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# Convergence to perfect competition of a dynamic matching and bargaining market with two-sided incomplete information and exogenous exit rate

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**Abstract**

Consider a decentralized, dynamic market with an infinite horizon and incomplete information in which buyers and sellers' values for the traded good are private and independently drawn. Time is discrete, each period has length $\delta$, and each unit of time a large number of new buyers and sellers enter the market. Within a period each buyer is matched with a seller and each seller is matched with zero, one, or more buyers. Every seller runs a first price auction with a reservation price and, if trade occurs, the seller and winning buyer exit with their realized utility. Traders who fail to trade either continue in the market to be rematched or exit at an exogenous rate. We show that in all steady state, perfect Bayesian equilibria, as $\delta$ approaches zero, equilibrium prices converge to the Walrasian price and realized allocations converge to the competitive allocation.



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## 1. Introduction

Asymmetric information and strategic behavior interfere with efficient trade. Nevertheless economists have long believed that for private goods' economies the presence of many traders

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overcomes both these imperfections and results in convergence to perfect competition. This paper contributes to a growing literature that shows the robust ability of simple market mechanisms to elicit cost and value information from buyers and sellers even as at it uses the information to allocate the available supply almost efficiently. In particular, for a two-sided incomplete information environment with independent private values, we show how a completely decentralized matching and bargaining market converges to a competitive outcome as each trader's ability to contact a sequence of possible trading partners increases.

Thus a market that for each trader is big over time—as opposed to big at a moment in time—overcomes the difficulties of asymmetric information and strategic behavior. This is a step towards a full understanding of why price theory with its assumptions of complete information and price-taking works as well as it does even in markets where the validity of neither of these assumptions is self-evident.

A description of our model and result is this. An indivisible good is traded in a market in which time progresses in discrete periods of length $\delta$ and generations of traders overlap. The parameter $\delta$ is the exogenous friction in our model that we take to zero. Every active buyer is randomly matched with an active seller each period. Depending on the luck of the draw, a seller may end up being matched with several buyers, a single buyer, or even no buyers. Each seller solicits a bid from each buyer with whom she is matched. If the highest of the bids is satisfactory to her, she sells her single unit of the good and both she and the successful buyer exit the market. A buyer or seller who fails to trade remains in the market and seeks a new match the next period unless for exogenous reasons he or she elects to exit the market without trading.

Each unit of time a large number of potential sellers (formally, measure 1) enters the market along with a large number of potential buyers (formally, measure $a$). Each potential seller independently draws a cost $c$ in the unit interval from a distribution $G_S$ and each potential buyer draws independently a value $v$ in the unit interval from a distribution $G_B$. Individuals' costs and values are private to them. A potential trader only enters the market if, conditional on his private cost or value, his equilibrium expected utility is positive. Potential traders who have zero probability of profitable trade in equilibrium elect not to participate.

If trade occurs between a buyer and seller at price $p$, then they exit with utilities $v - p$ and $p - c$ respectively that they discount back at rate $r$ to their times of entry. As in McAfee (1993), unsuccessful active traders face a risk of exiting whose source is exogenous. Specifically, each period each unsuccessful trader exits with probability $1 - \mathrm{e}^{-\delta\mu}$ where $\mu$ is the exit rate per unit of time. If $\delta$ is large (i.e., periods are long), then a trader who enters the market is impatient, seeking to consummate a trade and realize positive utility amongst the first few matches he realizes. If, however, $\delta$ is small (i.e., periods are short), then a trader can patiently wait through many matches looking for a good price with little concern about exiting with no gain.

Buyers with higher values find it worthwhile to submit higher bids than buyers with lower values. At the extreme, a buyer with a value 0.1 will certainly not submit a bid greater than 0.1 while a buyer with a value 0.95 certainly might. The same logic applies to sellers: low cost sellers are willing to accept lower bids than are higher cost sellers. This means high value buyers and low cost sellers tend quickly to realize a match that results in trade and exit. Low value buyers and high cost sellers may take a much longer time on average to trade and are likely to exit without trading. Consequently, among the buyers and sellers who are active in the market in a given period, low value buyers and high cost sellers may be overrepresented relative to the entering distributions $G_B$ and $G_S$.

We characterize equilibria for the steady state of this market and show that, as the period length goes to zero, all equilibria of the market converge to the Walrasian price and the competitive allocation. The Walrasian price $p_W$ in this market is the solution to the equation

$$G_S(p_W) = a\big(1 - G_B(p_W)\big), \tag{1}$$

i.e., it is the price at which the measure of entering sellers with costs less than $p_W$ equals the measure of entering buyers with values greater than $p_W$. If the market were completely centralized with every active buyer and seller participating in an enormous exchange that cleared each period's bids and offers simultaneously, then $p_W$ would be the market clearing price each period. Our result, carefully stated, is as follows. Given a $\delta > 0$, then each equilibria induces a trading range $[\underline{p}_\delta, \bar{p}_\delta]$ that simultaneously is the range of offers that sellers of different types make, the range of bids that buyers make, and the range of prices at which trades are actually transacted. We show that $\lim_{\delta\to 0} \underline{p}_\delta = \lim_{\delta\to 0} \bar{p}_\delta = p_W$, i.e., the trading range converges to the competitive price. That the resulting allocations give traders the expected utility they would realize in a perfectly competitive market follows directly.

The intuition why the trading range converges to a single price is easily stated. As the time period $\delta$ shrinks towards zero each trader expects to match an increasing number of times prior to exiting at a random time as a consequence of the exogenous exit rate. The result is a strong option value effect for every trader. Even if a buyer has a high value, he has an increasing incentive as $\delta$ decreases to bid low and hold out for an offer near the low end of the offer distribution. Therefore all serious buyers bid within an increasingly narrow range just above the minimum offer any seller makes. A parallel argument applies to sellers, with the net effect being, as $\delta$ becomes small, all bids and offers concentrate within an interval of decreasing length, i.e., the trading range converges to a single price. However, as we discuss below, what price the trading range converges to is not as obvious.

This paper is a companion to Satterthwaite and Shneyerov (2007). Both papers prove convergence of dynamic, decentralized markets with two-sided incomplete information to the Walrasian outcome as frictions are removed by letting time between trading opportunities approach zero. The difference in the two papers' models is that in this paper impatience is the consequence of an exogenous exit rate while in the latter it is the consequence of a participation cost. There are two reasons why the development of these parallel models is worthwhile.

The first reason is to demonstrate the robustness of convergence to the competitive price and allocation for different frictions that may exist within a market. For example, the rise of Internet enabled markets has driven the cost of participation in some—but certainly not all—markets almost to zero. Consequently here we eliminate participation costs as the source of friction and substitute an exogenous exit rate. One justification for this exogenous exit rate is that participating in a market with trivial participation costs still requires the scarce resource of attention. A trader when he decides to enter a market may know there is a significant probability that, if he is unsuccessful at trading quickly, his situation may change, preempt his attention, and force him to exit. He is therefore impatient to consummate the trade because exiting does not indicate that trade would no longer be of value. It only indicates that he can no longer give it attention. This interpretation of the fixed exit rate is consistent with a developing literature on the implications of attention scarcity, e.g., Falkinger (2007). The premise of this literature is that the allocation of attention across multiple projects, responsibilities, and unanticipated stimuli is a critical strategic decision each agent must make each day.

The second reason is that the two paper's models illuminate different aspects of why a matching and bargaining market with incomplete information may converge to perfect competition. It turns out that in both models the same intuition concerning option value applies as to why the distribution of prices at which trades occur converges to a single price. This congruence is absent when it comes to understanding why this limiting price must be the Walrasian price. This paper's strength is that it identifies a general mechanism that forces that price to be the Walrasian price, $p_W$, whenever transaction prices converge to a single price. To understand this, suppose to the contrary that prices converge to some lower price, $p_* < p_W$. If the period length $\delta$ is quite short, then a potential buyer may choose to enter if his value $v$ exceed $p_*$ because of the possibility of buying at or close to that price. Similarly, a potential seller may enter if her cost $c$ is less than $p_*$. With $p_*$ less than $p_W$ more potential buyers enter than potential sellers. This, however, need not prevent the market from clearing; all that is necessary is that buyers on average exit prior to trading more often than do sellers. This, to some extent, occurs automatically because buyers, being present in greater numbers, tend to wait longer than sellers to trade and therefore are more likely to exit due to the exogenous exit rate.

Nevertheless, as the period length approaches zero, we can show that the equilibrium trading range cannot converge to the price $p_* < p_W$. The quite general reasoning is that an active buyer whose value $v$ is relatively low necessarily has a low probability of trading in equilibrium and a high probability of exiting for exogenous reasons. If this were not so, then the market would not be in a steady state. The reason is that the excess buyers $p_*$ attracts into the market must exit exogenously for market clearing to occur. This, however, sets up a contradiction. As $\delta$ becomes small the trading range becomes extremely narrow as it converges to $p_*$. The low value buyer, who has a low probability of trading, can then increase his bid the very small amount that is just sufficient to make trade in the current period certain. Doing so is worthwhile—and breaks the putative equilibrium—because the cost in terms of the increased price he pays is small and is more than offset by the guarantee that trade occurs immediately with no possibility of unexpected exit for exogenous reasons. Thus, given that the trading range is collapsing to a single price, that price cannot be $p_*$, but must be $p_W$.

By contrast, in our participation cost model (Satterthwaite and Shneyerov, 2007) we identified a simpler, less general mechanism that guarantees convergence to $p_W$. In the steady state, traders have stationary strategies because, with a fixed per period participation cost, each period's optimization problem as to whether to be in the market and what price to bid/offer is unchanging. This means that every trader who enters the market stays until he trades; no exit occurs for other, exogenous reasons. Then, if $\delta$ is very small, every trader who enters ultimately gets to trade in a very small band around whatever price $p_*$ to which the trading range is shrinking. This price $p_*$ must be $p_W$ because that is the only price that clears the market. The strength of the participation cost paper (Satterthwaite and Shneyerov, 2007) that this paper does not share is that it proves existence in reasonable generality: provided the discount rate and the period length is sufficiently small, equilibria necessarily exist. In Section 4.4 we report that we have successfully proved existence when a restrictive assumption is made on the sellers' type distribution $G_S$, but that proof is not included here because of its length and lack of generality.

Our two papers build on a substantial literature that investigates the foundations of perfect competition using dynamic matching and bargaining games under full information. The strand of this literature to which our papers are most closely related includes Gale (1987) and Mortensen and Wright (2002).[1] These papers specify an explicit bargaining game between bilaterally

[1] The books of Osborne and Rubinstein (1990) and Gale (2000) contain excellent discussions of both their own and others' contributions to this literature.

matched traders and, conditional on that game, prove that the steady states of their papers' games converge to the Walrasian price and allocation as the market friction vanishes, just as is the case in this paper and our participation cost paper. A second, less closely related strand (Gale, 1986a; McLennan and Sonnenschein, 1991; Dagan et al., 2000) considers equilibria of matching and bargaining games at the limit where the market is frictionless. All three of these papers posit a fixed measure of consumers who begin the game with an endowment of multiple goods and trade each period according to a given protocol in order to improve their consumption bundles. In Gale and in McLennan and Sonnenschein matching is bilateral. In Dagan et al., as in our paper, matching is multilateral, a feature that enables them exploit the equivalence of core and Walrasian allocations in large markets. In all three of these papers each consumer exits (without being replaced) once he achieves a bundle that further trade will not improve. Only then does he realize the utility of the allocation he has obtained through his trades.

The primary difference between this literature and our papers is its complete information assumption: when two traders meet they reciprocally observe each other's cost/value. This—complete versus incomplete information—is fundamental, for our purpose is to determine if a decentralized markets can robustly elicit sufficient private valuation information at the same time it uses that information to assign the available supply almost efficiently.

Butters (1979), Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano (2002) are the most important prior bargaining and matching models that incorporate incomplete information, albeit one-sided in the cases of Wolinsky and of De Fraja and Sákovics.[2] Of these four, only Butters obtains convergence to perfect competition in the limit. Specifically, in an old, incomplete manuscript he analyzes a two-sided incomplete information model that is very similar to the one we study here and makes a great deal of progress towards proving a variant of the convergence theorem that we prove here. In contrast, the models of the other three papers fail to converge robustly to the Walrasian price. The reason for these failures is that the allocation problems those papers model are non-competitive in their fundamentals.

Our reason for asserting this may be seen by generalizing the static double auction model to a dynamic setting. In a double auction there are $m$ privately informed sellers, each of whom seeks to sell her one unit of the undifferentiated commodity and $n$ privately informed buyers, each of whom seeks to purchase one unit of the commodity. All traders simultaneously submit offers/bids from which the market maker computes a market clearing price $p$ and, at that price, assigns the $m$ units of supply to those $m$ traders who revealed through their offers/bids that they most value the available units. Satterthwaite and Williams (1989, 2002) and Rustichini et al. (1994) established that as $m$ and $n$ increase the market rapidly approaches the ex post efficient, competitive outcome.

The direct generalization of this model to our dynamic setting is this. Each unit of time measure 1 sellers and measure $a$ buyers choose whether to enter the market. Each of these traders has a private cost/value for a single unit of a homogeneous good and each faces a small exogenous probability of exit each period. The *baseline problem* is how in this dynamic setting the sellers' units of supply can be reallocated to those traders who most highly value them. Observe that this problem is intrinsically competitive because the traded good is homogeneous, the number of traders is always large, and the distributions $G_S$ and $G_B$ define well behaved supply and demand curves.

[2] Another example of a centralized trading institution is the system of simultaneous ascending-price auctions, studied in Peters and Severinov (2007). They also find robust convergence to the competitive outcome.

An immediate solution to this problem is each period to conduct a large static double auction involving all the active traders. This works because, with continua of traders, the double auction exactly computes the Walrasian price and uses it to mediate trade. The present paper's result is that decentralized matching and bargaining can with almost no loss of efficiency substitute for the large centralized market, provided the period length is small.

The models of Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano (2002), each has a non-competitive attribute.[3] To be specific, Wolinsky's model relaxes the homogeneous good assumption and does not fully analyze the effects of entry/exit dynamics. The entry/exit dynamics of De Fraja and Sákovics' model do not specify fixed measures of buyers and sellers entering the market each unit of time and therefore do not specify well defined demand and supply curves. Serrano's model is of a market that may initially be large but, as buyers and sellers successfully trade and no new traders enter, it becomes small and non-competitive over time, an effect that the discreteness of its prices aggravates.

