Strategy-proof Location on a Network*

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Abstract

We consider rules that choose a location on a graph (*e.g.* a road network) based on agents' single-peaked preferences. First, we characterize the class of *strategy-proof*, *onto* rules when the graph is a tree. Such a rule is based on a collection of generalized median voter rules (Moulin, 1980) satisfying a consistency condition. Second, we characterize such rules for graphs containing cycles. We show that while such a rule is not necessarily *dictatorial*, the existence of a cycle grants some agent an amount of decisive power, unlike the case of trees. Rules for this case can be described in terms of a *subclass* of such rules for trees. *Journal of Economic Literature* Classification Numbers: C72, D78.

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1 Introduction

We examine the problem of choosing a location on a network (i.e. graph) based on agents' preferences over such locations. For example, consider the problem of locating a public facility, such as a post office or library, on a given road network. The choice of location is to be based on the preferences of the local citizens (or of members of a government committee). While some citizens may prefer that the facility be located near their homes, others may prefer that it be located near a workplace or some other location. In any case, the preferences are to be solicited as "votes," and a location is to be determined as a result.

The above example involves a *physical* road network. A *virtual* example involves choosing a time of day for, say, a meeting for a group of people. In this case, the network is a cycle, around the face of a clock.[1] Some people may prefer to hold meetings in the morning, while others may prefer other times of day.

A choice rule is a systematic way (i.e. function) to map (elicited) preferences into locations. One could imagine many desirable properties for a choice rule to satisfy. In this paper, our objective is to characterize the class of rules that satisfy the well-known incentives property of *strategy-proofness*: an agent should never be able to manipulate the choice rule by misreporting his preferences to it. We successfully characterize the class of (onto) choice rules that satisfy this condition when agents' preferences over points on a graph are "quadratic" (i.e. symmetric, single-peaked).[2]

The significance of the results is as follows. First, the most important part of the contribution is clearly that we provide a description of the non-

[1] We thank Michael Schwarz for suggesting this example.

[2] In other words, preferences over points are inversely related to distance from a most preferred point). Our results are technically stronger using this domain, being robust to the choice of domain of single-peaked preferences, as we discuss in the Conclusion.

manipulable choice rules (or "voting rules") for situations in which a location must be chosen. Second, our characterization result provides a class of rules that is surprisingly related to a class of *strategy-proof* rules in exchange economies described by Barberà and Jackson [4]. This connection is discussed in the Conclusion.

The seminal paper in the related literature is by Moulin [17], who characterizes the class of generalized median voter schemes (g.m.v.s.) as the only *strategy-proof* rules (satisfying a "peaks-only" condition) when agents have single-peaked preferences over an interval. Ching [10] shows that the peaks-only requirement is redundant. This characterization has been generalized to multi-dimensional frameworks by various authors. Border and Jordan [9] restrict multi-dimensional preferences over Euclidean space to be separable and quadratic, and show that a *strategy-proof* rule must behave like a g.m.v.s. on each dimension. Barberà, Massó, and Serizawa [6] extend that result to full-dimensional *subsets* of Euclidean space, and show the same conclusion; furthermore, they describe which g.m.v.s.'s are actually feasible for a given subset.[3] For a much larger class of preferences, Zhou [24] provides an impossibility result for full-dimensional outcome spaces.[4]

In this paper, the range of a rule is the set of points on a graph. Therefore, while we are expanding upon the notion of an interval, we are not analyzing full-dimensional, convex subsets of Euclidean space. When travel is restricted to a road network, convex combinations of locations are typically not feasible. In a sense, our setting can be seen as a combinatorial generalization of the 1-dimensional case.

In a related paper, Danilov [12] considers a similar setting of tree networks, with single-peaked preferences that are not necessarily symmetric (i.e. defined by distance from a peak). Imposing a "peaks-only" condition, he

[3]Barberà, Gul, and Stacchetti [2] offer a similar result for a discrete setting. Also, see Peremans, et al. [19].

[4]Also see Barberà and Jackson [3]. See Thomson [22] for a more comprehensive survey.

shows that *strategy-proof* rules can be recursively decomposed into medians of constant and dictatorial rules. This result is related to the first half of this paper, in which we provide a closed-form characterization of such rules as described below.

Other work has been done regarding single-peaked preferences on graphs. Hansen and Thisse [15] and Demange [13] restrict attention to graphs that are trees, and derive existence results for that model concerning Condorcet winners and the core, respectively. Moulin [18] discusses welfarism on more general graphs. Ching and Thomson [11] and Vohra [23] examine fairness criteria for graphs that are trees, while Gordon and Péqueux [14] do so when the graph consists of exactly one cycle.

Our results arrive with two distinct flavors. In particular, the flavor of the result depends on whether or not the given graph contains a cycle. First, in Section 3, we discuss the case in which the graph does not contain a cycle (i.e. is a tree). We provide a complete characterization of the class of *strategy-proof*, *onto* rules. Naturally, since an interval is a special case of a tree, the result is an extension of the results of Moulin's [17] and Border and Jordan [9], characterizing generalized median voter schemes. Our characterization describes each *strategy-proof*, *onto* rule as a family of g.m.v.s.'s that together satisfy a feasibility condition we call *consistency*. We call such a rule an extended median voter scheme.

Second, in Section 4, we examine the case in which the graph consists of exactly one cycle (e.g. a circle). In complete contrast to the characterization for trees, we show that in this case, only dictatorial rules are both *strategy-proof* and *onto*.

Finally, in Section 5, we analyze general graphs that contain at least one cycle. For such graphs, we again provide a complete characterization. A *strategy-proof*, *onto* rule for this case is described by a blend of the previous two characterizations. First, there exists an agent who has dictatorial power on or between any cycles on the graph. However, when this agent's peak is not on or between any cycles on the graph, the rule behaves like a g.m.v.s., with the restriction that it choose a point that is closer to this agent's peak than any other point on or between cycles.

We also discuss how the rules in this last characterization can be alternatively described as a certain subclass of the set of extended median voter schemes. Therefore, the class of *strategy-proof*, *onto* rules for graphs with cycles can be thought of as a *subclass* of such rules for trees.

2 The Model

There is a set of agents, $N = \{1, 2, \ldots, n\}$, with arbitrary agents denoted i, j, etc. There is a "road network" represented by a graph, G, formalized below. A point (location) is to be chosen on G, based on the agents' preferences over points on G.

A (finite) *graph* is a closed, connected subset of Euclidean space, $G \subset \mathbb{R}^k$, that is composed of the union of a finite number of (closed) curves of finite length.[5] Each such curve is called an *edge*. Each of the two extremities of an edge is called a *vertex*. A vertex that lies on only one edge is called a *leaf* (or extreme point); the set of leaves is denoted $L \subset G$.

A *path* between two points on the graph, $x, y \in G$, is a minimal connected subset of G that contains x and y. Since a path is a curve, it has a well-defined length. The *distance* between any two points $x, y \in G$, denoted $d(x, y)$, is the minimum path-length between the two points. Denote the set of minimal-length paths between x and y by $[x, y] \equiv \{z \in G : d(x, z) + d(y, z) = d(x, y)\}$. Typically, $[x, y]$ is a single path.[6]

[5] For a more complete formalization of graphs and the lengths of curves, see Berge [8], especially p. 102.

[6] One exception would be if G is a circle with a circumference of 2, and $d(x, y) = 1$.

A *cycle* in G is the union of two paths in G whose intersection is equal to the set of both of their endpoints. As a distance normalization, we assume that for any graph with cycles, the distance around each cycle is at least 1, *i.e.* for any cycle $C \subset G$, there exist $x, y \in C$ such that $d(x, y) \geq 1/2$. Our analysis does not involve comparisons of different graphs with cycles, so this assumption is a normalization made without loss of generality.

An important class of graphs are those that contain no cycle. A graph is a *tree* if it contains no cycle.

Each agent has a quadratic preference relation over G: there exists a point, $p_i \in G$, called the peak of the preference relation, such that the agent's preferences are represented by the utility function $u(x) = -d(p_i, x)$.

