisfying $\rho_0 = 1$ and $\rho_t < \rho_{t-1}$ for every time $t > 0$. Finally, we define the function $\Phi : (0,1) \to \mathbb{R}$ by

$$\Phi(w) = (1-w)\log\left(\frac{1-w}{1-\beta}\right) + w\log\left(\frac{w}{\beta}\right), \quad w \in (0,1),$$

and we adopt the conventions

$$\Phi(0) = 0, \quad \infty \cdot 0 = 0 \quad \text{and} \quad \rho_{T+1} = 0 \quad \text{for all } \rho \in \mathcal{D}. \tag{9}$$

A simple duality result can now be stated.

[10] A process M is a **P-martingale relative to $\mathcal{S}$** if it is $\mathcal{S}$-adapted and $\mathbb{E}[M_{t+1} \mid \mathcal{S}_t] = M_t$ for all $t < T$.

Proposition 12 *Suppose the CE $U_0 : \mathcal{L}_{++} \to (0, \infty)$ admits the recursive representation of Theorem 11 with $\delta \geq 1$ and $\gamma_S > \gamma_R = 1$. Then, then for every $c \in \mathcal{L}_{++}$,*

$$\log U_0(c) = \min_{\substack{\rho \in \mathcal{D} \\ \xi \in \Xi}} \mathbb{E} \sum_{t=0}^{T} \xi_t \left[(\rho_t - \rho_{t+1}) \left(\log c_t + \frac{\log \xi_t}{\gamma_S - 1} \right) + \frac{\rho_t}{\delta - 1} \Phi \left(\frac{\rho_{t+1}}{\rho_t} \right) \right].$$

Note that for $\delta = 1$, conventions (9) imply that the minimizing ρ is achieved for $\rho_t = \beta^t$ (as in the example below). It is also worth noting that one can easily apply a change-of-measure formula (see, for example, Lemma 5.35 and Proposition B.24 in Skiadas, 2009) to transform the preceding expression to a corresponding multiple-prior representation, where each ξ corresponds to a prior over horse-race outcomes exactly as in the construction of the Proposition's proof.

Example 13 *We consider an economy with two agents. The preferences of agent $i \in \{1, 2\}$ are represented by the CE U_0^i on $\mathcal{L}_{++}$, which is assumed to take the form of Proposition 12, relative to a common filtration, prior $P \in \Delta_{\mathcal{R} \perp \mathcal{S}}$, and impatience parameter $\beta \in (0, 1)$. Moreover, both agents have unit EIS ($\delta = 1$) and unit CRRA toward roulette uncertainty ($\gamma_R = 1$). The only dimension in which the agent preferences differ is in their CRRA toward horse-race uncertainty, which is assumed to be one for agent one and $\gamma_S > 1$ for agent two. Using Proposition 12, the agent utilities can be expressed as*

$$\frac{\log U_0^1(c)}{1 - \beta} = \mathbb{E} \left[\sum_{t=0}^{T-1} \beta^t \log c_t + \frac{\beta^T}{1 - \beta} \log c_T \right],$$

$$\frac{\log U_0^2(c)}{1 - \beta} = \min_{\xi \in \Xi} \mathbb{E} \left[\sum_{t=0}^{T-1} \beta^t \left(\xi_t \log c_t + \frac{\xi_t \log \xi_t}{\gamma_S - 1} \right) + \frac{\beta^T}{1 - \beta} \left(\xi_T \log c_T + \frac{\xi_T \log \xi_T}{\gamma_S - 1} \right) \right].$$

The aggregate endowment $e \in \mathcal{L}_{++}$ is assumed to follow the dynamics

$$\log \frac{e_t}{e_{t-1}} = \mu + \sigma \varepsilon_t, \quad \mu, \sigma \in \mathbb{R},$$

where the ε_t are i.i.d. random variables of unit mean and unit variance. We are interested in the set of efficient (Pareto optimal) allocations of e between the two agents. We identify the set of all possible allocations with the set of all adapted processes that are valued in $(0, 1)$, denoted by $\mathcal{L}(0, 1)$. The allocation (c^1, c^2) corresponding to $p \in \mathcal{L}(0, 1)$ is defined by $c_t^1 = (1 - p_t) e_t$ and $c_t^2 = p_t e_t$, $t \in [0, T]$. Efficient allocations are parameterized by a relative agent weight $\alpha \in (0, \infty)$ as solutions to the central planner problem

$$\max_{p \in \mathcal{L}(0,1)} \alpha \log U_0^1((1 - p) e) + \log U_0^2(pe).$$

To solve this problem, we insert the above dual expression for $\log U_0^2$ and we interchange the order of maximization over $p \in \mathcal{L}(0,1)$ and the minimization over $\xi \in \Xi$ using the Minimax Theorem. Fixing any $\xi \in \Xi$, maximization over $\mathcal{L}(0,1)$ can be performed spot-by-spot given the problem's additive structure:

$$\min_{p_t \in (0,1)} \alpha \log \left((1-p_t) e_t \right) + \xi_t \log \left(p_t e_t \right).$$

The minimum is achieved at $p_t = \xi_t / (\alpha + \xi_t)$. Substituting this allocation into the central planner's problem, the determination of the process ξ reduces to solving

$$V_0(1) = \min_{\xi \in \Xi} \mathbb{E} \left[\sum_{t=1}^{T-1} \beta^{t-1} \left(F(\xi_t) + (\alpha + \xi_t) \log e_t \right) + \frac{\beta^{T-1}}{1-\beta} \left(F(\xi_t) + (\alpha + \xi_t) \log e_t \right) \right],$$

where

$$F(\xi_t) = \alpha \log \left(\frac{\alpha}{\alpha + \xi_t} \right) + \xi_t \log \left(\frac{\xi_t}{\alpha + \xi_t} \right) + \frac{\xi_t \log \xi_t}{\gamma_S - 1}.$$

Let $L_{++}(\mathcal{S}_{t-1})$ denote the set of all $\mathcal{S}_{t-1}$-measurable, strictly positive random variables. The Bellman equation corresponding to the preceding minimization problem is

$$V_{t-1}(x) = \min_{\xi_t \in \Xi(t,x)} \mathbb{E}_{t-1} \begin{cases} F(\xi_t) + (\alpha + \xi_t) \log e_t + \beta V_t(\xi_t), & \text{if } t < T, \\ \left(F(\xi_T) + (\alpha + \xi_T) \log e_T \right) / (1-\beta), & \text{if } t = T, \end{cases}$$

where x ranges over the elements of $L_{++}(\mathcal{S}_{t-1})$ with unit mean, and

$$\Xi(t,x) = \left\{ \xi_t \in L_{++}(\mathcal{S}_t) : \mathbb{E}\left[\xi_t \mid \mathcal{S}_{t-1} \right] = x \right\} = \left\{ \xi_t : \xi \in \Xi,\ \xi_{t-1} = x \right\}.$$

Direct calculation shows that the modified value function

$$J_t(x) = V_t(x) - \frac{\alpha + x}{1-\beta} \left(\frac{\mu}{1-\beta} + \log e_t \right)$$

solves the simplified Bellman equation

$$J_{t-1}(x) = \min_{\xi_t \in \Xi(t,x)} \mathbb{E}_{t-1} \left[F(\xi_t) + \frac{\sigma \xi_t \varepsilon_t}{1-\beta} + \beta J_t(\xi_t) \right], \quad t < T-1,$$

$$J_{T-1}(x) = \min_{\xi_T \in \Xi(T,x)} \frac{1}{1-\beta} \mathbb{E}_{T-1} \left[F(\xi_T) + \frac{\sigma \xi_T \varepsilon_T}{1-\beta} \right]$$

This problem can be handled with routine numerical methodology. Given a solution, corresponding Arrow-Debreu prices are also determined as the marginal utility of agent one (which is proportional to that of agent two).

