Production, Growth and Business Cycles: Technical Appendix*

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Abstract

The methods used in our two survey papers on real business cycles (King, Plosser and Rebelo [1988a,b]) are detailed in this document. Our presentation of the basic neoclassical model of growth and business cycles is broken into three parts. First, we describe the model and its steady state, discussing: the structure of the environment including government policy rules; the nature of optimal individual decisions and the dynamic competitive equilibrium; technical restrictions to insure steady state growth; comparable restrictions on preferences and policy rules; stationary levels and ratios in the steady state; and the nature of a transformed economy. Second, we detail methods for studying near steady-state dynamics, considering: the linear approximation approach; the rational expectations solution algorithm; the nature of alternative solutions; and the special case of the fixed labor model. Third, we discuss the computation of simulations, moments and impulse responses.

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The objective of this appendix is to provide a detailed analysis of a neoclassical economy that is sufficiently flexible to permit: (a) exogenous steady state growth; (b) distorting tax rules of various sorts; and (c) time varying government spending. Although we do not focus on all of these issues in the present discussion, other investigations in progress will utilize this framework. The appendix is divided into three main parts. Part A describes the artificial economy under study and analyses its steady state, Part B develops methods to study approximate dynamics around the steady state, and Part C derives a set of formulas for generating population moments. This technical appendix is designed to serve two functions. First , it develops the theoretical material in sections 2 and 3 of the main text in more depth. Second, it serves as a detailed guide to PC-MATLAB programs for computing dynamic equilibria, written by King and Rebelo in the Spring of 1987. Notation in programs and the technical appendix has been detailed as closely as feasible.

1. Steady State Growth in the Basic Neoclassical Model

1.1. Environmental and Policy Specifications

The economy is populated by many identical agents, of sufficient number that each perceives his influence on aggregate quantities to be insignificant.

1.1.1. Preferences.

Each agent has preferences

$$U = \sum_{t=0}^{\infty} \beta^t [u(C_t, L_t)], \quad 0 < \beta < 1, \tag{A1}$$

where the amount of consumption is C_t and the amount of leisure is L_t. The function $u(C_t, L_t)$ is assumed to be strictly increasing, concave, twice continuously differentiable and to satisfy Inada-type conditions that ensure that the optimal solution for C_t and L_t is always (if feasible) interior. Later in the text we impose further restrictions on β to guarantee that life-time utility U is finite.

1.1.2. Production Technology.

For each agent, output at a point in time is the result of a constant returns to scale production function,

$$F_t(K_t, N_t), \tag{A2}$$

where K_t is the predetermined capital stock and N_t is the quantity of labor input. We take the production function to be constant returns-to-scale because it admits natural aggregation. By writing the production function as $F_t(K_t, N_t)$ we permit general time variation, including the temporary "shocks" to production opportunities and permanent technological change that are discussed below. We assume that $F_t(\bullet)$ has standard neoclassical properties, i.e., it is concave, twice continuously differentiable, satisfies the Inada conditions and implies that both factors of production are essential.

1.1.3. Accumulation Technology.

In this simple neoclassical economy there is only one commodity that can either be consumed or invested, i.e., stored for use in production in the next period. The evolution of capital is thus

$$K_{t+1} = (1 - \delta_K)K_t + I_t, \tag{A3}$$

where I_t is gross investment (i.e. the amount of current output stored to be used in next period's production) and δ_K is the rate of depreciation.

1.1.4. Individual Resource Constraints.

In each period, an individual agent faces two resource constraints: (i) his total amount of time cannot exceed the endowment of unity and (ii) his total uses of goods (for consumption and investment) cannot exceed his disposable income, which derives from output less his net transactions with the government. These conditions are

$$L_t + N_t \leq 1, \tag{A4}$$

$$C_t + I_t \leq Y_t^d = (1 - \tau_t)Y_t + T_t, \tag{A5}$$

where τ_t is the tax rate on output and T_t is the level of transfer payments at date t. It is straightforward to extend the model to consider different tax rates on capital and labor income.

1.1.5. Policy Rules.

The government specifies a path for the per capita level of government purchases ($\underline{G}_t$) and taxes output at a rate that varies according to a policy rule which links this rate (τ_t) to the levels of exogenous variables–for example, the level of per capita government purchases ($\underline{G}_t$)–and to the level of endogenous variables in the economy. The general form of this rule is

$$\tau_t = \tau_t(A_t, \underline{G}_t, \underline{K}_t, \underline{N}_t), \tag{A6}$$

where τ_t is twice continuously differentiable and we indicate per capita quantities by an underbar. The government follows a balanced budget policy with the difference between the government expenditures and the output tax revenue being financed by lump sum taxes or transfers (T).

$$\tau_t \underline{Y}_t = \underline{G}_t + \underline{T}_t, \tag{A7}$$

1.1.6. Per Capita Resource Constraints.

The per capita resource constraints follow from the combination of private and government constraints.

$$\underline{L}_t + \underline{N}_t \leq 1, \tag{A8}$$

$$\underline{C}_t + \underline{I}_t + \underline{G}_t \leq \underline{Y}_t, \tag{A9}$$

1.2. Optimal Individual Decisions and The Competitive Equilibrium

Since all agents are identical, in competitive equilibrium there will be no intertemporal trade, so we can focus on the decision problem for an individual agent facing a sequence of resource constraints. The agent seeks to maximize (A1) subject to the sequence of constraints implied by (A2) through (A5), given sequences of tax rates and transfers. The Lagrangian associated with the optimization problem is

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \beta^t [u(C_t, 1 - N_t)] \\ &\quad + \sum_{t=0}^{\infty} \Lambda_t [(1 - \tau_t) F_t(K_t, N_t) + \underline{T}_t + (1 - \delta_K) K_t - C_t - K_{t+1}], \end{aligned} \tag{A10}$$

where K_0 is treated as given and Λ_t is the multiplier attached to the t period resource constraint. (The single constraint is obtained by combinations of (A2)-(A5)).

Using the notation $D_n f$ to denote the partial derivative of the function f with respect to its nth argument,[1] the efficiency conditions (for an interior optimum) are the following four equations,

$$\beta^t D_1 u(C_t, 1 - N_t) - \Lambda_t = 0, \tag{A11}$$

$$-\beta^t D_2 u(C_t, 1 - N_t) + \Lambda_t (1 - \tau_t) D_2 F_t(K_t, N_t) = 0, \tag{A12}$$

$$\Lambda_{t+1}[(1 - \tau_{t+1}) D_1 F_{t+1}(K_{t+1}, N_{t+1}) + (1 - \delta_K)] - \Lambda_t = 0, \tag{A13}$$

$$(1 - \tau_t) F_t(K_t, N_t) + \underline{T}_t + (1 - \delta_K) K_t - K_{t+1} - C_t = 0, \tag{A14}$$

for $t = 0, 1, 2, ...\infty$ and the "transversality condition", $\lim_{t\to\infty} \Lambda_t K_{t+1} = 0$.[2]

The system of equations (A11)-(A14) can be expressed either as a second-order difference equation in the capital stock or as a system of two first-order difference equations in the Lagrange multiplier and the capital stock. There is an infinite number of paths consistent with equations (A11)-(A14). However, only one of these paths is consistent with the initial value for the capital stock (K_0) and the "transversality condition". Finding this unique path amounts to determining the value of the two constants in the solution to the system of difference equations by using the two boundary conditions of the problem (the knowledge that at $t = 0$ the capital stock is K_0 and the "transversality condition").[3]

[1] For functions of a single variable we use Df and $D^i f$ $(i > 1)$ to denote respectively the first and the ith total derivative of f with respect to its argument.

[2] If our economy had a finite horizon, T, we would have to impose the restriction that $K_{T+1} \geq 0$. Otherwise it would be possible to consume an infinite amount by driving K_{T+1} to $-\infty$. The efficiency condition associated with the non-negativity constraint imposed on K_{T+1} is $\Lambda_T K_{T+1} = 0$. The "transversality condition" is the counterpart of this equation when the economy has an infinite horizon. For a detailed discussion of the conditions under which the Euler equations (A11)-(A14) and the transversality condition are sufficient or necessary for the associated capital path to be optimal, see Weitzman (1973) and Romer and Shinotsuka (1987).

[3] Engineers usually refer to this as a "two-point boundary value" problem, due to the fact that the boundary conditions are not contemporaneous (in this case we have one condition at $t = 0$ and the other at $t = \infty$).

The preceding optimal decisions are valid for arbitrary specifications of tax and spending sequences. In perfect foresight competitive equilibrium, given the policy specification (A6)-(A8), tax and transfer sequences depend on individual decisions, which in turn depend on tax rates. As in Romer (1986), equilibrium sequences can be obtained by combining the individual's efficiency conditions with aggregate consistency conditions, which in this case are the constraints of the government. Then, equilibrium sequences $[\{\underline{C}_t\}_{t=0}^{\infty}$, $\{\underline{N}_t\}_{t=0}^{\infty}$, $\{\underline{K}_t\}_{t=0}^{\infty}$ and $\{\underline{\Lambda}_t\}_{t=0}^{\infty}]$ satisfy the following four equations,

$$\beta^t D_1 u(\underline{C}_t, 1-\underline{N}_t) - \underline{\Lambda}_t = 0, \tag{A15}$$

$$\beta^t D_2 u(\underline{C}_t, 1-\underline{N}_t) - \underline{\Lambda}_t[1-\tau_t(A_t, \underline{G}_t, \underline{K}_t, \underline{N}_t)] D_2 F_t(\underline{K}_t, \underline{N}_t) \quad = 0, \tag{A16}$$

$$\underline{\Lambda}_{t+1}\{[1-\tau_{t+1}(A_{t+1}, \underline{G}_{t+1}, \underline{K}_{t+1}, \underline{N}_{t+1})] D_1 F_{t+1}(\underline{K}_{t+1}, \underline{N}_{t+1}) + (1-\delta_K)\} - \underline{\Lambda}_t = 0, \tag{A17}$$

$$F_t(\underline{K}_t, \underline{N}_t) + (1-\delta_K)\underline{K}_t - \underline{K}_{t+1} - \underline{C}_t - \underline{G}_t = 0, \tag{A18}$$

for all $t = 0, 1, 2, ...\infty$ and the "transversality condition", $\lim_{t\to\infty} \underline{\Lambda}_t \underline{K}_{t+1} = 0$.[4]

There is substantial generality in the specification of preferences and technology described in this section. In the next two sections, we explore the restrictions generated by the requirement that, in the absence of distorting taxes, the system displays steady state growth. Steady state growth is defined to be a situation in which C_t, Y_t, I_t, K_t and the wage rate grow at constant, but possibly differing rates. We begin with restrictions on the form of technical progress and on the form of the production function F.