Recently five interesting working papers have appeared that build on our two matching and bargaining papers with incomplete information. Lauermann (2006) is most closely related to this paper because he also assumes that the friction in the market is an exogenous probability of exit. His paper identifies a set of four conditions on trading outcomes that are sufficient to guarantee convergence independent of many of the market's details. The other four papers consider markets in which the friction is an explicit per period cost of participation and are therefore more closely related to Satterthwaite and Shneyerov (2007). Shneyerov and Wong (2007a) consider a model that is a replica of Mortensen and Wright (2002) but with private information, provide a simple necessary and sufficient condition for existence of a non-trivial equilibrium, and compare equilibrium outcomes to those in Mortensen and Wright. Shneyerov and Wong (2007b) then establish that the convergence rate is worst case optimal in the sense of Satterthwaite and Williams (2002). Atakan (2007a, 2007b) also considers markets in which meetings are pairwise and develops conditions that guarantee convergence. The important innovation that both of his papers pioneer is relaxation of the usual assumption that good being traded is homogeneous. The resulting models fit, to take two examples, the realities of the housing and labor markets much better than do the existing models with homogeneous goods.

Looking ahead, the next section formally states the model and our main result establishing that the Walrasian price emerges as the market becomes frictionless. Section 3 derives basic properties of equilibria. Section 4 utilizes the notation and basic results from Sections 2 and 3 to present a computed example illustrating our result and then to describe in detail the mechanisms that force convergence to the Walrasian price. Section 5 then proves our result and Section 6 both summarizes and discusses possible extensions.

## 2. Model and theorem

We study the steady state of a market with two-sided incomplete information and an infinite horizon. In it heterogeneous buyers and sellers meet once per period ($t = \cdots, -1, 0, 1, \ldots$) and trade an indivisible, homogeneous good. Every seller is endowed with one unit of the traded good for which she has cost $c \in [0, 1]$. This cost is private information to her; to other traders it is an independent random variable with distribution $G_S$ and density $g_S$. Similarly, every buyer seeks to purchase one unit of the good for which he has value $v \in [0, 1]$. This value is private to him;

[3] More complete discussions of these three papers are contained in Satterthwaite and Shneyerov (2007).

to others it is an independent random variable with distribution $G_B$ and density $g_B$. Our model is therefore the standard independent private values model. We assume that the two densities are bounded away from zero: a $\underline{g} > 0$ exists such that, for all $c, v \in [0, 1]$, $g_S(c) > \underline{g}$ and $g_B(v) > \underline{g}$. This implies that $0 < p_W < 1$, i.e., the Walrasian price as defined in Eq. (1) is interior to the interval $[0, 1]$.

The length of each period is $\delta$. Each unit of time a large number of potential sellers and a large number of potential buyers consider entering the market; formally each unit of time measure 1 of potential sellers and measure $a$ of potential buyers consider entry where $a > 0$. This means that each period measure $\delta$ of potential sellers and measure $a\delta$ of potential buyers consider entry. Only those potential traders whose expected utility from entry is positive elect to enter and become active traders.[4] Active buyers and sellers who do not leave the market through either trade or exogenous exit carry over and remain active in the next period.

Let the strategy of a seller, $S : [0, 1] \to [0, 1] \cup \{\mathcal{N}\}$, map her cost $c$ into either a decision $\mathcal{N}$ not to enter or the minimal bid that she is willing to accept. Similarly, let the strategy of a buyer, $B : [0, 1] \to [0, 1] \cup \{\mathcal{N}\}$, map his value $v$ into either a decision $\mathcal{N}$ not to enter or the bid that he places whenever he is matched with a seller. Traders who do not enter receive zero utility. Denote with $\mu > 0$ the exogenous exit rate.[5] Finally let $\zeta$ be the endogenous steady-state ratio of active buyers to active sellers in the market. Given this notation, a period consists of four steps:

1. Each potential trader decides whether to enter and become an active trader as a function of his type, i.e., a potential seller declines entry if $S(c) = \mathcal{N}$ and a potential buyer declines entry if $B(v) = \mathcal{N}$.
2. Every active buyer is matched with one active seller. His match is equally likely to be with any active seller and is independent of the matches other buyers realize. Since there are continua of buyers and sellers the matching probabilities are Poisson: the probability that a seller is matched with $k = 0, 1, 2, \ldots$ buyers is[6]

$$\xi_k = \frac{\zeta^k}{k!\mathrm{e}^{\zeta}}. \tag{2}$$

   Consequently a seller may end up being matched with zero buyers, one buyer, two buyers, etc.[7]

[4] In an earlier version of this paper we assumed that potential traders whose expected utility is zero did enter the market and become active. These traders had zero probability of trading and exited the market at the exogenous rate $\mu\delta$ per period. Modulo a plausible restriction on the equilibrium strategies of traders who have zero probability of trading we are able to show that the equilibrium trading range collapses to the Walrasian price.

[5] The presence of a positive exit rate (or something similar) is necessary if participation costs are zero. The reason is that every trader who enters must have a probability of either trading or exiting that, per unit of time, is bounded away from zero. Otherwise traders whose probability of trading is infinitesimal but positive would accumulate in the market and jeopardize the existence of a steady state. The presence of the exogenous exit rate does this directly. The presence of a small participation cost in Satterthwaite and Shneyerov (2007) does this indirectly, for it causes any potential trader who has a low or zero probability of trading to refuse entry because he cannot in expectation recover those expected costs.

[6] In a market with $M$ sellers and $\zeta M$ buyers, the probability that a seller is matched with $k$ buyers is $\xi_k^M = \binom{\zeta M}{k}(\frac{1}{M})^k(1 - \frac{1}{M})^{\zeta M-k}$. Poisson's theorem (see, for example, Shiryaev, 1995) shows that $\lim_{M\to\infty} \xi_k^M = \xi_k$.

[7] This one-to-several matching among sellers and buyers contrasts with the pairwise matching that, with the exception of Dagan et al. (2000), is to our knowledge universal in this literature. Matching in this manner skirts the conceptual issues that arise in a model with continua of buyers and sellers entering each period; see the discussion of this issue in McLennan and Sonnenschein (1991, footnote 4).

3. Traders within a match bargain in accordance with the rules of the buyers' bid double auction.[8]
   (a) Simultaneously every buyer announces a take-it-or-leave-it offer to the seller. A type $v$ buyer bids $B(v)$. At the time he submits his bid, he does not know how many other buyers he is bidding against; he only knows the endogenous steady-state probability distribution of how many buyers with whom he is competing.[9]
   (b) The seller reviews the bids she has received and accepts the highest one provided it is at least as large as her reservation cost, $S(c)$. If two or more buyers tie with the highest bid, then the seller uses a fair lottery to choose between them.
   (c) If trade occurs between a type $c$ seller and a type $v$ buyer at price $B(v) = p$, then the seller leaves the market with utility $p - c$ and the buyer leaves the market with utility $v - p$. Each seller, thus, runs an optimal auction; moreover their commitment to this auction is credible since the reservation value each sets stems from their dynamic optimization.[10]
4. Every active trader who fails to trade either remains active the next period with probability $e^{-\delta\mu}$ or, for exogenous reasons, exits with probability $1 - e^{-\delta\mu}$. Traders who exit without trading leave with zero utility.

Traders' time preference cause them to discount their expected utility at the rate $r \geqslant 0$ per unit time. This, together with the exit risk $\mu$ per unit time, induces impatience. Section 3.1 shows that $e^{-\delta\beta}$ is the overall rate per period at which each trader discounts his utility where $\beta = \mu + r$.

A seller who has low cost tends to trade within a short number of periods of her entry because most buyers with whom she might be matched have a value higher than her cost and therefore tend to bid sufficiently high to obtain agreement. A high cost seller, on the other hand, tends not to trade as quickly or, perhaps, not at all. As a consequence, in the steady state among the population of sellers who are active, high cost sellers are relatively common and low cost sellers are relatively uncommon. Parallel logic implies that, in the steady state, low value buyers are relatively common and high value buyers are relatively uncommon.

To represent this endogenous distribution of trader types, let $T_S$ be the measure of active sellers in the market after entry occurs and before matching, $T_B$ be the measure of active buyers after entry occurs and before matching, $F_S$ be the distribution of active seller types within the measure $T_S$, and $F_B$ be the distribution of active buyer types within the measure $T_B$. The buyer–seller ratio $\zeta$ is therefore $T_B/T_S$. The corresponding densities are $f_S$ and $f_B$ and, establishing useful notation, the right-hand distributions are $\bar{F}_S \equiv 1 - F_S$ and $\bar{F}_B \equiv 1 - F_B$. Define $W_S(c)$ and $W_B(v)$ to be the beginning-of-period, steady-state, net payoffs to a seller of type $c$ and the buyer of type $v$

$$\underline{c} \equiv S(0),$$

$$\bar{c} \equiv \sup_c \{c \mid W_S(c) > 0\},$$

$$\underline{v} \equiv \inf_v \{v \mid W_B(v) > 0\}, \quad \text{and}$$

8 See Satterthwaite and Williams (1989) for an analysis of the static buyers' bid double auction.

9 In Section 4.3 we discuss how allowing each buyer to observe how many rivals he faces might change our results.

10 We do not know if these auctions are the equilibrium mechanism that would result if we tried to replicate McAfee's (1993) analysis within our model.

$$\bar{v} \equiv B(1). \tag{3}$$

No seller enters whose cost is $\bar{c}$ or greater and no buyer enters whose value is $\underline{v}$ or less because a trader only becomes active if his expected utility from participating is positive. We show in the next section that active sellers' equilibrium reservation costs all fall in the interval $[\underline{c}, \bar{c})$, active buyers' equilibrium bids all fall in $(\underline{v}, \bar{v}]$, and that $[\underline{c}, \bar{c}] = [\underline{v}, \bar{v}] \equiv [\underline{p}, \bar{p}]$. We call this interval the trading range.

Our goal is to show that in all symmetric, steady-state equilibria the trading range converges to the Walrasian price and the allocation converges to the competitive allocation as the period length goes to zero. By a steady-state equilibrium we mean one in which every seller in every period plays a symmetric, time invariant strategy $S$, every buyer plays a symmetric, time invariant strategy $B$, and both these strategies are always optimal. Given the friction $\delta$, a market equilibrium $M_\delta$ consists of strategies $\{S, B\}$, traders' masses $\{T_S, T_B\}$, and distributions $\{F_S, F_B\}$ such that (i) $\{S, B\}$, $\{T_S, T_B\}$, and $\{F_S, F_B\}$ generate $\{T_S, T_B\}$ and $\{F_S, F_B\}$ as their steady state and (ii) no type of buyer can increase his expected utility (including the continuation payoff from matching in future periods if trade fails in the current period) by a unilateral deviation from the strategy $B$, (iii) no type of seller can commit to reject any bid above her full dynamic opportunity cost $c + \mathrm{e}^{-\beta\delta} W_S(c)$ and (iv) strategies $\{S, B\}$, masses $\{T_S, T_B\}$, and distributions $\{F_S, F_B\}$ are common knowledge among all active and potential traders. We assume that, for each $\delta > 0$, a market equilibrium exists in which each potential trader's ex ante probability of trade is positive.[11] Throughout the paper we only consider these equilibria; we never consider the trivial no trade equilibrium in which all potential sellers and buyers choose not to enter ($\lambda = \mathcal{N}$).

Note that beliefs are straightforward because there are continua of traders and matching is anonymous and independent. Therefore off-the-equilibrium path actions do not cause inference ambiguities. In addition, observe that for simplicity of exposition we assume traders use symmetric, pure strategies. At a cost in notation we could define trader-specific and mixed strategies and then prove that the anonymous nature of matching and the strict monotonicity of strategies implies they in fact must be symmetric and (essentially) pure. To see this, first consider the implication of anonymous matching for buyers. Even if different traders follow distinct strategies, every buyer would still draw his opponents from the same population of active traders.[12] Therefore, for a given value $v$, every buyer would have the identical best response correspondence. Second, as we show below, every selection from this correspondence is strictly increasing. This means that the best response is pure except at a measure zero set of values where jumps occur. Mixing can occur at these jump points, but does not affect other traders' strategies because the measure of the jump points is zero and has no consequence for other traders' maximization problems.

We may now state our main result.

**Theorem 1.** *Fix a sequence of symmetric, steady-state market equilibria $M_\delta$ indexed by $\delta$ in which $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$ and each potential trader's ex ante probability of trade is positive. Let $\{S_\delta, B_\delta\}$ be the strategies associated with the equilibrium that $\delta$ indexes, let $[\underline{c}_\delta, \bar{c}_\delta] = [\underline{v}_\delta, \bar{v}_\delta] = [\underline{p}_\delta, \bar{p}_\delta]$ be the trading range those strategies imply, and let $W_{S\delta}(c)$and $W_{B\delta}(v)$ be the resulting interim expected utilities of the sellers and buyers respectively. Then both the bidding and offering ranges converge to $p_W$:*

[11] In Section 4.4 we describe the limited existence results that we have obtained.

[12] This is strictly true because we assume a continuum of traders.

$$\lim_{\delta\to 0} \underline{c}_\delta = \lim_{\delta\to 0} \bar{c}_\delta = \lim_{\delta\to 0} \underline{v}_\delta = \lim_{\delta\to 0} \bar{v}_\delta = p_W. \tag{4}$$

*In addition, each trader's interim expected utility converges to the utility he would realize if the market were perfectly competitive*:

$$\lim_{\delta\to 0} W_{S\delta}(c) = \max[0, p_W - c] \tag{5}$$

*and*

$$\lim_{\delta\to 0} W_{B\delta}(v) = \max[0, v - p_W]. \tag{6}$$

Section 5 contains the theorem's proof.

This theorem is almost identical to Theorem 1 in our participation cost paper (Satterthwaite and Shneyerov, 2007); the only difference is that that theorem does not assert that $[\underline{c}_\delta, \bar{c}_\delta] = [\underline{v}_\delta, \bar{v}_\delta] = [\underline{p}_\delta, \bar{p}_\delta]$.The reason is that in the presence of participation costs traders' equilibrium strategies must recover their expected costs. This requires, in effect, an equilibrium bid–ask spread that does not arise in the exogenous exit rate model. To understand this specifically, consider active types only and begin with this paper's model. Here buyers and sellers' equilibrium strategies have congruent, fully overlapping ranges: $S_\delta : [0, \bar{c}_\delta) \to [\underline{c}_\delta, \bar{c}_\delta)$, $B_\delta : (\underline{v}_\delta, 1] \to (\underline{v}_\delta, \bar{v}_\delta]$, and $[\underline{c}_\delta, \bar{c}_\delta] = [\underline{v}_\delta, \bar{v}_\delta]$. This congruence is driven by the fact that market participation is free. To demonstrate this, suppose that in equilibrium the type $\underline{v}_\delta$ buyer bids such that $\underline{v}_\delta > B_\delta(\underline{v}_\delta) > S_\delta(0)$, i.e., $\underline{v}_\delta$, the lowest active buyer, bids more than the lowest reservation price that any seller offers. But this cannot be equilibrium behavior because a type $v'$ buyer for whom $v' \in (S_\delta(0), \underline{v}_\delta)$ can successfully enter with some bid $B(v') \in (S(0), v')$. Doing so has no cost and, with some positive probability, results in a trade that earns him $v' - B(v')$ profit. This contradicts $\underline{v}_\delta$ being the type of the lowest active buyer.