A Appendix: Continuous-Time Formulation

This appendix presents a continuous-time version of the recursive utility of Theorem 11, given Brownian/Poisson uncertainty. One reason for doing so is that applications often assume continuous time in order to benefit from the simplifications afforded by Ito's lemma. Even if one is not interested in continuous-time formalisms, however, it is still important to know how the utility function behaves quantitatively in a high-frequency setting. Skiadas (2013c) argues that ambiguity aversion represented by certain types of smooth recursive utility disappears as the frequency increases, given Brownian or Poisson information. This section explains why this is *not* the case when ambiguity aversion is represented as in Theorem 11. The continuous-time utility presented below takes a functional form within a class introduced in Schroder and Skiadas (2003, 2008) as part of their analysis of optimal consumption/portfolio choice.

The analysis that follows parallels that in Skiadas (2013c). In the first subsection, we consider a single node of a discrete tree that approximates a continuous-time tree with the roulette uncertainty represented by a Brownian motion and the horse-race uncertainty represented by either a Brownian motion or a Poisson process. The case of a Poisson source of risk is of interest because certain types of ambiguous uncertainty, like the arrival of disasters, is naturally modeled by Poisson uncertainty. The single-period conditional CE on the given node is approximated in the tradition of Arrow (1965, 1970) and Pratt (1964). In the second subsection, the single-period CE approximations is combined with a first-order approximation of the intertemporal aggregator to establish the continuous-time version of the recursive utility.

The theory of convergence of recursive utility along a sequence of discrete trees converging to a Brownian/Poisson filtration is still at an early stage, even in the case of EZW utility. The recent contribution by Kraft and Seifried (2011) is a first step, but much remains to be done. A rigorous convergence theory is highly technical and certainly outside the scope of this paper. Just as Duffie and Epstein (1992) did for EZW utility and Chen and Epstein (2002) did for recursive utility with a maxmin conditional CE, the argument given below gives a strong indication about what the limiting form of the utility function should look like.[11]

[11] Any rigorous convergence results would have to impose additional regularity conditions on the consumption process. Bounded log-consumption (as assumed in Kraft and Seifried (2011) for EZW utility) should work, but weaker exponential-moment conditions, as suggested in the existence theory for continuous-time EZW in Schroder and Skiadas (1999), would be more appropriate for applications. This is a highly technical

A.1 Small-Risk CE Approximations

Isolating a single period of length $h \in (0,1)$, in this section we compute approximations of the CE value of a random payoff $x(h)$ realized at the end of the period, for small h. In terms of our earlier dynamic model, the single-period uncertainty should be thought of as the conditional uncertainty following any given node on the information tree, $x(h)$ should be thought of as the ratio of the realized end-of-period utility to the beginning-of-period utility, and the CE should be thought of as the conditional CE used in the recursive representation of Theorem 11.

For any positive integer n, let Δ_n denote the set of all $p \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n p_i = 1$. The single-period CE is defined in terms of roulette probabilities $p \in \Delta_R$ and horse-race probabilities $q(h) \in \Delta_S$. As the notation indicates, $q(h)$ but not p can depend on the parameter h. The reason is that while roulette uncertainty will correspond to Brownian motion (whose discrete approximation involves probabilities that do not scale with frequency), horse-race uncertainty will also be given a Poisson specification in Proposition 15 below (where the probability of an arrival over the single period is approximately proportional to h). In terms of our earlier notation relating to the dynamic information tree, let $F = G \cap H$ be the spot designating the start of the given period, with its immediate successors denoted as in (1), and let $P \in \Delta_{\mathcal{R}\perp\mathcal{S}}$ be the underlying probability that is part of the representation of Theorem 11. Then $p_r = P[G_r \mid G]$ and $q_s = P[H_s \mid H]$. The corresponding conditional CE value $\nu_h(U)$ of a contingent end-of-period payoff $U : \{1,\ldots,R\} \times \{1,\ldots,S\} \to \mathbb{R}_{++}$ is defined as

$$\phi_{\gamma_S} \circ \nu_h(U) = \sum_{s=1}^{S} q_s(h)\, \phi_{\gamma_S} \circ \phi_{\gamma_R}^{-1}\left(\sum_{r=1}^{R} p_r \phi_{\gamma_R}(U_{rs}(h))\right), \tag{10}$$

for a CRRA γ_R associated with roulette risk and a CRRA γ_S associated with horse-race uncertainty. As before, the notation ϕ_γ is defined in equation (5). The CRRA parameters γ_R and γ_S do *not* vary with h.

We wish to approximate $\nu_h(x(h))$ for small h, assuming the payoff structure[12]

$$x(h) = 1 + \mu h + \sigma_R B^R(h) + \sigma_S B^S(h) + o(h), \tag{11}$$

for constant $\mu, \sigma_R, \sigma_S \in \mathbb{R}$ and random variables $B^R(h)$ and $B^S(h)$, representing roulette risk and horse-race uncertainty, respectively. Accordingly, we assume that $B^R(h)$ depends

topic left for future, more mathematically oriented contributions.

[12] We use the standard little-oh notation. Every occurence of $o(h)$ stands for some function $\varepsilon(h)$, not the necessarily the same function every time, such that $\lim_{h\downarrow 0} \varepsilon(h)/h = 0$.

only on the roulette state and $B^S(h)$ depends only on the horse-race state. We summarize this assumption by writing, for all $(r,s) \in \{1,\ldots,R\} \times \{1,\ldots,S\}$,

$$B_r^R(h) = B^R(h)(r,s) \quad \text{and} \quad B_s^S(h) = B_S(h)(r,s). \tag{12}$$

The factors B^R and B^S are normalized so that

$$\sum_{r=1}^{R} p_r B_r^R(h) = 0 \quad \text{and} \quad \sum_{r=1}^{R} p_r \left[B_r^R(h)\right]^2 = h + o(h), \tag{13}$$

and analogously,

$$\sum_{s=1}^{S} q_s(h) B_s^S(h) = 0 \quad \text{and} \quad \sum_{s=1}^{S} q_s(h) [B_s^S(h)]^2 = h + o(h). \tag{14}$$

In a dynamic setting, the pair $(B^R(h), B^S(h))$ should be thought of as a copy of i.i.d. increments of a martingale that generates the underlying information filtration. (Heuristically speaking, the single-period notation B corresponds to the infinitesimal dB in continuous time.) The first proposition below corresponds to the case in which this martingale becomes a two-dimensional standard Brownian motion in the limit as the frequency goes to infinity and therefore h goes to zero.