[4] The existence and uniqueness of a solution to this problem are not guaranteed in general. See Romer and Sasaki (1984) for an existence proof for an economy in which the competitive equilibrium is suboptimal.

1.3. Technological Restrictions to Ensure Steady State Growth

As is familiar from Swan (1963) and Phelps (1966), there must be restrictions on the form of technical progress if a steady state is to be feasible. (To discuss the feasibility of steady state growth, we consider the economy without taxes or government purchases). In particular, the form of technical progress–implicit in the notation $F_t(K_t, N_t)$–must be expressible in a labor augmenting form.

This may be (tediously) demonstrated as follows. Suppose that we write the production function as $Y_t = AF(X_{Kt}K_t, X_{Nt}N_t)$, where X_{Kt} represents capital augmenting technical progress and X_{Nt} represents labor augmenting technical progress. Let X_{Kt} and X_{Nt} grow respectively at the rates γ_{XK} and γ_{XN}.[5] The growth rate of Y, γ_Y, satisfies $\gamma_Y = [Y_{t+1}/Y_t] = [X_{K,t+1}K_{t+1}/X_{K,t}K_t][F(1, Z_{t+1})/F(1, Z_t)]$, where $Z_t = X_{Nt}N_t/X_{Kt}K_t$.

There are two cases to be explored. In the first case, the ratio Z is constant over time so that $\gamma_Y = \gamma_{XK}\gamma_K$, from the preceding expression for γ_Y since this expression–with Z constant over time–links the growth rate of Y to that of capital augmenting technical progress and that of capital. The resource constraint (A9) implies that $\gamma_K = [Y - C]/K + (1 - \delta_K)$. If $Y > C$, so that investment is strictly positive, in order for γ_K to be constant, then, it must be the case that Y/K and C/K are constant as well. The constancy of Y/K implies that $\gamma_K = \gamma_Y$, which in turn implies that $\gamma_Y = \gamma_{XK}\gamma_K = \gamma_K$. Equivalently, $\gamma_{XK} = 1$. Thus, the feasibility of steady state growth requires that there be no capital augmenting technical progress.

The second case is that the ratio $[F(Z_{t+1}, 1)/F(Z_t, 1)]$ is constant irrespective of the constancy of the ratio $Z = X_N N/X_K K$. In that case, the production function is Cobb-Douglas, so that it is always possible to write the technical

[5] Throughout our discussion, we use notation $\gamma_q = q_t/q_{t-1}$, *i.e.*, γ_q is one plus the growth rate of q; for expositional convenience, we sometimes refer to γ_q as the (gross) growth rate of q.

progress as labor augmenting. That is, $Y_t = A(X_{Kt}K_t)^{1-\alpha}(X_{Nt}N_t)^{\alpha}$ can always be written as $Y_t = AK_t^{1-\alpha}\widetilde{X}_{Nt}^{\alpha}$, where $\widetilde{X}_{Nt} = X_{Kt}^{(1-\alpha)/\alpha}X_{Nt}$.

Thus, technical change must be expressible in labor augmenting form. This implies that Hicks neutral technological progress, which corresponds to the case in which $X_{Nt} = X_{Kt}$, is only consistent with the feasibility of steady state growth when the production technology is Cobb-Douglas.

The observed long-run constancy of factor shares is a frequently employed rationale for the restriction to the Cobb-Douglas form (see, *e.g.*, Prescott (1986)). The preceding analysis implies, however, that with labor augmenting technological change, any constant returns-to-scale production function is compatible with constant long-run factor shares, *i.e.*, shares that are invariant to the scale of X.[6] However, in order to facilitate comparison of our analysis with other studies, we utilize the Cobb-Douglas form in our analysis in the main text. In view of the foregoing, though, we write that production function as

$$Y_t = A_t(K_t)^{1-\alpha}(N_tX_t)^{\alpha}, \tag{A19}$$

where X_t is a smoothly varying "trend" growth in labor augmenting technical change and A_t is a temporary displacement to total factor productivity.

1.3.1. Implications for Feasible Steady States

In order to determine what are the feasible steady state growth rates we start by examining the model's production and accumulation structures, under the

[6] This may be demonstrated as follows. The marginal product of a unit of time is $AX_tD_2F(K_t, NX_t)$ on a steady state path. Labor's share is $s_N = ANX_tD_2F(K_t, NX_t)]$. On a steady state path, $F(K_t/NX_t, 1)$ is constant and the $F(K_tNX_t) = NX_tF(K_t/NX_t)$. Hence, labor's share is invariant to the scale of X_t in steady state.

assumption that A_t is constant for all time, *i.e.*, $A_t = A$. These implications are as follows:

(i) Since the amount of time devoted to work (N) has to be between zero and 1, the only feasible constant growth rate for N is zero, that is, $\gamma_N = 1$.

(ii) From the commodity resource constraint, $C + I = Y$, it follows that if $I > O$ the steady state growth rates of consumption and investment must be equal to the growth rate of output and to each other, *i.e.*, $\gamma_C = \gamma_I = \gamma_Y$.[7][8]

(iii) From the accumulation equation ($K_{t+1} = (1 - \delta_K)K_t + I_t$), it follows that the growth rate of gross investment and capital must be equal in the steady state, $\gamma_K = \gamma_I$

(iv) Any constant returns to scale production function–along with fixity of $N - -$implies that $\gamma_Y = \gamma_K = \gamma_X$ in a steady state, since $Y_t = A_t F(K_t, X_t) = X_t A_t F(K_t/X_t, 1)$. Thus constancy of K/X is necessary for a steady state, unless the production function is Cobb-Douglas.[9] Thus, K and X grow at the same rate. Further, with K/X constant, Y and X grow at a common rate.

Collecting these results, we can conclude that the growth rates of all commodity quantity variables are equal to the rate of growth of labor augmenting technical change.

[7] This should not be regarded as a general feature, since it is peculiar to models in which capital is simply stored consumption good.

[8] It is straightforward to demonstrate this result, though more tedious in discrete time than in the familiar continuous time case. Define $s_{ct} = C_t/Y_t$ and $s_{it} = I_t/Y_t$. Then, taking ratios of output in two adjacent periods, it follows that $(Y_{t+1}/Y_t) = C_{t+1}/Y_t) + (I_{t+1}/Y_t) = \gamma_{c,t+1}s_{ct} + \gamma_{I,t+1}s_{it} = \gamma_{T,t+1}$. Since $s_{it} + s_{ct} = 1$ for all t, it follows that $(\gamma_{c,t+1} - \gamma_{I,t+1})s_{ct} = (\gamma_{Y,t+1} - \gamma_{I,t+1})$. Thus, if there is steady state growth (all γ's are independent of t), then the above condition can be fulfilled only by one of the two following situations: (a) s_c is constant if it is not true that $\gamma_C - \gamma_I = \gamma_Y - \gamma_I = 0$; or $(b)\gamma_C = \gamma_I = \gamma_Y$. But, if s_c is constant, then it follows that C and Y are growing at the same rate, which contradicts the requirement of condition (a). Thus, condition (b) must hold in any steady state.

[9] If the production function is Cobb-Douglas, then $\log(\gamma_Y) = \alpha log(\gamma_X) + (1 - \alpha)\log(\gamma_K)$. In conjunction with the requirement that $\gamma_I = \gamma_K = \gamma_Y$, this then leads to the conclusion that $\gamma_Y = \gamma_K = \gamma_X$.

$$\gamma_Y = \gamma_C = \gamma_K = \gamma_I = \gamma_X. \tag{A20}$$

Thus, this model has a unique steady state growth rate, although no restrictions have so far been placed on the levels of any quantity variable. Furthermore, the technology places strong restrictions on the relationship between the steady state growth rates of the different variables.

1.3.2. Implications for Steady State Marginal Products

First, the marginal product of capital, $AD_1F(K_t, NX_t)$, is constant over time, since $\gamma_K = \gamma_X$, and constant returns-to-scale implies that this is homogeneous of degree zero in X, so that it may be written $AD_1F(K_t/X_t, N)$.[10] Second, the marginal product of a unit of time, $X_tAD_2F(K_t, NX_t)$, grows at rate γ_X, since $AD_2F(K_t, NX_t)$ is also homogeneous of degree zero in X.[11]

1.3.3. Implications for Local Elasticities

The constant returns to scale structure also has implications for elasticities of marginal products near the steady state. To avoid considering scale directly, define $k_t = K_t/X_t$ and $y_t = Y_t/X_t$. Then, the production function may be written as $y_t = A_tF(k_t, N_t)$. Let the shares of capital and labor be s_K and s_N, which are invariant to scale as discussed above. Under constant returns to scale, of course, $s_K + s_N = 1$. Let the elasticity of the marginal product of capital, $A_tD_1F(k_t, N_t)$ with respect to capital be denoted ξ_{kk} and that with respect to labor ξ_{kN}. Similarly, let ξ_{NN} and ξ_{Nk} denote elasticities of the marginal product of labor. It is a standard result of production theory[12] that these elasticities arguments can be

[10] This is easy to see in the Cobb-Douglas case, since $AD_1F(K_t, NX_t) = (1-\alpha)AK_t^{-\alpha}(NX_t)^{\alpha}$.

[11] In the Cobb-Douglas case, $X_tAD_2F(K_t, NX_t) = \alpha AK_t^{1-\alpha}(N)^{\alpha-1}X_t^{\alpha}$.