Note that preference relations are uniquely defined by their peaks. Arbitrary peaks are denoted p_i, p'_j, etc. In standard fashion, for a list of peaks $p \in G^n$ for n agents, the list obtained by replacing agent i's peak p_i with p'_i is written (p'_i, p_{-i}).

A *(social choice) rule* is a function $f\colon G^n \to G$ mapping lists of agents' peaks into points on the graph. We often have reason to discuss the restriction of such a function to a subdomain. For any subgraph $G' \subset G$, the restriction of f to G' is the function $f|_{G'}\colon G'^n \to G$ such that for all $p \in G'^n$, $f|_{G'}(p) = f(p)$.

We are interested in *onto* rules that are also non-manipulable in the sense of being *strategy-proof*:

$$\forall p \in G^n, i \in N, p'_i \in G,\ \ d(p_i, f(p)) \leq d(p_i, f(p'_i, p_{-i})) \tag{1}$$

A standard result in the *strategy-proofness* literature states that any *strategy-proof*, *onto* rule satisfies what is known as *unanimity*: for all $p \in G^n$ and $i \in N$, if for all $j \in N$, $p_j = p_i$, then $f(p) = p_i$. The proof is straightforward, and left to the reader (e.g., see Barberà and Peleg [7]).

We conclude this section by providing a few more preliminary results regarding *strategy-proof* rules on graphs. If the graph is assumed to be a tree, stronger versions of these lemmas can be obtained. Since such results follow from the main results of Section 3, we do not state them here.

The first such result is similar to one by Border and Jordan [9] stating that *strategy-proofness* implies what they call *uncompromisingness*—moving an agent's peak closer to the chosen location should not change the choice of location. At this point we can show that on graphs in general, this conclusion is true in a *neighborhood* of the originally chosen location.

LEMMA 1 (LIMITED UNCOMPROMISINGNESS) *Let f be a* strategy-proof *rule for a graph G. For all $i \in N$ and all $x, p_i \in G$, there exists $\epsilon(x, p_i) > 0$ such that for all $p_{-i} \in G^{n-1}$ and all $p'_i \in [p_i, x]$, if $f(p) = x$ and $d(p'_i, x) < \epsilon(x, p_i)$, then $f(p'_i, p_{-i}) = f(p)$.*

Proof: Let $p'_i \in [p_i, f(p)]$, $x = f(p)$, and $y = f(p'_i, p_{-i})$. By eqn. (1) above, we have both $d(p_i, x) \leq d(p_i, y)$ and $d(p'_i, y) \leq d(p'_i, x)$. If p'_i is sufficiently close to $f(p)$, it follows from *strategy-proofness* that $x = f(p)$, regardless of the values of p_{-i}. □

Since a *strategy-proof*, *onto* rule must satisfy *unanimity*, this result can be used to show our second result: The point chosen by the rule must lie "between" the agents' peaks, in the sense that the shortest paths from the agents' peaks to the chosen point must jointly intersect only at the chosen point.

LEMMA 2 (NO INTERSECTING SHORTEST PATHS) *Let f be a* strategy-proof *rule for a graph G. For all $p \in G^n$, we have $\bigcap_{i \in N} [p_i, f(p)] = \{f(p)\}$.*

Proof: Suppose for a contradiction that $\bigcap_{i \in N} [p_i, f(p)] \neq \{f(p)\}$. Hence there exists $x \in \bigcap_{i \in N} [p_i, f(p)] \setminus \{f(p)\}$ such that for all $i \in N$, $x \in [p_i, f(p)]$. This implies that for all $i \in N$, $[x, f(p)] \subset [p_i, f(p)]$.

For all $i \in N$, let $\epsilon(f(p), p_i)$ be defined as in Lemma 1. Let $\tilde{p} \in [x, f(p)]$ satisfy $0 < d(\tilde{p}, f(p)) < \min_i \epsilon(f(p), p_i)$. For all $i \in N$, let $p'_i = \tilde{p}$. By repeated application of Lemma 1, $f(p') = f(p)$, contradicting *unanimity.* $\square$

The next lemma states that when all agents' peaks lie in a sufficiently small neighborhood, a *strategy-proof*, *onto* rule chooses an efficient point—a point lying in the union of the shortest paths between peaks. Recall that distance has been normalized so that the length of each cycle (if one exists) is at least 1.

LEMMA 3 (LIMITED EFFICIENCY) *Let f be a* strategy-proof *rule for a graph G. Let $p \in G^n$ be such that for all $i, j \in N$, $d(p_i, p_j) \leq 1/8$. Then $f(p) \in \bigcup_{i,j \in N}[p_i, p_j]$.*

Proof: Suppose in contradiction to the Lemma that $f(p) \notin \bigcup_{i,j \in N}[p_i, p_j]$.

Claim: For all $i, j \in N$, if $[p_i, f(p)] \cap [p_j, f(p)] = \{f(p)\}$, then, where $p'_j = p_i$, $f(p'_j, p_{-j}) \notin \bigcup_{\ell,m \in N}[p_\ell, p_m]$.

Suppose $i, j \in N$ are such that $[p_i, f(p)] \cap [p_j, f(p)] = \{f(p)\}$. Since $f(p) \notin [p_i, p_j]$ by assumption, $[p_i, f(p)] \cup [p_j, f(p)] \cup [p_i, p_j]$ contains a cycle. Let $d^i = d(p_i, f(p))$, $d^j = d(p_j, f(p))$, and $d^{ij} = d(p_i, p_j)$. As the minimum length of a cycle is 1, we have $d^i + d^j + d^{ij} \geq 1$. Since $d^{ij} \leq 1/8$, we have either $d^i \geq 7/16$ or $d^j \geq 7/16$ (or both). In the former case, the triangle inequality implies $d^j + d^{ij} \geq d^i$, hence $d^j \geq 5/16$ in all cases.

Letting $p'_j = p_i$ and $x = f(p'_j, p_{-j})$, *strategy-proofness* implies $d(p_j, x) \geq 5/16$. For any other agent $k \in N$, $d(p_j, p_k) + d(p_k, x) \geq d(p_j, x)$, so $d(p_k, x) \geq 3/16$. Therefore $x \notin \bigcup_{\ell,m \in N}[p_\ell, p_m]$, and the Claim is proven.

By Lemma 2, let $i, j \in N$ be such that $[p_i, f(p)] \cap [p_j, f(p)] = \{f(p)\}$. For all $k \in N$, let $p'_k = p_i$. By the Claim, we have $f(p'_j, p_{-j}) \notin \bigcup_{\ell,m \in N}[p_\ell, p_m]$.

Repeating the argument, by Lemma 2, there must exist $k \in N$ such that $[p_i, f(p'_j, p_{-j})] \cap [p_k, f(p'_j, p_{-j})] = \{f(p'_j, p_{-j})\}$. By the Claim, we have $f(p'_k, p'_j, p_{-jk}) \notin \bigcup_{\ell,m \in N}[p_\ell, p_m]$.

This argument can be repeated until we have $f(p') \notin \bigcup_{\ell,m \in N}[p_\ell, p_m]$, which contradicts *unanimity*. $\square$

3 Rules for Trees

In this section, we characterize the class of *strategy-proof*, *onto* rules for trees. This characterization is, naturally, a generalization of Border and Jordan's [9] characterization of such rules on lines. A technical detail is the fact that when talking about one dimension, Border and Jordan deal with the real line, while we deal with finite intervals. It can be shown, though, that their results also hold on intervals.[7] In terms of our model, Border and Jordan show that if a graph G consists of a single edge (i.e., is a single curve), then each *strategy-proof*, *onto* rule is a generalized median voter scheme (g.m.v.s.) as introduced by Moulin [17] and defined below.

For any $x, y \in G$, consider the restriction of a *strategy-proof*, *onto* rule f to $[x, y]$, denoted $f|_{xy} \colon [x, y]^n \to G$. Note that by Lemma 2, since G is a tree, if f is *strategy-proof* and *onto*, we must have $f|_{xy}(p) \in [x, y]$. Therefore we actually have $f|_{xy} \colon [x, y]^n \to [x, y]$.