Proposition 14 *Suppose $q = q(h)$ does* not *vary with h, and therefore neither does the CE $\nu = \nu_h$ defined in* (10)*. Suppose further that the payoff $x(h)$ is defined by* (11) *for constant $\mu, \sigma_R, \sigma_S \in \mathbb{R}$ and random variables $B^R(h)$ and $B^S(h)$ satisfying* (12), (13) *and* (14). *Then*

$$\nu(x(h)) = 1 + \left(\mu - \frac{\gamma_R}{2}\sigma_R^2 - \frac{\gamma_S}{2}\sigma_S^2\right) h + o(h). \tag{15}$$

Note that for $\gamma_R = \gamma_S$, equation (15) reduces to the familiar Arrow-Pratt approximation of constant-CRRA expected utility.

Another type of approximation arises if the limiting martingale involves unpredictable jumps. We illustrate with a simple case in which a martingale whose increments are i.i.d. copies of the horse-race factor $B_S(h)$ converges to a compensated Poisson process with arrival rate λ as the frequency goes to infinity. Roulette uncertainty is again assumed to be of the Brownian type (although it could alternatively be assumed to be another Poisson-type factor, with the obvious modifications to the approximation).

Proposition 15 *Suppose that $S = 2$ and for some constant $\lambda > 0$,*

$$q_1(h) = 1 - q_2(h) = \lambda h + o(h) \quad \text{and} \quad B_1^S(h) = 1 - q_1(h), \quad B_2^S(h) = -q_1(h). \tag{16}$$

Suppose further that the payoff $x(h)$ is defined by (11) *for constant $\mu, \sigma_R, \sigma_S \in \mathbb{R}$ and random variables $B^R(h)$ and $B^S(h)$ satisfying* (12) *and* (13)*. (Note that* (14) *follows from* (16)*.) Then the CE ν_h defined in* (10) *satisfies*

$$\nu_h(x(h)) = 1 + \left(\mu - \frac{\gamma_R}{2}\sigma_R^2 - \left(\sigma_S - \frac{(1+\sigma_S)^{1-\gamma_S} - 1}{1-\gamma_S}\right)\lambda\right)h + o(h).$$

In contrast to approximation (15) the last approximation does not reduce to an Arrow-Pratt approximation if $\gamma_R = \gamma_S$, because higher than second moments of the Poisson risk source are not negligible. Skiadas (2013c) offers further related discussion.

A.2 Recursive Utility as a BSDE

While last section's results are entirely rigorous, the transition to the continuous-time limit requires a convergence theory that is beyond the scope of this paper. As already discussed in the introductory remarks, we instead proceed heuristically to motivate a continuous-time recursive utility consistent with last section's CE approximations.

Taking as given some probability space, we assume that the underlying filtration over the continuous time set $[0, T]$ is the standard filtration generated by two stochastically independent sources of risk, forming the column vector $B = \left(B^R, B^S\right)'$. The process B^R, which represents roulette risk, is assumed to be a standard Brownian motion. The process B^S, which represents horse-race uncertainty, is stochastically independent from B^R and is assumed to be either a standard Brownian motion or a compensated Poisson process with arrival rate $\lambda > 0$. We fix a reference consumption plan c with corresponding utility process U. For every time $t < T$, c_t represents a time-t consumption rate, while c_T represents a terminal lump-sum consumption level. As in the discrete model, we set $U_T = c_T$. The discrete-time interpretation of utility units in terms of equivalent perpetuities applies here, too. (In particular, we think of the terminal consumption c_T as being equivalent to a continuous perpetuity that begins at time T and pays c_T per unit of time for ever after.)

The continuous-time version of the recursive utility of Theorem 11 can be expressed heuristically as

$$U_{t-} = \phi_\delta^{-1}\left(\left(1 - e^{-\beta dt}\right)\phi_\delta(c_t) + e^{-\beta dt}\phi_\delta\left(\upsilon_t\left(U_{t+dt}\right)\right)\right), \tag{17}$$

where U_{t-} denotes the time-t utility value just prior to any time-t jump, dt is a time infinitesimal analogous to the quantity h of the discrete-time analysis, and

$$\upsilon_t\left(U_{t+dt}\right) = \phi_{\gamma_S}^{-1}\left(\mathbb{E}_t\left[\phi_{\gamma_S} \circ \phi_{\gamma_R}^{-1}\left(\mathbb{E}_t\left[\phi_{\gamma_R}\left(U_{t+dt}\right) \mid B_{t+dt}^S\right]\right)\right]\right),$$

where $\mathbb{E}_t$ denotes conditional expectation given the history of B up to time t. Our objective is to transform this heuristic utility specification to a BSDE that can be given a rigorous mathematical meaning.

We will apply last section's CE approximations heuristically by letting $h = dt$ and $x\left(dt\right) = U_{t+dt}/U_{t-}$. Terms that are order $o\left(dt\right)$ are treated as zero (for example, $e^{-\beta dt} = 1 - \beta dt$). The analog to expression (11) for $x\left(h\right)$ are the utility Ito dynamics:

$$\frac{dU_t}{U_{t-}} = \mu_t dt + \sigma_t dB_t. \tag{18}$$

Here $\sigma_t = \left(\sigma_t^R, \sigma_t^S\right)$, where σ_t^R is the volatility of the roulette Brownian motion, and σ_t^S is either the volatility of the horse-race Brownian motion or the time-t jump size of the horse-race Poisson jump, conditionally on there being a time-t jump. Using the fact that the CE is homogeneous of degree one, the approximations of Propositions 14 and 15 translate to

$$\frac{\upsilon_t\left(U_{t+dt}\right)}{U_{t-}} = \mathbb{E}_t\left[\frac{U_{t+dt}}{U_{t-}}\right] - \mathcal{R}\left(\sigma_t\right)dt = 1 + \left(\mu_t - \mathcal{R}\left(\sigma_t\right)\right)dt, \tag{19}$$

where the $\mathcal{R}\left(\sigma_t\right)dt$ term represents a relative risk/ambiguity-aversion adjustment to the risk-neutral CE under prior P, and is specified as follows.

- If $\left(B^R, B^S\right)$ is a two-dimensional standard Brownian motion, then

$$\mathcal{R}\left(\sigma\right) = \frac{\gamma_R}{2}\left(\sigma^R\right)^2 + \frac{\gamma_S}{2}\left(\sigma^S\right)^2.$$

- If B^R is a standard Brownian motion and B^S is a compensated Poisson process with arrival rate λ, then

$$\mathcal{R}\left(\sigma\right) = \frac{\gamma_R}{2}\left(\sigma^R\right)^2 + \left(\sigma^S - \frac{\left(1+\sigma^S\right)^{1-\gamma_S} - 1}{1-\gamma_S}\right)\lambda.$$

Now substitute (19) into (17), take a first-order Taylor expansion of ϕ_δ around U_{t-}, solve for μ_t and insert the resulting expression back into the utility dynamics (18) to find

$$\frac{dU_t}{U_{t-}} = -\left(\beta u_\delta\left(\frac{c_t}{U_t}\right) - \mathcal{R}(\sigma_t)\right)dt + \sigma_t dB_t, \quad U_T = c_T, \tag{20}$$

where

$$u_\delta(x) = \frac{x^{1-\delta}-1}{1-\delta} \quad (= \log x \text{ if } \delta = 1). \tag{21}$$

Equation (20) is a BSDE to be solved jointly in (U, σ). This is a fixed-point problem, whose solution requires regularity conditions. We refer to Delong (2013) and Skiadas (2008, 2010) for appropriate references to the BSDE literature. The resulting utility specification, with or without jumps, is within the broader class introduced in Schroder and Skiadas (2008) in the context of optimal consumption/portfolio choice. In the case that B^S is a standard Brownian motion, the above specification already appears in Schroder and Skiadas (2003) and reduces to the continuous-time version of EZW utility of Duffie and Epstein (1992) if $\gamma_R = \gamma_S$.