Obviously the argument just made breaks down in the participation cost model where each trader incurs a cost each period he is in the market. With participation costs buyers' strategies retain the structure $S^P_\delta : [0, \bar{c}_\delta) \to [\underline{c}_\delta, \bar{c}_\delta]$. Buyers' strategies, however, take on a modified structure: $B^P_\delta : (\underline{v}_\delta, 1] \to (\underline{p}_\delta, \bar{p}_\delta]$ where $[\underline{p}_\delta, \bar{p}_\delta] \neq [\underline{c}_\delta, \bar{c}_\delta]$, $\underline{p}_\delta \in (\underline{c}_\delta, \bar{c}_\delta]$, and $\bar{p}_\delta \in (\bar{c}_\delta, 1]$.[13] In words, the presence of participation costs causes the range of buyers' strategies to slide up relative to the range of the sellers' strategies, reducing their overlap. The extreme case is the "full trade" equilibrium, which the participation cost paper (Theorem 2) shows must exist if $\delta$ is sufficiently small. There the range of the buyers' strategies slides up so far that all overlap but one point is eliminated: $\underline{p}_\delta = \bar{c}_\delta$ and $[\underline{c}_\delta, \bar{c}_\delta] \cap [\underline{p}_\delta, \bar{p}_\delta] = \underline{p}_\delta$.[14]

That these differences in equilibrium structure have no effect on either models' convergence to the competitive price is remarkable. Apparently the underlying mechanisms forcing convergence—even in the presence of incomplete information—are robust.

## 3. Basic properties of equilibria

In this section we derive several basic properties that all equilibria must satisfy. These properties—formulas for the probability of trade, establishment of the strict monotonicity of

[13] This characterization is from Lemma 6 in Satterthwaite and Shneyerov (2007).

[14] This difference in the structure of equilibrium strategies across the two models can be seen by comparing the top panels of each paper's Fig. 1.

strategies, and necessary conditions for a strategy pair $(S, B)$ to be an equilibrium—enable us to compute examples of equilibria and provide the foundations for the proof of our main result.

### 3.1. Discounted ultimate probability of trade

An essential construct for our analysis is the discounted ultimate probability of trade. It allows a trader's expected gains from participating in the market to be written as simply as possible. Given any period, let $\rho_S(\lambda)$ be the probability that a seller who offers $\lambda \in [0,1] \cup \{\mathcal{N}\}$ trades that period and let $\rho_B(\lambda)$ be the probability that a buyer who bids $\lambda$ trades that period. Observe that $\rho_S$ is non-increasing and $\rho_B$ is non-decreasing for $\lambda \in [0,1]$ because all trade is mediated by the buyers' bid double auction. Let $\bar{\rho}_S(\lambda) = 1 - \rho_S(\lambda)$ and $\bar{\rho}_B(\lambda) = 1 - \rho_B(\lambda)$. If a trader chooses $\lambda = \mathcal{N}$, then he does not enter and $\rho_S(\mathcal{N}) = \rho_B(\mathcal{N}) = 0$. In Section 3.3 we derive explicit formulas for $\rho_B$ and $\rho_S$.

Define recursively $P_B(\lambda)$ to be a buyer's discounted ultimate probability of trade if he bids $\lambda$:

$$\begin{aligned} P_B(\lambda) &= \rho_B(\lambda) + \bar{\rho}_B(\lambda)\mathrm{e}^{-\mu\delta}\mathrm{e}^{-r\delta} P_B(\lambda) \\ &= \rho_B(\lambda) + \bar{\rho}_B(\lambda)\mathrm{e}^{-\beta\delta} P_B(\lambda) \end{aligned}$$

where $\beta = \mu + r$. Therefore

$$P_B(\lambda) = \frac{\rho_B(\lambda)}{1 - \mathrm{e}^{-\beta\delta} + \mathrm{e}^{-\beta\delta}\rho_B(\lambda)}. \tag{7}$$

Observe that the formula incorporates both the trader's risk of having to exit and his time discounting into the calculation. The parallel recursion for sellers implies that

$$P_S(\lambda) = \frac{\rho_S(\lambda)}{1 - \mathrm{e}^{-\beta\delta} + \mathrm{e}^{-\beta\delta}\rho_S(\lambda)}. \tag{8}$$

$P_B$ is non-decreasing and $P_S$ is non-increasing.

### 3.2. Strategies are strictly increasing

This subsection demonstrates the most basic property that our equilibria satisfy: equilibrium strategies for active traders are strictly increasing. As a preliminary step, we first show that the sets of buyers and sellers active in the market are well behaved: buyers come from an interval at the top of the [0, 1] set of possible types and sellers come from an interval at the bottom.

**Claim 2.** *In any equilibrium $\underline{v} < 1$, $\bar{c} > 0$,*

$$(\underline{v}, 1] = \{v \mid W_B(v) > 0\}, \tag{9}$$

$$[0, \bar{c}) = \{c \mid W_S(c) > 0\}, \tag{10}$$

*$S(c) = \mathcal{N}$ for $c \in [\bar{c}, 1]$, and $B(v) = \mathcal{N}$ for $v \in [0, \underline{v}]$.*

**Proof.** Since we have ruled out the uninteresting no trade equilibrium, every potential trader has a positive ex ante probability of trade: $\int_0^1 \rho_B[B(v)]g_B(v)\,\mathrm{d}v > 0$ and $\int_0^1 \rho_S[S(c)]g_S(c)\,\mathrm{d}c > 0$. These inequalities hold only if $\underline{v} < 1$ and $\bar{c} > 0$, establishing the first part of the claim.

A buyer who every period employs strategy $B(v) \in [0,1] \cup \{\mathcal{N}\}$ gets equilibrium payoff $W_B(v) = vP_B[B(v)] - D_B(B(v))$ where $D_B(\lambda)$ is the discounted expected equilibrium payment from using strategy $\lambda \in [0,1] \cup \{\mathcal{N}\}$. Note that $D_B(v) = 0$ if $B(v) = \mathcal{N}$. By definition

$$W_B(v) = \max_{\lambda \in [0,1] \cup \{\mathcal{N}\}} v P_B(\lambda) - D_B(\lambda),$$

and, for all $\lambda \in [0,1] \cup \{\mathcal{N}\}$, $v P_B(\lambda) - D_B(\lambda)$ is differentiable in $v$. Application of Milgrom and Segal's (2002, Theorem 2) envelope theorem for arbitrary choice sets enables us to write

$$W_B(v) = W_B(0) + \int_0^v P_B\big[B(x)\big]\,\mathrm{d}x. \tag{11}$$

Therefore $W_B(\cdot)$ is continuous and non-decreasing on $[0,1]$. Then, by the definition of $\underline{v}$, $W_B(v) > 0$ for $v \in (\underline{v}, 1]$ and $W_B(v) = 0$ for $v \in [0, \underline{v})$. $W_B(\underline{v}) = 0$ because $W_B(\cdot)$ is continuous, which establishes (9). Finally, $B(v) = \mathcal{N}$ for $v \in [0, \underline{v})$ because a trader only enters if he has strictly positive utility. The proof of (10) and $S(c) = \mathcal{N}$ for all $c \in [\bar{c}, 1]$ is exactly parallel. □

**Claim 3.** *B is strictly increasing on* $(\underline{v}, 1]$

**Proof.** $W_B(v) = \sup_{\lambda \geqslant 0}(v - \lambda) P_B(\lambda) = (v - B(v)) P_B(B(v))$ is the upper envelope of a set of affine functions. It follows that, for $v \in (\underline{v}, 1]$, $W_B$ is a continuous, increasing, and convex function that is differentiable almost everywhere.[15] Convexity implies that $W_B'$ is non-decreasing on $[\underline{v}, 1]$. By the envelope theorem $W_B'(\cdot) = P_B[B(\cdot)]$; $P_B[B(\cdot)]$ is therefore non-decreasing on $[\underline{v}, 1]$ at all differentiable points. Milgrom and Segal's (2002) Theorem 1 implies that at non-differentiable points $v' \in [\underline{v}, 1]$

$$\lim_{v \to v'-} W_B'(v) \leqslant P_B\big(B(v')\big) \leqslant \lim_{v \to v'+} W_B'(v).$$

Thus $P_B[B(\cdot)]$ is everywhere non-decreasing on $[\underline{v}, 1]$.

Pick any $v$, $v' \in (\underline{v}, 1]$ such that $v < v'$. Since $P_B[B(\cdot)]$ is everywhere non-decreasing, $P_B[B(v)] \leqslant P_B[B(v')]$ necessarily. We first show that $B$ is non-decreasing on $(\underline{v}, 1]$. Suppose, to the contrary, that $B(v) > B(v')$. The rules of the buyer's bid double auction imply that $P_B(\cdot)$ is non-decreasing; therefore $P_B[B(v)] \geqslant P_B[B(v')]$. Consequently $P_B[B(v)] = P_B[B(v')]$. But this gives $v'$ incentive to lower his bid to $B(v')$, since by doing so he will buy with the same positive probability but pay a lower price. This contradicts $B$ being an optimal strategy and establishes that $B$ is non-decreasing. If $B(v') = B(v)$ $(= \lambda)$ because $B$ is not strictly increasing, then any buyer with $v'' \in (v, v')$ will raise his bid infinitesimally from $\lambda$ to $\lambda' > \lambda$ to avoid the rationing that results from a tie. This proves that $B$ is strictly increasing. □

**Claim 4.** *S is continuous, strictly increasing on* $[0, \bar{c})$*, and*

$$S'(c) = 1 - \mathrm{e}^{-\beta\delta} P_S\big[S(c)\big] > 0$$

*for almost all* $c \in [0, \bar{c})$.

**Proof.** Our assumption in Section 2's definition of a market equilibrium $M_\delta$ that sellers do not have commitment power to reject bids above their dynamic opportunity cost mandates that

$$S(c) = c + \mathrm{e}^{-\beta\delta} W_S(c) \tag{12}$$

[15] An increasing function is differentiable almost everywhere.

for all $c \in [0, \bar{c})$ where $W_S(c)$ is the equilibrium payoff to a seller with cost $c$. In a stationary equilibrium $W_S(c) = D(S(c)) - cP_S(S(c))$ where $P_S[S(c)]$ is her discounted ultimate probability of trading when her offer is $S(c)$ and $D(S(c))$ is the expected equilibrium payment to the seller with cost $c$. Milgrom and Segal's Theorem 2 implies that $W_S$is continuous and can be written, for $c \in [0, \bar{c}]$, as

$$W_S(c) = W_S(\bar{c}) + \int_c^{\bar{c}} P_S\big(S(x)\big)\,\mathrm{d}x \tag{13}$$

$$= \int_c^{\bar{c}} P_S\big(S(x)\big)\,\mathrm{d}x \tag{14}$$

where the second line follows from the definition of $\bar{c}$ and the continuity of $W_S$. This immediately implies that $W_S$ is strictly decreasing (and therefore almost everywhere differentiable) because the definition of $\bar{c}$ implies that $P_S(S(c)) > 0$ for all $c \in [0, \bar{c})$. It, when combined with Eq. (12), also implies that $S$ is continuous. Therefore, for almost all $c \in [0, \bar{c})$,

$$S'(c) = 1 - \mathrm{e}^{-\beta\delta} P_S\big[S(c)\big] > 0$$

because $W_S'(c) = -P_S[S(c)]$. This, together with the continuity of $S$, establishes that $S$ is strictly increasing for all $c \in [0, \bar{c})$. □

**Claim 5.** $\underline{p} = \underline{v} = \underline{c} = S(0) = \lim_{v\downarrow\underline{c}} B(v)$ *and* $\bar{p} = \bar{c} = \bar{v} = B(1) = \lim_{c\uparrow\bar{v}} S(c)$.

**Proof.** Given that $S$ is strictly increasing, $S(0) = \underline{c}$ is the lowest offer any seller ever makes. A buyer with valuation $v < \underline{c}$ does not enter the market because he can only hope to trade by submitting a bid at or above $\underline{c}$ and such a bid would be above his valuation. $S$ is continuous by Claim 4, so a buyer with valuation $v > \underline{c}$ will enter the market with a bid $B(v) \in (\underline{c}, v)$ because he can make profit with positive probability. Therefore $\lim_{v\downarrow\underline{c}} B(v) = \underline{v} = \underline{c}$.

By definition $\bar{c} \equiv \sup_c\{c \mid W_S(c) > 0\}$. That Eq. (12) and continuity of $W_S$ imply that $\lim_{c\uparrow\bar{v}} S(c) = \bar{c}$ can be seen as follows. A seller with cost $c \geqslant \bar{v} = B(1)$ will not enter the market, so $\bar{c} \leqslant B(1)$. Suppose $\bar{c} = \lim_{c\uparrow\bar{v}} S(c) < \bar{v} \equiv B(1)$. Then a seller with cost $c' \in (\bar{c}, B(1))$ can enter and, with a positive probability, earn a profit with an offer $S(c') \in (\bar{c}, B(1))$. This, however, is a contradiction: $\sup_c\{c \mid W_S(c) > 0\} \geqslant c' > \bar{c} \equiv \sup_c\{c \mid W_S(c) > 0\}$. Therefore $\lim_{c\uparrow\bar{v}} S(c) = \bar{c} = \bar{v} = B(1)$. □

These findings are summarized as follows.

**Proposition 6.** *Suppose that* $\{B, S\}$ *is a stationary equilibrium. Then* $S$ *and* $B$ *are strictly increasing over* $[0, \bar{c})$ *and* $(\underline{v}, 1]$. *Additionally* $S$ *is continuous and, for almost all* $c \in [0, \bar{c})$,

$$S'(c) = 1 - \mathrm{e}^{-\beta\delta} P_S\big[S(c)\big] > 0.$$

*The strategies also satisfy the boundary conditions* $\underline{p} = \underline{v} = \underline{c} = S(0) = \lim_{v\downarrow\underline{c}} B(v)$ *and* $\bar{p} = \bar{c} = \bar{v} = B(1) = \lim_{c\uparrow\bar{v}} S(c)$. *Finally,* $S(c) = \mathcal{N}$ *for* $c \in [\bar{c}, 1]$ *and* $B(v) = \mathcal{N}$ *for* $v \in [0, \underline{v}]$.

The strict monotonicity of $B$ and $S$ allows us to define their inverses, $V$ and $C$, on the domain $[\underline{p}, \bar{p}]$: $V(\lambda) = \inf\{v\colon B(v) > \lambda\}$ and $C(\lambda) = \inf\{c\colon S(c) > \lambda\}$. These functions are used frequently below.