Such a function $f|_{xy}$ is called a *generalized median voter scheme on* $[x, y]$ (g.m.v.s.) if there exist $2^{|N|}$ points in $[x, y]$, $\{\alpha^{xy}_S\}_{S \subseteq N}$, such that

1. $S \subset T \subset N$ implies $d(\alpha^{xy}_S, x) \leq d(\alpha^{xy}_T, x)$,[8]

2. $\alpha^{xy}_\emptyset = x$ and $\alpha^{xy}_N = y$,

[7] To sketch the proof, let f be a *strategy-proof*, *onto* rule on $[0, 1]$. Define a *strategy-proof*, *onto* rule on the real line, g, where (i) if all agents' peaks are less than 0, g chooses the maximum peak, (ii) if all agents' peaks are greater than 1, g chooses the minimum peak, and (iii) otherwise, the rule coincides with f (restricting agents' preferences to $[0, 1]$). If f is not one of the rules characterized by Border and Jordan (defined below), then neither is g, which is a contradiction.

[8] This condition is not necessary in the definition, but imposing it rules out redundant parameterizations of certain generalized median voter schemes.

3. for all $p \in [x,y]^n$, $f|_{xy}(p)$ is the unique point satisfying

$$d(f|_{xy}(p), x) = \max_{S \subset N} \min\{(d(p_i, x))_{i \in S}, d(\alpha_S^{xy}, x)\}$$

Proposition 1 (Border and Jordan [9]) *Suppose G contains exactly two vertices, so $G = [x,y]$, and let f be a* strategy-proof, onto *rule for G. Then f is a generalized median voter scheme on G.*

Condition (2) in the definition of a g.m.v.s. is a consequence of the *onto* requirement. Dropping this condition yields well-defined g.m.v.s.'s that fail to be *onto*.

An observation which is important in our results is the following. The definition of a g.m.v.s. on a line involves arbitrarily picking a direction (i.e., a "left" and a "right"). For example, in the definition, the roles of x and y could be reversed to yield

there exist $2^{|N|}$ points in $[x,y]$, $\{\alpha_S^{yx}\}_{S \subseteq N}$, such that

1. *$S \subset T \subset N$ implies $d(\alpha_S^{yx}, y) \leq d(\alpha_T^{yx}, y)$,*
2. *$\alpha_\emptyset^{yx} = y$ and $\alpha_N^{yx} = x$,*
3. *for all $p \in [x,y]^n$, $f(p)$ is the unique point satisfying*

$$d(f(p), y) = \max_{S \subset N} \min\{(d(p_i, y))_{i \in S}, d(\alpha_S^{yx}, y)\}$$

In other words, in the original statement of the Proposition, x is "left" and y is "right." However, this choice was arbitrary. The important observation for our purposes is that there is a relationship between these two sets of parameters.

Suppose the two sets of parameters $\{\alpha_S^{xy}\}$ and $\{\alpha_S^{yx}\}$ each describe a given g.m.v.s. Then it is easy to see that for all $S \subseteq N$,

$$\alpha_S^{xy} = \alpha_{N \setminus S}^{yx}$$

This follows from the definition of a g.m.v.s.

Proposition 1 can be shown to have implications for *strategy-proof*, *onto* rules for trees. In particular, for the situations in which agents' peaks are restricted to being on a given path on a tree, a *strategy-proof*, *onto* rule must behave like a g.m.v.s. along that path. In other words, for any interval $[x, y] \in G$, the restriction of a *strategy-proof*, *onto* rule to the domain $[x, y]^n$ is a g.m.v.s.

PROPOSITION 2 (G.M.V.S.'S ON ANY PATH) *Let f be a* strategy-proof, onto *rule for a tree G. For all $x, y \in G$, $f|_{xy}$ is a g.m.v.s. on $[x, y]$ with parameters $\{\alpha_S^{xy}\}_{S \subseteq N}$.*

Proof: Fix $x, y \in G$ and let $p \in [x, y]^n$. By Lemma 2, we have $f(p) \in [x, y]$. Therefore the range of $f|_{[x,y]}$ is contained in $[x, y]$. Furthermore, by unanimity, $f|_{[x,y]}$ is onto $[x, y]$. In other words, $f|_{[x,y]}$ is a *strategy-proof*, *onto* rule from the domain $[x, y]^n$ onto $[x, y]$. Therefore by Proposition 1, $f|_{[x,y]}$ is a g.m.v.s. □

Proposition 2 shows that we can describe the behavior of *strategy-proof*, *onto* rules on any single path of a tree. For the remainder of this section, for a given tree G, rule f, and points $x, y \in G$, the parameters $\{\alpha_S^{xy}\}_{S \subset N}$ are understood to be those described in Proposition 2.

Even though *strategy-proof*, *onto* rules behave like g.m.v.s.'s when restricted to paths, one cannot begin to construct a *strategy-proof*, *onto* rule for trees by arbitrarily choosing a g.m.v.s. for each path on the tree. The critical issue is that for any two paths that intersect on an interval, the two corresponding g.m.v.s.'s on those two paths must not contradict each other

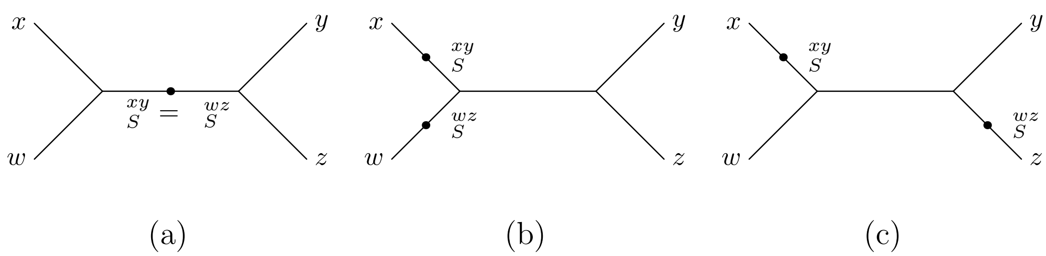

Figure 1: The *consistency* condition among generalized median voter schemes along different paths of a tree. The condition is satisfied in (a) and (b), but not in (c).

on that intersection. This implies that if a *strategy-proof*, *onto* rule f must coincide with a family of g.m.v.s.'s corresponding to various paths on the tree, then those g.m.v.s.'s must be "self-consistent" in some way—they must agree on the intersection of their domains. We formalize this *consistency* notion next.

Let $\{\alpha_S^{xy}\}_{S \subseteq N}$ and $\{\alpha_S^{wz}\}_{S \subseteq N}$ parameterize two g.m.v.s.'s defined on two paths, $[x, y]$ and $[w, z]$, on a tree G. Call these two g.m.v.s.'s (or these two sets of parameters) *consistent* if for all $S \subseteq N$,

$$\text{if } [x, y] \cap [w, z] \cap [x, w] \text{ contains at most one point} \tag{2}$$

$$\text{then } [x, y] \cap [w, z] \cap [\alpha_S^{xy}, \alpha_S^{wz}] \text{ contains at most one point} \tag{3}$$

As an example, see Figure 1. Since $[x, y] \cap [w, z] \cap [x, w]$ is a single point, the hypothesis of the consistency condition, eqn. (2), is satisfied. Note that if the positions of points w and z were exchanged, eqn. (2) would not be satisfied. In Figures 1a and 1b, *consistency*—in particular, eqn. (3)—is satisfied. In Figure 1c, it is not.

3.1 A Characterization

With this *consistency* notion we have just defined, we can describe the class of rules for trees that are *strategy-proof* and *onto*. Underlying the structure of such rules is, associated with each path on the tree, a g.m.v.s. that is consistent with any other such g.m.v.s. associated with any other path. With respect to these g.m.v.s.'s, a *strategy-proof*, *onto* rule chooses, for any profile of peaks, the unique point $a \in G$ on the tree such that, for any path which includes a, given the restriction of agents' preferences to that path, that point would be chosen by the g.m.v.s. associated with that path.

A formal definition is given below. First, in order to demonstrate that such rules are well-defined, we show that for any family of consistent g.m.v.s.'s and any profile of peaks, such a unique point a exists. Recall that L is the set of leaves (extreme points) for G. For all $x, y \in L$ and $p_i \in G$, let the unique point in $[x, y]$ closest to p_i be denoted $p_i^{xy} = \arg\min_{z \in [x,y]} d(z, p_i)$.

