Bribing and Signalling in Second Price Auctions*

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Abstract

We examine whether a two-bidder, second-price auction for a single good (with private, independent values) is immune to a simple form of collusion, where one bidder may bribe the other to commit to stay away from the auction (i.e. submit a bid of zero). In either of two cases—where the potential bribe is fixed or allowed to vary—the only robust equilibria involve bribing. In the fixed-bribe case, there is a unique such equilibrium. In the variable bribes case, all robust equilibria involve low briber-types revealing themselves through the amount they offer, while all high types offer the same bribe; only one such equilibrium is continuous. Bribing in all cases causes inefficiency.

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1 Introduction

Our goal is to examine the extent to which an auction mechanism is immune to a simple form of collusion in which one bidder may bribe another to leave the auction. Specifically, we consider a second-price (or English) auction where two buyers have independent private valuations for a good. Before the auction begins, one of the buyers has the opportunity to offer the other a bribe in exchange for the other's commitment to remove himself from the auction (or bid zero). We analyze two versions of this model: one in which the amount of the bribe is exogenously fixed (e.g. representing a fixed, legal "favor" one bidder can do for the other, which cannot be regulated), and one in which the amount is chosen by the briber.

With respect to a given equilibrium concept for this extended game, we examine whether the second-price auction is "bribe-proof" in the following sense. We say that the auction is *strongly* bribe-proof if bribing does not occur in *any* equilibrium of the extended game. We show that the second-price auction fails this requirement under any reasonable equilibrium concept: under both fixed and variable bribes, there exists a robust equilibrium in which bribing occurs. We say that the auction is *weakly* bribe-proof if bribing does not occur in *some* equilibrium of the extended game. We show that the second-price auction fails weak bribe-proofness under standard refinements such as D1. While there is a sequential equilibrium in which no bribe is offered, this equilibrium turns out not to be robust.

The concept of bribe-proofness is practical in the sense that bribing agreements represent a minimal, simple form of collusion. A bribing contract like ours is relatively easy to enforce; participation in the auction is often verifiable, and the contract does not rely on post-auction conditional payments.

The bribing contracts we study certainly do not represent all possible collusive arrangements. However, the availability of even these can induce collusion and inefficiencies in the second-price auction. One interpretation of our results is that, in the private-values auction environment, no efficient and strategy-proof mechanism is resistant to very simple forms of bidder collusion, even if the buyers have incomplete information regarding each others' valuations.

1.1 Related literature

Bribing contracts have been analyzed by Schummer (2000) in the context of dominant strategy implementation. In a general collective decision problem, he calls a mechanism bribe-proof if, given player i's type, player j has no incentive to pay i to commit to misreport his type, even when j reports truthfully. Schummer (2000) shows that only constant mechanisms are bribe-proof. In this paper, we extend this type of analysis to a Bayesian setting, where players do not know each others' types, and where the decision problem of allocating an object is being solved with a second-price auction.

Our paper also contributes to a growing literature on collusion in auctions, including Graham and Marshall (1987), Mailath and Zemsky (1991), McAfee and McMillan (1992), and Marshall and Marx (2002).[1] These authors model collusion by assuming that a subset of buyers congregates before the auction, and play some kind of "collusive mechanism" or "knock-off auction." Graham and Marshall (1987) show that a group of bidders can collude in an incentive compatible and ex-ante budget balanced way by simply asking low-valuation bidders in the group to drop out of the auction. Payments are made to all group members before determining who should drop out, while after the auction, the group's high-valuation member makes a payment back to the group only if the manipulation produced ex-post gains for him. Mailath and Zemsky (1991) provide a more sophisticated mechanism that also achieves ex-post budget balance, and identify the optimal collusive contract subject to this constraint.

The main difference between our extended game and the way collusion is modelled in this literature is that we consider a different (and particular) bribing stage. Instead of the agents jointly designing a collusive side-contract,

[1] Laffont and Martimort (2000) have a two-agent public goods setup where the modelling of collusion is similar to that of this literature.

one of our agents is fixed as having the opportunity to offer a contract to the other agent.[2] This is important because in our model the "designer" of the mechanism, bidder j, has private information, and his goal is not the maximization of the joint surplus, but rather his own. The result of this difference is that in our game, signalling is an issue, and the bribing equilibrium is not efficient.

In previous work on bidding rings, the ring serves as a device to siphon profits from the seller to the ring members, and overall efficiency is not lost (under ex-ante symmetry) as a consequence. In our model, though, bribing leads to a loss in social surplus. We do not assert that our way of modelling collusion is better, but we think that it is an interesting alternative, especially, that inefficiencies arising from bribing have not been considered before.

1.2 Outline of Results

In Section 3 we start with a model in which the briber may only offer an exogenously fixed bribe amount b. We show that in this model, there are precisely two equilibria in pure strategies: (i) a bribing equilibrium in which high briber types offer the bribe, and low acceptor types accept it, and (ii) a no-bribing equilibrium in which the bribe is never offered.

Since the bribing equilibrium has full support on the action space, it is robust to the usual equilibrium refinements of signalling games. We argue that the no-bribing equilibrium, however, is not robust. First, we show that it fails the iterated deletion of dominated strategies if and only if the amount of the bribe is sufficiently large compared to a certain function of the distribution of types. Second, regardless of the distribution of types, the no-bribing equilibrium does not survive standard equilibrium refinements (such as D1 or Perfect Sequential Equilibrium).

In Section 4 we turn to the case of variable bribe amounts, i.e. where the briber may choose to offer any amount. Here, among all equilibria in

[2]Furthermore, we restrict the set of available contracts to "bribing contracts," that is, a transfer from j to i conditional on i bidding zero.

which bribing occurs, we show that there is a unique one that is continuous. It is a "mostly-separating" equilibrium, in the sense that any briber type below a certain threshold offers a unique amount b as a function of his type, while all other types offer the same amount. All bribes are accepted with positive probability, and the highest bribe is always accepted. We also show that any equilibrium satisfying the D1 criterion must look like the continuous equilibrium: a unique bribe offer is made by any low briber type, and the same offer is made by all high briber types.

In the bribing equilibria of both models, inefficiency occurs with positive probability. Proofs are collected in the Appendix.

2 The Bribing Contract

Consider a second-price (Vickrey) auction for a single indivisible good, with two risk-neutral bidders i and j. The buyers have private valuations, $\theta_i, \theta_j \in [0, 1]$, drawn independently according to the same differentiable c.d.f. F. We assume that $0 < F'(x) < \infty$ for all $x \in [0, 1]$. Everything is commonly known except the valuations, which are privately known by the buyers who hold them.

We modify the second-price auction to model *bribing* in the following way. After the buyers learn their valuations, but before the auction starts, bidder j has an opportunity to offer a bribe b to bidder i in exchange for i's commitment not to bid. If i accepts the bribe, he is committed to making a bid of 0 in the auction; we are assuming that the bribing contract is enforceable. If i rejects the bribe or if j doesn't offer a bribe in the first place, then the game proceeds as a second-price auction.

We provide results for two cases: (Section 3) when b is given exogenously, so j decides whether to bribe but not how much to offer; and (Section 4) when j may also choose the amount of the bribe b. One interesting aspect of this game is that buyer j's decision whether or not to offer a bribe (and the amount offered) reveals information regarding his type. This signalling effect adds much complication to a Bayesian model, and makes out-of-equilibrium beliefs an important issue.[3] In a second-price auction, however, if a bribe is offered but it is declined, then the players' beliefs about each other's type becomes irrelevant, since bidders have an incentive to bid truthfully regardless of their information.

Formally, the game we describe above involves three stages: a stage where bidder j decides whether to offer a bribe, a stage where i decides whether to accept an offer (if made), and the second-price auction stage. In order to simplify the presentation, however, we do not explicitly model the bidders' behavior in the auction stage. We assume that bidders bid truthfully in the second-price auction, except of course when bidder i accepts a bribe, in which case he is forced to bid zero.[4] If a bribe b is offered and accepted, then the payoffs to i and j are b and $\theta_j - b$, respectively. Otherwise, the payoff to the bidder with the highest type is $\max(\theta_i, \theta_j) - \min(\theta_i, \theta_j)$, while the other bidder receives zero. We formalize the definitions of strategies and equilibrium concepts in each of the following two sections.

3 Fixed Bribe Amount

In this section we assume that the amount that the briber j can offer, $b \leq \mathbf{E}(\theta_i)$, is exogenously fixed, and that he chooses only whether to offer it. In this model, a (pure) *strategy* for bidder j prescribes for each type θ_j a decision of whether to offer the bribe b. Hence it can be represented by the set $\mathbb{B} \subseteq [0, 1]$ of types that offer the bribe. A strategy for bidder i prescribes for each type θ_i a decision of whether to accept b if offered; it can

[3]In our model, out-of-equilibrium beliefs affect what would happen if j offered a bribe that is not expected.

[4]This assumption is innocuous since our most interesting results concern equilibria in undominated strategies. It does, however, rule out equilibria (in weakly dominated strategies) in which bidder j threatens to bid the maximum amount in the second-price auction, forcing i (if he believes this threat) to accept the bribe regardless of his type. Since our goal is to determine when bribing equilibria are the only "reasonable" ones, our results are not weakened by this assumption.

be represented by the set $\mathbb{A} \subseteq [0,1]$ of types that would accept the bribe if it were offered. We assume that $\mathbb{A}$ and $\mathbb{B}$ are measurable.

A *sequential equilibrium* is a pair of strategies $(\mathbb{A}, \mathbb{B})$ and a posterior belief distribution, μ, which satisfy the usual consistency and rationality conditions for each type.[5]

Some of our results involve equilibria whose description includes a partition of the set of types. As in many such games with a continuum of types, a pair of equilibria may exist which differ only in the behavior of a single (borderline) type. In order to describe such equilibria more concisely, we introduce the following notation. For any $0 \leq a \leq 1$, we write $[0, a\rangle$ to mean "$[0, a]$ or $[0, a)$." Similarly, $\langle a, 1]$ means "$[a, 1]$ or $(a, 1]$." This notation facilitates the description of "essentially unique" equilibria, where certain types on interval boundaries may behave in indeterminate (and irrelevant) ways.

Our first result describes the structure of all sequential equilibria in the model with a fixed bribe b. Strategies are described by sets which are 2-partitions of $[0,1]$.