### 3.3. Explicit formulas for the probabilities of trading

Focus on a seller of type $c$ who in equilibrium has a positive probability of trade. In a given period she is matched with zero buyers with probability $\xi_0$ (recall Definition 2 in Section 2) and with one or more buyers with probability $\bar{\xi}_0 = 1 - \xi_0$. Suppose she is matched and $v^*$ is the highest type buyer with whom she is matched. Since by Proposition 6 each buyer's bid function $B$ is increasing, she accepts his bid if and only if $B(v^*) \geqslant \lambda$ where $\lambda$ is her reservation price. The distribution from which $v^*$ is drawn is $F_B^*$: for $v \in [\underline{v}, 1]$,

$$F_B^*(v) = \frac{1}{\bar{\xi}_0} \sum_{i=1}^{\infty} \xi_i \big[F_B(v)\big]^i = \frac{1}{\bar{\xi}_0} \sum_{i=1}^{\infty} \frac{\zeta^i}{\mathrm{e}^{\zeta} i!} \big[F_B(v)\big]^i = \frac{\mathrm{e}^{\zeta F_B(v)} - 1}{\mathrm{e}^{\zeta} - 1} \tag{15}$$

where $F_B$ is the steady-state distribution of buyer types and $\{\xi_0, \xi_1, \xi_2, \ldots\}$ are the probabilities (defined in Eq. (2)) with which each seller is matched with zero, one, two, or more buyers.[16] Note that this distribution is conditional on the seller being matched. Thus if a seller has reservation price $\lambda$, her probability of trading in a given period is

$$\rho_S(\lambda) = \bar{\xi}_0\big[1 - F_B^*\big(V(\lambda)\big)\big]. \tag{16}$$

A similar expression obtains for $\rho_B(\lambda)$, the probability that a buyer submitting bid $\lambda$ successfully trades in any given period. In order to derive this expression, we need a formula for $\omega_k$, the probability that the buyer is matched with $k$ rival buyers. If $T_B$ is the mass of active buyers and $T_S$ is the mass of active sellers, then $\omega_k T_B$, the mass of buyers participating in matches with $k$ rival buyers, equals $k+1$ times $\xi_{k+1} T_S$, the mass of sellers matched with $k+1$ buyers:

$$\omega_k T_B = (k+1)\xi_{k+1} T_S.$$

Solving, substituting in the formula for $\xi_{k+1}$, and recalling that $\zeta = T_B/T_S$ shows that $\omega_k$ and $\xi_k$ are identical:

$$\omega_k = \frac{(k+1)}{\zeta}\xi_{k+1} = \frac{(k+1)}{\zeta} \frac{\zeta^{k+1}}{(k+1)!\mathrm{e}^{\zeta}} = \xi_k. \tag{17}$$

The implication of this is that the distribution of bids from other buyers that a buyer must beat is exactly the same distribution of bids that each seller receives when she is matched with at least one buyer.[17]

Turning back to $\rho_B$, a buyer who bids $\lambda$ and is the highest bidder has probability $F_S(C(\lambda))$ of having his bid accepted. This is just the probability that the seller with whom the buyer is

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[16] This formula follows from the facts that:

$$\sum_{i=1}^{\infty} \frac{x^i}{i!} = \mathrm{e}^x - 1 \quad \text{and} \quad \bar{\xi}_0 = \sum_{i=1}^{\infty} \xi_i = \frac{\mathrm{e}^{\zeta} - 1}{\mathrm{e}^{\zeta}}.$$

[17] The congruence of these distributions is a consequence of the number of sellers being Poisson. See Myerson's (1999) discussion of games in which the number of participants is uncertain.

matched has a low enough reservation price so as to accept his bid. If a total of $j+1$ buyers are matched with the seller with whom the buyer is matched, then he has $j$ competitors and the probability that all $j$ competitors bid less than $\lambda$ is $[F_B(V(\lambda))]^j$. Therefore the probability that the bid $\lambda$ is successful in a particular period is

$$\begin{aligned}\rho_B(\lambda) &= F_S\big(C(\lambda)\big)\sum_{j=0}^{\infty}\omega_j\big[F_B\big(V(\lambda)\big)\big]^j\\ &= F_S\big(C(\lambda)\big)\sum_{j=0}^{\infty}\xi_j\big[F_B\big(V(\lambda)\big)\big]^j\\ &= F_S\big(C(\lambda)\big)\sum_{j=0}^{\infty}\frac{\zeta^j}{\mathrm{e}^{\zeta}\,j!}\big[F_B\big(V(\lambda)\big)\big]^j\\ &= F_S\big(C(\lambda)\big)\mathrm{e}^{-\zeta\bar{F}_B(V(\lambda))},\end{aligned}$$

recalling that $\Sigma_{j=0}^{\infty}x^j/j! = \mathrm{e}^x$ and $\bar{F}_B(x) \equiv 1 - F_B(x)$.

### *3.4. Necessary conditions for strategies and steady-state distributions*

In this subsection the goal is to write down a set of necessary conditions that are sufficiently complete so as to form a basis for calculating Section 4.1's example and, also, to create a foundation for Section 5's proof of Theorem 1.

The necessary conditions apply only to active trader types, i.e. for $v \in (\underline{v}, 1]$ and $c \in [0, \bar{c})$. We first derive fixed point conditions that traders' strategies must satisfy. Consider sellers first. Substituting (13),

$$W_S(c) = \int_c^{\bar{c}} P_S\big(S(x)\big)\,\mathrm{d}x \tag{18}$$

into (12) gives a fixed point condition sellers' strategies must satisfy:

$$S(c) = c + \mathrm{e}^{-\beta\delta}\int_c^{\bar{c}} P_S\big(S(x)\big)\,\mathrm{d}x. \tag{19}$$

Turning to buyers, we have from (11):

$$W_B(v) = \int_{\underline{v}}^{v} P_B\big[B(x)\big]\,\mathrm{d}x. \tag{20}$$

But also,

$$W_B(v) = \max_{\lambda\in[0,1]\cup\{\mathcal{N}\}}(v-\lambda)P_B(\lambda) = \big(v - B(v)\big)P_B\big(B(v)\big).$$

Substituting (20) into this and solving gives a fixed point condition buyers' strategies must satisfy:

$$B(v) = v - \frac{1}{P_B[B(v)]} \int_{\underline{v}}^{v} P_B\big[B(x)\big]\,\mathrm{d}x. \tag{21}$$

In our model, the distributions $\{F_B, F_S\}$ are endogenously determined by traders' strategies. In any steady state, the numbers of entering and leaving traders must be equal. This gives rise to three necessary conditions. First, in the steady state, for each type $v \in (\underline{v}, 1]$, the density $f_B$ must be such that the mass of buyers entering equals the mass of buyers leaving:

$$a\delta g_B(v) = T_B f_B(v)\big\{\rho_B\big[B(v)\big] + \bar{\rho}_B\big[B(v)\big]\big(1 - \mathrm{e}^{-\mu\delta}\big)\big\} \tag{22}$$

where the left-hand side is the measure of type $v$ buyers of who enter each period and the right-hand side is the measure of type $v$ buyers who exit each period. Note that it takes into account that within each period successful traders exit prior to traders who exit for exogenous reasons. It also takes into account that $T_B$ is defined to be the measure of buyers who are active immediately after entry has occurred within the period. Second, the analogous steady-state condition for the density $f_S$ is, for $c \in [0, \bar{c})$,

$$\delta g_S(c) = T_S f_S(c)\big\{\rho_S\big[S(c)\big] + \bar{\rho}_S\big[S(c)\big]\big(1 - \mathrm{e}^{-\mu\delta}\big)\big\}. \tag{23}$$

Third, trade always occurs between pairs consisting of one seller and one buyer. Therefore, given a cohort of buyers and sellers who enter during a given unit of time, the mass of those buyers who ultimately end up trading must equal the mass of sellers who ultimately end up trading:

$$a \int_{\underline{v}}^{1} \tilde{P}_B(v) g_B(v)\,\mathrm{d}v = \int_{0}^{\bar{c}} \tilde{P}_S(c) g_S(c)\,\mathrm{d}c \tag{24}$$

where $\tilde{P}_B(v)$ and $\tilde{P}_S(c)$ are, respectively, the ultimate trading probabilities for a type $v$ buyer and a type $c$ seller. Notice the distinction between $P_B$ and $\tilde{P}_B$. The former is the *discounted* ultimate probability of trade while the latter is the *undiscounted* ultimate probability of trade. Examination of the derivation of $P_B$ (Eq. (7)) shows that the formulas for $\tilde{P}_B(v)$ and $\tilde{P}_S(c)$ are

$$\tilde{P}_B(\lambda) = \frac{\rho_B(\lambda)}{1 - \mathrm{e}^{-\mu\delta} + \mathrm{e}^{-\mu\delta}\rho_B(\lambda)} \geqslant P_B(\lambda) \tag{25}$$

and

$$\tilde{P}_S(\lambda) = \frac{\rho_S(\lambda)}{1 - \mathrm{e}^{-\mu\delta} + \mathrm{e}^{-\mu\delta}\rho_S(\lambda)} \geqslant P_S(\lambda). \tag{26}$$

Together the fixed point conditions (19) and (21), the expected utility formulas (18) and (20), the steady-state conditions (22) and (23), and the overall mass balance equation (24) form a useful set of necessary conditions for equilibria of our model.

## 4. Discussion

### 4.1. A computed example

These necessary conditions (18)–(24) supplemented with boundary conditions enable us to compute an illustrative example of an equilibrium for our model and to show how, as $\delta$ is reduced, the equilibrium converges towards the perfectly competitive limit. The boundary conditions are

$$S(0) = \underline{c}, \quad \lim_{c \uparrow \bar{v}} S(c) = \bar{c}, \quad \lim_{c \uparrow \bar{v}} W_S(c) = 0,$$
$$\lim_{v \downarrow \underline{c}} B(v) = \underline{v}, \quad B(1) = \bar{v}, \quad \lim_{v \downarrow \underline{c}} W_B(v) = 0$$

where $\bar{c} = \bar{v} = \bar{p}$ and $\underline{v} = \underline{c} = \underline{p}$. Our computation specifies that traders' private values are drawn from the uniform distribution $(g_S(c) = g_B(v) = 1)$ on the unit interval, the mass of buyers entering each unit of time exceeds the mass of sellers entering by 10% ($a = 1.1$), the exit rate is one per unit time ($\mu = 1.0$), and the discount rate is zero ($r = 0.0$). The Walrasian price for these parameter values is $p_W = 0.524$. We computed the equilibrium by fitting sixth degree Chebyshev polynomials to the set of conditions using the method of collocation.[18]

Fig. 1 graphs equilibrium strategies $S$, $B$ and steady-state densities $f_S$, $f_B$ for these parameter values.[19] The left column of the figure graphs strategies and densities for period length $\delta = 0.2$; the right column does the same for period length $\delta = 0.1$. Visual inspection of these equilibria shows the flattening of strategies that occurs as the period length shortens and each trader's option to wait another period for a better deal becomes more valuable. Thus, as $\delta$ is cut in half, the trading range $[\underline{p}, \bar{p}]$ narrows from [0.387, 0.570] down to [0.449, 0.550], which is almost a halving of its width from 0.182 to 0.100. In both equilibria the buyer-seller ratio is $\zeta = 1.570$. Observe that for both period lengths the trading range includes the Walrasian price. Inspection of the densities shows that, as the period length shortens, sellers with costs just below $\bar{c}$ and buyers with values just above $\underline{v}$ tend to accumulate within the market.

Relative inefficiency is the expected gains that the traders would realize if the market were perfectly competitive divided into the expected gains that the traders *fail* to realize in the equilibrium of the double auction market. Comparing these two computed equilibria, cutting $\delta$ in half cuts the relative inefficiency $I$ of the equilibrium by slightly less than half: $I = 0.106$ for $\delta = 0.2$ and $I = 0.056$ for $\delta = 0.1$. Thus the decrease in relative inefficiency appears to be roughly linear with respect to $\delta$. The driver of this rate is the exit rate $\mu$ that forces $\delta\mu$ proportion of each trader type to leave the market discouraged, irrespective of their potential gains from trade. If as $\delta$ is reduced traders' strategies were to remain invariant—an increasingly good approximation as $\delta$ becomes small—then cutting $\delta$ in half reduces by half the waste that exit before trade generates.[20]

### 4.2. Sources of inefficiency

Fig. 1 highlights how this paper's model with its friction of an exogenous exit rate creates equilibrium inefficiency First, traders may have to wait before trading. This happens in several ways: a seller may fail to be matched with any buyers, a buyer may fail to outbid competing buyers, and a seller may set a reservation price higher than the highest bid she receives. Whatever the cause, a trader who fails to trade in the current period has $1 - e^{-\delta\mu}$ probability of exiting rather than carrying over to the next period. This wastes the gains from trade that he would have realized if he had not exited. Second, in any equilibrium the set of seller types who enter intersects with the set of buyer types who enter:

[18] See Judd (1998).

[19] We do not know if this equilibrium is unique.

[20] As $\delta$ is reduced from 0.2 to 0.1 the relative inefficiency is not quite reduced by half. An important reason for this is that reducing $\delta$ has no effect on the gains from trade that are realized by traders who succeed in trading the very period they enter.

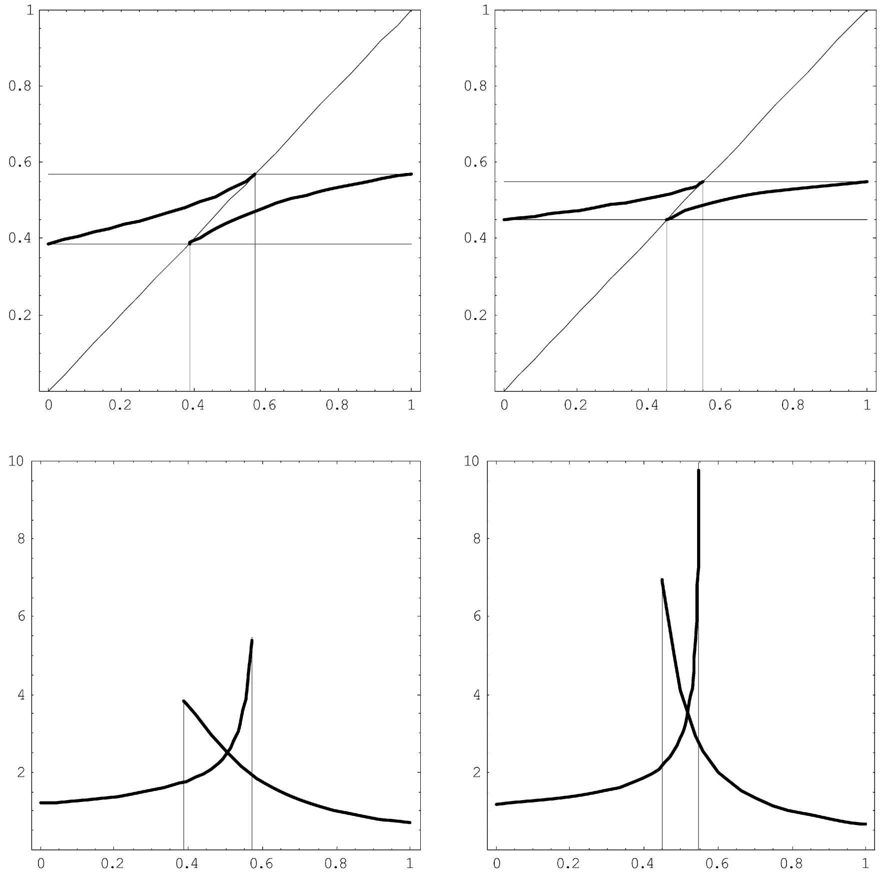


Fig. 1. This figure graphs two equilibria for the case in which $g_S$ and $g_B$ are uniform, $a = 1.1$, $\mu = 1.0$, and $r = 0.0$. The lower and upper curves in each top panel graph the bidding strategy $B(v)$ and the offer strategy $S(v)$ respectively. Each lower panel graphs the densities of types in the market: the increasing curve is the sellers' type density $F_S(c)$ and the decreasing curve is the buyers' type density $F_B(v)$. The two panels on the left side illustrate an equilibrium with period length $\delta = 0.20$. It has relative inefficiency $I = 0.095$ and masses of active traders $T_S = 0.201$ and $T_B = 0.316$. The two panels on the right side illustrate an equilibrium with period length $\delta = 0.10$. It has relative inefficiency $I = 0.0513$ and masses of active traders $T_S = 0.106$ and $T_B = 0.166$.

$$[0, \bar{c}_\delta) \cap (\underline{v}_\delta, 1] \neq [\underline{p}_\delta, \bar{p}_\delta].$$

This means that in every equilibrium trades occur between types that would not occur in a competitive equilibrium, e.g., a type $v$ buyer and type $c$ seller may trade even though $v > c > p_W$. Note that in some equilibria (including both that are graphed in Fig. 1) $p_W \in [\underline{p}_\delta, \bar{p}_\delta]$ so that every type who should enter does enter.[21] In such equilibria these two sources of inefficiency are the only sources of inefficiency.