PROPOSITION 3 *For all distinct $x, y \in L$, let $g^{xy}\colon [x, y]^n \to [x, y]$ be a g.m.v.s. on $[x, y]$. Furthermore, for all $w, x, y, z, \in L$, let g^{xy} and g^{wz} be consistent. Then for all $p \in G^n$, there exists a unique point, a, such that for all $x, y \in L$, $[x, y] \ni a$ implies $g^{xy}(p^{xy}) = a$.*

Proof: Within this proof, we shall call any point, a, that satisfies the conditions of the proposition a *critical* point. First we show the existence of a critical point for a given family of consistent g.m.v.s.'s, $\{g^{xy}\}_{x,y \in L}$.

Let $x, y \in L$, and let $a^1 = g^{xy}(p^{xy})$. If a^1 is critical, we are done. Otherwise there exist $w, z \in L$ such that (i) $a^1 \in [w, z]$, (ii) $a^2 \equiv g^{wz}(p^{wz}) \neq a^1$, and, without loss of generality, (iii) $[x, y] \cap [w, z] \cap [x, w]$ contains at most one point (choosing the orientation of w and z to satisfy the hypothesis of the consistency condition; otherwise, reverse the labels of w and z). See Figure 2.

We show that $[a^1, a^2] \cap [x, y] = \{a^1\}$. Suppose not. Then there exists $b \in [x, y]$ such that $b \neq a^1$ and $[b, a^1] \subset [a^1, a^2]$. Note that $[b, a^1] \subset [x, y] \cap [w, z]$.

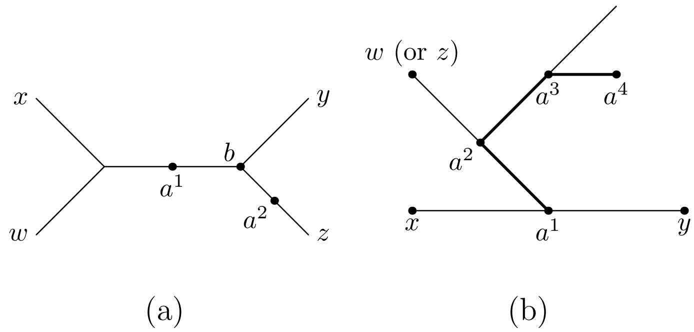

Figure 2: Proof of Proposition 3. In (a), a contradiction is reached. In (b), we have $[a^1, a^2] \subsetneq [a^1, a^3]$, etc., so the construction of a finite sequence of points a^k ends with a critical point.

Therefore by the uncompromisingness of a g.m.v.s. (in the sense of Border and Jordan [9]), both $g^{xy}(p^{ba^1}) = a^1$ and $g^{wz}(p^{ba^1}) = b$, which violates consistency.

If a^2 is critical, we are done. Otherwise there exist $w', z' \in L$ such that (i) $a^2 \in [w', z']$ and (ii) $a^3 \equiv g^{w'z'}(p^{w'z'}) \neq a^2$. We can similarly show that $[a^2, a^3] \cap [w, z] = \{a^2\}$. Since $[a^1, a^2] \subset [w, z]$ and G is a tree, this implies $[a^1, a^2] \subsetneq [a^1, a^3]$.

Similarly, if a^3 is not critical, there exists $a^4 = g^{w''z''}(p^{w''z''})$ such that $[a^1, a^3] \subsetneq [a^1, a^4]$. Since there is a finite number of leaves, there are a finite number of candidate critical points a^k. Since at each step $[a^1, a^k] \subsetneq [a^1, a^{k+1}]$, the process must end with a critical point.

To show uniqueness, let a and a' be critical points. There exist $x, y \in L$ such that $a, a' \in [x, y]$. Then by definition, $g^{xy}(p^{xy}) = a = a'$. □

Given this uniqueness result, we can now define the class of rules we characterize. A rule f is an *extended generalized median voter scheme* (e.m.v.s.) if

(i) for all $w, x, y, z \in G$, $f|_{xy}$ and $f|_{wz}$ are consistent g.m.v.s.'s, and

(ii) for all $p \in G$, $f(p)$ is the unique point a such that for all $x, y \in L$,

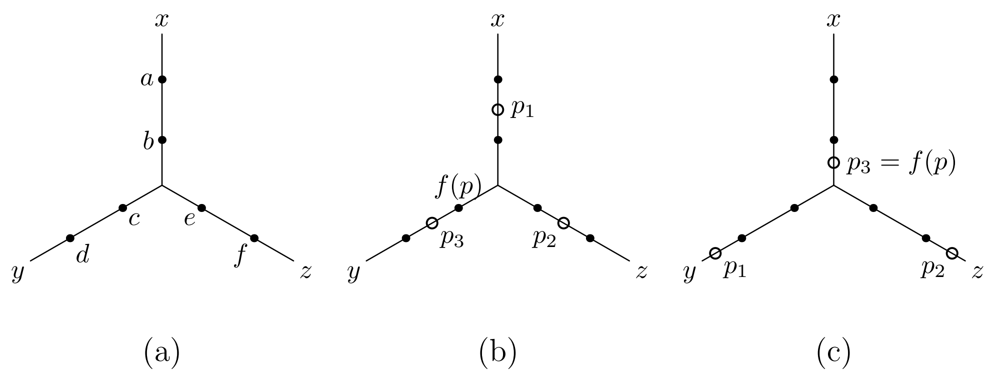

Figure 3: Extended median voter scheme of Example 1.

$a \in [x, y]$ implies $f|_{xy}(p^{xy}) = a$.

Proposition 3 implies that any consistent family of g.m.v.s.'s defines an e.m.v.s.

Example 1 We provide an example of an e.m.v.s. for a simple tree with three leaves. Consider the graph given in Figure 3a. We define an e.m.v.s. for $N = \{1, 2, 3\}$ by specifying the parameters of its consistent g.m.v.s.'s.

Let $\alpha_1^{xy} = \alpha_2^{xy} = a$, $\alpha_{1,2}^{xy} = b$, $\alpha_3^{xy} = c$, and $\alpha_{1,3}^{xy} = \alpha_{2,3}^{xy} = d$. Let $\alpha_3^{xz} = e$, and $\alpha_{1,3}^{xz} = \alpha_{2,3}^{xz} = f$. By consistency, this choice of parameters implies the locations of the remaining parameters $\{\alpha_S^{zy}\}$, etc.

For all $p \in G^3$, if all three peaks lie on a single path, it is as straightforward to find $f(p)$ as it is to calculate the outcome of a g.m.v.s. If the peaks do not lie on a common path, $f(p)$ is calculated by finding the unique point to satisfy condition (ii) in the definition of an e.m.v.s. Figures 3b and 3c provide two examples of such profiles, p, and the corresponding outcome $f(p)$. ■

Our characterization result for trees is the following result.

Theorem 1 *For any tree G, a rule f is* strategy-proof *and* onto *if and only if it is an e.m.v.s.*

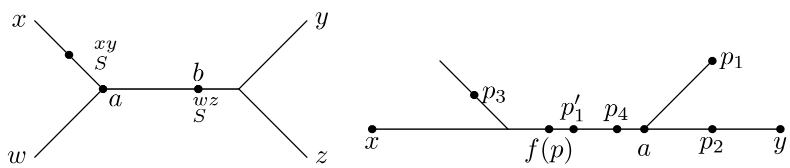

Figure 4: Proof of Theorem 1.

Proof: Let f be a *strategy-proof*, *onto* rule for a tree G. By Proposition 2, for each $x, y \in L$, $f|_{xy}$ is a g.m.v.s. on $[x, y]$ with parameters $\{\alpha_S^{xy}\}_{S \subseteq N}$.

Step 1. We show that for each $w, x, y, z \in L$, $f|_{xy}$ and $f|_{wz}$ are consistent.

Suppose not, i.e., that $[x, y] \cap [w, z] \cap [x, w]$ contains at most one point, but for some $S \subseteq N$, $[x, y] \cap [w, z] \cap [\alpha_S^{xy}, \alpha_S^{wz}] = [a, b]$, where $a \neq b$. Choose the labels a and b so that $d(a, x) < d(b, x)$. Without loss of generality, suppose that $a \in [\alpha_S^{xy}, b]$ (otherwise reverse the labels of the pairs (x, y) and (w, z)). See Figure 4a.