Proposition 1 *In any sequential equilibrium, the set of types that offer a bribe is of the form $\mathbb{B} = \langle B, 1]$ and the set of types that accept the bribe is of the form $\mathbb{A} = [0, A\rangle$; furthermore $B < 1$ implies $b < B < A \leq 1$.*

Proof: For a given equilibrium, denote the set of types that offer the bribe as $\mathbb{B}$, and the set of types that accept the bribe as $\mathbb{A}$. When $\mathbb{B}$ is non-empty, if player i accepts the bribe then it must exceed the profit he would get in the auction, given that $\theta_j \in \mathbb{B}$. In other words, if $\theta_i \in \mathbb{A}$ then

$$b \geq \mathbf{E}_{\theta_j}[(\theta_i - \theta_j)\mathbf{1}_{\{\theta_j \leq \theta_i\}} \mid \theta_j \in \mathbb{B}] \tag{1}$$

where $\mathbf{1}_X$ is the indicator function for event X. If this inequality holds for some θ_i then it holds for any $\theta_i' < \theta_i$. Therefore $\mathbb{A} = [0, A\rangle$. If $\mathbb{B}$ is empty then a similar argument (in which the posterior based on F is replaced by the out-

[5] See Fudenberg and Tirole (1991), Section 8.3.

of-equilibrium beliefs) shows that for any beliefs supporting the sequential equilibrium, $\mathbb{A}$ must be an interval.

To show that $\mathbb{B}$ is also an interval, define $B = \inf \mathbb{B}$. If $B = 1$ then we are done. Otherwise, since i can infer $\theta_j \geq B$ from the fact that the bribe was offered, he has an incentive to accept the bribe if his type is less than $B + b$. This follows because i's profit in the second-price auction is at most $\theta_i - B \leq b$. Therefore $A \geq \min\{1, B + b\} > B$.

For any $\theta_j \in \mathbb{B}$, the payoff from offering the bribe must be at least as great as his unconditional payoff in the second-price auction, that is,

$$F(A)(\theta_j - b) + \mathbf{E}_{\theta_i}[(\theta_j - \theta_i)\mathbf{1}_{(A<\theta_i\leq\theta_j)}] \geq \mathbf{E}_{\theta_i}[(\theta_j - \theta_i)\mathbf{1}_{(\theta_i\leq\theta_j)}]. \qquad (2)$$

Differentiating both the left and right hand sides,

$$\frac{\partial LHS(\theta_j)}{\partial\theta_j} = \max\{F(A), F(\theta_j)\} \geq F(\theta_j) = \frac{\partial RHS(\theta_j)}{\partial\theta_j}.$$

When $\theta_j < A$, the left hand side increases in θ_j strictly faster than the right hand side does. Therefore, for any $\theta_j \in \mathbb{B}$ for which $B \leq \theta_j < A$, and any $\theta'_j > \theta_j$, eqn. (2) holds strictly with respect to θ'_j. This implies $\theta'_j \in \mathbb{B}$, and therefore $\mathbb{B}$ is of the form $\langle B, 1]$. Furthermore, eqn. (2) cannot hold at $\theta_j = b$, hence $B > b$. □

Observe that the inequality $B < A$ implies inefficiency: if an equilibrium exists in which the bribe is offered with positive probability ($B < 1$), then with positive probability it will be accepted in situations where $\theta_i > \theta_j$, and bidder j will win receive the object.

3.1 The Bribing Equilibrium

Our next result states that regardless of the distribution and the amount of the bribe $b \leq \mathbf{E}(\theta_i)$, an essentially unique bribing equilibrium exists. By the previous result, high types offer the bribe while low types accept it, and inefficiency occurs with positive probability.

Proposition 2 *For any* $b \in (0, \mathbf{E}(\theta_i)]$, *there exists a sequential equilibrium in which bribing occurs. Moreover, all equilibria in which a bribe is offered with positive probability are essentially equivalent: there exist* A^b, B^b *such that in any equilibrium where bribing occurs, the sets of bribers and acceptors are* $\langle B^b, 1]$ *and* $[0, A^b\rangle$, *respectively.*

The proof of this result amounts to showing that for all $b \in (0, \mathbf{E}(\theta_i)]$, there is a unique $A, B \in [0, 1]$ such that for all $\theta_i \in [0, A\rangle$, inequality (1) holds, and for all $\theta_j \in \langle B, 1]$, inequality (2) holds. The details of this proof can be found in the Appendix. For the case in which the bidders' types are uniformly distributed, Figure 1 depicts the bribing equilibrium with respect to various levels of b. Inefficiency occurs whenever $B \leq \theta_i < \theta_j \leq A$, which occurs with probability $(A - B)/2$ in the uniform case.

The bribing equilibrium is robust to any refinement on out-of-equilibrium beliefs because both actions of player j are used in equilibrium with positive probability. Since it is essentially unique, the only other possible equilibrium play of the extended game is one where none of the briber types offer b. Such "no-bribing equilibrium" can be supported, for example, when bidder i believes that only type $\theta_j = 0$ would offer a bribe. In this case, his optimal strategy is to accept the bribe when his type is such that $\theta_i \in [0, b\rangle$. Then, bidder j never would benefit from offering the bribe in the first place, hence bribing would not occur.

These out-of-equilibrium beliefs are unreasonable, though, since for types $\theta_j < b$, offering the bribe is a strictly dominated strategy (as long as it is accepted with positive probability). In Sections 3.2 and 3.3 we examine whether *any* beliefs that can support the no-bribing equilibrium are consistent with standard refinements. In Section 3.2, we show that under some condition on F, the bribing equilibrium is the unique equilibrium to satisfy an iterated dominance condition or the Intuitive Criterion. In Section 3.3 we show that it is is the unique sequential equilibrium satisfying Cho and Sobel's (1990) D1 criterion. It is also the unique Perfect Sequential Equilibrium (Grossman

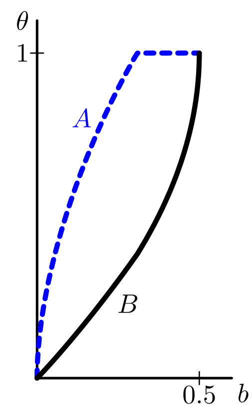

Figure 1: Equilibrium values of A and B under the uniform distribution, for any given bribe level $b \leq \mathbf{E}(\theta) = 0.5$.

and Perry (1986)) of the extended game.[6]

3.2 Iterative Dominance

In this section we examine the consequences of iteratively deleting weakly dominated strategies. Proposition 3 provides a necessary and sufficient condition under which this refinement rules out equilibria in which bribing never occurs ("no-bribing equilibria"). Since "order matters" when eliminating weakly dominated strategies, for simplicity we restrict attention to the case of eliminating *every* weakly dominated strategy in each round of deletion. We call this *maximal elimination of weakly dominated strategies.*

To describe the result, it is helpful to define the briber-type who would be indifferent between offering the bribe and not offering it, given that *every* acceptor type $\theta_i \in [0, 1]$ would accept the bribe.

Definition 1 For any $b \in [0, \mathbf{E}(\theta_i)]$ define θ^b to satisfy

$$\theta^b - b = \int_0^{\theta^b} (\theta^b - \theta_i)\, dF(\theta_i). \tag{3}$$

[6]In addition to other refinements, it is also unique in satisfying one introduced in a working paper version of the current paper.

One can check that θ^b is unique and well-defined by this equation.

Our result is that a no-bribing equilibrium survives this refinement if and only if the amount of the bribe, b, is greater than the expected price type $\theta_j = \theta^b + b$ would pay, conditional on winning the second-price auction.

Proposition 3 *For all $b \in (0, \mathbf{E}(\theta_i))$, there exists a no-bribing sequential equilibrium that survives the iterated maximal elimination of weakly dominated strategies if and only if*

$$b > \mathbf{E}[\theta_i \mid \theta_i \leq \theta^b + b] \tag{4}$$

If the distribution function is convex, for example, then condition (4) fails to hold, ruling out the no-bribing equilibrium. On the other hand, for example, if $F(x) = x^\alpha$ with $0 < \alpha < 1$, then eqn. (4) holds for small b.

Remark 1 Reasoning similar to that used in the proof of Proposition 3 can be used to show that there is a no-bribing equilibrium satisfying Cho and Kreps' (1987) Intuitive Criterion if and only if eqn. (4) is satisfied. Roughly speaking, the Intuitive Criterion requires the acceptor to form out-of-equilibrium beliefs that place no probability on any briber type who could not hope to gain a payoff higher than his equilibrium payoff, as long as the acceptor plays some best response strategy. Since a best response for the acceptor must involve an interval $[0, A]$ of accepting types, no briber with type $\theta_j \leq \theta^b$ could hope to do better offering the bribe than he does when not offering it (as in equilibrium). Hence, (out-of-equilibrium) beliefs for the acceptor must have support only on $[\theta^b, 1]$, and a conclusion similar to that of Proposition 3 is reached. It may also be noted that in our model, the Intuitive Criterion is equivalent to the (stronger) iterated version of that condition, defined by Fudenberg and Tirole (1991, p. 449).

3.3 D1 and Perfect Sequential Equilibrium

We consider the consequences of restricting the players' out-of-equilibrium beliefs according to Cho and Sobel's (1990) D1 criterion, and Grossman and Perry's (1986) Perfect Sequential Equilibrium in our extended game.

In the fixed bribes model, the only type of equilibrium in which one of the players has an out-of-equilibrium action is a no-bribing equilibrium. Therefore, for brevity, we define the consequences of D1 only for an equilibrium in which bribing does not occur.

Cho and Sobel's D1 criterion is based on the idea that if one sender type is "more likely" to benefit from using an out-of-equilibrium action than a second sender type, then the first should get "infinitely more weight" in the receiver's posterior beliefs after observing that out-of-equilibrium action. Here, if one briber type θ'_j would benefit from offering a bribe (out of equilibrium) against more of i's best responses than θ_j would, then θ_j should receive no weight in i's posterior beliefs after receiving the bribe offer.

To make this concept precise in our setting, we say that θ'_j *is more likely to benefit from bribing than* θ_j *is* when the following is true: For all $A \in [0, 1]$, if θ_j is (weakly) better off bribing (compared to not) when exactly $\theta_i \in [0, A]$ accept, then θ'_j is also better off bribing when exactly $\theta_i \in [0, A]$ accept. In other words, the set of i's best responses (which must be of the form $[0, A\rangle$) that would induce θ_j to offer a bribe is a (weak) subset of the best responses that would induce θ'_j to do so. Formally, for all $A \in [0, 1]$, if eqn. (2) holds for θ_j, then it holds for θ'_j.