B Appendix: Proofs

This appendix contains proofs omitted from the main text. Some needed facts, terminology and notation are reviewed below.

Suppose that $\succ$ is any relation on $(0,\infty)^N$ for some positive integer N. The following key result on SI additive representations is Theorem 17 of Skiadas (2013b). Recall that

$$\Delta_N = \left\{ p \in (0,1)^N : \sum_{n=1}^N p_n = 1 \right\}$$

and for any scalar δ, the function $u_\delta : (0,\infty) \to \mathbb{R}$ is defined by (21).

Lemma 16 *Assuming $N > 2$, the relation $\succ$ is a scale invariant, separable, increasing and continuous preference order if and only if there exist $p \in \Delta_N$ and $\gamma \in \mathbb{R}$ such that*

$$a \succ b \quad \iff \quad \sum_{n=1}^N p_n u_\gamma(a_n) > \sum_{n=1}^N p_n u_\gamma(b_n), \quad \text{for all } a, b \in (0,\infty)^N. \tag{22}$$

The parameters p and γ are unique.

A **certainty equivalent (CE)** on $(0,\infty)^N$ is any increasing, continuous function of the form $\nu : (0,\infty)^N \to (0,\infty)$ satisfying $\nu(\alpha \mathbf{1}) = \alpha$ for all $\alpha \in (0,\infty)$. The notation $\mathbf{1}$ refers to any vector all of whose components are equal to one, the dimensionality being implied by the context. The CE ν is said to be **scale invariant (SI)** if $\nu(\alpha x) = \alpha \nu(x)$ for all $\alpha \in (0,\infty)$ and $x \in (0,\infty)^N$. The CE ν is said to **represent** $\succ$ if for all $x, y \in (0,\infty)^N$, $\nu(x) > \nu(y)$ is equivalent to $x \succ y$.

The following simple facts are well known. (See, for example, Chapter 3 of Skiadas (2009).) A CE ν representing $\succ$ exists if and only if $\succ$ is an increasing and continuous preference order, in which case the representation is unique and is given by $\nu(x) = \inf\{\alpha \in (0,\infty) : \alpha\mathbf{1} \succ x\}$. (Equivalently, $\nu(x)$ is the unique value in $(0,\infty)$ such that $\nu(x)\mathbf{1} \sim x$.) If the CE ν represents $\succ$, then $\succ$ is SI if and only if ν is SI.

B.1 Proof of Theorem 11

We assume that $\succ$ is a binary relation on $\mathcal{L}_{++}$ satisfying Conditions 1 through 9. We will show existence and uniqueness of the claimed representation of $\succ$ and we will characterize ambiguity aversion by the condition $\gamma_S \geq \gamma_R$. The verification of the necessity of Conditions 1-9 is tedious but straightforward and is omitted.

Notational conventions: Throughout this proof, $F = G \cap H$ denotes a generic time-t spot, with $G \in \mathcal{R}_t^0$ and $H \in \mathcal{S}_t^0$. The immediate successor spots of F are denoted as in (1). For any $\alpha \in (0,\infty)$, we write α for the process that is identically equal to α. For any adapted process x, the common value that x takes at spot F is denoted x_F.

Step 1 (conditional utility processes) Condition 1 implies the existence of a unique SI CE $U_0 : \mathcal{L}_{++} \to (0,\infty)$ representing $\succ$. For any given spot $F \in \mathcal{F}_t^0$, let $\mathcal{L}_{++}^F$ denote the set of all $c \in \mathcal{L}_{++}$ such that $c = c_{F\times\{t,\ldots,T\}}\mathbf{1}$. Let the function $U_F : \mathcal{L}_{++}^F \to (0,\infty)$ be defined by

$$U_F(c) = h_F^{-1} \circ U_0\left(c_{F\times\{t,\ldots,T\}}\mathbf{1}\right), \quad \text{where} \quad h_F(\alpha) = U_0\left(\alpha_{F\times\{t,\ldots,T\}}\mathbf{1}\right).$$

By construction, $U_F(\alpha) = \alpha$ for all $\alpha \in (0,\infty)$. Moreover, with $\succ^F$ defined in (4),

$$a \succ^F b \quad \Longleftrightarrow \quad U_F(a) > U_F(b), \quad a, b \in \mathcal{L}_{++}.$$

We refer to these two conditions by saying that U_F is the CE representation of $\succ^F$, although technically speaking this is true only after restricting U_F and $\succ^F$ to consumption plans on the subtree rooted at spot F. Fixing any consumption plan c, the values $U_F(c)$ as F ranges over all spots are recursively related to each other in the following steps.

Step 2 (reduction to contingent annuities) We use the following characterization of the dynamic-consistency Condition 3. The straightforward proof can be found in Lemma 6.2 of Skiadas (2009).

Lemma 17 *Assuming that $\succ$ is increasing and continuous, Condition 3 is equivalent to the condition: For any $F \in \mathcal{F}_t^0$ and $a, b \in \mathcal{L}_{++}$ such that $a(\omega, s) = b(\omega, s)$ for all $(\omega, s) \notin F \times \{t, \ldots, T\}$, $a \succ^F b$ if and only if $a \succ b$.*

Let us fix any reference consumption plan $c \in \mathcal{L}_{++}$ and spot $F \in \mathcal{F}_t^0$. We show that

$$c \sim^F c_F 1_{F \times \{t\}} + \sum_{rs} U_{F_{rs}}(c) \, 1_{F_{rs} \times \{t+1, \ldots, T\}}. \tag{23}$$

For this step only, we enumerate the spots F_{rs} as $F_1, F_2, \ldots, F_N$, where $N = R \times S$. Let

$$x_i = U_{F_i}(c) \, 1_{F_i \times \{t+1, \ldots, T\}} - c 1_{F_i \times \{t+1, \ldots, T\}}, \quad i = 1, \ldots, N.$$

Adding x_i to c replaces the value of c at each spot of the subtree rooted at F_i with the same value $U_{F_i}(c)$, thus replacing c with a contingent annuity. We show inductively that

$$c \sim^F c + \sum\nolimits_{i=0}^{n} x_i, \quad n = 0, 1, \ldots, N. \tag{24}$$

The root of the induction is $c \sim^F c$. For the inductive step, we assume that $c \sim^F b$, where $b = c + \sum_{i=0}^{n-1} x_i$, with $b = c$ for $n = 0$. Condition 2 on the irrelevance of past or unrealized consumption and the fact that U_{F_n} is the CE representation of $\succ^{F_n}$ (see Step 1) imply that $c \sim^{F_n} c + x_n$. Since c equals b on the subtree rooted at F_n, we also have $b \sim^{F_n} b + x_n$. By Lemma 17, the last condition implies $b \sim b + x_n$, which in turn implies that $b \sim^F b + x_n$. The last equation combined with the inductive hypothesis $c \sim^F b$ gives $c \sim^F b + x_n$, proving (24). The claim (23) follows from (24) by letting $n = N$.