Let $p \in G^n$ be such that $i \in S$ implies $p_i = b$, and $i \notin S$ implies $p_i = a$. Since $[a, b] \subset [x, y] \cap [w, z]$, we have $f(p) = f|_{xy}(p^{xy}) = f|_{wz}(p^{wz})$.

By the definition of an *onto* g.m.v.s., we have $f|_{wz}(p^{wz}) = b$. Similarly, we have $f|_{xy}(p^{xy}) = a$, which is a contradiction.

Step 2. We show that f is an e.m.v.s.

Since $\{f|_{xy}\}_{x,y \in L}$ is a consistent family of g.m.v.s.'s, Proposition 3 implies that for all $p \in G^n$, there exists a unique point, $g(p)$, such that for all $x, y \in L$, $g(p) \in [x, y]$ implies $f|_{xy}(p^{xy}) = g(p)$. We need to show that $f = g$.

Suppose not. Then, for some $p \in G^n$ and some $x, y \in L$, we have $f(p) \in [x, y]$ and $f(p) \neq a \equiv f|_{xy}(p^{xy})$.

Let $M = \{i \in N : a \in [p_i, f(p)]\}$. See Figure 4b. Let ϵ be sufficiently small so as to satisfy the conditions of Lemma 1 (i.e., in the notation of that lemma, $\epsilon < \epsilon(f(p))$ for all $i \in N$). For all $i \in M$, Let $p_i' \in [f(p), a]$ satisfy

$0 < d(p'_i, f(p)) < \epsilon$. For all $i \notin M$, Let $p'_i = f(p)$.

By Lemma 1 (or simply by *strategy-proofness*), $f(p'_i, p_{-i}) = f(p)$. Thus by repeating this argument, $f(p') = f(p)$. Since $p' \in [x, y]^n$, $f|_{xy}(p')$ is well-defined. Furthermore, we have shown

$$f|_{xy}(p') = f(p) \tag{4}$$

Note that $i \in M$ implies $a \in [p_i^{xy}, f(p)]$. Therefore, by the definition of a g.m.v.s. (in particular, the uncompromisingness property, in the sense of Border and Jordan [9]), $f|_{xy}(p_M^{xy}, p'_{N \setminus M}) = f|_{xy}(p^{xy})$. Since for all $i \in M$, $p'_i \in [p_i^{xy}, f(p)]$, we similarly have $f|_{xy}(p') = f|_{xy}(p^{xy})$, contradicting eqn. (4). Therefore a *strategy-proof*, *onto* rule must be an e.m.v.s.

Proving that an e.m.v.s. is both *strategy-proof* and *onto* is straightforward, and is left to the reader. □

4 Rules for a Single Cycle

In this section, we consider the case in which the graph G consists of a single cycle. More generally, for graphs that contain a cycle, we describe the behavior of the restriction of a *strategy-proof*, *onto* rule to a single cycle. We show that the restriction of such a rule to a cycle is dictatorial.

The reasoning behind the proof is as follows. First, along "short" paths on G, a *strategy-proof*, *onto* rule must behave like a generalized median voter scheme. This is completely analogous to Proposition 2 above, and is stated below as Proposition 4.

Consider a cycle $C \subset G$. This cycle is composed of the union of many overlapping, "short" paths. Each pair of g.m.v.s.'s for these paths must be consistent. As we show in the proof of Theorem 2, the cyclic structure implies that each such g.m.v.s. must be dictatorial. In fact, this notion is used to prove a stronger statement in Theorem 2, concluding this section: there must be a dictator on the entire cycle.

The following result is analogous to Proposition 2 for trees, so we omit the proof.

PROPOSITION 4 (GENERALIZED MEDIAN VOTER SCHEMES) *For any graph G, let f be a* strategy-proof*,* onto *rule. For all $x, y \in G$ such that $d(x, y) \leq 1/8$, $f|_{xy}$ is a g.m.v.s. on $[x, y]$ with parameters $\{\alpha_S^{xy}\}_{S \subseteq N}$.*

To present the next set of results, we refer to the parameters, $\{\alpha_S^{xy}\}_{S \subseteq N}$, described in Proposition 4. The following lemma states that whenever $[x, y]$ lies within a cycle, these parameters lie at the extreme points of the interval. Furthermore, for each coalition, and for any pair of intervals, $[x, y]$ and $[w, z]$, that lie on the same cycle, the direction in which its parameter lies (i.e., the "right" or "left" of the interval) is consistent across the two intervals. In essence, this implies that on intervals of length less than 1/8 within a given cycle, a *strategy-proof*, *onto* rule can be described in terms of right- and left-coalitions.[9]

To be precise, we need to introduce notation to refer to *direction* around a cycle (e.g. clockwise vs. counter-clockwise). For any cycle $C \subset G$, we call $\succeq$ the "clockwise" operator. To avoid a tedious description of an intuitively simple operator, we informally define it as follows. Imagine fixing a (clockwise) direction on the cycle; there are two choices of direction, and the choice is arbitrary. For all $x, y \in C$ such that $d(x, y) \leq 1/8$, we say $x \succeq y$ if x lies beyond y in a clockwise direction. For example, on a clock, 4:00 $\succeq$ 3:00; 1:00 $\succeq$ 12:00; 3:00 and 9:00 are not comparable because the distance between them is greater than 1/8 (since the distance around a cycle is at least 1).

[9]This terminology should not be confused with the literature's standard description of generalized median voter schemes in terms of "right/left coalition systems."

LEMMA 4 (RIGHT-/LEFT-COALITIONS) *Suppose G contains a cycle $C \subset G$, and let f be a* strategy-proof, onto *rule for G. There exists a family of right-coalitions, $\mathcal{S} \subset 2^N$, such that for all $S \subset N$,*

(i) if $S \in \mathcal{S}$, then for all $x, y \in C$ such that $d(x,y) \leq 1/8$ and $x \succeq y$, we have $\alpha_S^{xy} = x$

(ii) if $S \notin \mathcal{S}$, then for all $x, y \in C$ such that $d(x,y) \leq 1/8$ and $x \succeq y$, we have $\alpha_S^{xy} = y$

Proof: Let $S \subset N$ and let $x, y \in C$ be such that $x \succeq y$ and $d(x,y) \leq 1/16$. Let $\alpha = \alpha_S^{xy}$. We will show that if $\alpha \neq y$, then for all $v, w \in C$ such that $v \succeq w$ and $d(v,w) \leq 1/8$, we have $\alpha_S^{vw} = v$, i.e., we will show that if $\alpha \neq y$, then S is a "right-coalition."

Let $z \in C$ be such that $y \succeq z$ and $d(x,z) \leq 1/8$. For all $i \in N$, let $p_i = x$ if $i \in S$ and $p_i = y$ otherwise. By definition of the α-parameters, we have $f(p) = \alpha$. Since $f(p) \neq y$, we also have

$$\alpha_S^{xz} = \alpha \tag{5}$$

For all $i \in N$, let $p'_i = y$ if $i \in S$ and $p'_i = z$ otherwise. By Proposition 4, $f(p) \succeq \min\{(p_i)_{i \in S}, \alpha_S^{xz}\} = y$. Therefore Lemma 3 implies $f(p) = y$. Hence,

$$\alpha_S^{yz} = y \tag{6}$$

Repeating the arguments that lead to eqns. (5) and (6), we can show that for all $w \in C$ such that $w \succeq z$ and $d(w,y) \leq 1/8$, we have $\alpha_S^{yw} = y$ and $\alpha_S^{zw} = z$.

Therefore, by choosing an appropriate sequence of points around C, the same arguments show that for all $v, w \in C$ such that $v \succeq w$ and $d(v,w) \leq 1/8$, we have $\alpha_S^{vw} = v$.

If instead we had $\alpha = y$, then we would have shown that S is a "left-coalition." □

The next lemma states that the restriction of a *strategy-proof*, *onto* rule to a cycle always selects an agent's peak.