We say that θ'_j *is strictly more likely to benefit from bribing than* θ_j *is* if θ'_j is more likely to benefit from bribing than θ_j is, but the converse is not true.

Let $D(\theta_j)$ be the acceptor type (if it exists) such that θ_j is indifferent between offering the bribe and not offering it when the bribe is accepted by precisely the set of types $\theta_i \in [0, D(\theta_j)]$; formally, let $D(\theta_j)$ be such that

$$F(D(\theta_j))(\theta_j - b) + \mathbf{E}_{\theta_i}[(\theta_j - \theta_i)\mathbf{1}_{\{D(\theta_j)<\theta_i\leq\theta_j\}}] = \mathbf{E}_{\theta_i}[(\theta_j - \theta_i)\mathbf{1}_{\{\theta_i\leq\theta_j\}}]. \quad (5)$$

if it exists. Note that $D(\theta_j)$ is defined only for sufficiently high types, in which case it is unique. In particular, it is defined if and only if $\theta_j \geq \theta^b$; furthermore $D(\theta^b) = 1$. It is clear that θ_j' is strictly more likely to benefit from bribing than θ_j is if and only if $D(\theta_j') < D(\theta_j)$.

The definition of the D1 criterion, in the spirit of Cho and Sobel's definition, is the following. When bidder i receives an out-of-equilibrium bribe offer, the support of his beliefs about bidder j's type must be restricted to the set of types $\theta_j \geq \theta^b$ that minimize $D(\theta_j)$.

Using eqn. (5), it can be shown (as we do in Lemma 1, Section 4.2, in the context of the more general variable bribes model) that $D()$ is continuous on its support, is strictly decreasing as long as $\theta_j < D(\theta_j)$ and is constant otherwise. Hence, it has a unique fixed point, $\theta^* = D(\theta^*)$, so the minimizers of $D()$ are $[\theta^*, 1]$. The types that would benefit most from bribing (in the no-bribing equilibrium) are $\theta_j \in [\theta^*, 1]$.

According to D1, bidder i must attach probability one to the event that the out-of-equilibrium bribe has been offered by $\theta_j \in [\theta^*, 1]$. Therefore, the offer would be accepted by (at least) all $\theta_i \in [0, \theta^* + b\rangle$. But this means that briber type $\theta_j = \theta^*$ would strictly prefer to deviate from the equilibrium and offer the bribe (since he would be indifferent when it is accepted by $\theta_i \in [0, \theta^*]$). This is the essence of the proof that for any $b < \mathbf{E}(\theta_i)$, there does not exist a no-bribing equilibrium that satisfies D1.

A no-bribing equilibrium also cannot be a Perfect Sequential Equilibrium. In a sender-receiver game, for a given equilibrium and out-of-equilibrium message m, Grossman and Perry (1986) call the beliefs of the receiver (upon seeing m) *consistent* with the equilibrium and the prior distribution of the sender's type, if there exists a mixed strategy α of the receiver that is a best response given these beliefs, and the beliefs are generated from the prior applying Bayes' rule conditional on the sender's type being in the set of types that benefit from sending m when the receiver's response to m is α. A Perfect Sequential Equilibrium is a sequential equilibrium that satisfies this additional consistency requirement.

In the no-bribing equilibrium of our extended game, Grossman and Perry's consistency implies the following regarding the acceptor's beliefs when he is unexpectedly offered b. His beliefs must come from the prior distribution applying Bayes' rule conditional on $\theta_j \in \mathbb{B}$ for some $\mathbb{B} \subseteq [0,1]$. By Proposition 1, his best response is to accept if and only if $\theta_i \in [0, A\rangle$, where $A \leq 1$. Given this response, player j will be better off deviating from the no-bribing equilibrium with type θ_j if and only if

$$F(A)(\theta_j - b) + \mathbf{E}_{\theta_i}[(\theta_j - \theta_i)\mathbf{1}_{\{A<\theta_i\leq\theta_j\}}] \geq \mathbf{E}_{\theta_i}[(\theta_j - \theta_i)\mathbf{1}_{\{\theta_i\leq\theta_j\}}].$$

Again from Proposition 1, we know that this inequality is satisfied by types θ_j belonging to some interval $\langle B, 1]$, and therefore by consistency, $\mathbb{B} = \langle B, 1]$. From Proposition 2, we know that for a given $b > 0$ there exist unique A and B satisfying the consistency requirement. In fact, we can conclude that in the no-bribing equilibrium of our game, but off the equilibrium path (i.e. if b is offered), Grossman and Perry's consistency requires the acceptor to behave as in the (unique) bribing equilibrium. Since briber types $\theta_j > B$ are strictly better off in the bribing equilibrium than in the no-bribing equilibrium, they would deviate and offer the bribe if they had consistent beliefs in the no-bribing equilibrium. This establishes that a no-bribing equilibrium cannot be a Perfect Sequential Equilibrium.

Proposition 4 *The only equilibrium that satisfies either Cho and Sobel's (1990) D1 criterion or Grossman and Perry's (1986) Perfect Sequential Equilibrium is the bribing equilibrium of Proposition 2.*

Since we have explained the ideas behind the proofs here, a more formal proof is omitted.

4 Variable Bribes

In this section, we examine the model in which j may offer any amount b to bidder i. As a simplification, we equate the act of offering a bribe of $b = 0$

with the act of offering no bribe.[7] Therefore, a strategy for j is simply a function mapping types into offers, $b\colon [0,1] \to \mathbb{R}_+$. A strategy for i specifies a measurable set of accepting types for each offer $b \in \mathbb{R}_+$, $\mathbb{A}(b) \subseteq [0,1]$. A sequential equilibrium is defined analogously to the previous section (with i's beliefs over types θ_j conditional on receiving any offer $b \in \mathbb{R}_+$).

Certain results from the previous section carry over to this one. In particular, bidder i's equilibrium strategies must be such that any offer $b \in \mathbb{R}_+$ is accepted by sets of the form $\mathbb{A}(b) = [0, A(b)\rangle$. In addition, we have the following.

Proposition 5 *In any sequential equilibrium, j's strategy $b(\theta_j)$ is weakly monotonic in θ_j. Furthermore, j's equilibrium payoff,*

$$\pi^e(\theta_j) = F(A(b(\theta_j)))(\theta_j - b(\theta_j)) + \mathbf{1}_{\{\theta_j > A(b(\theta_j))\}} \int_{A(b(\theta_j))}^{\theta_j} (\theta_j - x)dF(x), \quad (6)$$

is continuous and strictly increasing in θ_j.

This result implies that in any equilibrium, if two bribes $b > b'$ are both offered in equilibrium, then $A(b) > A(b')$.

The type of equilibrium behavior described in the fixed-bribe model can be supported in this model with an appropriate specification of (out-of-equilibrium) beliefs for i. For example, for any given $b < \mathbf{E}(\theta_i)$, the bribing equilibrium described in Proposition 2 can be extended to this model by specifying that whenever a different bribe $b' \neq b$ is offered, i believes that $\theta_j = 0$ with probability 1. The no-bribing equilibrium described in Section 3.2 applies with similar beliefs. Such beliefs are, of course, unappealing, and do not survive typical refinements used in signalling games with continuous type spaces.

On the other hand, there may exist an equilibrium in which j's strategy $b()$ is continuous. Under the assumption that F is log-concave, we prove

[7] Under any reasonable equilibrium concept, this assumption changes nothing in the analysis.

that there is such an equilibrium, and that it is unique (up to the specification of i's out-of-equilibrium behavior). Furthermore, under our refinement, any equilibrium bribing function must at least partially agree with this continuous function.

For the remainder of the section, we make the assumption that F is log-concave: $d[F/F']/d\theta \geq 0$. This is weaker than the widely used assumption that F' is log-concave.[8]

4.1 Continuous Equilibrium Bribing Function

In the continuous bribing equilibrium, $b()$ is strictly increasing on some interval $[0, \bar{\theta})$, and is constant on $[\bar{\theta}, 1]$. Therefore, if i receives some offer $b(\theta_j) < b(\bar{\theta})$, then j's type θ_j is perfectly revealed. In this case, i accepts the offer only if the bribe $b(\theta_j)$ exceeds his (perfectly anticipated) payoff in the auction, $\theta_i - \theta_j$, i.e. when $\theta_i < \theta_j + b(\theta_j)$.

In order for j to have the incentive to reveal his type (e.g. not to pretend to be a slightly higher type), a local incentive compatibility condition must be satisfied. An increase in the amount of bribe offered must be exactly offset by the increase in the set of types θ_i who would accept it. This leads to a differential eqn. (7) characterizing the bribing function.

Proposition 6 *Suppose F is log-concave. In any sequential equilibrium in which bribing occurs, if j's bribing strategy function $b()$ is continuous, then it is the unique solution to the following equation satisfying $b(0) = 0$.*

$$b'(\theta_j) = \begin{cases} \dfrac{F'(\theta_j + b(\theta_j))(\theta_j - b(\theta_j))}{F(\theta_j + b(\theta_j)) - F'(\theta_j + b(\theta_j))(\theta_j - b(\theta_j))} & \text{if } \theta_j + b(\theta_j) < 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Conversely, there exists a sequential equilibrium in which j's (continuous)

[8] Such distributions also satisfy the so-called monotone hazard rate condition. Bagnoli and Bergstrom (1989) provide an extensive list of distributions satisfying these types of conditions.

strategy $b()$ *is described by eqn. (7), with* $b(0) = 0$.

From eqn. (7), it follows that $b()$ is strictly increasing up to some $\bar{\theta}$, after which it is constant, where $\bar{\theta} + b(\bar{\theta}) = 1$.

The equilibrium is robust to any reasonable refinement of out-of-equilibrium beliefs: The only out-of-equilibrium bribe that can occur is $b > b(\bar{\theta})$. Even if all types θ_i accept this offer, no θ_j could benefit from offering it because all types θ_i already accept the smaller (equilibrium) bribe $b(\bar{\theta})$.

