e-Companion to An Ascending Vickrey Auction for Selling Bases of a Matroid

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A e-Companion

A.1 Matroid Proof(s)

Proof of Lemma 4. For (i): If e is a coloop then $C^\star$ is a cocircuit of $\mathcal{M}/e$ but $\mathcal{M}/e = \mathcal{M} \setminus e$ so $C^\star$ is a cocircuit of $\mathcal{M} \setminus e$ thereby validating the claim. So we assume now that e is not a coloop.

If $C^\star$ is a coloop of $\mathcal{M}$, then, as $C^\star \cup \{e\}$ is codependent in $\mathcal{M}$, the set $C^\star$ is codependent in $\mathcal{M} \setminus e$. As it contains only one element, clearly $C^\star$ is a cocircuit of $\mathcal{M} \setminus e$. So we assume now, that $C^\star$ contains at least two elements.

Now consider an element $f \in C^\star$. Notice $C^\star \setminus f$ is coindependent in $\mathcal{M}$; as $C^\star$ is codependent in $\mathcal{M}$ so is $C^\star \cup e$, therefore $C^\star$ is codependent in $\mathcal{M} \setminus e$.

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Case 1: If $(C^\star \cup e) \setminus f$ is coindependent in $\mathcal{M}$, then $C^\star \setminus f$ is coindependent in $\mathcal{M} \setminus e$. But now we have that $C^\star$ is codependent and $C^\star \setminus f$ is coindependent in $\mathcal{M} \setminus e$; hence there has to be a cocircuit in $\mathcal{M} \setminus e$ contained in $C^\star$ through f.

Case 2: If on the other hand $(C^\star \cup e) \setminus f$ is codependent in $\mathcal{M}$, then it contains a cocircuit $D^\star$ through e. As $\{e\}$ is not a coloop, $|D| \geq 2$; let g be an element of $D^\star \setminus e$; notice $g \in C^\star$. By strong cocircuit elimination, there is a cocircuit $D'^\star \subseteq (C^\star \cup D^\star) \setminus g$ containing e. So $D'^\star \subseteq (C^\star \cup e) \setminus g$ and $D'^\star \setminus \{e, g\} \subsetneq C^\star$ is codependent in $\mathcal{M} \setminus e$.
But $C^\star \setminus f$ is coindependent, so $(C^\star \cup e) \setminus f$ contains only the circuit $D^\star$ with $g \in D^\star$ and $(C^\star \cup e) \setminus \{f, g\}$ is coindependent in $\mathcal{M}$. Hence $C^\star \setminus \{f, g\}$ is coindependent in $\mathcal{M} \setminus e$. On comparing (in $\mathcal{M} \setminus e$) the coindependent set $C^\star \setminus \{f, g\}$ with the codependent set $D'^\star \setminus \{e, g\} \subsetneq C^\star$ notice that $(D'^\star \setminus \{e, g\}) \setminus (C^\star \setminus \{f, g\}) \subseteq \{f\}$. This shows that the cocircuit $D'^\star \setminus \{e, g\}$ contains f and is contained in $C^\star$.

For (ii): As $\{e\}$ is not a coloop, $\{e\} \neq C^\star$. As $e \cup (C^\star - e)$ is codependent in $\mathcal{M}$, the set $C^\star - e \neq \emptyset$ is codependent in $\mathcal{M} \setminus e$. As for any subset $I^\star \subsetneq (C^\star - e)$ the set $I^\star + e$ is coindependent in $\mathcal{M}$, the set $I^\star$ is coindependent in $\mathcal{M} \setminus e$. So in fact, $(C^\star - e)$ is *minimally* codependent in $\mathcal{M} \setminus e$. □

A.2 Proof of Theorem 7

To prove the relation between VCG-sequences and condensed ones, we need a few auxiliary results.

First, we show that contracting a b_i or removing an element that is placed in D_i does not affect future iterations of the algorithm. For contracting b_i the dual of Proposition 3 yields the following.

Lemma 19. *Given a matroid $\mathcal{M}$, a set $B \subseteq E(\mathcal{M})$, and $b \in B$, we have $\{C^\star \in \mathcal{C}^\star(\mathcal{M})\colon C^\star \cap B = \emptyset\} = \{C^\star \in \mathcal{C}^\star(\mathcal{M}/b)\colon C^\star \cap (B - b) = \emptyset\}$.*

This gives us the next result.