This may be compared with the sources of inefficiency in our participation cost paper (Satterthwaite and Shneyerov, 2007). In that model's "full trade equilibria" every match results in a

[21] We have not been able to rule out that equilibria exist in which $p_W \notin [\underline{p}_\delta, \bar{p}_\delta]$.

trade.[22] Nevertheless there are again two sources of inefficiency. As in this paper's model inefficiencies in matching occur when a seller fails to match with any buyer, when a buyer is outbid by other buyers, and when a seller sets her reservation price higher than the best bid any buyer submits to her. All such traders carry over to the next period, incurring wasteful participation costs $\delta\kappa$ where $\kappa$ is cost of participating in the market per unit time. The cumulative participation costs this waiting generates is analogous to the waste that exit without trade generates in this paper's model.

Additionally, in the participation cost model's full trade equilibria a gap necessarily exists between $\bar{c}_\delta$, the highest cost seller who enters, and $\underline{v}_\delta$, the lowest value seller who enters:

$$[0, \bar{c}_\delta) \cap (\underline{v}_\delta, 1] = \emptyset.$$

The Walrasian price necessarily falls within this gap: $p_W \in (\bar{c}_\delta, \underline{v}_\delta)$. Therefore potential sellers with types $c \in (\bar{c}_\delta, p_W)$ and potential buyers with types $v \in (p_W, \underline{v}_\delta)$ should enter and trade, but do not. Their potential gains from trade are lost.

The conclusion from this comparison is that while both the exogenous exit rate model and the participation cost model generate equilibrium inefficiency as a result of waiting, each has a distinct source of inefficiency that the other does not share: too much entry in the case of an exogenous exit rate and too little entry in the case of participation costs.

### 4.3. Mechanisms of convergence

*Law of one price.* As mentioned in the introduction, option value forces the trading range $[\underline{p}_\delta, \bar{p}_\delta]$ to collapse to a single price as $\delta \to 0$. A basic intuition for this is straightforward. Let

$$W^B(v, \rho_B, \delta) = \sum_{i=0}^{\infty} \left(\mathrm{e}^{-\beta\delta} \bar{\rho}_B\left(\lambda^*\right)\right)^i \rho_B\left(\lambda^*\right)\left(v - \lambda^*\right); \tag{27}$$

it is the expected present value of a type $v$ buyer who, facing period length $\delta$ and current period trading probabilities $\rho_B$, makes the bid $\lambda^*$ that is optimal. Note that $W^B(v, \rho_B, \delta)$ is decreasing in $\delta$.

Consider an equilibrium when the period length is $\delta' > 0$. A type $v$ active buyer selects his current period bid $\lambda' = B_{\delta'}(v)$ that maximizes his expected net present value:

$$B_{\delta'}(v) \in \arg\max_{\lambda \in [0,v]} \left[\rho_{B\delta'}(\lambda)(v - \lambda) + \bar{\rho}_{B\delta'}(\lambda)\mathrm{e}^{-\beta\delta'} W^B\left(v, \rho_{B\delta'}, \delta'\right)\right].$$

The product $\mathrm{e}^{-\beta\delta'} W^B(v, \rho_{B\delta'}, \delta')$ is his opportunity cost—his option value—of successfully trading in the current period.

Now, suppose that the period length shortens to $\delta'' < \delta'$, every buyer realizes this is so, but every buyer continues to act as if the trading probabilities remain unchanged at $\rho_{\delta'}$. This increases his opportunity cost of trading now, $\mathrm{e}^{-\beta\delta''} W^B(v, \rho_{B\delta'}, \delta'') > \mathrm{e}^{-\beta\delta'} W^B(v, \rho_{B\delta'}, \delta')$, and causes him to revise his bid downward to $\lambda'' < B_{\delta'}(v)$. That is, every buyer type revises his bid downward towards the bottom of the original trading range $[\underline{p}_\delta, \bar{p}_\delta]$. In particular, a type 1 buyer bids less than $B_{\delta'}(1) = \bar{v}_{\delta'} = \bar{p}_{\delta'}$, which chops off the top of the trading range. A symmetric argument implies that as $\delta$ shrinks $S_\delta(0) = \underline{c}_\delta = \underline{p}_\delta$ increases. Together these two arguments suggest that

[22] They prove the existence of full trade equilibria and computationally construct examples of them. They, however, have been unable to rule out the existence of other classes of equilibria.

as $\delta$ decreases to $\delta''$ from $\delta'$ the trading range $[\underline{p}_\delta, \bar{p}_\delta]$ gets shortened at both ends and forces the trading range to converge to a single price.

This intuition is incomplete, for it neglects the effect that reducing $\delta$ has (i) on the trading probabilities $\rho_{S\delta}$ and $\rho_{B\delta}$ and (ii) the endogenous, steady-state distributions $F_{S\delta}$ and $F_{B\delta}$ of active traders' types. If, as $\delta$ shrinks, active buyers as a whole increasingly make bids that are accepted only by sellers whose costs relative to all active sellers are low, then low cost sellers rapidly exit from the market while high cost sellers accumulate. This tendency is apparent in the bottom row of Fig. 1. The density of low cost sellers and high value buyers decreases slightly as $\delta$ decreases. More noticeably, the density of high cost sellers and low value buyers increases dramatically as $\delta$ decreases. These endogenous changes decrease the incentive of buyers to bid aggressively low and sellers to set their reservation values aggressively high. Conceivably, this skewing of $F_{S\delta}$ and $F_{B\delta}$ could cancel the option value effect and stall convergence. However, the proof of Proposition 7 shows that this cannot happen. Moreover with only moderate customization this proof applies both to this paper's model and to our participation cost model (Satterthwaite and Shneyerov, 2007). The conclusion is that in our matching and bargaining models the option value of waiting increases as the per period friction decreases and drives convergence towards one price.

This intuition is useful in considering the possible effects of a plausible modification of our model. We assume in our model that a trader chooses his bid/offer to maximize unconditionally his expected gains over all possible match sizes. It would arguably be a more attractive assumption to follow Dagan et al. (2000) and allow buyers and sellers to tailor their strategies to whatever match is realized, i.e., each trader would condition his bid/reservation cost on the realization of the number of buyers competing within his match. If we adopted this change, then in a given equilibrium each seller and each buyer's strategy would be an infinite sequence of functions: $\{S^1_\delta(c), S^2_\delta(c), \ldots, S^k_\delta(c), \ldots\}$ and $\{B^0_\delta(c), B^1_\delta(c), \ldots, B^\ell_\delta(c), \ldots\}$ where $k$ is the number of buyers with whom the seller is matched with and $\ell$ is the number of other buyers with whom the buyer is bidding against.

Our belief is that doing this would, in the end, make no difference in our result even while considerably increasing the complexity of our notation and proof. The reason is that, as $\delta$ becomes small, the discount factor $\mathrm{e}^{-\delta\beta} = \mathrm{e}^{-\delta(\mu+r)}$ approaches one, waiting becomes cheap and, exactly as above, traders will only accept a small difference between the best possible price and the price available in the current match. To be more specific, suppose that the trading range $[\underline{p}_\delta, \bar{p}_\delta]$ does not shrink to a point as $\delta$ approaches to zero, i.e., $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = \varepsilon > 0$ where now $\bar{p}_\delta \equiv \sup_{\ell\in\{0,1,2,\ldots\}}\{B^0_\delta(1), B^1_\delta(1), B^2_\delta(1), \ldots, B^\ell_\delta(1), \ldots\}$. An active buyer's maximization problem is:

$$B_\delta(v) = \arg\max_{\lambda\in[0,1]} P_{B\delta}(\lambda)(v-\lambda).$$

Formula (7) for the discounted ultimate probability of trade has the property that if $\rho_{B\delta}(\lambda) > 0$, no matter how small, $\lim_{\delta\to 0} P_{B\delta}(\lambda) = 1$. This means that, if $\delta$ is small, then an active type $v$ buyer can wait, with only a very small chance of exiting before trading, for the conjunction of two favorable, independent events: a match in which (i) he is the only buyer bidding for the seller's unit of supply and (ii) the seller has a cost that is close to zero. His option value driven, optimal bid is therefore, for sufficiently small $\delta$ and all possible match sizes $\ell$, to set $B^\ell_\delta(v)$ very close to $\underline{p}_\delta$ (i.e., less than $\varepsilon$ above $\underline{p}_\delta$) because $\rho_{B\delta}(B^\ell_\delta(v)) > 0$ whenever $B^\ell_\delta(v) > \underline{p}_\delta$. But then

$$\bar{p}_\delta \equiv \sup_{\ell\in\{0,1,2,\ldots\}} \left\{B^0_\delta(1), B^1_\delta(1), B^2_\delta(1), \ldots, B^\ell_\delta(1), \ldots\right\} < \underline{p}_\delta + \varepsilon,$$

which contradicts $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = \varepsilon$. Thus, it appears, in the model in which traders condition on the number of other traders in the match, option value causes the trading range to be progressively truncated from both ends as $\delta$ approaches zero, just as it does in our model in which traders do not condition on the match size.

*Convergence to the Walrasian price*. The necessity of supply to equal demand in the steady state causes the trading range $[\underline{p}_\delta, \bar{p}_\delta]$ to converge to $p_W$. To see this, suppose it converges to a price $p_*$ less than $p_W$. This attracts more buyers into the market, i.e., potential buyers with values in the interval $(\underline{p}, p_W)$ who would not enter in a competitive equilibrium find it worthwhile to enter. This excess entry of buyers requires that there be a compensating increased probability of exit prior to trading; otherwise the measure of buyers who are active would increase and the market would not be in steady state.

Let $\delta$ be small so that the length of $[\underline{p}_\delta, \bar{p}_\delta]$ is small. Consider a type $v'$ buyer whose value falls approximately midway in the interval $[\bar{p}_\delta, 1]$ so that $\bar{p}_\delta \ll v' \ll 1$. Necessarily this buyer's ultimate probability of trading, $P_{B\delta}(B_\delta(v))$, is less than one. But by increasing his bid from $B_\delta(v') = \lambda < \bar{p}_\delta$ to $\lambda' = \bar{p}_\delta$ he can increase his probability of ultimately trading from $P_{B\delta}(B_\delta(v'))$ to one. Given that $\delta$ is small, this increase in bid from $\lambda$ to $\lambda'$ increases the price that he pays only a tiny bit even as it guarantees that he will trade with certainty. The reason is that $\lambda, \lambda' \in [\underline{p}_\delta, \bar{p}_\delta]$ and the interval $[\underline{p}_\delta, \bar{p}_\delta]$ is narrow. Therefore, for small enough $\delta$, it pays the type $v$ buyer to deviate upward from his equilibrium bid, contradicting the hypothesis that in the sequence of equilibria the trading range shrinks to $p_* < p_W$.

This intuition underlies the construction of Claim 8's proof that $\lim_{\delta\to 0} \bar{v}_\delta \geqslant p_W$. A tighter rendition of this intuition follows. Consider the following hypothetical equilibrium that we have crafted to illustrate this mechanism. The distributions $G_B$ and $G_S$ of potential entrants' types are uniform on $[0, 1]$. The Walrasian price for the market is $p_W = 0.5$, but suppose as $\delta$ shrinks to zero the prices at which transaction take place converge to $p_* = 0.25$, i.e., $\lim_{\delta\to 0} \bar{v}_\delta = \lim_{\delta\to 0} \underline{v}_\delta = p_*$. Further suppose that when the period length is $\delta'$ the range of transaction prices is $[\underline{v}_{\delta'}, \bar{v}_{\delta'}] = [0.23, 0.27]$. Can this putative equilibria with its narrow trading range actually be an equilibrium?

No! To show this we choose a type $v' = 0.35$ buyer and show that this trading range implies upper and lower bounds on the discounted ultimate probability of trade, $P_{B\delta'}(B_{\delta'}(v'))$, that are mutually inconsistent. The upper bound is

$$P_{B\delta}\big(B_\delta\big(v'\big)\big) \leqslant \frac{G_S(\bar{v}_\delta)}{a\bar{G}_B(v')} \tag{28}$$

where $\bar{G}_B(v') \equiv 1 - G_B(v')$. To see this first recall (25), $P_{B\delta}(B_\delta(v')) \leqslant \tilde{P}_{B\delta} B_\delta(v')$, i.e., the discounted probability of trade is no more than the undiscounted probability of trade. To prove (28) it is therefore sufficient to show $\tilde{P}_{B\delta}(B_\delta(v')) \leqslant G_S(\bar{v}_\delta)/a\bar{G}_B(v')$ or, rewritten,

$$a\tilde{P}_{B\delta}\big(B_\delta\big(v'\big)\big)\bar{G}_B\big(v'\big) \leqslant G_S(\bar{v}_\delta). \tag{29}$$

Observe that $a\bar{G}_B(v')$ is the measure of buyers that (i) enter each period and have valuation at least $v'$ and (ii) $\tilde{P}_{B_\delta}(B_\delta(\cdot))$ is non-decreasing because $\tilde{P}_{B\delta}$ is non-decreasing and $B_\delta$ is increasing. Therefore the measure of buyers with valuations $v > v'$ who enter each period and ultimately trade is at most $a\tilde{P}_{B_\delta}(B_\delta(v'))\bar{G}_B(v')$, i.e., it is a lower bound on demand. Supply is the measure of sellers who enter each period and ultimately trade. It is bounded above by $G_S(\bar{v}_\delta)$, the measure of traders who enter each period and who may or may not trade. Mass balance, Eq. (24), states that demand must equal supply in a steady-state equilibrium. Its satisfaction requires that the lower bound on demand must be less than the upper bound on supply, which is exactly inequality (29). Upper bound (28) is therefore established.

