Supplement to "Bribing and Signalling in Second Price Auctions"
by P. Eső and J. Schummer (May 2003)

In this supplement, we consider three modifications to our model of collusion, which were suggested by an Associate Editor, and are mentioned briefly in the conclusion of the paper. These modifications of the model are:

1. Alternative bribing contracts (e.g. simultaneous offers).
2. Adding a reserve price to the second price auction.
3. Imperfect bribing transfers (e.g. burning money instead of bribing with costly bidding).

Modifications to the asymmetric, take-it-or-leave-it bribing contract are not in the spirit of our agenda, which is to describe a "simplest" type of collusion and ask whether mechanisms are immune to this minimal style of manipulation. Nevertheless, it is interesting to investigate some of these modifications and see whether properties of our bribing equilibrium generalize. In particular, an important question is whether our main results—the existence of a unique collusive equilibrium and, to a lesser extent, the inefficiency caused by collusion—are robust to simple changes in the rules of bribing. We argue here that these results are indeed robust.

The results in this document are preliminary, and not intended for publication. We gladly invite others to build on these results.

1. Alternatives to our bribing contract

One obvious candidate for an alternative bribing contract (with exogenous bribes) is one where *both* players can simultaneously offer the same fixed bribe to the other player in exchange for the other's commitment to stay away from the auction. Here, there is an important modelling difficulty (which did not arise in our asymmetric setup): the rules of the game must specify what happens if both players decide to offer the bribe (a bribing "tie"). Clearly, one of the two offers must be somehow invalidated.

The easy way out for the modeler is to assume that such "ties" are broken so that the bribe offered by the player with the higher type will prevail. This tie-breaking rule is *deus ex machina*, but it makes finding the unique robust symmetric equilibrium straightforward. In the equilibrium, both players will offer the bribe if and only if their type exceeds a certain threshold, θ^*, such that $b = E[\theta_i \mathbf{1}_{\theta_i \leq \theta^*}]$. Clearly, if the other player uses this strategy, type θ^* is indifferent between bribing and not (he wins against the same types of the other player and he either pays b up front or in expectation). It is easy to check that the no-bribing equilibrium is not robust to usual refinements, and the above equilibrium is the unique symmetric equilibrium where bribing occurs. (First, one can prove that the set of types that bribe is an interval; second, the type at the lower endpoint of this interval is indifferent between bribing and not bribing; third, this type, θ^*, is the unique solution to $b = E[\theta_i \mathbf{1}_{\theta_i \leq \theta^*}]$.) In this model, contrary to ours, collusion is efficient, but note that this is the result of the artificially efficient tie-breaking rule.

A more attractive tie-breaking rule would be to randomly invalidate one of the offers. From now on, we will call this version *the model with simultaneous bribe offers*. Proposition 1 from the paper (the interval-property of any equilibrium) immediately generalizes: in any equilibrium, of the extended game, the set of types that offer the bribe is of the form $\langle B, 1]$, and the set of types that accept the bribe is of the form $[0, A\rangle$; furthermore, $B < 1$ implies $b < B < A \leq 1$. We omit the proof as it is the same as that of Proposition 1.

More interestingly, the counterpart to Proposition 2 is the following.

Proposition 2a *In the model with simultaneous bribe offers (and random tie-break), for any $b \in (0, \mathbf{E}(\theta_i)]$, there exists a sequential equilibrium in which bribing occurs. Moreover, all equilibria in which a bribe is offered with positive probability are essentially equivalent: there exist A^b, B^b such that in any equilibrium where bribing occurs, the sets of bribers and acceptors are $\langle B^b, 1]$ and $[0, A^b\rangle$, respectively.*

Proof: The condition determining the lowest type that offers the bribe, B, is

$$\frac{1}{2}\left(F(A) + F(B)\right)(B - b) + \frac{1}{2}(1 - F(B))b = (1 - F(B))b + \int_0^B (B - x)dF(x).$$

It expresses that type B is indifferent between offering the bribe (left-hand side) and not (right-hand side). The condition determining A is formally the same as the one in the paper,

$$b \geq \frac{\int_B^A (A - x)dF(x)}{1 - F(B)},$$

which holds as equality if $A < 1$. Like in the proof of Proposition 2 of the paper, but now based on these two conditions, define

$$b_1(A,B) = \frac{(F(A)+F(B))\,B - 2\int_0^B (B-x)dF(x)}{1+F(A)},$$
$$b_2(A,B) = \frac{\int_B^A (A-x)dF(x)}{1-F(B)}.$$

Note that

$$\frac{\partial b_1}{\partial A} = \frac{F'(A)\left[(1+F(A))\,B - (F(A)+F(B))\,B + 2\int_0^B (B-x)dF(x)\right]}{(1+F(A))^2} > 0,$$

and clearly, $\partial b_1/\partial B > 0$. In the paper we show $\partial b_2/\partial A > 0$ and $\partial b_2/\partial B < 0$.