It may be readily checked that when types are distributed uniformly, the continuous bribing function is piecewise linear, specifically,

$$b(\theta_j) = \begin{cases} \frac{1}{2}\theta_j & \text{if } \theta_j \in [0, \frac{2}{3}) \\ \frac{1}{3} & \text{if } \theta_j \in [\frac{2}{3}, 1] \end{cases} .$$

When F is not log-concave, a continuous bribing function b satisfying (7) may or may not exist. Tedious difficulties arise when F is such that $b'(\theta_j)$ is infinite for some θ_j. Without the log-concavity assumption, it may be that any $b()$ satisfying eqn. (7) is discontinuous, in which case there is no sequential equilibrium where j has a continuous strategy. Intuitively, this happens when some type θ_j sees "increasing returns" from increasing the amount of his bribe; his increased expenditure is more than offset by the fast increase in the density of types θ_i that accept the higher bribe.

4.2 D1 in the Variable Bribes Model

In this section, we show that the briber's strategy in any equilibrium satisfying Cho and Sobel's (1990) D1 criterion must agree with the function described in Proposition 6, eqn. (7), on some interval $[0, \hat{\theta}_j\rangle$, and be constant (and accepted with probability one) otherwise. Furthermore, there is a discontinuity at $\hat{\theta}_j$ unless the bribing function completely coincides with eqn. (7). See Figure 2.

The intuition for why D1 rules out most discontinuities (and pooling on bribes other than one accepted with certainty) is similar to that offered in

Section 3.3 for the fixed-bribe model. Essentially, there can be no out-of-equilibrium bribe $\hat{b}$ and briber type θ_j such that both $b(\theta_j) < \hat{b}$, and that θ_j would strictly prefer to deviate to $\hat{b}$ if all types $\theta_i \in [0,1]$ would accept $\hat{b}$. If there were, then sufficiently high briber types would gain "more often" from offering it than would lower types, causing bidder i to accept it often enough to break the equilibrium.

This result—and the one for the fixed-bribe model—rely on the following lemma.

Lemma 1 *Fix an equilibrium, $b(), A()$, and an out-of-equilibrium bribe, $\hat{b}$. Denote j's equilibrium payoff by $\pi^e()$, and suppose that for some $\theta_j \in [0,1]$, $b(\theta_j) < \hat{b}$ and $\pi^e(\theta_j) < \theta_j - \hat{b}$.*[9] *For all $\theta_j \in [0,1]$, define $D(\theta_j)$ to solve*

$$F(D(\theta_j))(\theta_j - \hat{b}) + \mathbf{1}_{\{\theta_j > D(\theta_j)\}} \int_{D(\theta_j)}^{\theta_j} (\theta_j - x) dF(x) = \pi^e(\theta_j) \tag{8}$$

whenever such a solution exists.

Then $D()$ is continuous on its support, which is an interval containing 1. Furthermore, it equals 1 at the lower endpoint of its support. For all θ_j such that $D(\theta_j)$ exists and $b(\theta_j) < \hat{b}$, we have $dD(\theta_j)/d\theta_j \leq 0$, with strict inequality if and only if $\theta_j < D(\theta_j)$.

The interpretation of $D(\theta_j)$ (given an equilibrium and a deviation $\hat{b}$) is that it is the "least number of acceptor types" that briber type θ_j needs to accept $\hat{b}$ in order to make his deviation profitable. Under the hypothesis of the Lemma, there is a briber type such that he is willing to deviate if all types of player i accept $\hat{b}$. For all briber types higher than this one (and as long as $b(\theta_j) < \hat{b}$), the "threshold" number of acceptors required for a profitable deviation is weakly decreasing (and strictly decreasing if $\theta_j < D(\theta_j)$).

We formalize our main result of this section as follows.

[9] In the fixed bribes model, the result is applied to a no-bribing equilibrium, $b(\theta_j) = 0$, and $\hat{b}$ is the fixed bribe. Then the hypothesis of the lemma holds for all $\theta_j > \theta^b$.

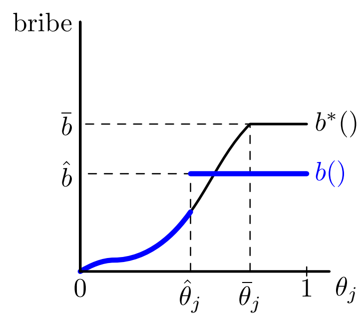

Figure 2: If an equilibrium satisfies D1, then the bribing function $b()$ must (i) coincide with the continuous one of Proposition 6 (denoted $b^*()$) on some $[0, \hat{\theta}_j\rangle$, and (ii) be constant on $\langle\hat{\theta}_j, 1]$.

Proposition 7 *Suppose F is log-concave. If a sequential equilibrium satisfies D1, then the bribing function $b()$ is such that (i) for some $\hat{\theta}_j \leq \bar{\theta}$,*

$$b(\theta_j) = \begin{cases} b^*(\theta_j) & \text{if } \theta_j < \hat{\theta}_j \\ \hat{b} \equiv \hat{\theta}_j - F(\hat{\theta}_j + b^*(\hat{\theta}_j))(\hat{\theta}_j - b^*(\hat{\theta}_j)) & \text{if } \theta_j > \hat{\theta}_j \end{cases}$$

where $b^*()$ is the continuous bribing function described in Proposition 6, and (ii) $\hat{b} \geq 1 - \mathbf{E}[\theta_j \mid \theta_j \geq \hat{\theta}_j]$.

Conversely, any function $b()$ satisfying these conditions is part of a sequential equilibrium satisfying D1.

The requirement that $\hat{b} = \hat{\theta}_j - F(\hat{\theta}_j + b^*(\hat{\theta}_j))(\hat{\theta}_j - b^*(\hat{\theta}_j))$ means that type $\theta_j = \hat{\theta}_j$ is indifferent between two situations: offering $\hat{b}$ when it is always accepted, and offering $b^*(\hat{\theta}_j)$ when bidder i "knows" j's type. This follows from a standard continuity argument. Requirement (ii) implies that type $\theta_i = 1$ would rather accept $\hat{b}$ than compete in the auction against the distribution of types $\theta_j \in \langle\hat{\theta}_j, 1]$. Hence that offer is always accepted: $A(\hat{b}) = 1$.

Finally, these two requirements combined imply that the discontinuity in $b()$ (at $\hat{\theta}_j$, if it exists) cannot occur arbitrarily close to $\theta_j = 0$. As $\hat{\theta}_j$ approaches zero, so does $\hat{b} = \hat{\theta}_j - F(\hat{\theta}_j + b^*(\hat{\theta}_j))(\hat{\theta}_j - b^*(\hat{\theta}_j))$. For small $\hat{\theta}_j$, this would contradict requirement (ii). Therefore, for any fixed distribution F, possible values for $\hat{\theta}_j$ in Proposition 7 are bounded away from zero.

5 Conclusion

We have examined a simple, specific form of collusion among two bidders in a second price auction, where one of the bidders is permitted to pay the other to commit to leave the auction (or bid zero). Regardless of whether the bribing bidder is permitted to offer only an exogenously fixed payment or is permitted to choose any payment, a robust equilibrium exists in which bribing occurs. In these equilibria, the object is allocated inefficiently with positive probability.

Equilibria in which bribing does not occur are not robust to intuitive refinements. Therefore, depending on the solution concept that an auctioneer uses, he may be unable to rationalize the use of this auction even on the basis of the *existence* of a collusion-free equilibrium.

Our approach differs from much of the collusion literature in a few ways. Foremost, we do not model a "collusion design" problem for the agents. In that literature, it is typical to assume that an uninterested third party designs and administers a revelation mechanism, making or receiving payments from bidders based on various information.[10] The third-party design assumption is a way to escape the issue of information transmission in the design stage. If one makes the more-realistic assumption that bidders already have some idea about their types at the design stage, then when a bidder proposes the use of a particular collusive mechanism, information about his type could be inferred from that proposal.[11] This type of inference is a part of what we model. In our model, there is common knowledge about whether bidder j desired to seek collusion, while that information exists about i whenever j

[10]See Marshall and Marx (2002) for a case-by-case analysis of the types of post-auction information that could be used.

[11]See Jackson and Wilkie (2001) for one approach to modelling this issue, where agents simultaneously propose and commit to contingent transfer mechanisms, and final payments are the sum of those proposals.

makes the offer.

We close by observing that our results are robust with respect to certain changes in our basic model.[12] For example, the existence of a reserve price r in the second-price auction does not change the flavor of our results. In the fixed bribe case, Propositions 1, 2, and 4 continue to hold (for $b \leq \mathbf{E}(\max\{\theta_i, r\}) - r$). Proposition 3 holds after redefining type θ^b. The results in the variable bribe case hold with a slightly modified version of eqn. 7, and initial condition $b(r) = 0$.

Another class of modifications to the model involves allowing both bidders to offer bribes. There are many ways to formalize such a model, and the results would depend on how such modelling choices are made. If bidders are permitted to offer bribes simultaneously, for example, then the results of the model would depend on how simultaneous offers are resolved. In the fixed bribe model, if they are resolved by randomly invalidating one of the offers, then bribing equilibria result which have a structure like the equilibrium in our model. On the other hand, if simultaneous offers are resolved endogenously (e.g., through further communication) then (depending on the precise model) efficient collusion may result. This is not surprising, given the observations of Graham and Marshall (1987) that members of a bidder ring can collude efficiently after truthfully revealing their types to each other.

Finally, some of our results in the fixed bribe case carry over to a similar model where bidding is costly, and bidder j may "burn" money instead of transferring it.[13] The burned money may represent, for example, the cost of placing an advertisement in a newspaper, as a signal of intention to bid in the auction. Preliminary numerical analysis shows that equilibria exist in this model for nondegenerate pairs of parameters (the cost of burning b, and the cost of bidding c).

[12] While we do not wish to formalize such changes here, preliminary notes on the following ideas are available upon request.

[13] We thank an Associate Editor for making this observation.

Appendix

This appendix contains the proofs of our results.

Proof of Proposition 2: For any $b \in (0, \mathbf{E}(\theta_i)]$, Proposition 1 implies that in any bribing equilibrium, briber and acceptor types are of the form $\langle B, 1]$ and $[0, A\rangle$. Since $B \geq b$, standard continuity arguments imply that type $\theta_j = B$ must be indifferent between offering the bribe and not, i.e., by eqn. (2) and $B \leq A$,

$$F(A)(B - b) = \int_0^B (B - x)\, dF(x). \tag{9}$$

Note that this holds even if $B = 1$ since bribing is occurring by assumption.