The lower bound on $P_{B\delta}(B_\delta(v'))$ follows from a revealed preference argument. In equilibrium the type $v'$ buyer must prefer bidding $B_\delta(v')$ to bidding $B_\delta(1)$:

$$P_{B_\delta}\big(B_\delta(v')\big)\big(v' - B_\delta(v')\big) \geqslant P_{B\delta}\big(B(1)\big)\big(v' - B_\delta(1)\big).$$

Observe that $P_{B\delta}(B(1)) = 1$ because every period every buyer is matched with a seller and a type 1 buyer is never outbid. Therefore a lower bound on $P_{B_\delta}(v')$ is

$$P_{B_\delta}\big(B_\delta(v')\big) \geqslant \frac{(v' - B_\delta(1))}{(v' - B_\delta(v'))} \geqslant \frac{v' - \bar{v}_\delta}{v' - \underline{v}_\delta},$$

where the second inequality is a consequence of $B_\delta(1) = \bar{v}_\delta$ and $B_\delta(v') \in [\underline{v}_\delta, \bar{v}_\delta]$.

We now apply these bounds to the type $v' = 0.35$ buyer. Recall that $G_B$ and $G_S$ are uniformly distributed on [0, 1]. Computation for the putative $\delta'$ equilibrium gives an upper bound,

$$P_{B\delta'}\big(B_{\delta'}(0.35)\big) \leqslant \frac{G_S(\bar{v}_{\delta'})}{\bar{G}_B(v')} = \frac{0.27}{0.65} = 0.42,$$

that is less than the lower bound,

$$P_{B\delta'}\big(B_{\delta'}(0.35)\big) \geqslant \frac{.35 - .27}{.35 - .23} = 0.67.$$

Therefore the hypothesized sequence of equilibria in which the trading range converges to $p_* < p_W$ cannot exist.

### 4.4. Existence of equilibria

While we have been unable to prove that in general an equilibrium exists for our model, we have been able to prove existence whenever the type distribution $G_S$ is concave.[23] This class of distributions does include the uniform distribution that we used for construction of our example above, but except for constructing examples concavity is not an economically plausible assumption to impose on type distributions. This proof, which parallels Athey (2001), is both long and tedious. Given its length and lack of generality we decided not to include it here. We do point out that our participation cost paper, Satterthwaite and Shneyerov (2007), does include a satisfactory existence result. Moreover the recent working papers of Shneyerov and Wong (2007a) and of Atakan (2007a, 2007b) also include existence results.

## 5. Proof of the theorem

### 5.1. The law of one price

In this subsection we prove that the trading range converges to a single price as the period length approaches zero: $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = 0$. As the discussion above indicates, this is driven by option value—when the period length is short a trader can wait until he or she draws a favorable price—complicated by the fact that distributions $F_S$ and $F_B$ of active traders' costs/values is

[23] Adam Szeidl collaborated with us on this proof.

endogenous. This proof is based on the proof of Lemma 10 in Satterthwaite and Shneyerov (2007). The differences, which occur in step 1, stem from the market friction here being the exogenous exit rate rather than being a participation cost.

**Proposition 7.** *Consider any sequence of equilibria for which $\delta_n \to 0$. Then* $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = \lim_{\delta\to 0}(\bar{v}_\delta - \underline{v}_\delta) = \lim_{\delta\to 0}(\bar{c}_\delta - \underline{c}_\delta) = 0$.

**Proof.** The proof is by contradiction: pick a small $\varepsilon$ and suppose $\bar{p}_\delta - \underline{p}_\delta > \varepsilon > 0$ along a subsequence for which $\delta \to 0$. Define

$$b'_\delta = \bar{p}_\delta - \frac{1}{3}\varepsilon,$$
$$b''_\delta = \bar{p}_\delta - \frac{2}{3}\varepsilon \tag{30}$$

and note that $b''_\delta > \underline{p}_\delta + \frac{\varepsilon}{3}$. Also define two probabilities. First

$$\phi_\delta = F_{S\delta}\bigl(C_\delta(b'_\delta)\bigr)$$

is the equilibrium probability that a randomly chosen active seller would accept a bid that is equal to $b'_\delta$. Second, let

$$\begin{aligned}\psi_\delta &= \sum_{k=0}^{\infty} \xi_{k\delta}\bigl[F_{B\delta}\bigl(V_\delta(b'_\delta)\bigr)\bigr]^k \\ &= \sum_{k=0}^{\infty} \omega_{k\delta}\bigl[F_{B\delta}\bigl(V_\delta(b'_\delta)\bigr)\bigr]^k.\end{aligned}$$

The first series is the equilibrium probability that a seller receives either no bid or the highest bid she receives is less or equal to $b'_\delta$ (see Eqs. (15) and (16)). The second series is the probability that a buyer who bids $b'_\delta$ outbids all other buyers who are matched with the same seller. The equality of these two probabilities follows from the equality $\xi_{k\delta} = \omega_{k\delta}$ that Eq. (17) established. Given these definitions the proof consists of three steps.

**Step 1.** *The fraction of sellers for whom $S_\delta(c) \leqslant b'_\delta$ does not vanish as $\delta \to 0$, i.e., $\underline{\phi} \equiv \underline{\lim}_{\delta\to 0}\phi_\delta > 0$.*

Pick a single period $t$. Let $m_\delta^+$ be the mass of sellers who enter the market at the beginning of period $t$ and for whom $b''_\delta \leqslant S_\delta(c) \leqslant b'$. Partition this mass of sellers into three masses:

- $m_\delta^-$ is the mass of sellers within $m_\delta^+$ who exit by trading during period $t$.
- $m_\delta^=$ is the mass of sellers within $m_\delta^+$ who for exogenous reasons exit without trading during period $t$.
- $m_\delta^0$ is the mass of sellers within $m_\delta^+$ who do not exit during period $t$.

Finally, let $m_\delta$ be the *steady-state* mass of active sellers for whom $b''_\delta \leqslant S_\delta(c) \leqslant b'_\delta$.

Suppose step 1's conclusion is not true. Then $\phi_\delta \to 0$ along a subsequence. Fix this subsequence. The hypothesis that $\phi_\delta \to 0$ implies that the mass of sellers entering each period who offer no more than $b'_\delta$ approaches zero.[24] Therefore $m_\delta \to 0$ as $\delta \to 0$. We show now that $m_\delta \to 0$ entails $\underline{c}_\delta \to b''_\delta$. This establishes a contradiction because Proposition 6 states that $\underline{c}_\delta = \underline{p}_\delta$ and by construction $b''_\delta > \underline{p}_\delta + \frac{\varepsilon}{3}$.

In the mass $m^+_\delta$ pick a seller $c''_\delta$ for whom

$$S_\delta\left(c''_\delta\right) = b''_\delta.$$

Note that her reservation price is as low as any other seller in $m^+_\delta$. Such a seller $c''_\delta$ always exists because $S_\delta$ is continuous (see Proposition 6) and $\underline{g}$ is a lower bound on the density of entering sellers. Her probability trading within period $t$ is $\rho_{S\delta}(b''_\delta)$; it is as high as the trading probability of any other seller in $m^+_\delta$ because $\rho_{S\delta}$ is non-increasing. Therefore $\rho_{S\delta}(b''_\delta)$ is at least as great as $m^-_{N\delta}/m^+_\delta$, which for any seller in $m^+_\delta$ is the ex ante probability of trading within period $t$. Thus

$$\rho_{S\delta}\left(b''_\delta\right) \geqslant \frac{m^-_\delta}{m^+_\delta}. \tag{31}$$

Now make the trivial observation $P_{S\delta}(b''_\delta)$, the discounted ultimate trading probability of a type $c''_\delta$ seller who trades either in period $t$ or after, is greater than $\rho_{S\delta}(b''_\delta)$. Therefore

$$P_{S\delta}\left(b''_\delta\right) \geqslant \rho_{S\delta}\left(b''_\delta\right) \geqslant \frac{m^-_\delta}{m^+_\delta}. \tag{32}$$

If counterfactually in period $t$ no sellers in the mass $m^+_\delta$ traded, then in period $t$ the mass that would exit for exogenous reasons would be $(1 - \mathrm{e}^{-\mu\delta})m^+_\delta$ because $(1 - \mathrm{e}^{-\mu\delta})$ is the probability that an active trader who has not traded in a period will exit for exogenous reasons. But some sellers within mass $m^+_\delta$ do successfully trade; therefore $m^=_\delta \leqslant (1 - \mathrm{e}^{-\mu\delta})m^+_\delta$. By definition $m^+_\delta = m^-_{N\delta} + m^=_{N\delta} + m^0_{N\delta}$, which implies that

$$\begin{aligned} m^+_\delta &\leqslant m^-_{N\delta} + \left(1 - \mathrm{e}^{-\mu\delta}\right)m^+_\delta + m^0_{N\delta} \\ &\leqslant m^-_{N\delta} + \left(1 - \mathrm{e}^{-\mu\delta}\right)m^+_\delta + m_\delta \end{aligned}$$

where the last inequality follows from the fact that the surviving mass $m^0_\delta$ from period 1 cannot exceed the steady-state mass $m_\delta$ of sellers who make offers in the interval $[b''_\delta, b'_\delta]$. Solving gives

$$m^-_\delta \geqslant \mathrm{e}^{-\mu\delta} m^+_\delta - m_\delta.$$

Substitute into (32) to get:

$$P_{S\delta}\left(b''_\delta\right) \geqslant \frac{m^-_{N\delta}}{m^+_\delta} \geqslant \left(\mathrm{e}^{-\mu\delta} - \frac{m_\delta}{m^+_\delta}\right). \tag{33}$$

Remember that $m^+_\delta$ is the mass of sellers who enter the market in period 1 and for whom $b''_\delta \leqslant S_\delta(c) \leqslant b'$. The slope of $S_\delta$ is at most one (see the formula in Proposition 6), the mass of potential entering sellers each period is 1, and the density $g_S$ has lower bound $\underline{g}$. Consequently

$$m^+_\delta \geqslant \frac{\varepsilon}{3}\underline{g}$$

[24] If it were possible for the steady-state mass $T_S$ of sellers to be infinite, then $\phi_\delta \to 0$ would not imply $m_\delta \to 0$. But $T_S$ must be finite because the mass of potential sellers per unit time is 1 and the exogenous exit rate per unit time $\mu$, is positive.

because $m_\delta^+$ is minimized when the slope of $S_\delta$ is maximal, the density $g_S$ is minimal, and $b'_\delta - b''_\delta = \varepsilon/3$. Substituting this into (33) gives the lower bound we need:

$$P_{S\delta}\bigl(b''_\delta\bigr) \geqslant \mathrm{e}^{-\mu\delta} - \frac{m_\delta}{\frac{\varepsilon}{3}\underline{g}}. \tag{34}$$

Recall that this step's argument is based on the hypothesis that $\phi_\delta \to 0$ as $\delta \to 0$. Further recall that $\phi_\delta \to 0$ implies that $m_\delta \to 0$ as $\delta \to 0$. Therefore

$$\lim_{\delta\to 0} P_{S\delta}\bigl(b''_\delta\bigr) \geqslant \lim_{\delta\to 0}\left[\mathrm{e}^{-\beta\delta} - \frac{m_\delta}{\frac{\varepsilon}{3}\underline{g}}\right] = 1,$$

i.e., the type $c''_\delta$ seller's discounted probability of trade approaches 1 from below. Proposition 6 states that, for almost all $c \in [0, \bar{c})$,

$$S'_\delta(c) = 1 - \mathrm{e}^{-\beta\delta} P_{S\delta}\bigl[S_\delta(c)\bigr].$$

Since $P_{S\delta}(S_\delta(c)) \geqslant P_{S\delta}(b''_\delta)$ for $c \leqslant c''_\delta$ and $S_\delta$ is increasing on $[0, \bar{c})$, it follows that, for all seller types $c \in [0, c''_\delta]$, $P_{S\delta}[S_\delta(c)] \to 1$ and

$$\lim_{\delta\to 0} S'_\delta(c) = 0.$$

Consequently, because $S_\delta$ is continuous,

$$\underline{c}_\delta = S_\delta(0) \to b''_\delta.$$

This, however, is a contradiction, for $\underline{c}_\delta = \underline{p}_\delta$ by Proposition 6 and $b''_\delta > \underline{p}_\delta + \varepsilon/3$ by construction. Therefore it cannot be that $\phi_\delta \to 0$.

**Step 2.** *If the ratio of buyers to sellers $\zeta_\delta$ is bounded away from* 0*, then the probability $\psi_\delta$ that the highest bid in a given meeting is less than $b'_\delta$ is also bounded away from* 0. Proof of this step stands alone and does not depend on step 1's result.

Formally, if $\underline{\lim}_{\delta\to 0}\zeta_\delta > 0$, then $\underline{\psi} \equiv \underline{\lim}_{\delta\to 0}\,\psi_\delta > 0$. Suppose not. Then $\psi_\delta \to 0$ and $\zeta_\delta \to \underline{\zeta} > 0$ along a subsequence. Fix this subsequence and recall that by construction

$$b''_\delta > \underline{p}_\delta + \frac{\varepsilon}{3}. \tag{35}$$

First, we show that the seller with cost $c''_\delta$ such that $S(c''_\delta) = b''_\delta$ prefers to enter. Since $\zeta_\delta \to \underline{\zeta}$ and $\psi_\delta \to 0$, for all $\delta$ sufficiently small, the probability that she meets a buyer for whom $B(v) \geqslant b'_\delta = b''_\delta + \varepsilon/3$ is at least $\frac{1}{2}(1 - \mathrm{e}^{-\underline{\zeta}})$. This is because, with $\psi_\delta \to 0$, (i) almost every bid she receives is greater than $b'_\delta$ and (ii) her probability of getting at least one bid is approaching $1 - \mathrm{e}^{-\underline{\zeta}}$, i.e., $\lim_{\delta\to 0}\rho_{S\delta}(b'_\delta) = 1 - \mathrm{e}^{-\underline{\zeta}}$. Therefore, using formula (8), her discounted probability of trading with a buyer for whom $B_\delta(v) \geqslant b'_\delta$ approaches 1:

$$\lim_{\delta\to 0} P_{S\delta}\bigl(b'_\delta\bigr) = \lim_{\delta\to 0}\frac{\rho_{S\delta}(b'_\delta)}{1 - \mathrm{e}^{-\beta\delta} + \mathrm{e}^{-\beta\delta}\rho_{S\delta}(b'_\delta)} = 1.$$

Consequently, the type $c''_\delta$ seller's profit as $\delta \to 0$ is at least $\varepsilon/3$ and she chooses to enter.