Step 3 (reduction to deterministic consumption) Given any $F \in \mathcal{F}_t^0$, $\succ_{R \times S}^F$ is an increasing, continuous preference order on $X_{R \times S}$ and therefore has a CE representation $\nu_{R \times S}^F : X_{R \times S} \to (0, \infty)$. Consider any reference $c \in \mathcal{L}_{++}$ and let $U_{F_{**}}(c)$ denote the element x of $X_{R \times S}$ defined by $x_{rs} = U_{F_{rs}}(c)$ for $(r, s) \in \{1, \ldots, R\} \times \{1, \ldots, S\}$. The definition of $\succ_{R \times S}^F$ and (23) imply that

$$c \sim^F c_F 1_{F \times \{t\}} + \nu_{R \times S}^F \left(U_{F_{**}}(c) \right) 1_{F \times \{t+1, \ldots, T\}}.$$

Defining the function $f : (0, \infty)^2 \to (0, \infty)$ by $f(x, y) = U_F \left(x 1_{F \times \{t\}} + y 1_{F \times \{t+1, \ldots, T\}} \right)$, it follows that

$$U_F(c) = f\left(c_F, \nu_{R \times S}^F \left(U_{F_{**}}(c) \right) \right). \tag{25}$$

We refer to f as the *intertemporal aggregator.* Condition 4 implies that the preceding definition of f does not depend on the specific nonterminal spot used in its definition.

Equation (25) is the essential utility recursion. In the following steps we derive the functional structure on the intertemporal aggregator f and the conditional CEs $\nu^F_{R\times S}$.

Step 4 (intertemporal aggregator structure) By Lemma 16 and Conditions 1 and 5, there exist unique time weights $w_t \in (0,\infty)$ such that $\sum_{t=0}^T w_t = 1$ and a unique scalar $\delta \in [0,\infty)$ such that the CE representation of the restriction of $\succ$ to the set $\mathcal{L}^{\mathcal{T}}_{++}$ of all deterministic plans is given by

$$U_0(c) = \phi_\delta^{-1}\left(\sum_{t=0}^T w_t \phi_\delta(c_t)\right), \quad c \in \mathcal{L}^{\mathcal{T}}_{++}.$$

For each $c \in \mathcal{L}^{\mathcal{T}}_{++}$, let the deterministic process U^c be defined by

$$U^c_t = \phi_\delta^{-1}\left(D_t \sum_{s=t}^T w_s \phi_\delta(c_s)\right), \quad \text{where} \quad D_t = \frac{1}{\sum_{s=t}^T w_s}. \tag{26}$$

Dynamic consistency, in the form of Lemma 17, implies that

$$U_F(c) = U_t(c), \quad c \in \mathcal{L}^{\mathcal{T}}_{++}, \quad F \in \mathcal{F}^0_t. \tag{27}$$

Indeed, if $a, b \in \mathcal{L}^{\mathcal{T}}_{++}$ are equal at all times prior to t, then $a \succ^F b$ is equivalent to $a \succ b$, which is equivalent to $U_0(a) > U_0(b)$, which is in turn equivalent to $U^a_t > U^b_t$ (since $a_s = b_s$ for $s < t$). By construction, it is also true that $U^\alpha_t = \alpha$ for every $\alpha \in (0,\infty)$, and therefore $c \mapsto U^c_t$ is the CE representation of each $\succ^F$, $F \in \mathcal{F}^0_t$, restricted to deterministic plans.

Combining (25) and (27) it follows that

$$U^c_t = f\left(c_t, U^c_{t+1}\right), \quad t = 0,\ldots,T, \quad c \in \mathcal{L}^{\mathcal{T}}_{++}.$$

On the other hand, the definition of U^c_t in (26) implies the recursion

$$U^c_t = \phi_\delta^{-1}\left((1-\beta_t)\phi_\delta(c_t) + \beta_t \phi_\delta\left(U^c_{t+1}\right)\right), \quad \text{where} \quad \beta_t = \frac{D_t}{D_{t+1}} \in (0,1).$$

The two recursions are consistent if and only if there exists $\beta \in (0,1)$ such that $\beta_t = \beta$ for all t, and the intertemporal aggregator takes the constant-EIS form

$$f(x,y) = \phi_\delta^{-1}\left((1-\beta)\phi_\delta(x) + \beta\phi_\delta(y)\right). \tag{28}$$

Step 5 (conditional CE structure) Recall that $\nu^F_{R\times S}$ is the CE representation of $\succ^F_{R\times S}$. Let ν^F_R and ν^F_S denote the respective CE representations of $\succ^F_R$ and $\succ^F_S$. Condition 7 implies that $\succ_R^F$ depends on $F = G \cap H$ only through the event G, that is, for any $H' \in \mathcal{S}_t^0$ whose intersection with G is nonempty, $\succ_R^{G \cap H'} = \succ_R^F$. Given this fact, we henceforth write $\succ_R^G$ and ν_R^G in place of $\succ_R^F$ and ν_R^F. Analogously, we write $\succ_S^H$ and ν_S^H in place of $\succ_S^F$ and ν_S^F.

Condition 9 implies the separability of both $\succ_R^G$ and $\succ_S^H$. Applying Lemma 16 to these two preference orders, we find that there exist unique $p^G \in \Delta_R$ and $\gamma_G \in \mathbb{R}$ such that

$$\nu_R^G(x) = \phi_{\gamma_G}^{-1}\left(\sum_{r=1}^{R} p_r^G \phi_{\gamma_G}(x_r)\right), \quad x \in X_R, \tag{29}$$

and unique $p^H \in \Delta_S$ and $\gamma_H \in \mathbb{R}$ such that

$$\nu_S^H(x) = \phi_{\gamma_H}^{-1}\left(\sum_{s=1}^{S} p_s^H \phi_{\gamma_H}(x^s)\right), \quad x \in X_S. \tag{30}$$

Condition 8 implies that the function $g : (0,\infty)^R \to (0,\infty)$ is well defined (for the given spot F) by

$$\nu_{R\times S}^F(x) = g\left(\nu_R^G\left(x_*^1\right), \ldots, \nu_R^G\left(x_*^S\right)\right). \tag{31}$$

To confirm this claim, consider any $y \in X_{R\times S}$ such that $\nu_R^G(x_*^s) = \nu_R^G(y_*^s)$ for every s. By monotonicity, $y_*^s + \varepsilon\mathbf{1} \succ_R^F x_*^s$ for all s and $\varepsilon > 0$. Condition 8 then implies that $y + \varepsilon\mathbf{1} \succ_{R\times S}^F x$ for all $\varepsilon > 0$. By continuity, this shows that $\nu_{R\times S}^F(y) \geq \nu_{R\times S}^F(x)$. The reverse inequality follows by symmetry, and therefore $\nu_{R\times S}^F(y) = \nu_{R\times S}^F(x)$, confirming that g is well-defined. Restricting (31) to $x \in X_S$ shows that $g = \nu_S^H$. Therefore, the combination of (31), (29) and (30) gives the representation

$$\nu_{R\times S}^F(x) = \phi_{\gamma_H}^{-1}\left(\sum_{s=1}^{S} p_s^H \phi_{\gamma_H} \circ \phi_{\gamma_G}^{-1}\left(\sum_{r=1}^{R} p_r^G \phi_{\gamma_G}(x_r^s)\right)\right), \quad x \in X_{R\times S}. \tag{32}$$

In the following step we show that the parameters γ_G and γ_H do not vary with G and H.