[12] See, for example, Ferguson (1964), pages 417-419, or Supplementary Note #1 at the back of this document.

simply described, using the local elasticity of substitution of labor for capital, $\zeta_{kN} = -d[\log(k/N)]/d[\log(D_1F(k_t, N_t)/D_2F(k_t, N_t))]$.

$$\xi_{kk} = -\xi_{kN} = -s_N/\zeta_{kN}, \tag{A21a}$$

$$\xi_{NN} = -\xi_{Nk} = -(1 - s_N)/\zeta_{kN}. \tag{A21b}$$

For the Cobb-Douglas case, $\zeta_{kN} = 1$ and $s_N = \alpha$, so that $\xi_{kk} = -\xi_{kN} = -\alpha$ and $\xi_{NN} = -\xi_{Nk} = -(1 - \alpha)$.

1.4. Restrictions on Preferences and Policy Rules

The next step involves checking whether this steady state path verifies the household's equilibrium conditions (A11)-(A14). Since we are interested in growing economies, we will focus on the case in which $\gamma_X > 1$. From above, we know that a technologically feasible steady state satisfies (A20), which specifies that consumption, investment, output, and capital all grow at the rate of labor augmenting technical progress. If the household equilibrium conditions turn out to be incompatible with the technologically feasible steady state growth rate, the steady state is of no interest to us, since it will never be the outcome of this economy's competitive equilibrium. Consequently, we restrict preferences so that the feasible steady state is, in fact, an optimal (and competitive equilibrium) outcome.

The efficiency condition (A13) or (A17), which are equivalent without taxes, specifies that the shadow price of capital will grow at a constant rate in the steady state, which we define to be γ_Λ. This growth rate is a function of the ratio $k_t = K_t/X_t$. That is, $\gamma_\Lambda[AD_1F(k, N) + (1 - \delta_K)] = 1$, where the absence of time dating of k and N indicates these are constant over time, which will hold in a steady state because X and K grow at the same rates and N is constant.

(i) Invariant Intertemporal Substitution in Consumption.

The optimality of a steady state restricts the utility function so that there is an intertemporal elasticity of substitution in consumption that is invariant to the scale of consumption.

The requirement for an efficient intertemporal plan (A11) implies that the growth rate of marginal utility is constant over time,[13]

$$\frac{D_1 u(C_t, L)}{D_1 u(C_{t+1}, L)} = \beta/\gamma_\Lambda = \beta[AD_1 F(k, N) + (1 - \delta_k)]. \tag{A22}$$

But, from the commodity resource constraint, a feasible steady state consumption program has the form $C_t = X_0(\gamma_X)^t[AF(k, N)+(1-\delta_K-\gamma_X)k]$. Thus, if marginal utility is to grow at a constant rate for all potential values of X_0, it must be the case that

$$\left[\frac{D_{11}u(C_{t+1}, L)C_{t+1}}{D_1 u(C_{t+1}, L)}\right]\frac{dX_0}{X_0} - \frac{D_{11}u(C_t, L)C_t}{D_1 u(C_t, L)}\frac{dX_0}{X_0} = 0. \tag{A23}$$

which implies that $\frac{D_{11}u(C_t,L)C_t}{D_1 u(C_t,L)}$ has to be constant over time. Define σ_c as the (absolute value) elasticity of marginal utility with respect to consumption. The preceding condition implies that σ_c is invariant to the scale of consumption. Integrating both sides of $C\ D_{11}u(C, L)/D_1 u(C, L) = -\sigma_c$ gives us the candidate momentary utility function forms $\widetilde{u}(C, L) = C^{1-\sigma_c}v_1(L) + v_2(L)$ for σ_c not equal to unity and $\widetilde{u}(C, L) = \log(C)v_1(L) + v_2(L)$, where $v_i(L)$ are arbitrary functions of L.[14] Strict concavity of the momentary utility function requires that $\sigma_c > 0$, which we impose.

[13] In these expressions, L is not dated since its growth rate is zero in a steady state.

[14] In the conventional neoclassical model L is fixed exogenously, so that the intertemporal requirement exhausts the implications of a steady state.

(ii) Invariance of Efficient Steady State Labor.

Within an efficient plan, the conditions (A15) and (A16) specify that the marginal rate of substitution between consumption and leisure equals the marginal rate of transformation. Taking logarithms, it follows that

$$\log(D_1 u(C_t, L_t)) + \log(X_t A D_2 F(K_t, N X_t)) = \log(D_2 u(C_t, L_t)). \tag{A24}$$

From the analysis above, we know that a steady state implies three conditions, which are important in the evolution of the expressions in the equation above. First, the marginal product of *effective* labor, $AD_2F(K_t, NX_t) = AD_2F(k_t, N)$, is constant in the steady state. Second, technology (X_t) grows at rate γ_X. Third, the marginal utility of consumption $D_1u(C_t, L_t)$ grows at rate $(\gamma_C)^{-\sigma_c}$, where σ_c is the elasticity defined above and where $\gamma_C = \gamma_X$ is the growth rate of consumption. Denote the elasticity of the marginal utility of leisure with respect to consumption as $\xi_{\ell c}$. In a steady state, it follows (from the differentiation) that this is constant over time and satisfies $1 - \sigma_c = \xi_{\ell c}$. This additional requirement that $1 - \xi_{\ell c} = \sigma_c$ yields further restrictions on admissible forms, which imply that $v_2(L)$ is constant in the non-log case and $v_1(L)$ is constant in the log case.[15] Since these constants do not affect preference orderings, we set them equal to unity in writing the utility function as the following.

$$\begin{aligned} u(C, L) &= \frac{1}{(1-\sigma_c)} C^{1-\sigma_c} v(L) \quad \text{if} \quad 0 < \sigma_c < 1 \text{ or } \sigma_c > 1, && \text{(A25a)} \\ u(C, L) &= \log(C) + v(L) \qquad\quad \text{if} \quad \sigma_c = 1. && \text{(A25b)} \end{aligned}$$

To insure that consumption and leisure are goods and that utility is concave, we need to impose some additional structure. When momentary utility is additively

[15] To check this, simply differentiate $\widetilde{u}$.

separable (A25b), all that we require is that $v(L)$ is increasing and concave. When momentary utility is multiplicatively separable, then we require that (i) $v(c)$ is increasing and concave if $\sigma_c < 1$ and decreasing and convex if $\sigma_c > 1$. Further, we require that $-\sigma_c[LD^2v(L)/Dv(L)] >$
$(1-\sigma_c)[LDv(L)/v(L)]$ to assure overall concavity of $u(\bullet)$.

There is economic content to these preference restrictions. Time is bounded, so that in a steady state it cannot grow. As discussed in the main text, for balanced growth to be optimal with labor supply chosen by agents, utility must be such that there are exactly offsetting income and substitution effects of the changes in real wages associated with sustained growth in productivity. In a static framework with no non-wage income, this invariance arises whenever utility is any concave transformation of $u(C,L) = C^{1-\sigma_c}v(L)/(1-\sigma_c)$, including–if utility is additively separable–$\log(C) + v(L)$.[16] The preference restrictions are the same on a steady state path in our model despite the presence of capital income because, in the steady state, variations in capital income are proportional to movements in labor productivity or to wage income.[17]

(iii) Conditions on government to guarantee steady state

Our objective is to analyze competitive equilibria that are suboptimal due to taxes and government spending, but we wish to maintain steady state growth in the face of these interventions. For this reason, we specify that there is a constant share of government purchases in national product, $s_g = (G/Y)$, and that there is a "normal level" of the tax rate, τ, that is invariant to the scale of the economy.

[16] In a static framework, (A25a) and (A25b) are both representative of the same preferences since they are monotonic transformations of each other. However, in a dynamic framework that is not the case since (A25a) or (A25b) only represent momentary utility. The utility function is $U = \sum_{t=0}^{\infty} \beta^t u(c_t, L)$.

[17] In a static framework the optimum number of leisure hours is determined by $\frac{1}{(1-L)+rK/w} v(L) = \frac{Dv(L)}{1-\sigma_c}$. It is clear that if $K = 0$ or rK/w is constant, L will also be constant. In our economy, the capital and labor share are constant implying constancy of rK/w.

In the analysis below, it is frequently convenient to consider the "wedge" function, defined as $\Omega_t = (1-\tau_t)$, and we let Ω denote the steady state wedge.

1.5. Levels and Ratios in Steady State

We can use the efficiency conditions together with the requirement that all variables grow at a constant rate to pin down the steady state value of several variables and ratios.

Ratios: From the requirement that $\log(\gamma_\Lambda/\beta) = -\sigma_c \log(\gamma_C) = -\sigma_c \log(\gamma_X)$ the level of the gross marginal product of capital is determined according to $[\Omega\ AD_1F(k,N) + (1-\delta_K)]\beta^* = \gamma_X$, where $\beta^* = \beta(\gamma_X)^{1-\sigma_c}$ and $k = K/X$. Since $s_K = kAD_1F(k,N)/AF(k,N)$, the output capital ratio is[18]

$$\frac{Y}{K} = \{[\gamma_X - \beta^*(1-\delta_k)]/[\beta^* s_K \Omega]\}. \tag{A26}$$

The fact that $\gamma_X \geq 1$ and $\beta^* < 1$ guarantee that Y/K is always positive.

Shares of output: From the per-capita resource constraint and the preceding conditions, it follows that the share of output devoted to gross investment (s_i) is

$$s_i = [\gamma_X - (1-\delta_K)]\frac{K}{Y} = \{[\gamma_X - (1-\delta_K)][\beta^*(s_K)\Omega]/[\gamma_X - \beta^*(1-\delta_K)]\}. \tag{A27}$$

Further, given that the share of government (s_g) is treated as exogenous, it follows that the share of consumption (s_c) is determined by the preceding as $s_c = 1 - s_i - s_g$.

[18] Under the Cobb-Douglas assumption this requirement yields the following expressions: (i) The effective labor-capital ratio, NX/K, is given by $(XN/K) = \{\gamma_X - \beta^*(1-\delta_K)]/[\beta^*(s_K)\Omega A)\}^{(1/s_N)}$; (ii) the output-capital ratio is $(Y/K) = [\gamma_X - \beta^*(1-\delta_K)]/[\beta^*(s_K)\Omega]\}$, and (iii) the output-labor ratio is $(Y/NX) = A^{(1/\alpha)}\{[\beta^*(s_K)\Omega]/[\gamma_X - \beta^*(1-\delta_K)]\}^{(s_K)/s_N}$. Note that these expressions contain the influence of the steady state wedge (Ω) on capital accumulation.