Second, since she chooses to enter, it must be that $c''_\delta \leqslant \bar{c}_\delta = \bar{p}_\delta$. Therefore the slope of $S$ for $c \in [0, c''_\delta)$ satisfies

$$S'(c) = 1 - e^{-\beta\delta} P_{S\delta}(c) \to 0$$

since $P_{S\delta}(S_\delta(c)) \geqslant P_{S\delta}(S_\delta(c''_\delta))$ and $P_{S\delta}(S_\delta(c''_\delta)) \to 1$. Therefore $\underline{c}_\delta \to b''_\delta$, a contradiction of inequality (35) and Proposition 6's conclusion that $\underline{c}_\delta = \underline{p}_\delta$.

**Step 3.** *For small enough $\delta$, a buyer for whom $v = 1$ prefers to deviate to bidding $b'_\delta$ instead of $\bar{p}_\delta$.* There are two cases to consider.

**Case 1.** $\underline{\lim}_{\delta\to 0}\zeta_\delta > 0$. We show, using both steps 1 and 2 of this proof, that bidding $\bar{p}_\delta$ cannot be equilibrium behavior for a type 1 buyer. Recall that $\phi_\delta$ is the probability that a seller will accept a bid less than $b'_\delta$ and that, according to step 1, $\underline{\phi} = \lim_{\delta\to 0}\phi_\delta > 0$. Additionally, recall that $\psi_\delta$ is the probability that the maximal rival bid a buyer faces in a given period is no greater than $b'_\delta$ and that, according to step 2, $\lim_{\delta\to 0}\psi_\delta = \underline{\psi} > 0$. For small enough $\delta > 0$, this second probability is bounded from below by $(1/2)\underline{\psi}$. It follows that, for small enough $\delta$, the buyer who bids $b'_\delta$ (i) wins over all his rival buyers with probability greater than $(1/2)\underline{\psi}$, and (ii) has his bid accepted by the seller with probability greater than $(1/2)\underline{\phi}$. Thus, for small enough $\delta$, $\rho_{S\delta}(b'_\delta) > \underline{\psi}\,\underline{\phi}/4$ and, again using formula (8), $\lim_{\delta\to 0} P_{S\delta}(b'_\delta) = 1$. Consequently deviating to $b'_\delta$ gives the type 1 buyer a profit of at least $1 - b'_\delta$, which is greater than $1 - \bar{p}_\delta$, the profit he would make with his equilibrium bid $B(1) = \bar{p}_\delta$. Therefore deviation to $b'_\delta$ is profitable for him.

**Case 2.** $\underline{\lim}_{\delta\to 0}\zeta_\delta = 0$. Fix a subsequence such that $\zeta_\delta \to 0$. The proof of this case relies only on the result in step 1 of this proof. The probability of meeting no rival buyers in a given period is $e^{-\zeta_\delta}$ and, since $\zeta_\delta \to 0$, this probability is at least $1/2$ for sufficiently small $\delta$. Therefore in any given period, for a type 1 buyer and for all small $\delta$, (i) the probability of meeting no rivals is at least $1/2$ and (ii) the probability of meeting a seller who would accept the bid $b'_\delta$ is at least $(1/2)\underline{\phi} > 0$. It follows that as $\delta \to 0$, his discounted probability of trading, $P_{S\delta}(b'_\delta)$, approaches 1. Therefore deviating to $b'_\delta$ gives him a profit of at least $1 - b'_\delta > 1 - \bar{p}_\delta$, which proves that a deviation to $b'_\delta$ is profitable for him. This completes step 3's proof.

Step 3 completes the proof that $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = 0$ because it contradicts the hypothesis that $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = \varepsilon > 0$. That $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = \lim_{\delta\to 0}(\bar{v}_\delta - \underline{v}_\delta) = \lim_{\delta\to 0}(\bar{c}_\delta - \underline{c}_\delta) = 0$ then follows directly from Proposition 6. □

### 5.2. *Convergence of the trading range to the Walrasian price*

The Walrasian price $p_W$ is the solution to the equation $G_S(p_W) = a\bar{G}_B(p_W)$. Recall from the beginning of Section 2 that $p_W \in (0, 1)$. In this subsection we prove three claims: $\lim_{\delta\to 0}\bar{v}_\delta \geqslant p_W$, $\lim_{\delta\to 0}\underline{c}_\delta \leqslant p_W$, and $\lim_{\delta\to 0} P_{S\delta}(S_\delta(c)) = 1$, $c \in [0, \bar{c})$. Together with Propositions 6 and 7's results that $[\underline{p}_\delta, \bar{p}_\delta] = [\underline{c}_\delta, \bar{c}_\delta] = [\underline{v}_\delta, \bar{v}_\delta]$ and $\lim_{\delta\to 0}(\bar{p}_\delta - \underline{p}_\delta) = 0$ these results immediately imply that the trading range collapses to the Walrasian price.

**Claim 8.** $\underline{\lim}_{\delta\to 0}\bar{v}_\delta \geqslant p_W$.

**Proof.** Let $v_* = \underline{\lim}_{\delta\to 0}\bar{v}_\delta$ and assume, contrary to the statement in the claim, that $v_* < p_W$. For the remainder of this proof, fix a subsequence $\bar{v}_\delta \to v_*$. Let $\tilde{v}_\delta = \bar{v}_\delta + \sqrt{\bar{v}_\delta - \underline{v}_\delta}$. Proposition 7's conclusion that $\lim_{\delta\to 0}(\bar{v}_\delta - \underline{v}_\delta) = 0$ implies that $\tilde{v}_\delta \in (\bar{v}_\delta, 1]$ for all small enough $\delta$. Revealed preference implies that

$$\pi_B\big(B_\delta(\tilde{v}_\delta), \tilde{v}_\delta\big) \geqslant \pi_B\big(B_\delta(1), \tilde{v}_\delta\big)$$
$$\big[\tilde{v}_\delta - B_\delta(\tilde{v}_\delta)\big]\ P_{B\delta}\big[B_\delta(\tilde{v}_\delta)\big] \geqslant \big[\tilde{v}_\delta - B_\delta(1)\big]\ P_{B\delta}\big[B_\delta(1)\big].$$

Therefore

$$\begin{aligned} P_{B\delta}\big[B_\delta(\tilde{v}_\delta)\big] &\geqslant \frac{\tilde{v}_\delta - B_\delta(1)}{\tilde{v}_\delta - B_\delta(\tilde{v}_\delta)} P_{B\delta}\big[B_\delta(1)\big] \\ &\geqslant \frac{\tilde{v}_\delta - B_\delta(1)}{\tilde{v}_\delta - \underline{v}_\delta} P_{B\delta}\big[B_\delta(1)\big], \end{aligned} \tag{36}$$

where the second inequality follows from $B_\delta$ being strictly increasing and, as a result, $B_\delta(\tilde{v}_\delta) \geqslant B_\delta(\underline{v}_\delta) = \underline{v}_\delta$. Note that

$$\begin{aligned} \frac{\tilde{v}_\delta - B_\delta(1)}{\tilde{v}_\delta - \underline{v}_\delta} &= \frac{\tilde{v}_\delta - \bar{v}_\delta}{\tilde{v}_\delta - \underline{v}_\delta} \\ &= \frac{1}{1 + \sqrt{\bar{v}_\delta - \underline{v}_\delta}}, \end{aligned} \tag{37}$$

where the first equality follows from $B_\delta(1) = \bar{v}_\delta$ and the second follows from the definition $\tilde{v}_\delta = \bar{v}_\delta + \sqrt{\bar{v}_\delta - \underline{v}_\delta}$. Combining (36) and (37) gives

$$P_{B\delta}\big[B_\delta(\tilde{v}_\delta)\big] \geqslant \frac{1}{1 + \sqrt{\bar{v}_\delta - \underline{v}_\delta}} P_{B\delta}\big[B_\delta(1)\big]. \tag{38}$$

Mass balance, Eq. (24) above, states that

$$\int_{\underline{v}_\delta}^{1} a g_B(x) \tilde{P}_{B\delta}\big[B_\delta(x)\big]\,\mathrm{d}x = \int_0^{\bar{v}_\delta} g_S(x) \tilde{P}_{S\delta}\big[S_\delta(x)\big]\,\mathrm{d}x. \tag{39}$$

Given that $\tilde{P}_{B\delta}[B_\delta(\cdot)]$ is non-decreasing and $\tilde{v}_\delta > \underline{v}$,

$$\begin{aligned} \int_{\underline{v}_\delta}^{1} a g_B(x) \tilde{P}_{B\delta}\big[B_\delta(x)\big]\,\mathrm{d}x &\geqslant \tilde{P}_{B\delta}\big[B_\delta(\tilde{v}_\delta)\big] \int_{\tilde{v}_\delta}^{1} a g_B(x)\,\mathrm{d}x \\ &= \tilde{P}_{B\delta}\big[B_\delta(\tilde{v}_\delta)\big] a \overline{G}_B(\tilde{v}_\delta). \end{aligned}$$

Because $\tilde{P}_{S\delta}[S_\delta(c)] \leqslant 1$ for all $c$,

$$\int_0^{\bar{v}_\delta} g_S(x) \tilde{P}_{S\delta}\big[S_\delta(x)\big]\,\mathrm{d}x \leqslant G_S(\bar{v}_\delta).$$

Therefore it follows from (39) that

$$\tilde{P}_{B\delta}\big[B_\delta(\tilde{v}_\delta)\big] a \overline{G}_B(\tilde{v}_\delta) \leqslant G_S(\bar{v}_\delta)$$

and, because $P_{B\delta}[\lambda] \leqslant \tilde{P}_{B\delta}[\lambda]$ for all $\lambda$,

$$P_{B\delta}\big[B_\delta(\tilde{v}_\delta)\big] a \overline{G}_B(\tilde{v}_\delta) \leqslant G_S(\bar{v}_\delta).$$

Then, by (38),

$$a\frac{1}{1+\sqrt{\bar{v}_\delta-\underline{v}_\delta}}P_{B\delta}\big[B_\delta(1)\big]\overline{G}_B(\tilde{v}_\delta)\leqslant G_S(\bar{v}_\delta). \tag{40}$$

A type $v=1$ buyer always trades immediately because $B(1)=\bar{v}=\bar{c}=S(\bar{c})$; therefore $P_{B\delta}[B_\delta(1)]=1$. Consequently

$$a\frac{1}{1+\sqrt{\bar{v}_\delta-\underline{v}_\delta}}\overline{G}_B(\tilde{v}_\delta)\leqslant G_S(\bar{v}_\delta).$$

Taking limits as $\delta\to 0$ and invoking continuity of $G_S$and $\overline{G}_B$, we obtain

$$a\lim_{\delta\to 0}\left(\frac{1}{1+\sqrt{\bar{v}_\delta-\underline{v}_\delta}}\right)\overline{G}_B\Big(\lim_{\delta\to 0}\tilde{v}_\delta\Big)\leqslant G_S\Big(\lim_{\delta\to 0}\bar{v}_\delta\Big). \tag{41}$$

By definition $\tilde{v}=\bar{v}_\delta+\sqrt{\bar{v}_\delta-\underline{v}_\delta}$; Proposition 7's result that $\lim_{\delta\to 0}(\bar{v}_\delta-\underline{v}_\delta)=0$ therefore implies $\lim_{\delta\to 0}\tilde{v}_\delta=\bar{v}_\delta$. Additionally, by hypothesis, $\lim_{\delta\to 0}\bar{v}_\delta=v_*$. Therefore (41) reduces to

$$a\overline{G}_B(v_*)\leqslant G_S(v_*).$$

This, however, is a contradiction because the maintained assumption that $v_*<p_W$ implies that $a\overline{G}_B(v_*)>a\overline{G}_B(p_W)=G_S(p_W)>G_S(v_*)$. □

**Claim 9.** $\lim_{\delta\to 0}P_{S\delta}(S_\delta(c))=1$ *for all* $c<p_W$.

**Proof.** Recall that $S_\delta(0)=\underline{c}_\delta$, and $S_\delta(\bar{c}_\delta)=\bar{c}_\delta$. The mean slope of $S_\delta$ over its domain $[0,\bar{c}_\delta]$ is

$$\frac{1}{\bar{c}_\delta}\int_0^{\bar{c}_\delta}S'_\delta(x)\,\mathrm{d}x=\frac{\bar{c}_\delta-\underline{c}_\delta}{\bar{c}_\delta}. \tag{42}$$

Proposition 6 implies that

$$P_{S\delta}\big(S_\delta(c)\big)=\mathrm{e}^{\beta\delta}\big(1-S'_\delta(c)\big)\quad\text{a.e. on }[0,\bar{c}_\delta],$$

so the average probability of ultimate trade is

$$\begin{aligned}\frac{1}{\bar{c}_\delta}\int_0^{\bar{c}_\delta}P_{S\delta}\big(S_\delta(c)\big)\,\mathrm{d}x&=\mathrm{e}^{\beta\delta}\left(1-\frac{1}{\bar{c}_\delta}\int_0^{\bar{c}_\delta}S'_\delta(x)\,\mathrm{d}x\right)\\&=\mathrm{e}^{\beta\delta}\left(1-\frac{\bar{c}_\delta-\underline{c}_\delta}{\bar{c}_\delta}\right).\end{aligned} \tag{43}$$

Taking limits shows that the average discounted probability of ultimate trade converges to 1:

$$\lim_{\delta\to 0}\frac{1}{\bar{c}_\delta}\int_0^{\bar{c}_\delta}P_{S\delta}\big(S_\delta(c)\big)\,\mathrm{d}x=\lim_{\delta\to 0}\mathrm{e}^{\beta\delta}\left(1-\frac{\bar{c}_\delta-\underline{c}_\delta}{\bar{c}_\delta}\right)=1$$

because $\lim_{\delta\to 0}(\bar{c}_\delta-\underline{c}_\delta)=0$ (Proposition 7), $\bar{c}_\delta=\bar{v}_\delta$ (Claim 5) and $\underline{\lim}_{\delta\to 0}\bar{v}_\delta\geqslant p_W$ (Claim 8). $P_{S\delta}(S_\delta(c))\leqslant 1$ for all $c\in[0,1]$. Therefore $P_{S\delta}(S_\delta(c))=1$ for all $c\in\lim_{\delta\to 1}[0,\bar{c}_\delta)=[0,p_w)$ because $P_{S\delta}(S_\delta(c))$ is non-increasing and the limiting average probability cannot equal 1 if $P_{S\delta}(S_\delta(c))<1$ for any interval in $[0,p_w]$. □

**Claim 10.** $\overline{\lim}_{\delta\to 0}\underline{c}_\delta \leqslant p_W$.

**Proof.** Verification of this claim follows the same logic as that of Claim 8. It does, however, require Claim 9's result that $\lim_{\delta\to 0} P_{S\delta}[S_\delta(0)] = 1$. Define $c_* = \overline{\lim}_{\delta\to 0}\underline{c}_\delta$ and suppose, contrary to the statement in the claim, that $c_* > p_W$. For the remainder of this proof, fix a subsequence $\underline{c}_\delta \to c_*$. Let $\tilde{c}_\delta = \underline{c}_\delta - \sqrt{\bar{c}_\delta - \underline{c}_\delta}$, noting that Proposition 7 implies $\tilde{c}_\delta \in [0, \bar{c}_\delta)$ for all small enough $\delta$. A seller who offers $S_\delta(v)$ and succeeds in trading does not realize $S_\delta(v)$ as her revenue. She realizes something more because the bid she accepts is at least as great as $S_\delta(v)$. Therefore, for each $\delta$ sufficiently small, a function $\phi_\delta : [\underline{c}_\delta, \bar{c}_\delta] \to [\underline{c}_\delta, \bar{c}_\delta]$ exists that maps, conditional on consummating a trade, the seller's offer into her expected revenue from the sale. Thus $\phi_\delta[S_\delta(c)]$ is a type $c$ seller's expected revenue given that she offers $S_\delta(c)$. Take note that $\phi_\delta[S_\delta(c)] \in [S_\delta(c), \bar{c}_\delta]$ because the expected revenue can be neither less than the seller's offer $S_\delta(c)$ nor more than the type 1 buyer's bid.