Lemma 5 (peak selection) *Suppose G contains a cycle $C \subset G$, and let f be a* strategy-proof, onto *rule for G. For all $p \in C^n$, $f(p) \in \{p_1, p_2, \ldots, p_n\}$.*

Proof: Let $p \in C^n$. Suppose in contradiction that $f(p) \notin \{p_1, p_2, \ldots, p_n\}$. By Lemma 1, there exists $p_1' \in [p_1, f(p)]$ such that $0 < d(p_1', f(p)) \leq 1/16$ and $f(p_1', p_{-1}) = f(p)$.

Similarly, there exists $p_2' \in [p_2, f(p)]$ such that $0 < d(p_2', f(p)) \leq 1/16$ and $f(p_1', p_2', p_{-1,2}) = f(p)$.

Repeating the construction for the other agents, we have $f(p') = f(p)$, $d(p_i', p_j') \leq 1/8$ for all $i, j \in N$, and $p_i' \neq f(p')$ for all $i \in N$, contradicting Lemma 4. □

Now we have our main result for the case of three agents.

Proposition 5 *Suppose G contains a cycle $C \subset G$, and let f be a* strategy-proof, onto *rule for G. If $|N| = 3$, then there exists $i \in N$ such that for all $p \in C^3$, $f(p) = p_i$.*

Proof: Let $p \in C^3$ be such that $d(p_i, p_j) = 1/3$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$. Assume without loss of generality (and by Lemma 5) that $f(p) = p_1$. We will show that for all $p \in C^3$, $f(p) = p_1$.

Let $\mathcal{S}$ be the set of coalitions described in Lemma 4. Note that by Proposition 4, it is sufficient to show that both (i) $\{1\} \in \mathcal{S}$ and (ii) if $S \in \mathcal{S}$ and $|S| = 1$, then $S = \{1\}$.[10] Notice also that if $S \in \mathcal{S}$, then $S \subset S' \subset N$ implies $S' \in \mathcal{S}$.

[10]These two conditions are what define a dictator for our class of median voter schemes.

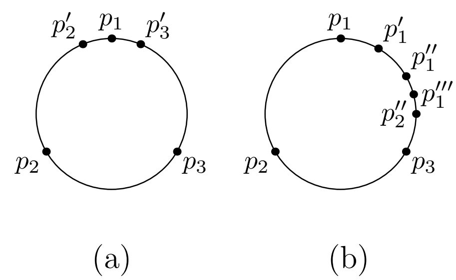

Figure 5: Proof of Proposition 5.

For $i \in \{2,3\}$, let $p'_i \in [p_i, p_1]$ be such that $0 < d(p'_i, p_1) \leq 1/16$ and $f(p_1, p'_2, p'_3) = f(p)$ (see Figure 5a). By Proposition 4, we have $\{3\} \notin \mathcal{S}$.

Let $p'_1, p''_1, p'''_1 \in [p_1, p_3]$ satisfy $d(p'_1, p_1) = 1/12$, $d(p''_1, p_1) = 1/6$, and $d(p'''_1, p_3) = 1/8$ (see Figure 5b). Since f satisfies *peak selection*, *strategy-proofness* implies $f(p'_1, p_2, p_3) = p'_1$. Similarly, we have $f(p''_1, p_2, p_3) = p''_1$ and $f(p'''_1, p_2, p_3)$. *Strategy-proofness* and *peak selection* also imply that for any $p''_2 \in [p'''_1, p_3]$, $f(p'''_1, p''_2, p_3) = p'''_1$. Therefore, $\{2,3\} \notin \mathcal{S}$, which implies $\{2,\} \notin \mathcal{S}$.

The symmetric argument, with $\tilde{p}_1 \in [p_1, p_2]$ satisfying $d(\tilde{p}_1, p_2) = 1/8$ and $p''_3 \in [p'''_1, p_2]$, demonstrates that $f(\tilde{p}_1, p_2, p''_3) = \tilde{p}_1$. Therefore, $\{1\} \in \mathcal{S}$. □

Now we prove the general result. The proof works by showing that if a *strategy-proof*, *onto* rule is *non-dictatorial* for the general case, then there must be such a rule for the 3-agent case, contradicting Proposition 5. The method of proof is similar to that found in Kalai and Muller [16], Aswal et al. [1], and Schummer [21].

Theorem 2 (cycle dictator) *Suppose G contains a cycle $C \subset G$, and let f be a* strategy-proof, onto *rule for G. There exists an agent $i \in N$ ("cycle dictator") such that for all $p \in C^n$, $f(p) = p_i$.*

Proof: The proof is by induction on $n = |N|$. Our method of proof requires having shown the result for the case $n = 3$, which was done in Proposition 5.

Proving the result for the case $n = 2$ is similar to the proof of Proposition 5, and is left to the reader.

Suppose that the result is true for n agents. We show that the result holds for $n + 1$ agents.

Let $f : C^{n+1} \to C$ be a *strategy-proof*, *onto* rule. Define two n-agent rules, g and g', as follows:

$$\forall p \in C^n,\ g(p_1, p_2, \ldots, p_n) = f(p_1, p_2, \ldots, p_n, p_n)$$

$$\forall p \in C^n,\ g'(p_1, p_2, \ldots, p_n) = f(p_1, p_1, p_2, \ldots, p_n)$$

That is, g is defined by creating a "copy" of agent n, placing that copy in the $n + 1$st position, and applying the rule f. Similarly, g' is defined by duplicating agent 1.

Step 1: g and g' are both *strategy-proof* and *onto*.

Since a *strategy-proof*, *onto* rule must satisfy *unanimity*, it follows that g is *onto*. It is also clear that agents 1 through $n - 1$ cannot manipulate the rule g. Thus, to demonstrate the *strategy-proofness* of g it suffices to prove that for all $p \in C^n$ and all $p'_n \in C$, $d(g(p), p_n) \leq d(g(p'_n, p_{-n}), p_n)$.

By the *strategy-proofness* of f, for all $p \in C^n$ and all $p'_n \in C$,

$$d(f(p_1, \ldots, p_n, p_n), p_n) \leq d(f(p_1, \ldots, p_n, p'_n), p_n) \leq d(f(p_1, \ldots, p'_n, p'_n), p_n)$$

Hence g is *strategy-proof*.

Similarly, g' is *strategy-proof* and *onto*.

Step 2: if $i < n$ is a cycle-dictator for g, then i is a cycle-dictator for f.

By the induction hypothesis above, there exists $i \in \{1, 2, \ldots, n\}$ such that for all $p \in C^n$, $g(p) = p_i$. Suppose $i \neq n$. We show that for all $p \in C^{n+1}$, $f(p) = p_i$.

Let $p \in C^{n+1}$. For all $j \in \{1, 2, \ldots, n\}$ with $j \neq i$, let $p'_j = f(p)$, and let

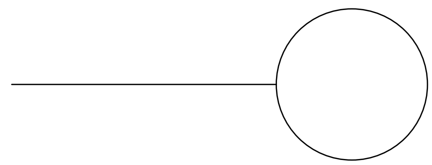

Figure 6: A simple graph with a cycle, admitting a *non-dictatorial*, *strategy-proof*, *onto* rule.

$p'_i = p_i$. Then by repeated application of Lemma 1, $f(p') = f(p)$. By the definitions of g and i, we also have $f(p') = g(p'_1, \ldots, p'_n) = p'_i = p_i$.

Step 3: if $i > 1$ is a cycle-dictator for g', then $i + 1$ is a cycle-dictator for f.

This follows as in Step 2.

Step 4: either $i < n$ is a g-dictator or $i > 1$ is a g'-dictator.

Let $p \in C^{n+1}$ satisfy $p_1 = p_2 \neq p_n = p_{n+1}$. Then $g(p_1, \ldots, p_n) = f(p) = g'(p_1, p_3, \ldots, p_{n+1})$. Therefore it cannot be that both $g(p_1, \ldots, p_n) = p_n$ and $g'(p_1, p_3, \ldots, p_{n+1}) = p_1$.