Step 6 (relating conditional CEs across spots) Suppose further that $t < T - 1$. Recall that $F = G \cap H$ and $G_1, \ldots, G_R$ are the immediate roulette spot successors to G. Letting each $\nu_R^{G_r}$ be defined as in Step 5, the definition of $\succ_{R\times R}^F$ implies that for all $x, y \in X_{R\times R}$, $x \succ_{R\times R}^F y$ if and only if

$$1_{F\times\{t\}} + \sum_r \nu_R^{G_r}(x_{r*})\, 1_{G_r\times\{t+1,\ldots,T\}} \succ^F 1_{F\times\{t\}} + \sum_r \nu_R^{G_r}(y_{r*})\, 1_{G_r\times\{t+1,\ldots,T\}}. \tag{33}$$

Again, Condition 7 implies that $\succ^F_{R\times R} \equiv \succ^G_{R\times R}$ depends on F only through G. Comparing (33) to the definition of $\succ^F_{R\times S}$ and using (29), it follows that $\succ^G_{R\times R}$ admits the CE representation

$$\nu^G_{R\times R}(x) = \phi^{-1}_{\gamma_G}\left(\sum_{r=1}^{R} p^G_r \phi_{\gamma_G}\left(\nu^{G_r}_R(x_{r*})\right)\right).$$

Each $\nu^{G_r}_R$ has an analogous representation to (29), resulting in

$$\nu^G_{R\times R}(x) = \phi^{-1}_{\gamma_G}\left(\sum_{r=1}^{R} p^G_r \phi_{\gamma_G} \circ \phi^{-1}_{\gamma_{G_r}}\left(\sum_{r'=1}^{R} p^{G_r}_{r'} \phi_{\gamma_{G_r}}(x_{rr'})\right)\right)$$

By Condition 9, $\succ^G_{R\times R}$ is separable and therefore has an additive representation according to Lemma 16. Since the restriction of that additive representation to X_R must be consistent with (29), it follows that there exists probability $p^G \in \Delta_{R\times R}$ such that

$$\nu^G_{R\times R}(x) = \phi^{-1}_{\gamma_G}\left(\sum_{r,r'} p^G_{rr'} \phi_{\gamma_G}(x_{rr'})\right).$$

Our earlier use of p^G in representation (29) is consistent, since the only way for the above two expressions for $\nu^G_{R\times R}$ to be compatible is that $\gamma_G = \gamma_{G_r}$ and $p^G_{rr'} = p^G_r p^{G_r}_{r'}$, and therefore $p^G_r = \sum_{r'=1}^{R} p^G_{rr'}$. Applying this argument at all roulette spots, it follows that there exists a common CRRA γ_R such that $\gamma_R = \gamma_G$ for every roulette spot G. The symmetric argument with horse-race uncertainty in place of roulette uncertainty shows that there exists γ_S such that $\gamma_S = \gamma_H$ for every horse-race spot H. Therefore, representation (32) can be refined to (as always, with $F = G \cap H$)

$$\nu^F_{R\times S}(x) = \phi^{-1}_{\gamma_S}\left(\sum_{s=1}^{S} p^H_s \phi_{\gamma_S} \circ \phi^{-1}_{\gamma_R}\left(\sum_{r=1}^{R} p^G_r \phi_{\gamma_R}(x_{rs})\right)\right), \quad x \in X_{R\times S}. \tag{34}$$

Step 7 (putting it all together) Establishing the Theorem's utility recursion is now a matter of putting together the earlier results and some simplifying notation. Given any $c \in \mathcal{L}_{++}$, Step 1 defines a value $U_F(c)$ for every spot F. Together these values define an adapted process U^c with $U^c_0 = U_0(c)$. Note that, because of (26), this notation is consistent with the definition of U^c in Step 4 in the case of deterministic c. Let us also define the probability P in terms of the $p^G \in \Delta_R$ and $p^H \in \Delta_S$ of representation (34) as follows. The restriction of P on $\mathcal{R}_T$ and $\mathcal{S}_T$ are defined recursively by multiplying through the transition probabilities

$$P[G_r \mid G] = p_r^G \quad \text{and} \quad P[H_s \mid H] = p_s^H,$$

for every roulette spot G with successor roulette spots $G_1, \ldots, G_R$, and every horse-race spot H with successor horse-race spots $H_1, \ldots, H_S$. The entire probability P on $\mathcal{F}_T = 2^\Omega$ is determined by the stochastic independence of $\mathcal{R}_T$ and $\mathcal{S}_T$: $P[F] = P[G]\,P[H]$ for every spot $F = G \cap H$, where $G \in \mathcal{G}_t^0$ and $H \in \mathcal{S}_t^0$. With U^c and P so defined, the Theorem's utility recursion is equivalent to recursion (25), for every nonterminal spot F, with the intertemporal aggregator given (uniquely) in (28) and the conditional CE given (also uniquely) in (34).

Step 8 (characterization of ambiguity aversion) Given the just derived utility representation, we show that $\succ$ is ambiguity averse if and only if $\gamma_S \geq \gamma_R$. Let us fix any reference spot F, nonempty $B \subseteq \{1, \ldots, R\}$, and any $x, y \in X_S$ such that $x \sim_S^F y$, which is equivalent to

$$L \equiv \sum_{s=1}^{S} p_s^H \phi_{\gamma_S}(x_s) = \sum_{s=1}^{S} p_s^H \phi_{\gamma_S}(y_s).$$

Letting $p_B^G = \sum_{r \in B} p_r^G$, the condition $x_B y \succeq_{R \times S}^F y$ is equivalent

$$\sum_{s=1}^{S} p_s^H \phi_{\gamma_S} \circ \phi_{\gamma_R}^{-1} \left(p_B^G \phi_{\gamma_R}(x_s) + \left(1 - p_B^G\right) \phi_{\gamma_R}(y_s) \right) \geq L. \tag{35}$$

Note that $\phi_{\gamma_S} \circ \phi_{\gamma_R}^{-1}$ is concave if $\gamma_S \geq \gamma_R$ and strictly convex if $\gamma_S > \gamma_R$. Therefore (35) is satisfied if and only if $\gamma_S \geq \gamma_R$.

B.2 Proof of Proposition 12

The proof is based on the recursive application of the following lemma.

Lemma 18 *For any $\alpha \in \mathbb{R}$ with $\alpha \neq 1$ and all $x \in (0, \infty)^N$,*

$$-\log\left(\sum_{n=1}^{N} p_n x_n^{1-\alpha}\right) = \min_{q \in \Delta_N} \left\{ \sum_{n=1}^{N} q_n (\alpha - 1) \log x_n + q_n \log\left(\frac{q_n}{p_n}\right) \right\}.$$

Proof. The claim is obtained by setting $z_n = (\alpha - 1) \log x_n$ in the following well-known variational identity (see Donsker and Varadhan (1975)):

$$-\log\left(\sum_{n=1}^{N} p_n \exp\left(-z_n\right)\right) = \min_{q \in \Delta_N} \left\{\sum_{n=1}^{N} q_n z_n + q_n \log\left(\frac{q_n}{p_n}\right)\right\}.$$

This is an instance of Fenchel-Legendre duality (see Section 12 of Rockafellar (1970)). ∎

Fixing the reference consumption plan c, we write $U = U^c$ for the utility process associated with c, which is computed by the utility recursion of Theorem 11.