There exists a bribing equilibrium for all $b \in [0, \mathbf{E}(\theta)]$ if and only if there exist $A \geq B$ such that $b = b_1(A,B) \geq b_2(A,B)$, with equality if $A < 1$. Existence can be shown by the method used in the proof of Proposition 2. Define

$$H = \{(A,B) : A \geq B,\ b_1(A,B) \geq b_2(A,B),\ \text{with "=" if } A < 1\}.$$

For all $B > 0$, there exists $A \geq B$ such that $(A,B) \in H$, because $b_1(B,B) > 0 = b_2(B,B)$. Moreover, for all $A < 1$, there exists a *unique* B with $(A,B) \in H$: this follows because $b_1(A,0) = 0 < b_2(A,0)$, $b_1(A,A) > b_2(A,A)$, and $b_1 - b_2$ is strictly decreasing in B. Finally, let $h(A) = \{B : (A,B) \in H\}$ be the (upper hemi-continuous) path from $(0,0)$ to $(1,1)$ defined by H. We claim that b_1 is strictly increasing along h: for $A' > A$, either $h(A') > h(A)$ and hence $b_1(A', h(A')) > b_1(A, h(A))$, or $h(A') < h(A)$ but then $b_1(A', h(A')) \geq b_2(A', h(A')) > b_2(A, h(A)) = b_1(A, h(A))$. Noting $b_1(0,0) = 0$, $b_1(1,1) = \mathbf{E}(\theta_j)$ completes the proof. ■

The bribing equilibrium in this model is inefficient just like it is in the model presented in the paper. Therefore, the inefficiency is *not* the result of the players' asymmetry with respect to being able to offer the bribe. Instead, it is the consequence of the fact that if types in $\langle B, 1]$ offer the bribe in equilibrium, then, when a bribe goes through, the receiver with type below $B + b$ has a strict incentive to accept the bribe. So, in some cases, a higher valuation buyer accepts a bribe from a lower valuation buyer. We also note that the same thing would happen in the variable bribes model with simultaneous offers as well: in the continuous equilibrium of this bribing game (we assert existence), high types are pooling on the highest bribe, which will lead to an inefficient outcome with positive probability.

Another variant on the rules of the bribing phase could be such that first, player 1 decides whether to bribe player 2, then, if he decided not to bribe or player 2 rejected the offer, player 2 could counter-bribe. (Offers and counteroffers could even be repeated finitely many times before the auction starts.) We conjecture that these variants of the bribing game have a collusive equilibrium as well. The analysis of these variants is more complicated only because there are several subgames and corresponding acceptance thresholds (A's) to keep track of, and, in the end, a solution to several equations in several unknowns would have to be found (unlike in the models considered above and in the paper, where we had two equations for A and B). In Section 3 of this Supplement we explore a similar, slightly more complicated model (burning money in an auction model with entry costs), where the nature of the problem can be seen very well.

2. Reservation price in the continuation auction

Propositions 1 and 2 can be generalized in the fixed-bribes model when there is a positive reserve price, r, in the continuation second-price auction. The proofs do not change substantially, but we include them for completeness.

Proposition 1b *Suppose there is a reserve price in the SPA. In any sequential equilibrium, the set of types that offer a bribe is of the form* $\mathbb{B} = \langle B, 1]$ *and the set of types that accept the bribe is of the form* $\mathbb{A} = [0, A\rangle$*; furthermore* $B < 1$ *implies* $b < B < A \leq 1$*.*