Also, either $A = 1$, or type $\theta_i = A$ is indifferent between accepting the bribe and not. By eqn. (1),

$$b \geq \frac{\int_B^A (A - x)\, dF(x)}{1 - F(B)}, \tag{10}$$

and if $A < 1$ then eqn. (10) holds with equality. (From Proposition 1, if $B = 1$ then $A = 1$ and eqn. (10) becomes $b \geq 0$; if $B < 1$ then $A > B$ and the right-hand side is positive.)

It is helpful to define the following functions for $A, B \in [0, 1]$, $A \geq B$.

$$b_1(A, B) = B - \frac{\int_0^B (B - x)\, dF(x)}{F(A)},$$

$$b_2(A, B) = \frac{\int_B^A (A - x)\, dF(x)}{1 - F(B)}.$$

Observe that since $B > 0$, $b_1(A, B) < B$.

To prove the existence of an equilibrium, we need to find A, B such that eqns. (9) and (10) hold, that is, $b = b_1(A, B) \geq b_2(A, B)$, with equality if $A < 1$.

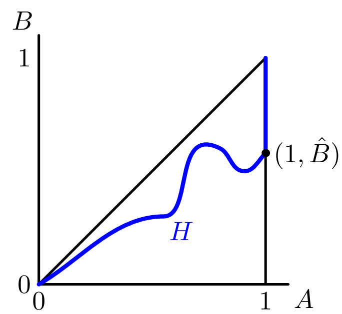

Figure 3: The set H in the proof of Proposition 2.

Define

$$H = \{(A, B) : A \geq B,\ b_1(A, B) - b_2(A, B) \geq 0, \text{ with equality if } A < 1\}.$$

We claim that (i) for all B, there exists A such that $(A, B) \in H$, and (ii) for all $A < 1$, there exists a unique B such that $(A, B) \in H$.

To see (i), first note that $(0, 0) \in H$. For $B > 0$, $b_1(B, B) - b_2(B, B) = E[\theta_i \mid \theta_i \leq B] - 0 > 0$ so either $(1, B) \in H$ or by continuity, $(A, B) \in H$ for some $A \in (B, 1)$. To see (ii) for $A \in (0, 1)$, note that $b_1(A, 0) - b_2(A, 0) = 0 - \int_0^A (A - x)\, dF(x) < 0$, while $b_1(A, A) - b_2(A, A) > 0$. Continuity implies that $b_1(A, B) - b_2(A, B) = 0$ for some $B \in (0, A)$, hence $(A, B) \in H$. Furthermore, this B is unique because $b_1(A, B) - b_2(A, B)$ is strictly decreasing in B (see eqns. (12) and (14) below).

Define the correspondence $h\colon [0, 1] \twoheadrightarrow [0, 1]$ such that $h(A) = \{B : (A, B) \in H\}$. By (ii), h is non-empty and if $A < 1$ then h is single-valued. It can be shown (e.g., by an application of the Maximum Theorem) that h is upper hemi-continuous. Therefore, for $A < 1$, $h(A)$ is a continuous function, and its graph, H, is connected. Define $\hat{B} = \lim_{A \uparrow 1} h(A) \in h(1)$. An example of h appears in Figure 3.

By differentiating b_1 and b_2, we find that for $0 < B < A \leq 1$,

$$\frac{\partial b_1}{\partial A} = \frac{F'(A)}{F(A)^2} \int_0^B (B - x)\, dF(x) > 0, \tag{11}$$

$$\frac{\partial b_1}{\partial B} = 1 - \frac{F(B)}{F(A)} > 0, \tag{12}$$

$$\frac{\partial b_2}{\partial A} = \frac{F(A) - F(B)}{1 - F(B)} > 0, \tag{13}$$

$$\frac{\partial b_2}{\partial B} = \frac{\left\{ \int_B^A (A - x)\, dF(x) - (A - B)(1 - F(B)) \right\} F'(B)}{(1 - F(B))^2} < 0 \tag{14}$$

where the last inequality follows by $\int_B^A (A - x)\, dF(x) < \int_B^A (A - B)\, dF(x) = (A - B)(F(A) - F(B)) \leq (A - B)(1 - F(B))$.

Consider any $A' > A$. If $h(A') > h(A)$ then by eqns. (11) and (12) we have $b_1(A', h(A')) > b_1(A, h(A))$. If $h(A') < h(A)$ then by eqns. (13) and (14) we have $b_2(A', h(A')) > b_2(A, h(A))$, which implies $b_1(A', h(A')) \geq b_2(A', h(A')) > b_2(A, h(A)) = b_1(A, h(A))$. Therefore, $b_1(A, h(A))$ is strictly increasing in A on $A \in [0, 1)$.

By continuity, $b_1(1, \hat{B}) = b_2(1, \hat{B})$. Therefore, eqns. (12) and (14) imply that $h(1) = [\hat{B}, 1]$, and $b_1(1, B)$ is strictly increasing in B.

Therefore, $b_1(A, B)$ is strictly increasing on H in the sense that if $A < 1$ then $b_1(A, B)$ is strictly increasing in A, and if $A = 1$ then $b_1(A, B)$ is strictly increasing in B. Since $b_1(0, h(0)) = 0$ (by $h(0) = 0$) and $b_1(1, 1) = \mathbf{E}(\theta_i)$, the strict monotonicity and continuity of $b \equiv b_1(A, B)$ along H implies that there exists a one-to-one mapping between $b \in [0, \mathbf{E}(\theta_i)]$ and $(A, B) \in H$, i.e. for all $b \in [0, \mathbf{E}(\theta_i)]$, there exist unique A, B solving eqns. (9) and (10). $\square$

Proof of Proposition 3: Recall that we assume that players bid their valuations in the second-price auction, if it is ever reached (which is consistent with deleting dominated strategies). We begin the process of iteratively deleting maximal sets of weakly dominated strategies whether or not eqn. (4) holds.

Round 1. In the first round of elimination, we delete any of j's strategies that prescribe type $\theta_j \leq b$ to offer the bribe. This is clear because by offering the bribe, such a type can only obtain a negative payoff if the bribe is accepted. For bidder i, we delete any strategy that prescribes type $\theta_i \leq b$ to reject the bribe because such a type cannot obtain a payoff higher than b in the second-price auction. It is straightforward to check that no other strategies can be eliminated in the first round.

Round 2. Subject to the first round of elimination, we delete any strategy for the briber that prescribes $\theta_j \leq \theta^b$ to offer the bribe. To see this, denote any set of acceptors by $\mathbb{A} \subseteq [0,1]$. From the first round, we must have $[0,b] \subseteq \mathbb{A}$. If $\theta_j \leq \theta^b$ offers the bribe then his profit would be

$$\mathbf{E}_{\theta_i}[(\theta_j - b)\mathbf{1}_{\{\theta_i \in \mathbb{A}\}} + (\theta_j - \theta_i)\mathbf{1}_{\{\theta_i \in [0,\theta_j]\setminus\mathbb{A}\}}] \leq \theta_j - b$$

where the inequality holds because $\theta_i > b$ for all $\theta_i \notin \mathbb{A}$. Since the LHS of eqn. (3) increases faster in θ^b than the RHS does, we have $\theta_j - b \leq \int_0^{\theta_j}(\theta_j - \theta_i)\, dF(\theta_i)$ for all $\theta_j \leq \theta^b$. Furthermore, for some admissible $\mathbb{A}$, the inequality is strict. Therefore for $\theta_j \leq \theta^b$ not offering the bribe dominates offering the bribe.

Continuing the second round of elimination, we delete any acceptor strategy that prescribes $\theta_i \leq 2b$ to reject the bribe. This follows because for any admissible briber strategy, the bribe is offered only by types $\theta_j > b$, limiting the acceptor's SPA-payoff to no more than $\theta_i - b$. It is again straightforward to check that no other strategies can be eliminated in the second round.

Round 3. Similarly, the acceptor's strategies that we delete in the third round of elimination are precisely those that prescribe $\theta_i \leq \theta^b + b$ to reject the bribe, because the briber's type is greater than θ^b if the bribe is offered.

Non-existence. As the first of two cases, suppose that eqn. (4) does not hold. Let $\theta_j = \min\{1, \theta^b + b\}$. If $\theta_j = 1$ then his payoff is $1 - b$ from offering the bribe and is $1 - \mathbf{E}(\theta_i)$ from not offering the bribe, so θ_j strictly prefers to bribe. If $\theta_j = \theta^b + b < 1$, then his payoff is at least $F(\theta_j)(\theta_j - b)$ from offering the bribe (because each $\theta_i \leq \theta^b + b$ accepts the bribe according to previous rounds of strategy deletion), and his payoff is $F(\theta_j)(\theta_j - \mathbf{E}[\theta_i \mid \theta_i \leq \theta_j])$ from not offering the bribe. Since eqn. (4) does not hold, for any admissible acceptor strategy, type θ_j's payoff from offering the bribe is weakly greater than that from not offering the bribe. Furthermore, since $\theta^b + b < 1$, this inequality is strict when i's strategy is to accept the bribe with any type. We conclude that for $\theta_j = \min\{1, \theta^b + b\}$, offering the bribe weakly dominates not offering the bribe, therefore the no-bribing equilibrium does not survive the iterated maximal elimination of weakly dominated strategies.

Existence. Second, suppose that eqn. (4) holds. We show that no briber strategies can be eliminated in the third round of deletion and that the no-bribing equilibrium can be supported. If the set of acceptors is exactly $[0, 1]$ then for all $\theta_j > \theta^b$, offering the bribe is strictly better than not offering it (because θ^b is indifferent and the LHS of eqn. (3) increases faster in θ^b than the RHS does).

On the other hand, if the set of acceptors is exactly $[0, \theta^b + b]$, which is also admissible even after the third round of deletion, then, for all $\theta_j > \theta^b$, offering the bribe is strictly worse than not offering it. To see this, consider the payoff difference between not offering and offering the bribe,

$$\mathbf{E}[(\theta_j - \theta_i)\mathbf{1}_{\{\theta_i \leq \theta_j\}}] - F(\theta^b + b)(\theta_j - b) - \mathbf{E}[(\theta_j - \theta_i)\mathbf{1}_{\{\theta^b + b \leq \theta_i \leq \theta_j\}}],$$

which has a derivative equal to

$$F(\theta_j) - F(\theta^b + b) - \max\{0, F(\theta_j) - F(\theta^b + b)\}.$$

This derivative is negative for $\theta_j < \theta^b + b$ and zero otherwise, so the payoff difference is minimized at $\theta_j = \theta^b + b$, where it equals

$$F(\theta^b + b)(\mathbf{E}[\theta_i \mid \theta_i \leq \theta^b + b] - b) > 0. \tag{15}$$

Hence for type $\theta_j > \theta^b$ not offering the bribe is strictly better than offering it. Therefore, we can delete no more briber strategies: For any $\theta_j > \theta^b$, either bribing or not bribing may be strictly better, depending on the set of accepting types (and we already established that any $\theta_j \leq \theta^b$ must not offer the bribe).