Revealed preference implies that

$$\begin{gathered}\pi_S\big(S_\delta(\tilde{c}_\delta), \tilde{c}_\delta\big) \geqslant \pi_S\big(S_\delta(0), \tilde{c}_\delta\big)\\ \big[\phi_\delta\big[S_\delta(\tilde{c}_\delta)\big] - \tilde{c}_\delta\big]\ P_{S\delta}\big[S_\delta(\tilde{c}_\delta)\big] \geqslant \big[\phi_\delta\big[S_\delta(0)\big] - \tilde{c}_\delta\big]\ P_{S\delta}\big[S_\delta(0)\big].\end{gathered} \tag{44}$$

Solving,

$$\begin{aligned}P_{S\delta}\big[S_\delta(\tilde{c}_\delta)\big] &\geqslant \frac{\phi_\delta[S_\delta(0)] - \tilde{c}_\delta}{\phi_\delta[S_\delta(\tilde{c}_\delta)] - \tilde{c}_\delta}\ P_{S\delta}\big[S_\delta(0)\big]\\ &\geqslant \frac{\underline{c}_\delta - \tilde{c}_\delta}{\bar{c}_\delta - \tilde{c}_\delta}\ P_{S\delta}\big[S_\delta(0)\big]\\ &= \frac{1}{1 + \sqrt{\bar{c}_\delta - \underline{c}_\delta}}\ P_{S\delta}\big[S_\delta(0)\big]\end{aligned} \tag{45}$$

where the second line follows from the fact that $\phi_\delta[S_\delta(c)] \in [\underline{c}_\delta, \bar{c}_\delta]$ for all active sellers and the third line follows by substituting in the definition for $\tilde{c}_\delta$.

As in the proof of Claim 8, the mass balance equation (39) must hold:

$$\int_{\underline{c}_\delta}^{1} a g_B(x) \tilde{P}_{B\delta}\big[B_\delta(x)\big]\,\mathrm{d}x = \int_0^{\bar{c}_\delta} g_S(x) \tilde{P}_{S\delta}\big[S_\delta(x)\big]\,\mathrm{d}x. \tag{46}$$

Since $P_{S\delta}[S_\delta(\cdot)]$ is decreasing and $P_{B\delta}[B_\delta(\cdot)] \leqslant 1$,

$$\begin{aligned}\int_0^{\bar{c}_\delta} g_S(x) \tilde{P}_{S\delta}\big[S_\delta(x)\big]\,\mathrm{d}x &\geqslant \tilde{P}_{S\delta}\big[S_\delta(\tilde{c}_\delta)\big] \int_0^{\tilde{c}_\delta} g_S(x)\,\mathrm{d}x\\ &= \tilde{P}_{S\delta}\big[S_\delta(\tilde{c}_\delta)\big] G_S(\tilde{c}_\delta)\end{aligned}$$

and

$$\int_{\underline{c}_\delta}^{1} a g_B(x) \tilde{P}_{B\delta}\big[B_\delta(x)\big]\,\mathrm{d}x \leqslant a\bar{G}_B(\underline{c}_\delta).$$

Therefore (46) and the inequality $\tilde{P}_{S\delta}[S_\delta(\tilde{c}_\delta)] \geqslant P_{S\delta}[S_\delta(\tilde{c}_\delta)]$ imply that

$$
\begin{aligned}
a\overline{G}_B(\underline{c}_\delta) &\geqslant \tilde{P}_{S\delta}\big[S_\delta(\tilde{c}_\delta)\big]G_S(\tilde{c}_\delta)\\
&\geqslant P_{S\delta}\big[S_\delta(\tilde{c}_\delta)\big]G_S(\tilde{c}_\delta).
\end{aligned}
$$

Substituting this into inequality (45) gives

$$
a\overline{G}_B(\underline{c}_\delta) \geqslant \frac{1}{1+\sqrt{\bar{c}_\delta-\underline{c}_\delta}}\, P_{S\delta}\big[S_\delta(0)\big]G_S(\tilde{c}_\delta). \tag{47}
$$

Taking limits as $\delta \to 0$ and invoking continuity of $G_S$ and $G_B$, we obtain

$$
a\bar{G}_B\Big(\lim_{\delta\to 0}\underline{c}_\delta\Big) \geqslant \lim_{\delta\to 0}\bigg(\frac{1}{1+\sqrt{\bar{c}_\delta-\underline{c}_\delta}}\bigg)\lim_{\delta\to 0}\big(P_{S\delta}\big[S_\delta(0)\big]\big)G_S\Big(\lim_{\delta\to 0}\tilde{c}_\delta\Big). \tag{48}
$$

Remember that $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{c}_\delta) = 0$ by Proposition 7, $\lim_{\delta\to 0}\tilde{c}_\delta = c_*$ by construction, $\lim_{\delta\to 0}\underline{c}_\delta = c_*$ by hypothesis, and $\lim_{\delta\to 0}(P_{S\delta}[S_\delta(0)]) = 1$ by Claim 9. Inequality (48) reduces to

$$
a\overline{G}_B(c_*) \geqslant G_S(c_*).
$$

This, however, is a contradiction because the maintained assumption $c_* > p_W$ implies $a\overline{G}_B(c_*) < a\overline{G}_B(p_W) = G_S(p_W) < G_S(c_*)$. □

**Proposition 11.** *Consider any sequence of equilibria for which* $\delta_n \to 0$. *Then* $\lim_{\delta\to 0}\bar{p}_\delta = \lim_{\delta\to 0}\underline{p}_\delta = \lim_{\delta\to 0}\bar{v}_\delta = \lim_{\delta\to 0}\underline{v}_\delta = \lim_{\delta\to 0}\bar{c}_\delta = \lim_{\delta\to 0}\underline{c}_\delta = p_W$.

**Proof.** Proposition 7, Claim 8, and Claim 10 immediately imply this. □

*5.3. Convergence of the equilibrium allocation to the perfectly competitive allocation*

All that remains to prove our main result, Theorem 1, is to show that convergence of the trading range, $[\underline{p}_\delta, \bar{p}_\delta]$, to the Walrasian price $p_W$ is sufficient to guarantee the efficient, competitive allocation. Doing so requires proof of one preliminary claim followed by a straightforward calculation of each trader's utility in the limit as $\delta \to \delta$.

**Claim 12.** $\lim_{\delta\to 0} P_{B\delta}(B_\delta(v)) = 1$ *for all* $v > p_W$.

**Proof.** Pick a buyer type $v$ for which $v > p_W$ and an arbitrarily small $\varepsilon > 0$ such that $v > p_W + 3\varepsilon$. For small enough $\delta$, $v > \bar{v}_\delta$, $|\bar{v}_\delta - p_W| < \varepsilon$, and, by Proposition 7, $\bar{v}_\delta - \underline{v}_\delta < \varepsilon$. Therefore $v - B_\delta(v) < v - p_W + 2\varepsilon$ because $B_\delta(v) \in [\underline{v}_\delta, \bar{v}_\delta]$. If instead type $v$ deviates to the bid $\lambda'_\delta = p_W + \frac{3}{2}\varepsilon$, then $v > \lambda'_\delta > \bar{v}_\delta$ and he is certain to trade immediately: $\rho_{B\delta}(\lambda'_\delta) = P_{B\delta}(\lambda'_\delta) = 1$. By revealed preference

$$
\begin{aligned}
P_{B\delta}\big[B_\delta(v)\big]\big(v - B_\delta(v)\big) &\geqslant P_{B\delta}\big(\lambda'_\delta\big)\bigg(v - p_W - \frac{3}{2}\varepsilon\bigg)\\
P_{B\delta}\big[B_\delta(v)\big](v - p_W + 2\varepsilon) &\geqslant 1\bigg(v - p_W - \frac{3}{2}\varepsilon\bigg)\\
P_{B\delta}\big[B_\delta(v)\big] &\geqslant \frac{(v - p_W - \frac{3}{2}\varepsilon)}{(v - p_W + 2\varepsilon)}.
\end{aligned}
$$

Obviously $P_{B\delta}[B_\delta(v)] \leqslant 1$. Therefore

$$1 \geqslant \lim_{\delta,\varepsilon\to 0} P_{B\delta}\big[B_\delta(v)\big] \geqslant \lim_{\varepsilon\to 0} \frac{(v - p_W - \frac{3}{2}\varepsilon)}{(v - p_W + 2\varepsilon)} = 1,$$

which proves the claim. □

**Proposition 13.** *Consider any sequence of equilibria for which $\delta_n \to 0$. Then each trader's interim expected utility converges for sellers to*

$$\lim_{\delta\to 0} W_{S\delta}(c) = \max[0, p_W - c]$$

*and for buyers to*

$$\lim_{\delta\to 0} W_{B\delta}(v) = \max[0, v - p_W].$$

**Proof.** In the perfectly competitive outcome any trader for whom trading at $p_W$ is profitable can do so instantly; therefore in a competitive market $W_B(v) = \max[v - p_W, 0]$ and $W_S(c) = \max[p_W - c, 0]$. To show that this is in fact the limiting equilibrium outcome for buyers, employ formula (11) for $W_B$ and let $\delta \to 0$:

$$\begin{aligned}
\lim_{\delta\to 0} W_{B\delta}(v) &= \lim_{\delta\to 0}\left\{W_{B\delta}(0) + \int_0^v P_{B\delta}\big[B_\delta(x)\big]\,\mathrm{d}x\right\} \\
&= \lim_{\delta\to 0}\left\{\max\left[0, W_{B\delta}(0) + \int_{\underline{v}_\delta}^v P_{B\delta}\big[B_\delta(x)\big]\,\mathrm{d}x\right]\right\} \\
&= \max\left\{0, \int_{\lim_{\delta\to 0}\underline{v}_\delta}^v \lim_{\delta\to 0}\big\{P_{B\delta}\big[B_\delta(x)\big]\big\}\,\mathrm{d}x\right\} \\
&= \max\left\{0, \int_{p_W}^v 1\,\mathrm{d}x\right\} \\
&= \max\{0, v - p_W\}.
\end{aligned}$$

The second line follows from Proposition 6: $W_{B\delta}(0) = 0$ and, for all $v < \underline{v}_\delta$, $P_{B\delta}[B_\delta(v)] = 0$. The fourth line follows from Proposition 11 and Claim 12:

$$\lim_{\delta\to 0} \underline{v}_\delta = p_W \quad \text{and} \quad \lim_{\delta\to 0} P_{B\delta}\big[B_\delta(v)\big] = 1.$$

An exactly parallel argument shows that $\lim_{\delta\to 0} W_{S\delta}(c) = \max\{0, p_W - c\}$. □

**Proof of theorem.** Propositions 11 and 13 together imply Theorem 1. □

## 6. Conclusions

In this paper we consider a simple, dynamic matching and bargaining market in which both sellers and buyers have incomplete information and risk being forced to exit at any moment due to an exogenous exit rate. We show that this market converges to the Walrasian price and competitive allocation as the length of the matching period goes to zero. This is significant for two reasons. First, given the ubiquity of private information, extension of the full information dynamic matching and bargaining models in this direction is critical because it shows that a decentralized market in which matching frictions are small can elicit private values and costs sufficiently well so as to allocate almost perfectly the market supply to the traders who most highly value that supply. Complete information is not necessary for efficiency in these models. Second, taking a broader perspective, this paper shows that in the presence of private information a fully decentralized market such as the one we model can deliver the same economic efficiency as a centralized market such as the $k$-double auction that Satterthwaite and Williams (1989, 2002) and Rustichini et al. (1994) studied.

In relation to our participation cost paper (Satterthwaite and Shneyerov, 2007) this paper indicates, first, that option value considerations drive convergence to a single price irrespective of whether the friction is an exogenous exit rate or a participation cost. The difference in the two papers is that this paper identifies a supply-demand mechanism that forces the price to which the market converges to be the Walrasian price. Our proof shows how, for example, if the market were to converge to a price below the Walrasian price, then buyers would be rationed and therefore have an incentive to deviate to a price closer to the Walrasian price. This, it appears, is a general argument that applies whenever the trading range converges to a single price. Our participation cost paper does not identify this mechanism because a simpler argument applies there. Together the two papers show why, even in the presence of incomplete information, matching and bargaining markets are robust with respect to the source of the market friction.

Nevertheless our model is a specific, not general, model of trade and therefore immediately raises a raft of further questions. Four stand out in our minds. First, existence of equilibria in which each potential trader has an ex ante positive probability of trade needs to be established in more generality than we have been able to do so. Second, it would be quite interesting to allow for finite numbers of traders as opposed to continua. Such a model, which would be substantially more realistic, would force us to look at transient distributions of types around the steady-state distributions because each entry cohort would be a finite sample of types that would not replicate the underlying continuous type distributions $G_S$ and $G_B$. Third, we would like to know if our results generalize to both correlated costs/values and to interdependent values with a common component and affiliated private signals as has been done in the case of static double auctions by Cripps and Swinkels (2006), Fudenberg et al. (2007), and Reny and Perry (2006). Showing this would be particularly significant if the stochastic process generating traders' cost and values resulted in a time varying Walrasian price. Convergence to that price as the period length approached zero would establish that fully decentralized dynamic matching and bargaining markets can effectively follow—and reveal—an unknown and changing competitive price. Fourth, an important complication of this question, which Wolinsky (1990) has explored in an imperfect information model, would be to allow some traders to enter with more accurate information than other traders about the underlying common component of traders' costs and valuations.

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