Lemma 20. *Consider a VCG-sequence $((C_1^\star, b_1, D_1), \ldots, (C_r^\star, b_r, D_r), (D_{r+1}))$ for $\mathcal{M}$. From any iteration i, the computation of the remaining sequence $[(C_k^\star, b_k, D_k),\ k \geq i]$ can be obtained from $\mathcal{M}' = \mathcal{M}/b_i$ or equally from $\mathcal{M}$.*

Proof. Lemma 19 implies that for any e, [every $C^\star \in \mathcal{C}(\mathcal{M})$ satisfying $f_C^\star = e$ intersects $\{b_1, \ldots, b_{k-1}\}$ in $\mathcal{M}$] if and only if [every $C^\star \in \mathcal{C}(\mathcal{M}')$ satisfying

$f^\star_C = e$ intersects $\{b_1, \dots, b_{k-1}\} \setminus \{b_i\}$ in $\mathcal{M}'$]. Hence all $C^\star_k$ and D_k chosen in one sequence also can be chosen in the other one. $\square$

Deleting arbitrary elements e is more delicate because cocircuits of $\mathcal{M}$ and $\mathcal{M}' = \mathcal{M} \setminus \{e\}$ might differ. Since e is not a coloop, by Cor. 5 it is clear that if $C^\star$ is a cocircuit of $\mathcal{M}$ then $C^\star \setminus \{e\}$ is the *union* of cocircuits of $\mathcal{M}'$. Hence it is conceivable that there is no cocircuit $C'^\star$ of $\mathcal{M}'$ with $f_{C^\star}, b_{C^\star} \in C'^\star$ and a more careful analysis becomes necessary utilizing the choice of earlier D_k, b_k.

Lemma 21. *Given a VCG-sequence* $((C^\star_1, b_1, D_1), \dots, (C^\star_r, b_r, D_r), (D_{r+1}))$ *consider iteration* 1 *and element* e *added to* D_1. *The computation of the remaining sequence from* $(1, e)$ *onwards can be carried out on the matroid* $\mathcal{M}' = \mathcal{M} \setminus e$ *such that* $(b'_k, D'_k, f_{C'^\star_k}) = (b_k, D_k, f_{C^\star_k})$ *for* $k \geq 1$.

Proof. Suppose the claim holds for some steps and in the next step in iteration $k \geq 1$ the element e' is put into D_k. Hence all cocircuits $C^\star \in \mathcal{M}$ with $f_{C^\star} = e'$ intersect $\{b_1, \dots, b_{k-1}\}$. Now if there were in $\mathcal{M}'$ a cocircuit $C'^\star$ with $f_{C'^\star} = e'$ disjoint from $\{b_1, \dots, b_{k-1}\}$ then either $C'^\star$ or $C'^\star \cup \{e\}$ is a cocircuit of $\mathcal{M}$, disjoint from $\{b_1, \dots, b_{k-1}\}$ and with second-best element e' contradicting the assumption.

Suppose instead that in the sequence the claim held sofar and in the next step in iteration $k \geq 1$ for the element e' there exists a cocircuit $C^\star_k$ of $\mathcal{M}$ with $f_{C^\star_k} = e'$ and disjoint from $\{b_1, \dots, b_{k-1}\}$. Since e is not a coloop, by Cor. 5 follows $C^\star_k - e$ is a *union* of cocircuits of $\mathcal{M}'$. Let $C'^\star$ be that part of $C^\star_k - e$ that contains b_k. If $f_{C^\star_k} \in C'^\star$ then $C'^\star$ has the same best and second-best element as $C^\star_k$. Otherwise, if $f_{C^\star_k} \notin C'^\star$ then either $C'^\star$ or $C'^\star \cup \{e\}$ is a cocircuit of $\mathcal{M}$ disjoint from $\{b_1, \dots, b_{k-1}\}$ and the second-best element has value less then e'. Consequentially, this cocircuit and its top element should have been chosen earlier; contradicting the sequence of events. $\square$

Lemma 22. *Given a VCG-sequence* $((C^\star_1, b_1, D_1), \dots, (C^\star_r, b_r, D_r), (D_{r+1}))$ *computed up to some* (i, e) *with* e *to be added to* D_i *with respect to the condensed rules and thereafter computed with respect to the uncondensed rules. The computation can be done with respect to the uncondensed rules from* (i, e) *on while the resulting sequence has the same* $b'_k, D'_k, f_{C'^\star_k}$ *as the original sequence.*

Proof. For $k < i$ we can set $(C'^\star_k, b'_k, D'_k) = (C^\star_k, b_k, D_k)$ and consider some e' (after e) to be added to D_k with $k \geq i$. Hence all cocircuits $C'^\star \in \mathcal{M}'$

with $f_{C'^\star} = e'$ intersect $\{b_1, \ldots, b_{k-1}\}$. Suppose there were a cocircuit $C^\star$ in $\mathcal{M}$ with $f_{C^\star} = e'$ disjoint from $\{b_1, \ldots, b_{k-1}\}$. Since e is not a coloop, Cor. 5 implies $C^\star - e$ is a *union* of cocircuits of $\mathcal{M}'$. Let $C'^\star$ be that part of $C^\star - e$ that contains $f_{C^\star}$. Then $f_{C'^\star} = e'$ is in $\mathcal{M}'$, contradicting the assumptions.