Therefore either Step 2 or Step 3 applies, and f is dictatorial on C. $\square$

5 Rules for Graphs with Cycles

Consider the case in which G consists of a cycle and a line segment intersecting the cycle at one of its endpoints (as in Figure 6). Clearly a dictatorial rule on G is both *strategy-proof* and *onto*. A *non-dictatorial*, *strategy-proof*, *onto* rule also exists for this graph. One such rule can be constructed as follows: for each profile of preferences, if at least one agent's peak lies on the cycle, choose the point on the cycle closest to agent 1's peak; otherwise, choose the peak of the agent closest to the cycle.

For this rule, agent 1 plays the role of "cycle dictator" from Theorem 2. On the line segment, the rule behaves just like a (*non-dictatorial*) generalized median voter scheme. However, this generalized median voter scheme has the additional feature that from the perspective of agent 1, the chosen location (on the line segment) is at least as good as any location on the cycle. In fact, this notion—that the cycle dictator likes the chosen location as much as any location on a cycle—is what helps to characterize the *strategy-proof*, *onto* rules for graphs.

Before we complete this characterization, we generalize Theorem 2 to more general cyclic subsets of graphs. We have shown that on any given cycle, a *strategy-proof*, *onto* rule must be dictatorial. This result extends to certain connected sets of cycles.

The following lemma says that each of the "cycle dictators" described by Theorem 2 (for each of the different cycles on the graph) are the same agent, i.e. there is one agent such that whenever all peaks are on the same cycle, that agent's peak is chosen.

LEMMA 6 (UNIQUE CYCLE DICTATOR) *Suppose $C, C' \subset G$ are two cycles, and let f be a* strategy-proof*,* onto *rule for G. There exists an agent $i \in N$ such that for all $p \in C^n \cup C'^n$, $f(p) = p_i$.*

Proof: Without loss of generality, we will assume that C and C' can be connected by a path whose interior intersects no cycles. If the conclusion of the Lemma holds for this case, then the conclusion holds in the general case by repeating the argument.

Therefore, describe a path connecting C and C' by letting $x \in C$, $x' \in C'$ be such that for any cycle C'', $C'' \cap [x, x'] \subset \{x, x'\}$. (If C and C' intersect, let $x = x'$.) See Figure 7.

By Theorem 2, there exist $i, j \in N$ such that for all $p \in C^n$ and all $p' \in (C')^n$, we have $f(p) = p_i$ and $f(p') = p_j$. Let $x^1, x^2, \ldots, x^\ell \in G$ be such that $x^1 \in C \setminus \{x\}$, $x^k \in C' \setminus \{x'\}$, for $k \in \{1, 2, \ldots, \ell - 1\}$, $d(x^k, x^{k+1}) \leq 1/8$, and if $\ell \geq 2$, $x^2, \ldots, x^{\ell-1} \in [x, x']$.

By Proposition 4, the restriction of f to $[x^k, x^{k+1}]^n$ must be a generalized

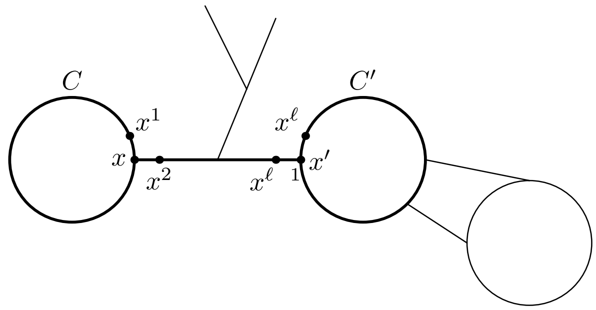

Figure 7: Proof of Lemma 6. Drawn with thick lines are two cycles, and a path between them that intersects no other cycle on its interior.

median voter scheme for each k. For each k, let $(\alpha_S^{x^k x^{k+1}})_{S \subseteq N}$ be the parameters for the rule in which we (arbitrarily) set the partial order $\preceq$ to satisfy $x^k \preceq x^{k+1}$.

Let $p_i = x^1$ and for all $k \neq i$, $p_k = x$. Since $p \in C^n$, we have $f(p) = p_1$, which implies that for all $S \subset N$ such that $i \notin S$, $\alpha_S^{x^1,x^2} = x^1$. As in the proof of Lemma 4, this implies that for all $m \in \{1, 2, \ldots, \ell - 1\}$, $\alpha_S^{x^m,x^{m-1}} = x^m$.

A symmetric argument shows that for all $m \in \{1, 2, \ldots, \ell{-}1\}$, $\alpha_{\{j\}}^{x^m,x^{m-1}} = x^{m+1}$. Therefore $i = j$. In fact this also shows that when peaks lie within the same interval of length less than 1/8, agent i's peak is chosen. □

The following lemma states that the unique cycle dictator described in Lemma 6 is a dictator over the minimal connected subgraph containing all cycles in the graph. We will refer to this (unique) minimal subgraph as the **cycles neighborhood**. In Figure 7, the cycles neighborhood consists of the part of the graph drawn with thick lines plus everything lying to the right of C'.

Lemma 7 (cycles neighborhood dictator) *Suppose that G contains at least one cycle. Let $\mathcal{C} \subseteq G$ be the minimal connected subgraph of G containing all of the cycles in G (i.e., the cycles neighborhood of G). There* *exists an agent $i \in N$ such that for all $p \in \mathcal{C}^n$, $f(p) = p_i$.*

Proof: Let $p \in \mathcal{C}$. Let $i \in N$ be the cycles dictator described in Lemma 6. Note that Lemma 2 implies that $f(p) \in \mathcal{C}$. If $f(p) = p_i$, we are done. Otherwise, for all $j \neq i$, let $p'_j = f(p)$. By repeated application of *strategy-proofness*, $f(p_i, p'_{-j}) = f(p)$.

By Lemma 1, for all p'_i sufficiently close to $f(p)$, we have $f(p') = f(p)$. Suppose there exists such a p'_i not equal to $f(p)$ which lies on a cycle also containing $f(p)$. Then by Lemma 6, we have $f(p') = p'_i$, contradicting Lemma 1.

Otherwise, since $f(p) \in \mathcal{C}$, there exists such a p'_i not equal to $f(p)$ such that p'_i and $f(p)$ lie on the same path between two cycles, and $d(p'_i, f(p)) \leq 1/8$. As shown at the end of the proof of Lemma 6, in this situation we must have $f(p') = p'_i$, contradicting Lemma 1. □

Our first main result of this section is that a *strategy-proof*, *onto* rule must choose a location along the unique path between the cycle dictator's peak and the cycles neighborhood. Therefore, whenever the cycle-dictator's peak lies in the cycle-neighborhood, (and, hence, when this path is a point,) that agent's peak is chosen. All of our characterizations for graphs with cycles are based upon this result.

THEOREM 3 (CYCLE DICTATOR'S RATIONALITY) *Suppose that G contains at least one cycle. Let $\mathcal{C} \subseteq G$ be the cycles neighborhood of G. There exists an agent $i \in N$ such that for all $p \in G^n$, $f(p) \in \bigcap_{x \in \mathcal{C}} [p_i, x]$.*

Proof: Let $i \in N$ be the cycle dictator described in Lemma 7. Without loss of generality, assume that for all $j \neq i$, $p_j = f(p)$ (as in the proof of Lemma 7). By Lemma 1, for $p'_i \in [p_i, f(p)]$ sufficiently close to $f(p)$, we have $f(p'_i, p_{-i}) = f(p)$. Therefore if $f(p) \in \mathcal{C}$, then with Lemma 7, we must have $f(p) \in \bigcap_{x \in \mathcal{C}} [p_i, x]$ (otherwise, there exists such a $p'_i \in \mathcal{C}$ such that $f(p'_i, p_{-i}) \neq p'_i$).

If $f(p) \notin \mathcal{C}$, then an argument similar to the one in the proof of Lemma 6 can be used, along a path from $f(p)$ to $\mathcal{C}$, to show that if $d(p_i', f(p)) \leq 1/8$, we must have $f(p) = f(p_i', p_{-i}) \in [p_i', x]$ for all $x \in \mathcal{C}$. $\square$

5.1 A Characterization

Reconsider the example of a graph given in Figure 6. According to Theorem 3, under any *strategy-proof*, *onto* rule, there exists an agent, say agent 1, such that for any profile of preferences, (i) if agent 1's peak is on the cycle, his peak is chosen, and (ii) otherwise, the chosen location must lie on the interval between his peak and the cycle.