Proof: For a given equilibrium, denote the set of types that offer the bribe as $\mathbb{B}$, and the set of types that accept the bribe as $\mathbb{A}$. When $\mathbb{B}$ is non-empty, if player i accepts the bribe then it must exceed the profit he would get in the auction, given that $\theta_j \in \mathbb{B}$. In other words, if $\theta_i \in \mathbb{A}$ then

$$b \geq \mathbf{1}_{\{r \leq \theta_i\}} \mathbf{E}_{\theta_j}[(\theta_i - \max\{\theta_j, r\}) \mathbf{1}_{\{\theta_j \leq \theta_i\}} \mid \theta_j \in \mathbb{B}] \tag{1}$$

where $\mathbf{1}_X$ is the indicator function for event X. If this inequality holds for some θ_i then it holds for any $\theta_i' < \theta_i$. Therefore $\mathbb{A} = [0, A\rangle$. If $\mathbb{B}$ is empty then a similar argument (in which the posterior based on F is replaced by the out-of-equilibrium beliefs) shows that for any beliefs supporting the sequential equilibrium, $\mathbb{A}$ must be an interval.

To show that $\mathbb{B}$ is also an interval, define $B = \inf \mathbb{B}$. If $B = 1$ then we are done. Otherwise, since i can infer $\theta_j \geq B$ from the fact that the bribe was offered, he has an incentive to accept the bribe if his type is less than $B + b$. This follows because i's profit in the second-price auction is at most $\theta_i - B \leq b$. Therefore $A \geq \min\{1, B + b\} > B$.

For any $\theta_j \in \mathbb{B}$, the payoff from offering the bribe must be at least as great as his unconditional payoff in the second-price auction, that is (since clearly $B > r$),

$$F(A)(\theta_j - b - r) + \mathbf{E}_{\theta_i}[(\theta_j - \theta_i)\mathbf{1}_{(A<\theta_i\leq\theta_j)}] \geq \mathbf{E}_{\theta_i}[(\theta_j - \max\{\theta_i, r\})\mathbf{1}_{(\theta_i\leq\theta_j)}]. \quad (2)$$

Differentiating both the left and right hand sides,

$$\frac{\partial LHS(\theta_j)}{\partial\theta_j} = \max\{F(A), F(\theta_j)\} \geq F(\theta_j) = \frac{\partial RHS(\theta_j)}{\partial\theta_j}.$$

When $\theta_j < A$, the left hand side increases in θ_j strictly faster than the right hand side does. Therefore, for any $\theta_j \in \mathbb{B}$ for which $B \leq \theta_j < A$, and any $\theta'_j > \theta_j$,

$$F(A)(\theta'_j - b) + \mathbf{E}_{\theta_i}[(\theta'_j - \theta_i)\mathbf{1}_{(A<\theta_i\leq\theta'_j)}] > \mathbf{E}_{\theta_i}[(\theta'_j - \theta_i)\mathbf{1}_{(\theta_i\leq\theta'_j)}]. \quad (3)$$

This implies $\theta'_j \in \mathbb{B}$, and therefore $\mathbb{B}$ is of the form $\langle B, 1]$. Furthermore, eqn. (3) cannot hold at $\theta_j = b$, hence $B > b$. ■

Proposition 2b *Suppose there is a reserve price in the SPA. For any $b \in (0, \mathbf{E}(\theta_i)]$, there exists a sequential equilibrium in which bribing occurs. Moreover, all equilibria in which a bribe is offered with positive probability are essentially equivalent: there exist A^b, B^b such that in any equilibrium where bribing occurs, the sets of bribers and acceptors are $\langle B^b, 1]$ and $[0, A^b\rangle$, respectively.*

Proof: For any $b \in (0, \mathbf{E}(\max\{\theta_i, r\}) - r]$, Proposition 1b implies that in any bribing equilibrium, briber and acceptor types are of the form $\langle B, 1]$ and $[0, A\rangle$. Since $B \geq b$, standard continuity arguments imply that type $\theta_j = B$ must be indifferent between offering the bribe and not, i.e., by eqn. (2) and $B \le A$,

$$F(A)(B - b - r) = \int_0^B (B - \max\{x, r\})\, dF(x). \tag{4}$$

Note that this holds even if $B = 1$ since bribing is occurring by assumption.

Also, either $A = 1$, or type $\theta_i = A$ is indifferent between accepting the bribe and not. By eqn. (1) (since $B \ge r$),

$$b \ge \frac{\int_B^A (A - x)\, dF(x)}{1 - F(B)}, \tag{5}$$

and if $A < 1$ then eqn. (5) holds with equality. (From Proposition 1b, if $B = 1$ then $A = 1$ and eqn. (5) becomes $b \ge 0$; if $B < 1$ then $A > B$ and the right-hand side is positive.)