To support the no-bribing equilibrium in which j never offers the bribe we specify i's out-of-equilibrium beliefs as follows. If the bribe is offered then i believes that $\theta_j = \theta^b + \varepsilon$ with probability 1. His best response is to accept if and only if $\theta_i \leq \theta^b + \varepsilon + b$. For small $\varepsilon > 0$, no θ_j wants to offer the bribe. This can be seen by perturbing eqn. (15) with ε. □

Proof of Proposition 5: Monotonicity of $b()$. Suppose that two bribe amounts, b and $b' < b$, are offered in equilibrium. From the arguments of Proposition 1, the set of types θ_i that accept b and b' are $[0, A(b)\rangle$ and $[0, A(b')\rangle$ respectively. Clearly $A(b') < A(b)$, otherwise no type would offer b.

Let the infimum type who offers b be denoted $\tilde{\theta}_j = \inf\{\theta_j : b(\theta_j) = b\}$. We show that $\theta_j > \tilde{\theta}_j$ implies that θ_j strictly prefers offering b to offering b', implying monotonicity.

If $\tilde{\theta}_j = 1$, we are done. If $\tilde{\theta}_j < 1$, denote the expected payoff to some type θ_j from offering b as

$$\pi(\theta_j, b) = F(A(b))(\theta_j - b) + \mathbf{1}_{\{\theta_j > A(b)\}} \int_{A(b)}^{\theta_j} (\theta_j - x)\, dF(x). \tag{16}$$

As with eqn. (2), we have $\partial\pi(\theta_j, b)/\partial\theta_j = \max\{F(A(b)), F(\theta_j)\}$. Therefore,

$$\frac{\partial[\pi(\theta_j, b) - \pi(\theta_j, b')]}{\partial\theta_j} = \max\{F(A(b)), F(\theta_j)\} - \max\{F(A(b')), F(\theta_j)\} \geq 0.$$

Incentive compatibility (and continuity) imply $\pi(\tilde{\theta}_j, b) - \pi(\tilde{\theta}_j, b') \geq 0$. Since $\tilde{\theta}_j < 1$, we have $\tilde{\theta}_j < A(b)$ (as in the second paragraph of the proof of Proposition 1). Therefore, $\partial[\pi(\tilde{\theta}_j, b) - \pi(\tilde{\theta}_j, b')]/\partial\theta_j > 0$, and so for all $\theta_j > \tilde{\theta}_j$, $\pi(\theta_j, b) - \pi(\theta_j, b') > 0$.

Continuity of $\pi^e()$. To see that $\pi^e()$ is strictly increasing, note that for all $\theta'_j < \theta_j$, $\pi^e(\theta'_j) < \pi(\theta_j, b(\theta'_j)) \leq \pi^e(\theta_j)$, where the first inequality follows from the definition of π and the fact that $\theta_j > b(\theta_j)$, and the second inequality follows from incentive compatibility.

We first show continuity approaching from the right. Suppose towards contradiction that for some $\theta_j \in [0, 1)$, there exists $\delta > 0$ such that for all $\varepsilon > 0$, $\pi^e(\theta_j) + \delta \leq \pi^e(\theta_j + \varepsilon)$. Observe that

$$\begin{aligned} & \pi^e(\theta_j + \varepsilon) - \pi(\theta_j, b(\theta_j + \varepsilon)) \\ = \;& F(A(b(\theta_j + \varepsilon)))\, \varepsilon + \mathbf{1}_{\{\theta_j + \varepsilon > A(b(\theta_j + \varepsilon))\}} \int_{A(b(\theta_j+\varepsilon))}^{\theta_j+\varepsilon} (\theta_j + \varepsilon - x)\, dF(x) \\ & - \mathbf{1}_{\{\theta_j > A(b(\theta_j + \varepsilon))\}} \int_{A(b(\theta_j+\varepsilon))}^{\theta_j} (\theta_j - x)\, dF(x) \\ = \;& F(A(b(\theta_j + \varepsilon)))\, \varepsilon + \mathbf{1}_{\{\theta_j > A(b(\theta_j + \varepsilon))\}} \int_{A(b(\theta_j+\varepsilon))}^{\theta_j} \varepsilon\, dF(x) \\ & + \mathbf{1}_{\{\theta_j > A(b(\theta_j + \varepsilon))\}} \int_{\theta_j}^{\theta_j+\varepsilon} (\theta_j + \varepsilon - x)\, dF(x) \\ & + \mathbf{1}_{\{\theta_j + \varepsilon > A(b(\theta_j + \varepsilon)) > \theta_j\}} \int_{A(b(\theta_j+\varepsilon))}^{\theta_j+\varepsilon} (\theta_j + \varepsilon - x)\, dF(x) \\ < \;& 4\varepsilon \end{aligned}$$

where the inequality follows because each of the four terms is no greater than ε and the last one is strictly less than ε. Therefore,

$$\lim_{\varepsilon \downarrow 0} \left[\pi^e(\theta_j + \varepsilon) - \pi(\theta_j, b(\theta_j + \varepsilon))\right] = 0.$$

But then for $\varepsilon > 0$ sufficiently small, $\pi(\theta_j, b(\theta_j + \varepsilon)) > \pi^e(\theta_j + \varepsilon) - \delta \geq \pi^e(\theta_j)$, contradicting incentive compatibility. Therefore π^e is continuous from the right.

To see continuity from the left, suppose towards contradiction that we have $\lim_{\theta'_j \uparrow \theta_j} \pi^e(\theta'_j) < \pi^e(\theta_j)$. Since $\pi(\theta'_j, b(\theta_j))$ is continuous in θ'_j, we have $\lim_{\theta'_j \uparrow \theta_j} \pi(\theta'_j, b(\theta_j)) = \pi(\theta_j, b(\theta_j)) = \pi^e(\theta_j)$. But then for θ'_j sufficiently close to θ_j, $\pi(\theta'_j, b(\theta'_j)) < \pi(\theta'_j, b(\theta_j))$, contradicting incentive compatibility. □

Proof of Proposition 6: Existence. When F is log-concave, eqn. (7) (and the requirement $b(0) = 0$) uniquely defines a continuous function. To see that, note that $b'(0) = 1/2$ by l'Hôpital's rule. Therefore, $0 < b'(\theta_j) < \infty$ on some interval $[0, \varepsilon)$.

Using local arguments, consider (locally) the inverse of $b(\theta_j)$, denoted $\Theta(b)$, defined by $\Theta(0) = 0$ and

$$\Theta'(b) = \frac{F(\Theta(b) + b)}{F'(\Theta(b) + b)(\theta_j(b) - b)} - 1$$

when $\Theta(b) + b \leq 1$.

We claim that $\Theta(b)$ is a well-defined, (weakly) increasing and continuous function, and $\Theta'(b) > 0$ almost everywhere. To see this, note that $\Theta'(0) = 2$. If for some $b > 0$, $\Theta'(b) = 0$, then $\Theta(b) > b$. Furthermore,

$$\Theta''(b) = \left(\frac{F(\Theta(b) + b)}{F'(\Theta(b) + b)}\right)' \frac{1}{\Theta(b) - b} + \frac{F(\Theta(b) + b)}{F'(\Theta(b) + b)} \frac{1}{(\Theta(b) - b)^2},$$

which is strictly positive because $(F/F')' \geq 0$ by log-concavity, and $\Theta(b) > b$. Therefore, $\Theta(b)$ is strictly increasing in a right-hand side neighborhood of b. That is, whenever $b'(\theta_j)$ becomes infinite, θ_j is only an inflexion point of $b()$, and $b()$ continues with a positive and finite derivative in the right-hand side neighborhood of θ_j. This demonstrates the existence of a unique continuous $b()$ from eqn. (7).

To finish the proof, we construct a strategy (and beliefs) for i, and show that it and $b()$ form a sequential equilibrium.

We show that $\theta_j > 0$ implies $b(\theta_j) < \theta_j$ to establish that it is rational for θ_j to offer $b(\theta_j)$. Since $b'(0) = 1/2$, $b(\theta_j) < \theta_j$ holds sufficiently close to $\theta_j = 0$. Let $\theta_j' = \min\{\theta_j : b(\theta_j) \geq \theta_j\}$ (assuming by contradiction that the set is nonempty). By continuity, $b(\theta_j') = \theta_j' > 0$. This implies $b'(\theta_j') = 0$. Since b' is also continuous, this implies $b(\theta_j' - \epsilon) \geq \theta_j - \epsilon$, a contradiction.

This implies that $b'(\theta_j)$ is positive whenever $\theta_j + b(\theta_j) < 1$ (i.e. whenever b' is not explicitly defined to be zero). Therefore, $b()$ is invertible on $[0, b(\bar{\theta}_j))$.

To construct the equilibrium, the acceptor i believes that an offer $\hat{b} < b(\bar{\theta})$ comes from type $\theta_j = b^{-1}(\hat{b})$; an offer of $b(\bar{\theta})$ comes from some type in $[\bar{\theta}, 1]$, where i's beliefs are a Bayesian update of F over that interval. Let i's beliefs for any out-of-equilibrium offer $\hat{b} > b(\bar{\theta})$ be the same posterior over $[\bar{\theta}, 1]$. An (obvious) best response for i is to accept an offer $\hat{b}$ if and only if $\theta_i \leq A(\hat{b})$ where

$$A(\hat{b}) = \begin{cases} b^{-1}(\hat{b}) + \hat{b} & \text{if } \hat{b} < b(\bar{\theta}_j) \\ 1 & \text{if } \hat{b} \geq b(\bar{\theta}_j) \end{cases}$$

Note that $A()$ is continuous and differentiable everywhere except $\hat{b} = b(\bar{\theta})$.