Steady State Level of Labor: Given a particular specification of preferences, it is possible to solve for the steady state level of labor using the equilibrium conditions (A15) and (A16), in combination with the preceding expressions. In a steady state, these expressions imply that $D_1u(C_t, 1-N)\Omega X_t A D_2 F(K_t, NX_t) = D_2u(C_t, 1-N)$. For the preferences specifications with $\sigma_c \neq 1$, we could divide both sides of the expression by $Y_t^{(1-\sigma_c)}$, so that it can be written $D_1u(s_c, 1-N)\Omega\frac{s_N}{N} = D_2u(s_c, 1-N)$ or $v(1-N)\Omega\frac{s_N}{N} = \frac{s_c}{1-\sigma_c}Dv(1-N)$. With $\sigma_c = 1$, it follows that $\frac{1}{s_c}\Omega\frac{\alpha}{N} = DV(1-N)$. Since Ω and s_N are parameters and s_c has been determined earlier, the appropriate one of these conditions determines the level of N. For most specifications of utility, we must solve this expression numerically, but there are a few exceptions. For example, if $u(C, L) = \log(C) + \theta_\ell \log(L)$ and the production function is Cobb-Douglas, then it is easy to show that this condition implies $N = (\alpha\ \Omega)/(\alpha\ \Omega + s_c\theta_\ell)$.

But, practically, we do not have information on preference parameters (such as θ_ℓ) that determine the steady state level of hours. Fortunately, we do not need to determine there parameters explicitly–in order to analyze near steady state dynamics all we need is to specify the number of hours worked per period in the steady state (see below). In any case, a value of steady state N combined with the preceding conditions on ratios yields the steady state paths of all of the system's variables.

1.6. The Transformed Economy

Having restricted technology, preferences and government behavior so that a steady state path exists and is consistent with the private agent's efficiency conditions, our objective is now to characterize the local dynamics around the steady state path. But before turning to the study of the dynamics of our economy it is convenient to transform its variables, expressing preferences and technology in terms of variables that will be constant in the steady state. Since all the original variables (except N_t) grow at the same rate as X_t in the steady state, this can be accomplished by deflating these variables by $X_t : c = C/X$, $k = K/X$, $g = G/X$, etc. Our economy expressed in terms of the transformed variables is identical to an economy in which technological progress is absent and growth rates are zero in the steady state with two exceptions.

First, the capital accumulation equation is altered as follows. In the level economy, $K_{t+1} = (1-\delta_K)K_t + I_t = (1-\delta_K)K_t + A_tF(K_t, N_tX_t) - C_t$ so that in the transformed economy

$$\begin{aligned} \gamma_K k_{t+1} &= (K_{t+1}/X_{t+1})(X_{t+1}/X_t) \\ &= (1-\delta_K)k_t + i_t = (1-\delta_K)k_t + A_tF(k_t, N_t) - c_t. \end{aligned} \tag{A28}$$

Second, the process of transforming consumption (by dividing by X_t) in the preference specification potentially alters the effective rate of time preference. That is,

$$\begin{aligned} U &= \sum_{t=0}^{\infty} \beta^t [u(C_t, L_t)] = \sum_{t=0}^{\infty} \beta^t [u(c_t X_t, L_t)] \\ &= \begin{cases} (X_0^{1-\sigma_C}) \sum_{t=0}^{\infty} (\beta^*)^t [\frac{1}{(1-\sigma_C)} c_t^{1-\sigma_C} v(L)] & \text{for } \sigma_c \neq 1 \\ \sum_{t=0}^{\infty} (\beta^*)^t [\log(c_t) + v(L_t) + \log(X_t)] & \text{for } \sigma_c = 1 \end{cases} . \end{aligned}$$

where we have defined $\beta^* = \beta(\gamma_X)^{1-\sigma_c}$. In order to ensure that lifetime utility (U) is finite, we assume that $\beta^* < 1$. Thus, we can express preferences in terms of $u(c_t, L_t)$ and the discount factor β^* for all values of σ_c, since the terms $X_0^{1-\sigma_c}$ and $\sum_{t=0}^{\infty} \beta^t [\log(X_t)]$ can be ignored since they do not affect the preference orderings on the transformed variables. That is, we may set $X_0 = 1$ or $\sum_{t=0}^{\infty} \beta^t [\log(X_t)] = 0$ in the preceding expressions. Consider maximizing transformed utility subject to the private resource constraint analogous to (A14) and then requiring that (A28) hold for the economy as a whole. The resulting equilibrium conditions analogous to (A15)-(A18) are most useful if the transformation to current valued multipliers is employed, *i.e.*,

$$
\begin{aligned}
D_1 u(c_t, 1 - N_t) - \lambda_t &= 0, & \text{(A29)} \\
D_2 u(c_t, 1 - N_t) - \lambda_t \Omega_t A D_2 F(k_t, N_t) &= 0, & \text{(A30)} \\
\beta^* \lambda_{t+1}[\Omega_{t+1} A D_1 F(k_{t+1}, N_{t+1}) + (1 - \delta_K)] - \lambda_t \gamma_X &= 0, & \text{(A31)} \\
AF(k_t, N_t) + (1 - \delta_K) k_t - \gamma_X k_{t+1} - c_t - g_t &= 0. & \text{(A32)}
\end{aligned}
$$

for all $t = 0, 1, \ldots$ and where $\lambda_t = \tilde{\Lambda}_t / (\beta^*)^t$.[19] The "transversality condition" can now be expressed as $\lim_{t \to \infty} (\beta^*)^t \lambda_t k_{t+1} = 0$.

2. Near Steady State Dynamics

The objective of this section is to analyze the local dynamics around the steady state, when the economy faces alternative deterministic sequences of "shocks". In the present setting, the shocks are taken to be proportionate variations in government spending (G_t) and total factor productivity (A_t) but it is easy enough to incorporate alternative displacements using the methods outlined in this section.

Our method is to analyze near steady state dynamics by considering the economy expressed in terms of transformed variables.

2.1. Linear Approximation

We now turn to analysis of approximate economies. As discussed in the body of the paper, our strategy is to approximate conditions (A29)-(A32) linearly around

[19] $\tilde{\Lambda}_t$ are the present valued Lagrange multipliers that correspond to the maximization problem for the transformed economy (not in the text). Recall that we used Λ_t to denote the Lagrange multipliers associated with the original (untransformed) economy.

a steady state and then to solve the resulting linear dynamic system. Our presentation of this approach is designed to facilitate understanding of the general method, which is presented in more detail in King (1987). Since the neoclassical model has only a single state (capital) variable, there are presentations that are more direct than the one that we undertake here. However, our presentation is designed to make clear that the methods we employ can be readily extended to analysis of economies with multiple state variables (see the examples in King (1987)).

Approximation of (A29)-(A32) near the stationary levels (k, N, c, g, and i) implied by $A_t = A$ yields expressions for percentage deviations from steady state levels, which we denote by a circumflex ($\hat{}$). The first two expressions imply that

$$\begin{aligned} \xi_{cc}\hat{c}_t - \xi_{c\ell}\{N/(1-N)\}\hat{N}_t &= \hat{\lambda}_t, & \text{(A33)} \\ \xi_{\ell c}\hat{c}_t - \xi_{\ell\ell}\{N/(1-N)\}\hat{N}_t &= \{\hat{\lambda}_t + \hat{A}_t + \xi_{Nk}\hat{k}_t + \xi_{NN}\hat{N}_t\} & \text{(A34)} \\ &\quad +\{\omega_A\hat{A}_t + \omega_k\hat{k}_t + \omega_N\hat{N}_t + \omega_g\hat{g}_t\}. \end{aligned}$$

In these expressions, there are a number of elasticities, all of which are evaluated at the stationary point $(k, N, c, y, etc.)$.

First, the ξ's on the left-hand side of (A33) and (A34) are elasticities of marginal utility. Generally, these may be shown to be $\xi_{cc} = cD_{11}u(c,L)/D_1u(c,L)$; $\xi_{c\ell} = LD_{21}u(c,L)/D_1u(c,L)$; $\xi_{\ell c} = cD_{12}u(c,L)/D_2u(c,L)$; and $\xi_{\ell\ell} = L\,D_{22}u(c,L)/D_2u(c,L)$. When the utility function is additively separable, (A25b), it is the case that $\xi_{cc} = -1$; $\xi_{c\ell} = 0$; $\xi_{\ell c} = 0$; and $\xi_{\ell\ell} = LD^2v(L)/Dv(L)$. When the utility function is multiplicatively separable, (A25a), it follows that $\xi_{cc} = -\sigma_c$; $\xi_{c\ell} = LDv(L)/v(L)$; $\xi_{\ell c} = 1 - \sigma_c$; and $\xi_{\ell\ell} = LD^2v(L)/Dv(L)$. There is an additional restriction which must be satisfied and is discussed in Supplementary Note #2.

Second, the ξ's on the right-hand side of (A34) are elasticities of the marginal product of labor, as given by (A21b); in the Cobb-Douglas case, these are $\xi_{Nk} = (1-\alpha)$ and $\xi_{NN} = \alpha - 1$. Third, the ω's are elasticities of the wedge function, $\Omega_t = (1-\tau_t)$, with respect to its arguments.

Differentiation of the intertemporal efficiency (A31) condition implies that

$$\{\hat{\lambda}_{t+1} + \eta_A \hat{A}_{t+1} + \eta_k \hat{k}_{t+1} + \eta_N \hat{N}_{t+1} + \eta_G \hat{g}_{t+1}\} = \hat{\lambda}_t. \tag{A35}$$

where η_A is the elasticity of the net after-tax marginal product of capital, $[\Omega(k,N,A,g)AD_1F(k,N)$-$(1-\delta_K)]$, with respect to A evaluated at the steady state, etc. The elasticities may be shown to be $\eta_A = [\gamma_X - \beta^*(1-\delta_K)][\omega_A + 1]/\gamma_X$; $\eta_k = [\gamma_X - \beta^*(1-\delta_K)][\omega_k + \xi_{KK}]/\gamma_X$; $\eta_N = [\gamma_X - \beta^*(1-\delta_K)][\omega_N + \xi_{KN}]/\gamma_X$ and $\eta_g = [\gamma_X - \beta^*(1-\delta_K)][\omega_g]/\gamma_X$, where again the ω's are the elasticities of the tax wedge. (Recall that under the Cobb-Douglas structure $\xi_{KK} = -\alpha$ and $\xi_{KN} = \alpha$). Finally, differentiation of the resource constraint implies

$$\begin{aligned} \hat{y}_t &= \{\hat{A}_t + s_N \hat{N}_t + s_K \hat{k}_t\} \\ &= s_c \hat{c}_t + s_g \hat{g}_t + s_i \phi \hat{k}_{t+1} - s_i(\phi - 1)\hat{k}_t. \end{aligned} \tag{A36}$$

where s_K, s_N, s_c, s_g and s_i are shares discussed earlier and $\phi = \gamma_X/[\gamma_X - (1 - \delta_K)] > 1$ given that $\gamma_X \geq 1$. The elasticities of production with respect to inputs (s_N and s_K) take on simple forms under the Cobb-Douglas production structure, $s_N = \alpha$ and $s_K = (1-\alpha)$.