Conversely, the following method will always produce a *strategy-proof*, *onto* rule for this graph: (i) if agent 1's peak is on the cycle, choose his peak, and (ii) otherwise, on the line segment, use an *onto* generalized median voter scheme that always chooses a point between agent 1's peak and the point on the line segment intersecting the cycle.[11] Given Border and Jordan's [9] characterization of generalized median voter schemes as the only *strategy-proof* rules for symmetric, single peaked preferences on a line segment, this method can be shown to provide the *only* way to construct *strategy-proof*, *onto* rules for this particular graph.

For more general graphs with cycles, a similar characterization holds, as we formalize below. That is, for any such graph, each *strategy-proof*, *onto* rule can be constructed by choosing an agent, say agent 1, such that (i) if agent 1's peak lies on the cycles neighborhood, choose his peak, and (ii) otherwise, if agent 1's peak lies on some "subtree," use any *strategy-proof*, *onto* rule for trees, *specific to that subtree*, that always chooses a point between agent 1's peak and the unique intersection of that subtree with the cycles neighborhood. In the second case (ii), we rely on the characterization

[11]The arguments of the generalized median voter scheme are the points on the line segments closest to the agents' peaks on the graph—their "peaks" on the line segment.

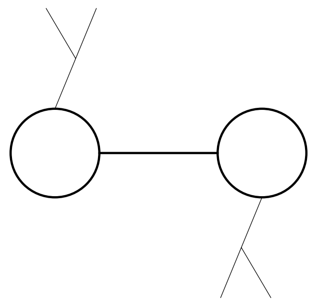

Figure 8: A graph with two maximal trees. The cycles neighborhood is the dumbbell shape drawn with thick lines.

results for trees from Section 3.1.

Our description of *strategy-proof* rules therefore depends on labeling the "subtrees" which, together with the cycles neighborhood, make up G. Let $\mathcal{C} \subset G$ be the cycles neighborhood of G. A tree $T \subset G$ is a **maximal tree of** G if (i) T contains more than one point, (ii) $T \cap \mathcal{C}$ contains at most one point, and (iii) there exists no tree $T' \subset G$ such that $T \subsetneq T'$ and $T' \cap \mathcal{C}$ contains at most one point. See Figure 8.

Our main characterization result for graphs with cycles is based on the ideas and results mentioned above, and can be roughly described as follows. First, by Theorem 3, the chosen location must lie between a prespecified agent's peak and the cycles neighborhood. Second, the restriction of a *strategy-proof* rule to a maximal tree of G must itself be a *strategy-proof* rule defined on that maximal tree. Therefore, by Theorem 1,[12] the rule must behave like an e.m.v.s. on that maximal tree.

THEOREM 4 *Let $\mathcal{C} \subset G$ be the cycles neighborhood of G and let $T_1, T_2, \ldots, T_k$ be the maximal trees of G. A rule $f\colon G^n \to G$ is* strategy-proof *and* onto *if and only if there exists $i \in N$ such that*

[12]To invoke Theorem 1, we need to prove that the restricted rule is *onto* the maximal tree. This can be done with the aid of Lemma 3.

1. *for all* $p \in G^n$, $p_i \in \mathcal{C}$ *implies* $f(p) = p_i$,

2. *for each* T_j, $1 \leq j \leq k$, *there exists an e.m.v.s. (on* T_j*),* $f_j\colon T_j^n \to T_j$, *such that for all* $p \in G^n$, $p_i \in T_j$ *implies*

   (a) $f(p) = f_j(\tilde{p})$, *where for all* $k \in N$, $\tilde{p}_k = \arg\min_{z \in T_j} d(z, p_k)$, *and*

   (b) *where* $\{x_j\} = \mathcal{C} \cap T_j$, *we have* $f(p) \in [p_i, x_j]$.

Proof: Follows from Theorems 1 and 3. □

It is interesting to observe that the class of rules described in Theorem 4 can be thought of as a subclass of e.m.v.s.'s in the following sense. The restriction of a *strategy-proof*, *onto* rule on a cyclic graph to the points *not* in the cycles neighborhood—in other words, $f|_{G \setminus \mathcal{C}}$—is, by Theorem 4, a collection of k e.m.v.s.'s, one for each maximal tree. However, due to condition 2b of the Theorem, it can be shown that these e.m.v.s.'s together form a single e.m.v.s. for the tree obtained from G by contracting each edge in the cycles neighborhood of G to a single point.

6 Conclusion

6.1 Summary

We have derived a characterization of the class of *strategy-proof*, *onto* rules that choose locations on networks (graphs), when agents' preferences over points on the graph are inversely related to distance from a most-preferred point on the graph (i.e., symmetric, single-peaked preferences). The flavor of the results depend on whether the graph contains a cycle. When the graph is a tree (no cycles), we describe the class of *strategy-proof*, *onto* rules as *extended median voter schemes*. This class of rules is, necessarily, a generalization of the class of generalized median voter schemes for an interval, described by Moulin [17]. However, the generalization is not straightforward in the sense that (i) it relies on a type of consistency as described in Section 3, and (ii) given consistent g.m.v.s.'s, when peaks do not lie on a single path, there is only one way to choose a location in a *strategy-proof* way.

When the graph contains at least one cycle, the class of *strategy-proof*, *onto* rules is more restricted. On the part of the graph containing cycles (i.e., the "cycles neighborhood"), one agent must exercise dictatorial power. On the other parts of the graph, though, this agent's power is diminished: there is some flexibility in the choice of a point between this agent's most preferred location and the cycles neighborhood. We in fact show that the class of rules for cyclic graphs is in a sense isomorphic to a subclass of the extended median voter schemes.

The results for cyclic graphs are partially negative and partially positive; one agent acts as a dictator on or between all cycles on the network, but exercises more limited power on other parts of the network. If the network is thought of as representing a highway network, with cycles around an urban center, and subtrees branching out into the suburbs, the rules can be, very roughly, described as follows: A given agent either chooses an *exact* location within the urban area or chooses a suburb in which the location should lie; then if a suburb was chosen, the remaining agents choose the exact location within the suburb (according to a generalized median voter scheme particular to that suburb).

6.2 Comments

There are two important issues upon which we comment: the choice of domain, and a connection to *strategy-proofness* results on other domains.

The results in this paper are based on the domain of single-peaked preferences that are *quadratic*. That is, preferences depend only on distance from the peak. Our result for trees can be used to show that the characterization of e.m.v.s.'s also holds on the (larger) domain of non-symmetric, single-peaked preferences, i.e., preferences that merely satisfy the condition that if p is an agent's most preferred point, then $a \in [b, p]$ implies that a is preferred to b.[13] This is the domain of preferences used, for example, by Danilov [12] and Moulin [17]. Therefore, Danilov's result provides a recursive definition of e.m.v.s.'s.

For any significantly larger class of preferences (that are not single-peaked for all agents), we are confident that an impossibility result would obtain. This finding would be consistent with the recent work on maximal domains, showing that on an interval, no interesting, significant superset of single-peaked preferences can allow reasonable *strategy-proof* rules (e.g. see Barberà, Massó, and Neme [5], and the papers they cite).

Our rules can, however, be extended to certain domains in which only one agents' preferences are always single-peaked.[14] An interesting example is the domain of exchange economies. In particular, Barberà and Jackson [4] characterize the class of *strategy-proof* rules for 2-agent, ℓ-good exchange economies as those that, essentially, allow trade only along ℓ directions from an endowment point.[15]

The directions must be such that one of the agents has single-peaked preferences over the entire range of the rule. The other agent then has single-peaked preferences over any one of the ℓ directions, but not necessarily single-peaked over the entire graph. Therefore, the range can be thought of as a tree with the endowment point representing a central node, and each direction of trade representing an edge from that node. Surprisingly, the

[13] The proof has a standard style of extending *strategy-proofness* characterizations to larger domains (see Schummer [20]). The extension does not require a peaks-only condition to be imposed; the condition is implied.

[14] We thank an anonymous referee for this observation.

[15] See their Theorem 5 for a more precise description.

strategy-proof rules characterized by Barberà and Jackson [4] are a subclass of e.m.v.s.'s.

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