For j, offering $\hat{b} > b(\bar{\theta})$ is strictly dominated by offering $b(\bar{\theta}_j)$. Therefore to check incentive compatibility, it suffices to check that no type θ_j prefers to offer $\hat{b} \leq b(\bar{\theta})$, i.e. where $\hat{b} = b(\hat{\theta}_j)$ for some $\hat{\theta}_j$.

To prove this, first consider the quantity

$$\pi(\theta_j, b(\hat{\theta}_j)) - \pi(\hat{\theta}_j, b(\hat{\theta}_j)) = F(\hat{\theta}_j + \hat{b})(\theta_j - \hat{\theta}_j) + \mathbf{1}_{\{\theta_j > \hat{\theta}_j + \hat{b}\}} \int_{\hat{\theta}_j + \hat{b}}^{\theta_j} (\theta_j - x) dF(x)$$

(see eqn. (16)) which can be written as

$$\begin{cases} \int_{\hat{\theta}_j}^{\hat{\theta}_j + b(\hat{\theta}_j)} F(\hat{\theta}_j + b(\hat{\theta}_j)) dx + \int_{\hat{\theta}_j + b(\hat{\theta}_j)}^{\theta_j} (\theta_j - x) F(x) dx & \text{if } \hat{\theta}_j + b(\hat{\theta}_j) \leq \theta_j \\ \int_{\hat{\theta}_j}^{\theta_j} F(\hat{\theta}_j + b(\hat{\theta}_j)) dx & \text{if } \hat{\theta}_j \leq \theta_j < \hat{\theta}_j + b(\hat{\theta}_j) \\ -\int_{\theta_j}^{\hat{\theta}_j} F(x + b(x)) dx & \text{if } \theta_j < \hat{\theta}_j. \end{cases}$$

Second, consider the quantity $\pi(\theta_j, b(\theta_j)) - \pi(\hat{\theta}_j, b(\hat{\theta}_j))$. Since

$$\frac{d}{d\theta_j} \pi(\theta_j, b(\theta_j)) = \begin{cases} F(\theta_j + b(\theta_j)) & \text{if } \theta_j < \bar{\theta} \\ 1 & \text{if } \theta_j \geq \bar{\theta} \end{cases}$$

(which can be verified either directly or via the Envelope Theorem), we have

$$\pi(\theta_j, b(\theta_j)) - \pi(\hat{\theta}_j, b(\hat{\theta}_j)) = \begin{cases} \int_{\hat{\theta}_j}^{\theta_j} F(\min\{x + b(x), 1\}) dx & \text{if } \hat{\theta}_j \leq \theta_j \\ -\int_{\theta_j}^{\hat{\theta}_j} F(x + b(x) \wedge 1) dx & \text{if } \theta_j < \hat{\theta}_j. \end{cases}$$

Comparing these two quantities reveals

$$\pi(\theta_j, b(\theta_j)) - \pi(\hat{\theta}_j, b(\hat{\theta}_j)) \geq \pi(\theta_j, b(\hat{\theta}_j)) - \pi(\hat{\theta}_j, b(\hat{\theta}_j))$$

implying incentive compatibility.

Uniqueness. Consider a sequential equilibrium in which j's offer strategy, $b(\theta_j)$, is continuous and where a positive bribe is offered by some type θ_j.

Since any positive bribe would be accepted with positive probability (à la Proposition 1), it is clear that $b(0) = 0$. Let $\theta'_j = \max\{\theta_j : b(\theta_j) = 0\}$. For any $\delta > 0$ the set of acceptors of a bribe $b_\delta \equiv b(\theta'_j + \delta) > 0$ includes the interval $[0, \theta'_j + b_\delta]$. Hence the payoff for θ'_j from offering b_δ is at least $F(\theta'_j + b_\delta)(\theta'_j - b_\delta)$ while his payoff in equilibrium is $F(\theta'_j)(\theta'_j - \mathbf{E}[x \mid x \leq \theta'_j])$. For δ sufficiently small, $b_\delta < \mathbf{E}[x \mid x \leq \theta'_j]$, therefore incentive compatibility requires $\theta'_j = 0$. Therefore $b(\theta_j)$ is strictly increasing at $\theta_j = 0$.

We extend this argument to prove that $b()$ can be constant only on some interval whose maximum is 1. For this, suppose that there exists a bribe b such that $\{\theta_j : b(\theta_j) = b\} = [\theta''_j, \theta'_j]$ where $\theta''_j < \theta'_j < 1$. This bribe is accepted by $\theta_i \in [0, A\rangle$ where $A < \theta'_j + b$ because $\theta''_j < \theta'_j$. Define $c = \theta'_j + b - A > 0$.[14] For any $\delta > 0$ (and $\theta'_j + \delta \leq 1$), the set of types that accept a bribe $b_\delta = b(\theta'_j + \delta)$ is an interval $[0, A_\delta\rangle$, where $A_\delta \geq \min\{\theta'_j + b, 1\}$. Hence $A_\delta \geq A + c$. If θ'_j offers b_δ then his payoff is $F(A_\delta)(\theta'_j - b_\delta)$. His equilibrium payoff is $F(A)(\theta'_j - b)$ when $A \geq \theta'_j$, and is at most $F(\theta'_j)(\theta'_j - b)$ when $A < \theta'_j$. As $\delta \to 0$, we have $b_\delta \to b$. However, $A_\delta - A \geq c > 0$ and $A_\delta - \theta'_j \geq \min\{b, 1 - \theta'_j\} > 0$, therefore type θ'_j has a strict incentive to offer b_δ instead of b for δ sufficiently close to 0. We conclude that if $b()$ is constant on a non-degenerate interval $[\theta''_j, \theta'_j]$ then it is constant on $[\theta''_j, 1]$ also.

We have established that the equilibrium bribe-function, $b()$, is strictly increasing on an interval $[0, \hat{\theta}]$ and constant on $[\hat{\theta}, 1]$. For any $\theta_j \in [0, \hat{\theta}]$, if j offers bribe $b(\theta_j)$, then it is accepted by types $\theta_i \in [0, \theta_j + b(\theta_j)]$. His equilibrium payoff is

$$\pi^e(\theta_j) = \pi(\theta_j, b(\theta_j)) = F(\theta_j + b(\theta_j))(\theta_j - b(\theta_j))$$

[14]Note that $A = 1$ would require $\theta'_j = 1$, otherwise any $\theta_j > \theta'_j$ should not offer any bribe greater than b. Therefore $A < 1$ also.

Furthermore, the Envelope Theorem implies

$$\frac{d}{d\theta_j}\pi(\theta_j, b(\theta_j)) = F(\theta_j + b(\theta_j))$$

Therefore,

$$F'(\theta_j + b(\theta_j))(1 + b'(\theta_j))(\theta_j - b(\theta_j)) = F(\theta_j + b(\theta_j))b'(\theta_j).$$

Therefore $b'()$ is defined by eqn. (7).

Finally, we show that $\hat{\theta} = \bar{\theta}$ where $\bar{\theta}$ is defined to be the lowest type such that $\beta(\theta_j) + \theta_j = 1$. Suppose that $\hat{\theta} < \bar{\theta}$. For $\delta \geq 0$ denote the set of types that accept $b(\hat{\theta} - \delta)$ by $[0, a_\delta]$. Notice that a_δ is discontinuous at $\delta = 0$ because $\hat{\theta} + b(\hat{\theta}) < 1$. Therefore, for sufficiently small $\delta > 0$, type $\hat{\theta} - \delta$ has a strict incentive to offer $b(\hat{\theta})$, which is a contradiction. On the other hand, if $\hat{\theta} > \bar{\theta}$ then type $\bar{\theta} + \varepsilon$ could strictly gain by offering $b(\bar{\theta})$, which is accepted by all types $\theta_i \in [0, 1]$, also a contradiction. □

Proof of Lemma 1: For θ_j such that $\theta_j - \hat{b} > \pi^e(\theta_j)$, we must have $D(\theta_j) < 1$ by definition. By (8) and the continuity of $\pi^e()$, $D()$ exists and is continuous in an open ball around θ_j.

Recall that $d\pi^e(\theta_j)/d\theta_j = \max\{F(A(b(\theta_j))), F(\theta_j)\}$. By (8) and the implicit function theorem,

$$\frac{dD(\theta_j)}{d\theta_j} = -\frac{\max\{F(D(\theta_j)), F(\theta_j)\} - \max\{F(A(b(\theta_j))), F(\theta_j)\}}{F'(D(\theta_j))\left(\min\{\theta_j, D(\theta_j)\} - \hat{b}\right)}. \tag{17}$$

Assume that $b(\theta_j) < \hat{b}$. We will show that the denominator in (17) is positive and the numerator is either positive (when $\theta_j < D(\theta_j)$), or zero (when $\theta_j \geq D(\theta_j)$).

First, note that $D(\theta_j) > \hat{b}$ because if only types below $\hat{b}$ accept $\hat{b}$ then the profit of the briber will be strictly less when offering $\hat{b}$ than his payoff would be by not bribing, so it cannot equal his equilibrium payoff. Combined with $\theta_j > \hat{b}$, this implies that the denominator in (17) is indeed positive.

If $\theta_j < D(\theta_j)$ then (8) becomes,

$$F(D(\theta_j))(\theta_j - \hat{b}) = F(A(b(\theta_j)))(\theta_j - b(\theta_j)) + \mathbf{1}_{\{\theta_j > A(b(\theta_j))\}} \int_{A(b(\theta_j))}^{\theta_j} (\theta_j - x) dF(x)$$
$$> F(A(b(\theta_j)))(\theta_j - \hat{b}),$$

because $\hat{b} > b(\theta_j)$ and the integral is non-negative. But then $D(\theta_j) > A(b(\theta_j))$ as well, so the numerator in (17) is positive.

If $\theta_j \geq D(\theta_j)$ then there are two cases. If $\theta_j < A(b(\theta_j))$ then (8) becomes

$$F(D(\theta_j))(\theta_j - \hat{b}) + \int_{D(\theta_j)}^{\theta_j} (\theta_j - x) dF(x) = F(A(b(\theta_j)))(\theta_j - b(\theta_j)).$$

But, using $\hat{b} < D(\theta_j)$, the above equality, and $\theta_j < A(b(\theta_j))$,

$$\begin{aligned} F(\theta_j)(\theta_j - \hat{b}) &\equiv F(D(\theta_j))(\theta_j - \hat{b}) + \textstyle\int_{D(\theta_j)}^{\theta_j} (\theta_j - \hat{b}) dF(x) \\ &> F(D(\theta_j))(\theta_j - \hat{b}) + \textstyle\int_{D(\theta_j)}^{\theta_j} (\theta_j - x) dF(x) \\ &= F(A(b(\theta_j)))(\theta_j - b(\theta_j)) \\ &> F(\theta_j)(\theta_j - b(\theta_j)), \end{aligned}$$

which contradicts $b(\theta_j) < \hat{b}$. So we conclude that $\theta_j < A(b(\theta_j))$ is impossible. On the other hand, if $\theta_j \geq A(b(\theta_j))$ then the numerator in (17) is zero, as claimed.