The solution algorithm will be discussed in more detail in material that follows (Section B.2 below), but it is desirable to provide an overview before preceding to some further details. (A33)-(A36) can be combined to eliminate consumption, effort, and output flows, yielding a difference equation system in capital stock

($\hat{k}$) and shadow price ($\hat{\lambda}$). This difference system can then be solved, subject to the transversality condition, to produce a unique solution sequence for the capital stock ($\hat{k}$) and shadow price ($\hat{\lambda}$), given specification of exogenous sequences for ($\hat{A}$) and ($\hat{G}$).

Solutions for flows and relative prices. With the solution sequence for $\hat{\lambda}$ and $\hat{k}$ in hand, (A33) and (A34) make it possible to compute solution sequences for the flow variables $\hat{N}_t$ and $\hat{c}_t$. Further, given these solutions and the following expressions, expressions for other variables of interest can be constructed. In particular, we can develop approximate solutions for output, labor productivity (real wages) and investment.

$$\hat{y}_t = \hat{A}_t + s_N \hat{N}_t + s_K \hat{k}_t, \tag{A37}$$

$$\hat{y}_t - \hat{N}_t = \hat{A}_t + (1 - s_N)\hat{N}_t + s_K \hat{k}_t, \tag{A38}$$

$$\hat{\imath}_t = \frac{1}{s_i}\hat{y}_t - \frac{s_c}{s_i}\hat{c}_t - \frac{s_g}{s_i}\hat{g}_t. \tag{A39}$$

2.2. Solution Algorithm

In discussion of the solution algorithm, we utilize the language of discrete time optimal control, since methods are based on the extensive applied mathematics literature which deals with linear optimal control problems. Expressions (A33) and (A34) relate the flow variables ("controls") $\hat{c}_t$ and $\hat{N}_t$ to the capital stock and shadow price (controlled state and co-state variables), as well as to the exogenous variables, technology shocks and government purchases (uncontrolled states) . These expressions may be written

$$M_{cc}\left[\begin{array}{c}\hat{c}_t\\ \hat{N}_t\end{array}\right]=M_{cs}\left[\begin{array}{c}\hat{k}_t\\ \hat{\lambda}_t\end{array}\right]+M_{ce}\left[\begin{array}{c}\hat{A}_t\\ \hat{g}_t\end{array}\right], \tag{A40}$$

where the M_{cc}, M_{cs}, and M_{ce} are 2x2 matrices. The mnemonic in the matrix notation is that M_{cc} relates controls to controls, M_{cs} relates controls to states, and M_{ce} relates controls to exogenous variables.

The expressions (A35) and (A36) relate variations in controlled state ($\hat{k}$) and co-state ($\hat{\lambda}$) to changes in controls and exogenous state variables. These expressions may be written as

$$M_{ss}(B)\left[\begin{array}{c}\hat{k}_{t+1}\\ \hat{\lambda}_{t+1}\end{array}\right]=M_{sc}(B)\left[\begin{array}{c}\hat{c}_{t+1}\\ \hat{N}_{t+1}\end{array}\right]+M_{se}(B)\left[\begin{array}{c}\hat{A}_{t+1}\\ \hat{g}_{t+1}\end{array}\right], \tag{A41}$$

where $M_{ss}(B)$, $M_{sc}(B)$, and $M_{se}(B)$ are matrix polynomials in the backshift operator B at most of power 1. The notation is continued from above so that M_{sc} relates states to controls, etc. The components of any first order matrix polynomial $M(B)$ are expressed as M_0 (the constant term) and M_1 (the term that multiplies B).

Since M_{cc} is invertible (as a consequence of strict concavity of momentary utility), the combination of these expressions implies:[20]

$$\left[M_{ss}(B)-M_{sc}(B)M_{cc}^{-1}M_{cs}\right]\left[\begin{array}{c}\hat{k}_{t+1}\\ \hat{\lambda}_{t+1}\end{array}\right] = \left[M_{se}(B)+M_{sc}(B)M_{cc}^{-1}M_{ce}\right]\left[\begin{array}{c}\hat{A}_{t+1}\\ \hat{g}_{t+1}\end{array}\right]$$

or

$$M_{ss}^{*}(B)\left[\begin{array}{c}\hat{k}_{t+1}\\ \hat{\lambda}_{t+1}\end{array}\right] = M_{se}^{*}(B)\left[\begin{array}{c}\hat{A}_{t+1}\\ \hat{g}_{t+1}\end{array}\right], \tag{A42}$$

where $M_{ss}^{*}(B)$ and $M_{se}^{*}(B)$ are first-order matrix polynomials in B. To convert this system to a normal difference equation form, premultiply by the inverse of M_{ss0}^{*}, yielding the fundamental dynamic system of the neoclassical model.

[20] In the presence of tax rules such as (A.6), further restrictions have to be imposed on τ_t to ensure invertibility of Mcc.

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{bmatrix} = -\left[M^*_{ss_0}\right]^{-1} M^*_{ss_1} \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} + \left[M^*_{ss_0}\right]^{-1} M^*_{se_0} \begin{bmatrix} \hat{A}_{t+1} \\ \hat{g}_{t+1} \end{bmatrix} + \left[M^*_{ss_0}\right]^{-1} M^*_{se_1} \begin{bmatrix} \hat{A}_t \\ \hat{g}_t \end{bmatrix}. \tag{A43}$$

Define the matrices $W = -[M^*_{ss_0}]^{-1}M^*_{ss_1}$, $R = [M^*_{ss_0}]^{-1}M^*_{se_0}$ and $Q = [M^*_{ss_0}]^{-1}M^*_{se_1}$. In the present application, these are all 2x2 matrices, since there are two elements of the state-costate vector and two exogenous variables, but the formal analysis below will be conducted as though there were an arbitrary number of elements of the state-costate vector $(2n_s)$ and n_e exogenous variables, in which case the W matrix would be $(2n_s) \times (2n_s)$ and the R and Q matrices are $(2n_s) \times (n_e)$. The system dynamics are governed by the characteristic roots and characteristic vectors of the W matrix. Let P be the matrix of eigenvectors and let μ be a diagonal matrix with the roots on the diagonal, ordered in ascending absolute value. Then, $P\mu P^{-1} = W$ and the solution to the fundamental difference equation at date t is given by

$$\begin{aligned} \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} &= P\ \mu^t P^{-1} \begin{bmatrix} \hat{k}_0 \\ \hat{\lambda}_0 \end{bmatrix} + \sum_{h=0}^{t} P\ \mu^h P^{-1} R \begin{bmatrix} \hat{A}_{t-h+1} \\ \hat{g}_{t-h+1} \end{bmatrix} \\ &\quad + \sum_{h=0}^{t} P\ \mu^h P^{-1} Q \begin{bmatrix} \hat{A}_{t-h} \\ \hat{g}_{t-h} \end{bmatrix}. \end{aligned} \tag{A44}$$

For undistorted economies, the characteristic roots of the matrix W satisfy the restrictions that $\mu_1 < (\beta^*)^{1/2} < 1 < (\beta^*)^{-1/2} < \mu_2$ (see below). Further, for economies that are not too different from the undistorted one, it will be the case that the roots μ_1 and μ_2 are such that one is unstable and one is stable, a requirement which we impose in our simulations by means of constraints on the tax function.[21] In this case, there will be a single choice of the initial shadow

[21] One of these constraints for the fixed labor model, is $\omega_K < -\xi_{kk}$. When $\omega_K = -\xi_{kk}$, $u_1 = 1$,

price $(\hat{\lambda}_0)$ which makes the solution for $\{\hat{\lambda}_t\}_{t=0}^{\infty}$ and $\{\hat{k}_t\}_{t=0}^{\infty}$ consistent with the transversality condition. This "saddle point" instability characteristic is familiar from the continuous time growth model of Cass (1965) and Koopmans (1965), in which the dynamics are in differential equation form.

To determine this initial condition on the shadow price, $(\hat{\lambda}_0)$, we follow Vaughan's (1970) line of argument, which has been applied to linear rational expectations models by Blanchard and Kahn (1980). We begin by partitioning the matrices P, μ, P^{-1}, R and Q as follows:

$$
\begin{aligned}
P &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} p_{11}^* & p_{12}^* \\ p_{21}^* & p_{22}^* \end{bmatrix}, \\
\mu &= \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \\
R &= \begin{bmatrix} R_{ke} \\ R_{\lambda e} \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_{ke} \\ Q_{\lambda e} \end{bmatrix},
\end{aligned}
$$

where–in this application–all of the elements of P, P^{-1} and μ are scalars and the elements of R and Q are (1x2) row vectors. More generally, the submatrices of P, P^{-1} and μ are $n_s \times\ n_s$ matrices and the submatrices of R and Q are $n_s \times\ n_s$. Since the characteristic vectors in P are only defined up to a linear transformation we will latter normalize $P_{11} = 1$ and $P_{12} = 1$ (this is different from the standard normalization $P_{1i}^2 + P_{2i}^2 = 1$).