Now consider the b_k chosen in iteration k. There is a $C_k^\star \in \mathcal{C}^\star(\mathcal{M}')$ with $f_{C_k^\star} = e'$ disjoint from $\{b_1, \ldots, b_{k-1}\}$. Either $C_k^\star$ or $C_k^\star \cup \{e\}$ is a cocircuit of $\mathcal{M}$; in the first case clearly the sequences agree. In the second case, since $v_e \leq v_{e'}$ they also agree. □

Proof of Theorem 7. We start with a sequence $\mathcal{K}^1 = ((C_1^\star, b_1, D_1), (C_2^\star, b_2, D_2), \ldots, (C_r^\star, b_r, D_r), (D_{r+1}))$ and apply Lemma 21 for all elements of iteration one and then Lemma 20 for the chosen element. This yields a second sequence, which has the same (b, D, f) as the original. Let the resulting sequence, starting with the second element be $\mathcal{K}^2 = ((C_2^{\star 2}, b_2^2, D_2^2), \ldots, (C_r^{\star 2}, b_r^2, D_r^2), (D_{r+1}))$. Now by the invoked lemmas, $\mathcal{K}^2$ is a VCG-sequence of $\mathcal{M}_2$. This can be iteratively repeated to obtain VCG-sequences $\mathcal{K}^i$ of $\mathcal{M}_i$. Clearly, the diagonal sequence $((C_1^{\star 1}, b_1^1, D_1^1), (C_2^{\star 2}, b_2^2, D_2^2), \ldots, (C_r^{\star r}, b_r^r, D_r^r), (D_{r+1}))$ is a condensed VCG-sequence of $\mathcal{M}$ and has the same (b, D, f) as $\mathcal{K}^1$.

Now for the opposite direction, consider a condensed VCG-sequence $\mathcal{K}^1 = ((C_1^\star, b_1, D_1), (C_2^\star, b_2, D_2), \ldots, (C_r^\star, b_r, D_r), (D_{r+1}))$ and apply Lemma 20 for the selected element and then Lemma 22 for all elements of iteration r. This yields a second sequence, which has the same (b, D, f) as the original. Let the resulting sequence be $\mathcal{K}^2 = ((C_1^{\star 2}, b_1^2, D_1^2), (C_2^{\star 2}, b_2^2, D_2^2), \ldots, (C_r^{\star 2}, b_r^2, D_r^2), (D_{r+1}))$. Now by the invoked lemmas, the first $r - 1$ component of $\mathcal{K}^2$ are determined as a condensed VCG-sequence, while the two last components are determined as a VCG-sequence; finally both sequences have the same (b, D, f) . This can be iteratively repeated to obtain the VCG-sequence $\mathcal{K}^r$ of $\mathcal{M}_{i+1}$ that has the same (b, D, f) as $\mathcal{K}^1$.

□

A.3 Proof of Theorem 13

Proof of Theorem 13. First we have to show, that Auction 3 determines a condensed VCG sequence if bidders behave truthfully.

We are going to do this, by showing inductively, that the sequence determined by Auction 3 with added line 12.5 could be equivalently determined by starting with $i = 1$ (and empty D_i) and then executing Procedure 2.

Now, let's have a look at the tuple $(C_1^\star, b_1, D_1)$ determined by Auction 3, where $C_1^\star$ is the cocircuit that led into the while condition ultimately increasing i to 2. We start with determining the set $F = \{f \in E \colon v_f = 0\}$ and the elements are ordered according to tie-breaking the same way as in Procedure 2. Notice that the term $\mathcal{M} \setminus (D_i \cup \{e\})$ in Procedure 2 equals $(\mathcal{M} \setminus D_i) \setminus e$ which matches the term $\mathcal{M} \setminus f_\ell$ in Line 6 of Auction 3 (because in the auction elements put into D_i are immediately removed from $\mathcal{M}$).

Now, beginning with $\ell = 1$ it is checked whether $\mathcal{M} \setminus f_\ell$ still fulfills the no-monopoly condition and the same happens in Procedure 2. If this is the case, then the inner part of the while-loop of Procedure 2 is carried out, and the while loop in Line 6 of Auction 3 is skipped; in both cases, the current element is added to D_i, in the auction removed from $\mathcal{M}$, and the next element is considered. Now, when $\ell = k$ in Auction 3 then the 'next element' might be slightly more complicated; in this case p is increased and the next batch of $F = \{f \in E \colon v_f = 0\}$ determined. Sooner or later both procedures will hit an element (the same) whose removal would violate the no-monopoly condition. At this time both chose a cocircuit $C_i^\star$ showing this and its best element (according to tie-breaking) b_i. Finally, in both procedures b_i is contracted in $\mathcal{M}$ and D_i is removed in Procedure 2 (in the auction this happend already).