Step 1 (dual representation of horse-race risk aversion) Recall the second step (7) of the utility recursion:

$$(\gamma_S - 1) \log \upsilon_t = -\log \mathbb{E}_t \left[\bar{U}_{t+1}^{1-\gamma_S}\right], \quad \text{where } \upsilon_t \equiv \upsilon_t\left(U_{t+1}\right). \tag{36}$$

To apply Lemma 18, let us isolate any time-t spot $F = G \cap H$, where $G \in \mathcal{R}_t^0$ and $H \in \mathcal{S}_t^0$. As always, we use the notation $F_{rs} = G_r \cap H_s$ for the immediate successor spots of F, as defined in (1). We apply Lemma 18 with $\alpha = \gamma_S$, $N = S$ and

$$p_s = P\left[G \cap H_s \mid G \cap H\right] = P\left[H_s \mid H\right],$$

where the second equality follows from the fact that $P \in \Delta_{\mathcal{R} \perp \mathcal{S}}$. Each $q \in \Delta_N = \Delta_S$ is represented by the ratios

$$\xi\left(H_s \mid H\right) \equiv \frac{q_s}{p_s}, \quad s = 1, \ldots, S. \tag{37}$$

Note that

$$\xi\left(H_s \mid H\right) > 0 \quad \text{and} \quad \sum_{s=1}^{S} \xi\left(H_s \mid H\right) P\left[H_s \mid H\right] = 1. \tag{38}$$

Conversely, if (38) is satisfied, then (37) defines a $q \in \Delta_S$ in terms of the $\xi\left(H_s \mid H\right)$. Suppose now we have made a selection $q \in \Delta_S$ for every nonterminal horse-race event H, defining the associated $\xi\left(H_s \mid H\right)$. We can then recursively piece together an $\mathcal{S}$-adapted process ξ by letting

$$\frac{\xi\left(H_s\right)}{\xi\left(H\right)} = \xi\left(H_s \mid H\right), \quad H \in \mathcal{S}_t^0, \quad H_s \in \mathcal{S}_{t+1}^0 \cap H; \quad \xi_0 = 1.$$

Conditions (38) are equivalent to the requirement that $\xi \in \Xi$.

Given these observations, and the assumption $\gamma_S > 1$, it is clear that Lemma 18 implies that (36) can be expressed, spot by spot, as

$$\log \upsilon_t = \min_{\xi \in \Xi} \mathbb{E}_t \left[\frac{\xi_{t+1}}{\xi_t} \log \bar{U}_{t+1} + \frac{1}{\gamma_S - 1} \frac{\xi_{t+1}}{\xi_t} \log \left(\frac{\xi_{t+1}}{\xi_t} \right) \right],$$

where a single $\xi \in \Xi$ simultaneously achieves the minimum at every nonterminal spot. Inserting the expression for $\log \bar{U}_{t+1}$ from the first step (6) of the utility recursion, using the fact that ξ_{t+1}/ξ_t is $\mathcal{S}_{t+1}$-measurable and the law of iterated expectations, we arrive to the expression:

$$\log \upsilon_t = \min_{\xi \in \Xi} \mathbb{E}_t \left[\frac{\xi_{t+1}}{\xi_t} \log U_{t+1} + \frac{1}{\gamma_S - 1} \frac{\xi_{t+1}}{\xi_t} \log \left(\frac{\xi_{t+1}}{\xi_t} \right) \right]. \tag{39}$$

Step 2 (adding preferences for intertemporal smoothing) We similarly transform the third step (8) of the utility recursion, which can be restated as

$$(\delta - 1) \log U_t = - \log \left((1 - \beta)\, c_t^{1-\delta} + \beta \upsilon_t^{1-\delta} \right).$$

Recall that $\mathcal{D}$ denotes the set of all strictly positive, predictable and strictly decreasing processes ρ where $\rho_0 = 1$. Let also D_t denote the set of every strictly positive $\mathcal{F}_t$-measurable $(0,1)$-valued random variable. For any $\rho \in \mathcal{D}$, we have $\rho_{t+1}/\rho_t \in D_t$ for all $t < T$. Conversely, given $d_t \in D_t$, $t = 0, \ldots, T-1$, a unique $\rho \in \mathcal{D}$ is determined by the recursion $\rho_{t+1}/\rho_t = d_t$, starting with $\rho_0 = 1$. For $\delta > 1$, the duality of Lemma 18 (with $N = 2$) can therefore be stated in this context as

$$\log U_t = \min_{\rho \in \mathcal{D}} \left\{ \left(1 - \frac{\rho_{t+1}}{\rho_t} \right) \log c_t + \frac{\rho_{t+1}}{\rho_t} \log \upsilon_t + \frac{1}{\delta - 1} \Phi \left(\frac{\rho_{t+1}}{\rho_t} \right) \right\}, \tag{40}$$

where a single $\rho \in \mathcal{D}$ achieves the minimum for all $t < T$. The same identity holds trivially if $\delta = 1$, because of conventions (9), with the minimizing ρ_{t+1}/ρ_t equal to β.

Combining (39) and (40), the utility recursion of Theorem 11 can be written as

$$\begin{aligned} \log U_t = \min_{(\rho,\xi) \in \mathcal{D} \times \Xi} & \left(1 - \frac{\rho_{t+1}}{\rho_t} \right) \log c_t + \frac{1}{\delta - 1} \Phi \left(\frac{\rho_{t+1}}{\rho_t} \right) \\ & + \frac{\rho_{t+1}}{\rho_t} \mathbb{E}_t \left[\frac{\xi_{t+1}}{\xi_t} \log U_{t+1} + \frac{1}{\gamma_S - 1} \frac{\xi_{t+1}}{\xi_t} \log \left(\frac{\xi_{t+1}}{\xi_t} \right) \right], \end{aligned}$$

where a common process pair (ρ, ξ) simultaneously achieves the minima at all nonterminal spots of the filtration.

Step 3 (unwinding the recursion) Multiplying both sides of the preceding recursion by $\rho_t \xi_t$ and applying the usual dynamic programming argument, we note that

$$\log U_0 = \min_{(\rho,\xi)\in\mathcal{D}\times\Xi} V_0(\rho,\xi), \tag{41}$$

where the process $V(\rho,\xi)$ (corresponding to $\rho\xi\log U$) is defined recursively by

$$V_t(\rho,\xi) = (\rho_t - \rho_{t+1})\,\xi_t \log c_t + \frac{\rho_t\xi_t}{\delta-1}\Phi\left(\frac{\rho_{t+1}}{\rho_t}\right) + \mathbb{E}_t\left[V_{t+1}(\rho,\xi) + \frac{1}{\gamma_S - 1}\rho_{t+1}\xi_{t+1}\log\left(\frac{\xi_{t+1}}{\xi_t}\right)\right],$$

with terminal condition $V_T(\rho,\xi) = \rho_T\xi_T\log c_T$. We now subtract $\mathbb{E}_t V_{t+1}(\rho,\xi)$ from both sides of this recursion, we take the unconditional expectation on both sides, and we add up the resulting expression from $t=0$ to $T-1$ to find (using $\rho_{T+1}=0$):

$$V_0(\rho,\xi) = \mathbb{E}\sum_{t=0}^{T}\xi_t\left[(\rho_t-\rho_{t+1})\log c_t + \frac{\rho_t}{\delta-1}\Phi\left(\frac{\rho_{t+1}}{\rho_t}\right)\right] + \frac{C_0(\rho,\xi)}{\gamma_S-1},$$

where

$$C_0(\rho,\xi) = \mathbb{E}\left[\sum_{t=0}^{T-1}\rho_{t+1}\xi_{t+1}\log\left(\frac{\xi_{t+1}}{\xi_t}\right)\right].$$