Consider briber type θ_j such that $\theta_j - \hat{b} > \pi^e(\theta_j)$. If $\theta_j < D(\theta_j)$ then, as we have established, $D(\theta'_j) > D(\theta_j)$ for all $\theta'_j < \theta_j$ in the support of $D()$. Since $\hat{b}$ is less than any θ'_j in the support of $D()$, by continuity, $D()$ must equal 1 at the lower endpoint of its support. If $\theta_j > D(\theta_j)$ then $D()$ is constant around θ_j; by continuity, there exists $\theta^*_j < \theta_j$ such that $\theta^*_j = D(\theta^*_j)$. But then, $D(\theta'_j) > D(\theta^*_j)$ for all $\theta'_j < \theta^*_j$, and again, $D()$ equals 1 at the lower endpoint of its support. Finally, $D()$ is well-defined for all $\theta'_j > \theta_j$, including $\theta'_j = 1$. □

Proof of Proposition 7: In a given equilibrium $b(), A()$, Cho and Sobel's D1 criterion requires that after a given deviation of the briber to offering $\hat{b}$, the acceptor must believe that j's type is one of those for which $D()$ as defined in eqn. (8) is minimal.

Suppose $b()$ is discontinuous at $\hat{\theta}_j$, and denote

$$b' = \lim_{\theta_j \uparrow \hat{\theta}_j} b(\theta_j) < \lim_{\theta_j \downarrow \hat{\theta}_j} b(\theta_j) = b''.$$

where the inequality follows from monotonicity.

Step 1 (no pooling to the left of $\hat{\theta}_j$). Suppose towards contradiction that $b^{-1}(b') = \langle \theta'_j, \hat{\theta}_j \rangle$ is a nondegenerate interval. In the arguments below, it is without loss of generality to assume $b(\hat{\theta}_j) = b'$ since by continuity of $\pi^e()$, the equilibrium payoff of $\hat{\theta}_j$ is the same as if he offered b' and pooled with some types lower than his own. Therefore, briber type $\hat{\theta}_j$ would deviate to some $\hat{b} = b' + \varepsilon$, $\varepsilon > 0$ small, provided that $A(\hat{b}) > \hat{\theta}_j + \hat{b}$. Hence $D(\hat{\theta}_j)$ exists with respect to such a deviation $\hat{b}$.

If $\hat{\theta}_j < D(\hat{\theta}_j)$ then, by Lemma 1, $D()$ is strictly decreasing at $\hat{\theta}_j$, and for all $\theta'_j < \hat{\theta}_j$, $D(\theta'_j) > D(\hat{\theta}_j)$. Hence, by D1, the acceptor must believe that an out-of-equilibrium $\hat{b}$ may only be offered by briber types higher than $\hat{\theta}_j$. Therefore, $A(\hat{b}) > \hat{\theta}_j + \hat{b}$, and a briber of type $\hat{\theta}_j$ has an incentive to deviate. If $\hat{\theta}_j \geq D(\hat{\theta}_j)$ then, by Lemma 1, $D()$ is constant around $\hat{\theta}_j$. By continuity, there exists $\theta^*_j \leq \hat{\theta}_j$ such that $\theta^*_j = D(\theta^*_j)$. By Lemma 1, for all $\theta'_j < \theta^*_j$, we have $D(\theta'_j) > D(\theta^*_j)$. According to D1, the acceptor must believe that $\hat{b}$ may only be offered by briber types higher than θ^*_j, hence $A(\hat{b}) > \theta^*_j + \hat{b}$. But then, a briber of type θ^*_j has a strict incentive to deviate to offering $\hat{b}$. We conclude that D1 rules out pooling to the left of any locus of discontinuity of $b()$.

Step 2 (pooling on $(\hat{\theta}_j, 1]$). Suppose towards contradiction that $b()$ is strictly increasing on a nondegenerate interval $(\hat{\theta}_j, \theta'_j)$. Then, for all θ_j in this range, $A(b(\theta_j)) = \theta_j + b(\theta_j)$, implying $\pi^e(\theta_j) = F(\theta_j + b(\theta_j))(\theta_j - b(\theta_j))$. Continuity implies $\pi^e(\hat{\theta}_j) = F(\hat{\theta}_j + b'')(\hat{\theta}_j - b'')$ regardless of whether $b(\hat{\theta}_j) =$

b''.

We first claim that for $\varepsilon > 0$ sufficiently small, briber type $\hat{\theta}_j$ would strictly prefer to deviate to offering $b'' - \varepsilon$ if that act would reveal his type. Recall that the derivative of $\pi^e()$ at $\hat{\theta}_j$ from the right is $d\pi^e(\hat{\theta}_j)/d\theta_j = F(\hat{\theta}_j + b'')$. Also,

$$\lim_{\delta \downarrow 0} \frac{b(\hat{\theta}_j + \delta) - b''}{\delta} = \frac{F'(\hat{\theta}_j + b'')(\hat{\theta}_j - b'')}{F(\hat{\theta}_j + b'') - F'(\hat{\theta}_j + b'')(\hat{\theta}_j - b'')}.$$

That is, $b()$ must satisfy the usual differential equation from the right due to (local) incentive compatibility. Furthermore,

$$0 < F(\hat{\theta}_j + b'') - F'(\hat{\theta}_j + b'')(\hat{\theta}_j - b'') = -\frac{d}{db}\left[F(\hat{\theta}_j + b)(\hat{\theta}_j - b)\right]\Big|_{b=b''},$$

where the inequality follows from the strict monotonicity of $b()$. Therefore, for $\varepsilon > 0$ sufficiently small, briber type $\hat{\theta}_j$ strictly prefers to deviate to $b'' - \varepsilon$ if this act reveals his type:

$$F(\hat{\theta}_j + b'' - \varepsilon)(\hat{\theta}_j - b'' + \varepsilon) > F(\hat{\theta}_j + b'')(\hat{\theta}_j - b'') = \pi^e(\hat{\theta}_j).$$

We complete the proof of Step 2 by arguing that according to D1, i must believe that j's type is at least $\hat{\theta}_j$ when a bribe $\hat{b} \in (b', b'')$ is offered. This is so because by continuity, $\pi^e(\hat{\theta}_j) = F(\hat{\theta}_j + b')(\hat{\theta}_j - b')$. Since $\hat{b} > b'$ and $F(D(\hat{\theta}_j))(\hat{\theta}_j - \hat{b}) \geq \pi^e(\hat{\theta}_j)$, we must have $D(\hat{\theta}_j) \geq \hat{\theta}_j + b' > \hat{\theta}_j$. Applying Lemma 1, we conclude that $D()$ is strictly decreasing at $\hat{\theta}_j$, so all $\theta_j < \hat{\theta}_j$ will have $D(\theta_j) > D(\hat{\theta}_j)$.

Step 3. We claim that $A(b'') = 1$. Suppose, to the contrary, $A(b'') < 1$. Then, briber type $\theta_j = 1$ would strictly prefer offering $\hat{b} = b'' + \varepsilon$, for $\varepsilon > 0$ sufficiently small, if it is accepted by (nearly) all acceptor types, hence $D(1) < 1$. By Lemma 1, $D()$ is constant around (to the left of) $\theta_j = 1$. By continuity, we can find $\theta_j^* < 1$ such that $\theta_j^* = D(\theta_j^*)$. As a consequence of D1, the acceptor must believe that the briber's type is greater than θ_j^* when

$\hat{b}$ is offered, hence $A(\hat{b}) \geq \theta_j^* + \hat{b}$. But this is a contradiction: briber type θ_j^* will strictly prefer to deviate to offering $\hat{b}$.

Converse. Finally, we show that any bribing strategy such as in Proposition 7 satisfies D1. Let $b' = \lim_{\theta_j \uparrow \hat{\theta}_j} b(\theta_j)$ and $b'' = \{b(\theta_j) : \theta_j > \hat{\theta}_j\}$. Player i's best response to j's equilibrium actions is obvious: $A(b) = b^{-1}(b) + b$ for all $b < b'$, and $A(b'') = 1$. We will now show that for all $\hat{b} \in (b', b'')$, D1 restricts i's beliefs to be concentrated on $\theta_j = \hat{\theta}_j$, hence $A(\hat{b}) = \hat{\theta}_j + \hat{b}$. This response makes it unprofitable for any briber type to deviate, so the proposed strategies indeed form an equilibrium.

By continuity of $\pi^e()$, $\hat{\theta}_j - b'' = F(\hat{\theta}_j + b')(\hat{\theta}_j - b')$. Therefore, for all $\hat{b} \in (b', b'')$, briber type would deviate to offering $\hat{b}$ provided sufficiently many acceptors accepted it, $D(\hat{\theta}_j) < 1$. Note that by $0 < b' < \hat{b}$,

$$F(\theta_j + b')(\hat{\theta}_j - b') > F(\hat{\theta}_j)(\hat{\theta}_j - \hat{b}),$$

hence $\theta_j < D(\hat{\theta}_j)$. By Lemma 1, for all $\theta_j < \hat{\theta}_j$, $D(\theta_j) > D(\hat{\theta}_j)$. On the other hand, for $\theta_j > \hat{\theta}_j$,

$$F(D(\theta_j))(\theta_j - \hat{b}) = \theta_j - b''.$$

Since $(\theta_j - b'')/(\theta_j - \hat{b})$ is increasing in θ_j (by $\hat{b} < b''$), we conclude that $D(\theta_j)$ is increasing for $\theta_j > \hat{\theta}_j$. Therefore, $D(\theta_j)$ is minimal at $\theta_j = \hat{\theta}_j$. By D1, following a deviation to $\hat{b}$, the acceptor must believe that $\theta_j = \hat{\theta}_j = \arg\min_{\theta'_j}\{D(\theta'_j)\}$. This completes the proof. □

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