After having done the case of $i = 1$ we assume next, that for $i - 1$ the sequences agree, and consider the next moment, i.e. Procedure 2 is started anew and we are in Auction 3 just leaving Line 10. Now we have to distinguish whether, the while loop in the auction is done another time or not. If it is, then there is another bidder getting a monopoly, if f_ℓ were removed, and a cocircuit $C^\star$ witnessing it. But then the very same cocircuit will do for Procedure 2 too. In both cases the same maximum element from $C^\star$ is chosen. Finally, in the procedure, D_i is empty, while in the auction we have not put anything into D_i but increase i next. So things agree.

Finally we have to consider the case that the while-condition of Line 6 is violated, because removing f_ℓ creates no monopoly. Here we have to distinguish, whether $f_\ell \in \mathcal{M}$ or not. The latter case is possible, if f_ℓ was awarded at $\ell' < \ell$ when $f_{\ell'}$ was critical, but after the set F was composed; in this case, the if-statement of Line 12.5 prevents inclusion of this element into D_i in the auction while removing it from $\mathcal{M}$ does not make a difference; as it is no longer part of $\mathcal{M}$ the procedure skips it automatically. If on the other hand $f_\ell \in \mathcal{M}$ then it puts f_ℓ into D_i and deletes f_ℓ from $\mathcal{M}$ and for the same reason the procedure puts f_ℓ into D_i. Now this continues in sync until either an element is found whose removal would create a monopoly in which case the auction and the procedure determine a cocircuit $C_i^\star$ and award its best element, or F is exhausted. If F is exhausted (which matters only for the auction) then p is increased in the auction, and a new set F determined. From then on, things continue as described above.

By Theorem 7 there is a corresponding VCG-sequence; hence (with Subsection 2.6) the efficient allocation is found. With Theorem 11 it follows that the p_i lead to Vickrey prices. □

A.4 Proof of Lemma 16 for the long-step auction

Proof of Lemma 16 for the long-step auction. The proof for the *long-step version* is quite similar. The only conceptual difference between the auctions is that (truthful) bidders allow the auctioneer to skip rounds (price levels p) in which F is empty in Line 4. Not surprisingly, bidders have neither an incentive to slow down this price search (especially given the added requirement in Line 4 for a bidder with $u_j = p$ to withdraw at least one element), nor an incentive to make the price "skip ahead." Let $s, \tilde{s}, \tilde{v}$ be now the same concepts in the long-step auction.

The only differences in the auction between using s and $\tilde{s}$ (aside from the ones already covered in the unit-step case) involve the augmented Line 14. Suppose the auction would have progressed identically under either strategy up to an instance of Line 14 where, using strategy s, bidder i would announce some u_i, while under $\tilde{s}$ he would announce some $\tilde{u}_i \neq u_i$. Observe that $\tilde{u}_i = \tilde{v}_f = \min_{e \in E_j(\mathcal{M})} \tilde{v}_e$ for some element f, where $E_j(\mathcal{M})$ denotes the remaining elements at that point in the auction.

If both $u_i > \min_{j \neq i} u_j$ and $\tilde{u}_i > \min_{j \neq i} u_j$, then this difference is inconsequential. The auction proceeds to the same price $p = \min_{j \neq i} u_j$ and, if other bidders are bidding truthfully, their behavior does not change. Furthermore, under $\tilde{s}$, bidder i withdraws no elements because $\min_{e \in E_j(\mathcal{M})} \tilde{v}_e > p$; hence this bidder does not change the outcome of this round of the auction by using $\tilde{s}$ rather than s.

If $u_i \leq \min_{j \neq i} u_j$, then the bidder is forced to declare at least one element f in Line 4 (at this round, under s). Therefore, $\tilde{u}_i = \tilde{v}_f = \min_{e \in E_j(\mathcal{M})} \tilde{v}_e = p = u_i$. Again, the auction continues equivalently at this point.

Finally, if $\tilde{u}_i = p \leq \min_{j \neq i} u_j < u_i$, then under s bidder i simply declared an element in Line 4 (at price p), even though he did not reveal p to be the value of this (or any) element in the previous execution of Line 15. While this can be inferred as inconsistent behavior, it does not change the outcome of the auction if he uses $\tilde{s}$ and declares $\tilde{u}_i = p$. $\square$