It is helpful to define the following functions for $A, B \in [0, 1]$, $A \ge B$.

$$b_1(A, B) = B - \frac{\int_0^B (B - \max\{x, r\})\, dF(x)}{F(A)} - r,$$
$$b_2(A, B) = \frac{\int_B^A (A - x)\, dF(x)}{1 - F(B)}.$$

Observe that since $B > 0$, $b_1(A, B) < B$.

To prove the existence of an equilibrium, we need to find A, B such that eqns. (4) and (5) hold, that is, $b = b_1(A, B) \ge b_2(A, B)$, with equality if $A < 1$.

Define

$$H = \{(A, B) : A \ge B \ge r,\ b_1(A, B) - b_2(A, B) \ge 0, \text{ with equality if } A < 1\}.$$

We claim that (i) for all B, there exists A such that $(A, B) \in H$, and (ii) for all $A < 1$, there exists a unique B such that $(A, B) \in H$.

To see (i), first note that $(r, r) \in H$. For $B > r$, $b_1(B, B) - b_2(B, B) > 0$ so either $(1, B) \in H$ or by continuity, $(A, B) \in H$ for some $A \in (B, 1)$. To see (ii) for $A \in (r, 1)$, note that $b_1(A, r) - b_2(A, r) = 0 - [\int_r^A (A - x)\, dF(x)]/(1 - F(B)) < 0$, while $b_1(A, A) - b_2(A, A) > 0$. Continuity implies that $b_1(A, B) - b_2(A, B) = 0$ for some $B \in (r, A)$, hence $(A, B) \in H$. Furthermore, this B is unique because $b_1(A, B) - b_2(A, B)$ is strictly decreasing in B (see eqns. (7) and (9) below).

Define the correspondence $h\colon [0,1] \twoheadrightarrow [0,1]$ such that $h(A) = \{B : (A,B) \in H\}$. By (ii), h is non-empty and if $A < 1$ then h is single-valued. It can be shown (e.g., by an application of the Maximum Theorem) that h is upper hemi-continuous. Therefore, for $r \geq A < 1$, $h(A)$ is a continuous function, and its graph, H, is connected. Define $\hat{B} = \lim_{A\uparrow 1} h(A) \in h(1)$.

By differentiating b_1 and b_2, we find that for $r \leq B < A \leq 1$,

$$\frac{\partial b_1}{\partial A} = \frac{F'(A)}{F(A)^2}\int_0^B (B - \max\{x,r\})\,dF(x) > 0, \tag{6}$$

$$\frac{\partial b_1}{\partial B} = 1 - \frac{F(B)}{F(A)} > 0, \tag{7}$$

$$\frac{\partial b_2}{\partial A} = \frac{F(A) - F(B)}{1 - F(B)} > 0, \tag{8}$$

$$\frac{\partial b_2}{\partial B} = \frac{\left\{\int_B^A (A-x)\,dF(x) - (A-B)(1-F(B))\right\} F'(B)}{(1-F(B))^2} < 0 \tag{9}$$

where the last inequality follows by $\int_B^A (A-x)\,dF(x) < \int_B^A (A-B)\,dF(x) = (A - B)(F(A) - F(B)) \leq (A-B)(1-F(B))$.

Consider any $A' > A$. If $h(A') > h(A)$ then by eqns. (6) and (7) we have $b_1(A', h(A')) > b_1(A, h(A))$. If $h(A') < h(A)$ then by eqns. (8) and (9) we have $b_2(A', h(A')) > b_2(A, h(A))$, which implies $b_1(A', h(A')) \geq b_2(A', h(A')) > b_2(A, h(A)) = b_1(A, h(A))$. Therefore, $b_1(A, h(A))$ is strictly increasing in A for $A \in [r, 1)$.

By continuity, $b_1(1, \hat{B}) = b_2(1, \hat{B})$. Therefore, eqns. (7) and (9) imply that $h(1) = [\hat{B}, 1]$, and $b_1(1, B)$ is strictly increasing in B.