This partitioning makes particularly apparent the saddlepoint instability of the neoclassical model, for we may write

$$
P\ \mu^h P^{-1} = \begin{bmatrix} p_{11}\mu_1^h p_{11}^* + p_{12}\mu_2^h p_{21}^* & p_{11}\mu_1^h p_{12}^* + p_{12}\mu_2^h p_{22}^* \\ p_{21}\mu_1^h p_{11}^* + p_{22}\mu_2^h p_{21}^* & p_{21}\mu_1^h p_{12}^* + p_{22}\mu_2^h p_{22}^* \end{bmatrix}.
$$

i.e., after a displacement there is no tendency for the economy to return to the steady state. In that case, the decrease in the tax rate associated with an increase in capital exactly offsets the decreasing returns to capital. When $\omega_K > -\xi_{kk}$ the steady state is no longer stable.

As h becomes large, the terms in this expression are dominated by μ_2^h which is growing at a rate greater than β.

Vaughan's approach involves multiplying the state-costate vector by P^{-1}, thus defining a new vector of state and co-state variables, which are denoted $\tilde{k}_t$ and $\tilde{\lambda}_t$.

$$\begin{bmatrix} \tilde{k}_t \\ \tilde{\lambda}_t \end{bmatrix} = P^{-1} \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix}.$$

Multiplying both sides of the fundamental dynamic equation by P^{-1} leads to

$$\begin{aligned} \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix} &= P^{-1} \begin{bmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{bmatrix}, \\ &= P^{-1} W \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} + P^{-1} R \hat{e}_{t+1} + P^{-1} Q \hat{e}_t, \qquad \text{(A45)} \\ &= \mu \begin{bmatrix} \tilde{k}_t \\ \tilde{\lambda}_t \end{bmatrix} + P^{-1} R \hat{e}_{t+1} + P^{-1} Q \hat{e}_t, \end{aligned}$$

where the second equality follows from $P^{-1}W = P^{-1}P\,\mu P^{-1} = \mu P^{-1}$. The system (A45) is comprised of two decoupled difference equations (or, with multiple state variables, vectors of difference equations). The transformed capital component of this system is given by

$$\tilde{k}_{t+1} = \mu_1 \tilde{k}_t + [p_{11}^* R_{ke} + p_{12}^* R_{\lambda e}]\, \hat{e}_{t+1} + [p_{11}^* Q_{ke} + p_{12}^* Q_{\lambda e}]\, \hat{e}_t. \qquad \text{(A46)}$$

Thus, given an initial value of $\tilde{k}_0$ (see below), this is a stable difference equation, since the elements of μ_1 are less than 1 in absolute value and, hence, the specification of the initial value $\tilde{k}_0$ fully determines the subsystem solution.

By contrast, the analogous decoupled difference equation for the transformed shadow price is unstable in the backward direction since the elements of μ_2 exceed 1 in absolute value. For this reason, as Vaughan points out, one needs to impose a terminal rather than an initial value boundary condition of the transformed shadow price. To do this, it is easiest to pre-multiply the subsystem by μ_2^{-1} so it may be written as

$$\tilde{\lambda}_t = \mu_2^{-1}\tilde{\lambda}_{t+1} - \mu_2^{-1}\left[p_{21}^* R_{ke} + p_{22}^* R_{\lambda e}\right]\hat{e}_{t+1} - \mu_2^{-1}\left[p_{21}^* Q_{ke} + p_{22}^* Q_{\lambda e}\right]\hat{e}_t. \tag{A47}$$

Solving this equation forward, using the terminal condition that $\tilde{\lambda}_t$ grows at rate less than μ_2^{-1} (implied by the transversality condition) yields the solution:

$$\tilde{\lambda}_t = -\left(\sum_{j=0}^{\infty}\mu_2^{-j-1}\left[p_{21}^* R_{ke} + p_{22}^* R_{\lambda e}\right]\hat{e}_{t+j+1} + \sum_{j=0}^{\infty}\mu_2^{-j-1}\left[p_{21}^* Q_{ke} + p_{22}^* Q_{\lambda e}\right]\hat{e}_{t+j}\right). \tag{A48}$$

To return to our original specification for the state and co-state variables, we utilize the inverse transformation

$$\begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} = P \begin{bmatrix} \tilde{k}_t \\ \tilde{\lambda}_t \end{bmatrix}.$$

This transformation is applied to the previously determined solutions for the transformed state and co-state variables, which may be written in matrix form as:

$$\begin{aligned} \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{\lambda}_{t+1} \end{bmatrix} &= \begin{bmatrix} \mu_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{\lambda}_t \end{bmatrix} \\ &+ \begin{bmatrix} p_{11}^* R_{ke} + p_{12}^* R_{\lambda e} \\ 0 \end{bmatrix}\hat{e}_{t+1} + \begin{bmatrix} p_{11}^* Q_{ke} + p_{12}^* Q_{\lambda e} \\ 0 \end{bmatrix}\hat{e}_t \\ &+ \begin{bmatrix} 0 \\ I \end{bmatrix} f_{t+1}, \end{aligned} \tag{A49}$$

where $f_t = -\left(\sum_{j=0}^{\infty}\mu_2^{-j-1}\left[p_{21}^* R_{ke} + p_{22}^* R_{\lambda e}\right]\hat{e}_{t+j+1} + \sum_{j=0}^{\infty}\mu_2^{-j-1}\left[p_{21}^* Q_{ke} + p_{22}^* Q_{\lambda e}\right]\hat{e}_{t+j}\right)$.

Thus, it follows that the solutions for the original variables take the forms

$$
\begin{aligned}
\begin{bmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{bmatrix} &= P \begin{bmatrix} \mu_1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} \\
&+ P \begin{bmatrix} (p_{11}^* R_{ke} + p_{12}^* R_{\lambda e}) \\ 0 \end{bmatrix} \hat{e}_{t+1} + P \begin{bmatrix} (p_{11}^* Q_{ke} + p_{12}^* Q_{\lambda e}) \\ 0 \end{bmatrix} \hat{e}_t + P \begin{bmatrix} 0 \\ I \end{bmatrix} f_{t+1}. \\
\begin{bmatrix} \hat{k}_{t+1} \\ \hat{\lambda}_{t+1} \end{bmatrix} &= \begin{bmatrix} p_{11}\mu_1 p_{11}^* & p_{11}\mu_1 p_{12}^* \\ p_{21}\mu_1 p_{11}^* & p_{21}\mu_1 p_{12}^* \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} + \begin{bmatrix} p_{11}(p_{11}^* R_{ke} + p_{12}^* R_{\lambda e}) \\ p_{21}(p_{11}^* R_{ke} + p_{12}^* R_{\lambda e}) \end{bmatrix} \hat{e}_{t+1} \qquad \text{(A50)} \\
&+ \begin{bmatrix} p_{11}(p_{11}^* Q_{ke} + p_{12}^* Q_{\lambda e}) \\ p_{21}(p_{11}^* Q_{ke} + p_{12}^* Q_{\lambda e}) \end{bmatrix} \hat{e}_t + \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} f_{t+1}.
\end{aligned}
$$

In contrast to the earlier general solution, this system is now stable in the backward direction.

Initial Conditions. The relationship between the solutions is given, compactly, by

$$
\begin{aligned}
\tilde{k}_t &= p_{11}^* \hat{k}_t + p_{12}^* \hat{\lambda}_t, \qquad \text{(A51a)} \\
\tilde{\lambda}_t &= p_{21}^* \hat{k}_t + p_{22}^* \hat{\lambda}_t. \qquad \text{(A51b)}
\end{aligned}
$$

The second of these conditions implies that

$$
\hat{\lambda}_t = [p_{22}^*]^{-1} \tilde{\lambda}_t - [p_{22}^*]^{-1} p_{21}^* \hat{k}_t \qquad \text{(A52)}
$$

which implies an initial condition for the shadow price, given the initial capital stock $\hat{k}_0$ and the initial condition on the transformed shadow price, which is given by the preceding analysis (A48). The combination of (A51a) and (A52) implies that

$$
\begin{aligned}
\tilde{k}_t &= p_{11}^* \hat{k}_t + p_{12}^* \hat{\lambda}_t = p_{11}^* \hat{k}_t + p_{12}^* \left[[p_{22}^*]^{-1} \tilde{\lambda}_t - [p_{22}^*]^{-1} p_{21}^* \hat{k}_t \right] \\
&= \left[p_{11}^* - p_{12}^* [p_{22}^*]^{-1} p_{21}^* \right] \hat{k}_t + p_{12}^* [p_{22}^*]^{-1} \tilde{\lambda}_t, \qquad \text{(A53)}
\end{aligned}
$$

which indicates that specification of the initial capital stock and the transformed shadow price yield an initial condition on the transformed capital stock.

2.3. Alternative Solutions

There are two alternative solutions that will prove useful in considering some alternative experiments with this class of models.

First, if one is interested in evaluating the effects of a particular sequence of exogenous variables, $\{\hat{e}_t\}_{t=0}^{\infty}$, then the computationally advantageous solution is to compute $\hat{\lambda}_o$ and then to use the general solution (A45) above, which can be computed recursively, to supply subsequent values of $\hat{k}$ and $\hat{\lambda}$. This is the natural procedure for certain hypothetical policy experiments, such as investigating the implications of a temporary rise in military spending for a war of specified duration (Wynne (1987)).

Second, working under a certainty equivalence assumption, one can study the stochastic processes generated by the model economy using a linear, Markov structure for the exogenous forcing variables. Then, an alternative approach is the natural one.