Using the fact that $\xi\in\Xi$ and $\rho_{T+1}=0$, and the law of iterated expectations, we have

$$\begin{aligned} C_0(\rho,\xi) &= \mathbb{E}\left[\sum_{t=0}^{T-1}\rho_{t+1}\xi_{t+1}\log\xi_{t+1} - \sum_{t=1}^{T}\rho_{t+1}\xi_{t+1}\log\xi_t\right] \\ &= \mathbb{E}\left[\sum_{t=1}^{T}\rho_t\xi_t\log\xi_t - \sum_{t=1}^{T}\rho_{t+1}\mathbb{E}_t\xi_{t+1}\log\xi_t\right] \\ &= \mathbb{E}\left[\sum_{t=1}^{T}(\rho_t-\rho_{t+1})\,\xi_t\log\xi_t\right]. \end{aligned}$$

Substituting into the last expression for $V_0(\rho,\xi)$, we conclude that equation (41) is the Proposition's claimed identity.

B.3 Proof of Proposition 14

We use the notation $\mathbb{E}_R U = \sum_{r=1}^{R} p_r U_r$ and $\mathbb{E}_S U = \sum_{s=1}^{S} q_s U_s$, and the abbreviations

$$B_R = B^R(h), \quad B_r = B_r^R, \quad B_S = B^S(h), \quad B_s = B_s^S.$$

Recall also the simplified notation in (12). Since $p_r B_r^2 \le \mathbb{E}_R B_R^2 = h + o(h)$ for every $r \in \{1, \ldots, R\}$, and analogously for B_S, there is a constant M such that

$$\max_{r,s}\left\{B_r^2, B_s^2, B_r B_s\right\} \le Mh + o(h). \tag{42}$$

We write the CE ν as

$$\nu(U) = f_S^{-1}\mathbb{E}_S\left[f_S \circ f_R^{-1}\mathbb{E}_R f_R(U)\right], \quad U \in X,$$

where

$$f_R(y) = \frac{y^{1-\gamma_R} - \gamma_R}{1-\gamma_R} \quad \text{and} \quad f_S(y) = \frac{y^{1-\gamma_S} - \gamma_S}{1-\gamma_S}.$$

A second-order Taylor expansion of f_R around one, using (42) and the identities $f_R(1) = f_R'(1) = 1$ and $f_R''(1) = -\gamma_R$, gives (state by state)

$$f_R(x(h)) = 1 + \alpha h + \sigma_R B_R + \sigma_S B_S - \frac{\gamma_R}{2}\left(\sigma_R^2 B_R^2 + \sigma_S^2 B_S^2 + 2\sigma_R\sigma_S B_R B_S\right) + o(h).$$

Taking expectations with respect to roulette uncertainty and using (13), we have (at every horse-race state)

$$\mathbb{E}_R f_R(x(h)) = 1 + \left(\alpha - \frac{\gamma_R}{2}\sigma_R^2\right)h + \sigma_S B_S - \frac{\gamma_R}{2}\sigma_S^2 B_S^2 + o(h).$$

Letting $F = f_S \circ f_R^{-1}$, it is easily shown that $F(1) = F'(1) = 1$ and $F''(1) = \gamma_R - \gamma_S$. A second-order Taylor expansion of F around one, again using (42), gives

$$f_S \circ f_R^{-1}\mathbb{E}_R f_R(x(h)) = 1 + \left(\alpha - \frac{\gamma_R}{2}\sigma_R^2\right)h - \frac{\gamma_S}{2}\sigma_S^2 B_S^2 + \sigma_S B_S + o(h).$$

Taking expectations with respect to horse-race uncertainty, using (14), we have

$$\mathbb{E}_S f_S \circ f_R^{-1}\mathbb{E}_R f_R(x(h)) = 1 + \left(\alpha - \frac{\gamma_R}{2}\sigma_R^2 - \frac{\gamma_S}{2}\sigma_S^2\right)h + o(h).$$

Finally, since $f_S^{-1}(1) = (f_S^{-1})'(1) = 1$, a first-order Taylor expansion of f_S^{-1} around one gives the claimed CE approximation (15).

B.4 Proof of Proposition 15

We use the same notation and abbreviations as in the proof of Proposition 14. Here the bound (42) applies only to the roulette factor: $\max_r \left\{B_r^2\right\} \leq Mh + o(h)$.

At the first horse-race state we have $B_1 = 1 - \lambda h + o(h)$ and therefore

$$\mathbb{E}_R f_R\left(U^1(h)\right) = \mathbb{E}_R f_R\left(1 + \sigma_S + (\alpha - \lambda\sigma_S)h + \sigma_R B_R + o(h)\right) = f_R(1+\sigma_S) + o(1), \tag{43}$$

where $o(1)$ is a function of h that goes to zero as $h \downarrow 0$. At the second horse-race state, we have $B_2 = -\lambda h + o(h)$ and therefore (at every roulette state)

$$\begin{aligned} f_R\left(U^2(h)\right) &= f_R\left(1 + (\alpha - \lambda\sigma_S)h + \sigma_R B^R + o(h)\right) \\ &= 1 + (\alpha - \lambda\sigma_S)h + \sigma_R B_R - \frac{\gamma_R^2}{2}\sigma_R^2\left(B^R\right)^2 + o(h). \end{aligned}$$

Applying the roulette expectation operator,

$$\mathbb{E}_R f_R\left(U^2(h)\right) = 1 + \left(\alpha - \frac{\gamma_R^2}{2}\sigma_R^2 - \lambda\sigma_S\right)h + o(h). \tag{44}$$

Applying $F = f_S \circ f_R^{-1}$ to (43) and (44), we find (since $F(1) = F'(1) = 1$)

$$F\left(\mathbb{E}_R f_R\left(U^1(h)\right)\right) = f_S(1+\sigma_S) + o(1)$$

and

$$F\left(\mathbb{E}_R f_R\left(U^2(h)\right)\right) = 1 + \left(\alpha - \frac{\gamma_R^2}{2}\sigma_R^2 - \lambda\sigma_S\right)h + o(h).$$

Now we apply the horse-race expectation operator:

$$\begin{aligned} \mathbb{E}_S F\left(\mathbb{E}_R f_R(x(h))\right) &= f_S(1+\sigma_S)\lambda h + \left(1 + \left(\alpha - \frac{\gamma_R^2}{2}\sigma_R^2 - \lambda\sigma_S\right)h\right)(1 - \lambda h) + o(h) \\ &= 1 + \left(\alpha - \frac{\gamma_R^2}{2}\sigma_R^2 - \left(\sigma_S - \frac{(1+\sigma_S)^{1-\gamma_S} - 1}{1-\gamma_S}\right)\lambda\right)h + o(h). \end{aligned}$$

Finally, since $f_S^{-1}(1) = (f_S^{-1})'(1) = 1$, a first-order Taylor expansion of f_S^{-1} around one gives the claimed approximation of $\nu_h(x(h)) = f_S^{-1}\mathbb{E}_S F\left(\mathbb{E} f_R(x(h))\right)$.

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