Therefore, $b_1(A, B)$ is strictly increasing on H in the sense that if $A < 1$ then $b_1(A, B)$ is strictly increasing in A, and if $A = 1$ then $b_1(A, B)$ is strictly increasing in B. Since $b_1(r, h(r)) = 0$ (by $h(r) = r$) and $b_1(1,1) = \mathbf{E}(\max\{\theta_i, r\}) - r$, the strict monotonicity and continuity of $b \equiv b_1(A, B)$ along H implies that there exists a one-to-one mapping between $b \in [0, \mathbf{E}(\theta_i)]$ and $(A, B) \in H$, i.e. for all $b \in [0, \mathbf{E}(\max\{\theta_i, r\}) - r]$, there exist unique A, B solving eqns. (4) and (5). ■

For the variable bribes model, the result is as follows.

Proposition 5b *Suppose F is log concave. In any sequential equilibrium in which bribing occurs, if j's bribing strategy function $b()$ is continuous, then it is the unique* *solution to the following equation satisfying* $b(r) = 0$*.*

$$b'(\theta_j) = \begin{cases} \dfrac{F'(\theta_j + b(\theta_j))(\theta_j - b(\theta_j)) - r}{F(\theta_j + b(\theta_j)) - F'(\theta_j + b(\theta_j))(\theta_j - b(\theta_j)) - r} & \text{if } \theta_j + b(\theta_j) < 1 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Conversely, there exists a sequential equilibrium in which j*'s (continuous) strategy* $b()$ *is described by eqn. (10), with* $b(r) = 0$*.*

We omit the proof, since it is essentially identical to the one without a reserve price.

3. Alternative setup: burning money in a model with small entry fees

Let $\theta = \theta_i, \theta_j$ be two buyers' valuations distributed independently on $[\underline{\theta}, \bar{\theta}]$ according to cdf F. The two buyers participate in a second price auction with an entry fee $c \in (0, \bar{\theta})$. Assume that buyer j has a prior move: he may publicly burn b units of money before both simultaneously decide whether to participate.

We will look for a collusive equilibrium of this game where types $\theta_j \in \langle B, \bar{\theta}]$ of buyer j burn money. There are two subgames to consider.

Subgame B: buyer j burns money ($\theta_j \in \langle B, \bar{\theta}]$). Then he obviously participates in the SPA, and so will buyer i with type $\theta_i \in \langle A, 1]$ such that

$$\frac{\int_B^A (A - x) dF(x)}{1 - F(B)} - c \leq 0, \tag{11}$$

where (11) holds as equality when $A < 1$. This equation is reminiscent from the conditions characterizing the bribing equilibrium in our paper (c is in the role of b), but the similarities end there.

Subgame NB: buyer j does not burn money. Then $\theta_j \leq B$ is deduced, and the buyers play a SPA with entry fee c and *asymmetric* type-distributions, $\theta_i \in [\underline{\theta}, \bar{\theta}]$ with cdf F, and $\theta_j \in [\underline{\theta}, B\rangle$ with cdf $F/F(B)$.

Claim: There is an equilibrium of subgame NB where both buyers bid their valuation conditional on participating; i participates with $\theta_i \in \langle L_i, \bar{\theta}]$ and j with

$\theta_j \in \langle L_j, B \rangle$, where

$$\frac{F(L_j)}{F(B)} L_i + \mathbf{1}_{\{L_i > L_j\}} \int_{L_j}^{L_i} (L_i - x) \frac{dF(x)}{F(B)} = c, \tag{12}$$

$$F(L_i) L_j + \mathbf{1}_{\{L_j > L_i\}} \int_{L_i}^{L_j} (L_j - x) dF(x) = c. \tag{13}$$

Proof. Denote $F_i = F$ and $F_j = F/F(B)$. In the proposed equilibrium, if type θ_i bids b_i, then his payoff is

$$U_i(\theta_i, b_i) = F_j(L_j)\theta_i - c + \mathbf{1}_{\{b_i > L_j\}} \textstyle\int_{L_j}^{b_i} (\theta_i - x) dF_j(x). \tag{14}$$

First, suppose $\theta_i \geq L_i$. Clearly, if he bids at all, it is optimal for him to bid $b_i = \theta_i$. (If $\theta_i > L_j$ then the the last term in (14) is maximized by setting $b_i = \theta_i$; if $\theta_i \leq L_j$ then the same term is negative for all $b_i > L_j$, and zero for all $b_i \leq L_j$, so $b_i = \theta_i < L_j$ is again optimal.) He is better off bidding than not bidding since $U_i(\theta_i, \theta_i)$ is strictly increasing in θ_i and $U_i(L_i, L_i) = 0$. Therefore, $\theta_i \geq L_i$ has no incentive to deviate.