In this approach, the sequences $\{\hat{e}_{t+j}\}_{j=0}^{\infty}$ in the preceding solutions are replaced by $E\{\hat{e}_{t+j}\}_{j=0}^{\infty}|I_t$, where I_t contains the history of $\hat{e}$. Then, the discounted sums present in $\tilde{\lambda}_t$ can be collapsed to specified linear functions of I_t using the Hansen-Sargent (1980, 1981) formulas for forecasting of discounted sums. Denote this solution as $\tilde{\lambda}(I_t)$. Then, the combination of the capital component of the fundamental difference equation

$$\begin{aligned}\hat{k}_{t+1} &= [p_{11}\mu_1 p_{11}^* + p_{12}\mu_2 p_{21}^*]\,\hat{k}_t + [p_{11}\mu_1 p_{12}^* + p_{12}\mu_2 p_{22}^*]\,\hat{\lambda}_t(I_t) \\ &\quad + R_{ke}E\,[\hat{e}_{t+1}]\,|I_t + Q_{ke}\hat{e}_t, \qquad \text{(A45a)}\end{aligned}$$

and the preceding relation between transformed and untransformed shadow prices,

$$\hat{\lambda}_t = [p_{22}^*]^{-1}\tilde{\lambda}_t - [p_{22}^*]^{-1}p_{21}^*\hat{k}_t, \qquad \text{(A45b)}$$

yields the following equilibrium law of motion for the capital stock.

$$\begin{aligned} \hat{k}_{t+1} &= [p_{11}\mu_1 p_{11}^{-1}]\hat{k}_t + [p_{11}\mu_1 p_{12}^* + p_{12}\mu_2 p_{22}^*] \, [p_{22}^*]^{-1} \, \tilde{\lambda} \, [I_t] \\ &\quad + R_{ke} E \, [\hat{e}_{t+1}] \, | I_t + Q_{ke}\hat{e}_t, \end{aligned} \tag{A55}$$

where we use the formula $[p_{11}]^{-1} = p_{11}^* - p_{12}^* \, [p_{22}^*]^{-1} \, p_{21}^*$, which is obtained from the standard formula for inversion of a partitioned matrix.

Computation of $\tilde{\lambda}(I_t)$:

The transformed shadow price equation (A49) can be expressed in the form:

$$\tilde{\lambda}_t = \mu_2^{-1}\tilde{\lambda}_{t+1} + z_1\hat{e}_{t+1} + z_0\hat{e}_t,$$

where $\tilde{\lambda}_t$ is a $ns \times 1$ vector, μ_2 is an $ns \times ns$ matrix, $\hat{e}_t$ is a $ne \times 1$ vector, and z_0, z_1 are $ns \times ne$ matrices. Under the certainty equivalence assumption, this difference equation is assumed to hold in expectation form,

$$\tilde{\lambda}_t = \mu_2^{-1} E_t\tilde{\lambda}_{t+1} + z_1 E_t\hat{e}_{t+1} + z_0 E_t\hat{e}_t, \tag{A56}$$

For scalar versions of this type of expression, Hansen and Sargent (1980, 1981) have provided formulas, under the assumption that the exogenous variables are governed by ARMA processes. The forecasting formulas to be employed will utilize the fact that μ_2 is a diagonal matrix, so that the Hansen-Sargent formulas can be employed on an appropriate "line-by-line" basis.

First, partition z_0 and z_1 into ns row vectors, denoted z_{0i} and z_{1i}, $i = 1, ..., ns$. Then, a representative equation can be written as

$$\tilde{\lambda}_{it} = \mu_{2i}^{-1} E_t\tilde{\lambda}_{i,t+1} + z_{1i} E_t\hat{e}_{t+1} + z_{0i}\hat{e}_t, \tag{A57}$$

This can be solved forward in the standard manner and Hansen-Sargent formulas applied. To take the concrete (and easy) example that we utilize throughout our analysis, suppose that the vector of exogenous variables is an AR(1),

$$\hat{e}_t = \rho \hat{e}_{t-1} + \epsilon_t, \tag{A58}$$

where ρ is an $ne \times ne$ matrix. Then, it follows that $z_{1i} E_t \hat{e}_{t+1} + z_{0i} \hat{e}_t = [z_{1i}\rho + z_{0i}]\hat{e}_t$ and that

$$\tilde{\lambda}_{it} = [z_{1i}\rho + z_{0i}][I - \mu_{2i}^{-1}\rho]^{-1}\hat{e}_t. \tag{A59}$$

Application of this result to each of the ns equations yields the desired forecasting formula.

In combination with the law of motion for the exogenous variables, these expressions form a linear system comprising the state ($\hat{k}$), co-state ($\hat{\lambda}$) and the exogenous variables. Thus, multivariate linear system methods can be used for computation of (i) impulse responses; (ii) population moments; and (iii) stochastic simulations of the form reported in the text. The flow chart presented in Figure 1 summarizes the operations required to obtain these three types of output.

[Insert Figure 1 Here]

That is, in the specific case under study with a single capital stock and a first order autoregressive processes for the forcing variables, there will be a linear system that expresses the optimal evolution of the state variable, k, which is under the control of agents, and the exogenous state variables A and g.

$$s_{t+1} = \begin{bmatrix} \hat{k}_{t+1} \\ \hat{A}_{t+1} \\ \hat{g}_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_1 & \pi_{kA} & \pi_{kg} \\ 0 & \rho_{AA} & \rho_{Ag} \\ 0 & \rho_{gA} & \rho_{gg} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{A}_t \\ \hat{g}_t \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{A,t+1} \\ \epsilon_{g,t+1} \end{bmatrix}, \tag{A60}$$

where ϵ_A and ϵ_g are shocks to the system that are serially uncorrelated but are potentially contemporaneously correlated. The coefficients $\rho_{AA}, \rho_{Ag}, \rho_{gA}$, and ρ_{gg} govern the model's exogenous dynamics; under the maintained assumption that the shocks to the system are temporary in form, we require that the exogenous process for ($\hat{A}$ and $\hat{g}$) is stationary. The coefficients π_{kA} and π_{kg} are given by combining (A56) and (A60) above, involving both forecasting and technological parameters. For convenience, we write this solution in the vector autoregressive form as $s_{t+1} = M\ s_t + \epsilon_t$.

The other variables of the system, (the shadow price, consumption, effort, investment, output, productivity, etc.) can all be written as linear functions of s_t, using (i) the preceding relation, $\hat{\lambda}_t = p_{22}^{*-1}\tilde{\lambda}(I_t) - p_{22}^{*-1}p_{21}^*\hat{k}_t$; (ii) the preceding expressions for optimal controls, (A40), and the relations governing flows and relative prices, (A37)-(A39). Letting the vector of variables of interest be described as z_t, these linear restrictions can be expressed as $z_t = \Pi\ s_t$.

2.4. Detailed Example: The Standard Neoclassical Model

As a concrete application of the forgoing, consider the basic neoclassical model with fixed labor and public expenditures financed by lump sum taxes. Then, it follows that we eliminate (A34) and treat $\hat{N}_t = 0$ in all other equations above. The reduction to a single control makes it relatively easy to carry out the algebra that corresponds to our general solution strategy. The equation (A40) relating the single control to the state, co-state and exogenous variables, written in the general form is

$$M_{cc}\hat{c}_t = M_{cs}\begin{bmatrix}\hat{k}_t \\ \hat{\lambda}_t\end{bmatrix} + M_{ce}\begin{bmatrix}\hat{A}_t \\ \hat{g}_t\end{bmatrix},$$

where $M_{cc} = -\sigma_c$, $M_{cs} = [0\ 1]$ and $M_{ce} = [0\ 0]$. The equation relating states to controls and exogenous variables is

$$M_{ss}(B)\begin{bmatrix}\hat{k}_{t+1}\\ \hat{\lambda}_{t+1}\end{bmatrix} = M_{sc}(B)\hat{c}_{t+1} + M_{se}(B)\begin{bmatrix}\hat{A}_{t+1}\\ \hat{g}_{t+1}\end{bmatrix},$$

where $M_{ss}(B) = \begin{bmatrix}\eta_k & 1\\ \phi s_i & 0\end{bmatrix} + \begin{bmatrix}0 & -1\\ q & 0\end{bmatrix} B$ with $q = -[s_i(\phi - 1) + (1 - \alpha)]$. $M_{sc}(B) = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}0\\-s_c\end{bmatrix} B$ and $M_{se}(B) = \begin{bmatrix}-\eta_A & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 1 & -s_g\end{bmatrix} B$. The fundamental dynamic equation is

$$M^*_{ss}(B)\begin{bmatrix}\hat{k}_{t+1}\\ \hat{\lambda}_{t+1}\end{bmatrix} = M^*_{se}(B)\begin{bmatrix}\hat{A}_{t+1}\\ \hat{g}_{t+1}\end{bmatrix},$$

where $M^*_{ss}(B) = [M_{ss}(B) - M_{sc}(B)[M_{cc}]^{-1}M_{cs}]$ has a particularly simple form because $M_{sc_0} = [0 \;\; 0]$.

$$M^*_{ss}(B) = \begin{bmatrix}\eta_k & 1\\ \phi s_i & 0\end{bmatrix} + \begin{bmatrix}0 & -1\\ q & -s_c/\sigma_c\end{bmatrix} B.$$

Further, $M_{ce} = [0 \;\; 0]$ implies that

$$M^*_{se}(B) = \left[M_{se}(B) + M_{sc}(B)M_{cc}^{-1}M_{ce}\right] = M_{se}(B).$$

The state-costate transition matrix is

$$W = -[M^*_{ss_0}]^{-1}M^*_{ss_1} = -(s_i\phi)^{-1}\begin{bmatrix}q & -(s_c/\sigma_c)\\ -\eta_k q & (s_c/\sigma_c)\eta_k - s_i\phi\end{bmatrix}.$$

The other expressions in the fundamental difference system are as follows.

$$R = [M^*_{ss_0}]^{-1}M^*_{se_0} = \begin{bmatrix}0 & 0\\ -\eta_A & 0\end{bmatrix}.$$

Thus, the partitioning of this matrix as outlined above involves $R_{ke} = [0 \quad 0]$ and $R_{\lambda e} = [-\eta_A 0]$. Similarly,

$$Q = [M^*_{ss_0}]^{-1} M^*_{se_1} = -(s_i\phi)^{-1} \begin{bmatrix} -1 & s_g \\ \eta_k & -s_g\eta_k \end{bmatrix}$$

can be partitioned into $Q_{ke} = \begin{bmatrix} (s_i\phi)^{-1} & -s_g(s_i\phi)^{-1} \end{bmatrix}$ and $Q_{\lambda e} = \begin{bmatrix} -\eta_k(s_i\phi)^{-1} & s_g\eta_k(s_i\phi)^{-1} \end{bmatrix}$.

The Decomposition $W = P\ \mu P^{-1}$.