Second, suppose $\theta_i < L_i$. If he bids $b_i \leq L_j$ then

$$U_i(\theta_i, b_i) = F_j(L_j)\theta_i - c < F_j(L_j)L_i - c \leq 0.$$

If he bids $b_i > L_j$ then there are two cases. If $\theta_i < L_j$ then $\int_{L_j}^{b_i} (\theta_i - x) dF_j(x) < 0$, and

$$U_i(\theta_i, b_i) < F_j(L_j)\theta_i - c < F_j(L_j)L_i - c \leq 0.$$

If $\theta_i \geq L_j$ (which implies $L_i > L_j$) then

$$\textstyle\int_{L_j}^{b_i} (\theta_i - x) dF_j(x) \leq \int_{L_j}^{\theta_i} (\theta_i - x) dF_j(x) < \int_{L_j}^{L_i} (L_i - x) dF_j(x),$$

and so $U_i(\theta_i, b_i) < F_j(L_j)L_i - c + \int_{L_j}^{L_i} (L_i - x) dF_j(x) = 0$. Therefore, in any case, $\theta_i < L_i$ is worse off bidding than not bidding, and has no reason to deviate.$\blacksquare$

Now that we know exactly what happens in either of the two possible subgames after j's initial move, we can complete the characterization of the equilibrium by using that type $\theta_j = B$ of buyer j must be indifferent between entering either subgame. An additional condition is that this type should make non-negative surplus.

If type $\theta_j = B$ burns b then he will go to the auction and get $F(A)B - b - c$. (Note that by equation 11, $B < A$.) If he doesn't burn b and $B < L_j$ then he is not going to the auction, and will get overall zero payoff. If $B > L_j$ then his payoff will be $F(L_i)B + \int_{L_i}^{B}(B - x)dF(x) - c$. Indifference requires

$$F(A)B - b - c = \mathbf{1}_{\{B>L_i\}} \left(F(L_i)B + \int_{L_i}^{B} (B - x)dF(x) - c \right) \geq 0. \qquad (15)$$

Given b and c, we have four equations, (11)-(13), (15), in four unknowns, A, B, L_i, L_j.

We conjecture that a solution exists for a positive Lebesgue-measure set of parameters $(b, c) \in [0, E(\theta)]^2$. This conjecture is based on observations made in special cases. For example, direct calculation reveals that a solution exists when F is uniform, $b = .4$ and $c = .2$; in the equilibrium, $B = .6$, $A = 1$, $L_i = .2$, and $L_j = .6$. We can say that the equations are compatible under certain distributions for some parameter values.

Uniform case. When F is uniform, we can actually prove that a collusive equilibrium exists in a positive Lebesgue-measure set of parameter values.

Under uniform F, it is easy to see that $L_i < L_j$, and the four conditions become

$$\begin{aligned} \frac{1}{2}(A - B)^2 &\leq (1 - B)c, \\ L_iL_j &= cB, \\ L_iL_j + \frac{1}{2}(L_j - L_i)^2 &= c, \\ AB - b &= L_iB + \frac{1}{2}(B - L_i)^2. \end{aligned}$$

There is a continuum of solutions to these equations (the first condition holding as an equation as well), where

$$\begin{gathered} b = (1 - 2c)\left(1 + c - \sqrt{c - c^2}\right) - \tfrac{1}{2}\left(1 - c - \sqrt{c - c^2}\right)^2, \\ A = 1,\ B = 1 - 2c, \\ L_i = -c + \sqrt{c - c^2},\ L_j = c + \sqrt{c - c^2}. \end{gathered}$$

(To see this, plug $A = 1$ into the four equations. The first equation yields $B = 1-2c$, then the second and third equations give $L_i = -c + \sqrt{c - c^2}$, $L_j = c + \sqrt{c - c^2}$, and the fourth equation gives the formula for b.)

For c sufficiently small, the resulting values satisfy $b > 0$, $L_j \leq B$, and $B - b - c > 0$ (the last condition means that type $\theta_j = B$ prefers to participate), and so the solution forms an equilibrium. In the neighborhood such (b, c) pairs, by continuity (and some hand-waving), there must exist solutions where $A < 1$ (but close to 1). Therefore, we find an open set of (b, c) values that are compatible with a solution to the four equations.