The first step involves computing the characteristic roots associated with the system of difference equations. These are the solutions to $|W - ZI| = 0$. Since in this case W is a 2×2 matrix, it is easy to prove that the two characteristic roots, denoted by μ_i, are given by:

$$\begin{aligned} \mu_1 + \mu_2 &= w_{11} + w_{22}, \\ \mu_1\mu_2 &= w_{11}w_{22} - w_{12}w_{21}. \end{aligned}$$

where w_{ij} denotes the (i, j) element of W. That implies that:

$$\begin{aligned} \mu_1 + \mu_2 &= -(q + s_c\eta_k/\sigma_c - \phi s_i)/\phi s_i = \frac{1}{\beta^*} - \frac{s_c\eta_k}{\sigma_c s_i \phi} + 1, \\ \mu_1\mu_2 &= -q/\phi s_i = \frac{(\phi - 1)s_i + (1 - \alpha)}{\phi s_i} = \frac{1}{\gamma}[(1 - \delta_K) + (1 - \alpha)Y/K] = 1/\beta^*. \end{aligned}$$

The characteristic vectors, x_i, are the solutions to $(W - \mu_i I)x_i = 0$. To simplify the computations we normalize x_{1i} to be equal to one, instead of using the traditional normalization $x_{1i}^2 + x_{2i}^2 = 1$. The characteristic vectors are then of the form $\begin{bmatrix} 1 & (\mu_i - w_{11})/w_{12} \end{bmatrix}$. Using the fact that $w_{11} = \mu_1\mu_2$, it follows that the matrix of characteristic vectors is given by:

$$\begin{aligned} P &= \begin{bmatrix} 1 & 1 \\ \frac{\mu_1(1-\mu_2)}{w_{12}} & \frac{\mu_2(1-\mu_1)}{w_{12}} \end{bmatrix} \qquad \text{and} \\ P^{-1} &= (|P|)^{-1} \begin{bmatrix} \frac{\mu_2(1-\mu_1)}{w_{12}} & -1 \\ \frac{-\mu_1(1-\mu_2)}{w_{12}} & 1 \end{bmatrix}. \end{aligned}$$

with $|P| = \frac{\mu_2 - \mu_1}{w_{12}}$. Using equation (A49) and the fact that $\frac{P^*_{21} - \eta_k P^*_{22}}{P^*_{22}} = \frac{1-\mu_2}{w_{12}}$ we can write the solution for $\tilde{\lambda}_t$ as:

$$\tilde{\lambda}_t = \frac{P^*_{22}}{\mu_2} \frac{1}{1 - \mu_2^{-1} B^{-1}} \left\{ \eta_A \hat{A}_{t+1} + \frac{1-\mu_2}{\phi s_i w_{12}} \hat{A}_t - \frac{s_g}{\phi s_i} \frac{(1-\mu_2)}{w_{12}} \hat{g}_t \right\}.$$

Using (A55a) we can write

$$\hat{k}_{t+1} = \mu_1 \hat{k}_t + [\mu_1 P^*_{12} + \mu_2 P^*_{22}] \, [P^*_{22}]^{-1} \, \tilde{\lambda}_t - \eta_k \frac{\hat{A}_t}{\phi s_i} + \eta_k \frac{s_g}{\phi s_i} \hat{g}_t.$$

Substituting $\tilde{\lambda}_t$ and rearranging terms:

$$\hat{k}_{t+1} = \mu_1 \hat{k}_t + \frac{1}{\mu_2 \phi s_i} \frac{1}{1 - \mu_2^{-1} B^{-1}} \left(\left[\frac{s_c}{\sigma_c} \eta_A - 1 + B \right] \hat{A}_{t+1} + s_g (1 - B) \hat{g}_{t+1} \right).$$

Figure 2 may help in the interpretation of the following qualitative statements about the roots. For the undistorted economy we have been considering, it is unambiguous that the roots satisfy $\mu_1 < 1 < (\beta^*)^{-1} < \mu_2$. The larger is the (positive) ratio $(-s_c \eta_k / \sigma_c s_i \phi)$, then the smaller is μ_1 and the larger is μ_2, since an increase in this ratio shifts the dotted line outward.[22] (This ratio is positive, since the elasticity of the (after-tax) marginal product of capital with respect to the capital stock (η_k) is negative.)

[Insert Figure 2 Here]

[22] The two equations that determine the roots in the economy with a constant output tax are: $\mu_1 \mu_2 = \frac{(1-\delta_k)\beta^*(\Omega - 1) + \gamma_X}{\gamma_X \Omega \beta^*}$ and $\mu_1 + \mu_2 = \mu_1 \mu_2 - \frac{s_c \eta_k}{\phi s_i \sigma_c} + 1$.

3. Simulations, Moments and Impulse Response Functions

The linear character of the (approximate) dynamic system produced in the preceding section implies that we can draw upon the extensive body of material dealing with analysis of these systems.[23] For the purpose of the present study, the key elements are as follows.

First, the linear system's state variables evolve according to

$$s_{t+1} = M \; s_t + \epsilon_t, \tag{A61}$$

and the remainder of variables of interest are governed by

$$z_t = \Pi \; s_t. \tag{A62}$$

Simulations:

Given specification of the "innovations" in the stochastic processes for the exogenous variables, ϵ_{At} and ϵ_{Gt}, it is straightforward to generate simulated time paths given: (i) initial state values, s_o; and (ii) sequences of the ϵ_t for a given sample interval, $t = 1, ...T$. In our view, to avoid dependence on initial conditions, s_o should be generated as a draw from a tri-variate random variable, with probability distribution given by the stationary distribution of s. Recursive computation of the T period simulation is direct.

Impulse Response Functions

It is also straightforward to use the system (A61-A62) to generate impulse response functions. The programs that we employ undertake this in a direct manner, specifying ϵ_1, and then utilizing the formula (A61) recursively. (A62) then gives the other variables of interest, directly from the computed state responses.

[23] See, for instance Harvey (1981) and Chow (1975).

Population Moments

Computation of population moments is also straightforward. We employ the following procedure. First, we decompose the system matrix M as follows.

$$M = PM\ DM\ PM^{-1},$$

where PM is the matrix of eigenvectors of M and DM contains the eigenvalues on its diagonal (and zeros elsewhere). Then, we define transformed states as

$$s_t^* = PM^{-1} s_t,$$

and transformed innovations as

$$\epsilon_t^* = PM^{-1} \epsilon_t.$$

Then, the covariance between any two elements, s_{jt}^* and s_{it}^*, is given by

$$E[s_{jt}^* s_{it}^*] = [1 - dm_j dm_i]^{-1} E[\epsilon_{jt}^* \epsilon_{it}^*],$$

where dm_i is the ith diagonal elements of the matrix DM. Calculation of the variance-covariance matrix of the original (untransformed) state variables then is given by reversing the transformation:

$$E[s_t s_t'] = PM\ E[s_t^* s_t^{*\prime}] PM^{-1}.$$

The first order autoregressive form of the linear system makes it particularly easy to compute autocovariances at any desired lead or lag $j > 0$:

$$\begin{aligned} \text{lags} &: \quad E[s_t s_{t-j}^{'}] = M^j E[s_t s_t'], \\ \text{leads} &: \quad E[s_t s_{t+j}^{'}] = E[s_t s_t'](M')^j. \end{aligned}$$

Moments of the other variables of interest can also readily be calculated

$$E[z_t z'_{t\pm j}] = \Pi \ E[s_t s'_{t\pm j}]\Pi'.$$

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Supplementary Note #1:
Marginal Product Elasticities

This note reports the details of elasticity calculations behind equations (A21a,b). The details follow Ferguson, C.E., *Microeconomic Theory*, 3rd edition, 1972, 414-425.

Let the production function $Y = F(K, N)$: Dividing by the quantity of labor input and defining a production ($f(k)$) function for output per unit of labor input:

$$Y/N = F(K/N, 1) = f(k),$$

where $k = K/N$. Then, it is straightforward to verify that the marginal products of labor and capital are

$$\begin{aligned} \partial Y/\partial N &= f(k) - kDf(k), \\ \partial Y/\partial K &= Df(k). \end{aligned}$$

Now, define the marginal rate of technical substitution of labor for capital as

$$mrs_{K,N} = \frac{\partial Y/\partial N}{\partial Y/\partial K} = \frac{f(k) - kDf(k)}{Df(k)}.$$

It follows that the derivative of $mrs_{K,N}$ with respect to k is just

$$-\frac{f(k)D^2f(k)}{(Df(k))^2}$$

so that the elasticity of substitution of capital is just

$$\zeta_{KN} = d\log(K/N)/d\log(mrs_{K,N}) = -\frac{Df(k)[f(k) - kDf(k)]}{kf(k)D^2f(k)}.$$

Note that the elasticity of substitution of labor for capital is equal to that of capital for labor, since $\log(x) = -\log(1/x)$.

The relevant elasticities of the marginal products are computed as follows:

$$\begin{aligned}
\xi_{NN} &= [\frac{\partial}{\partial N}(\frac{\partial Y}{\partial N})]N/[\frac{\partial Y}{\partial N}] = \frac{k^2 D^2 f(k)}{f(k) - kDf(k)} = -\xi_{NK}, \\
\xi_{KK} &= [\frac{\partial}{\partial K}(\frac{\partial Y}{\partial K})]K[(\frac{\partial Y}{\partial K})] = \frac{kD^2 f(k)}{Df(k)} = -\xi_{KN}.
\end{aligned}$$

where the right -hand equalities may either be verified directly or viewed as a property of the zero degree homogeneity of the marginal products (as derivatives of the production function, which is homogeneous of degree one).

Now, define factor shares as follows:

$$\begin{aligned}
s_N &= \frac{N(\partial Y/\partial N)}{Y} = \frac{f(k) - kDf(k)}{f(k)}, \\
s_K &= \frac{K(\partial Y/\partial K)}{Y} = \frac{kDf(k)}{f(k)}.
\end{aligned}$$

Then, it follows that

$$\zeta_{KN} = \frac{kDf(k)}{f(k)} \frac{f(k) - fDf(k)}{k^2 D^2 f(k)} = s_K/\xi_{NN}.$$

Similarly,

$$\zeta_{KN} = \frac{f(k) - kDf(k)}{f(k)}/[\frac{kD^2 f(k)}{Df(k)}] = s_N/\xi_{KK}.$$

Thus, it follows that

$$\begin{aligned}
\xi_{KK} &= -\xi_{KN} = s_N/\zeta_{KN}, \\
\xi_{NN} &= -\xi_{NK} = s_K/\zeta_{KN}.
\end{aligned}$$

Thus, determination of the values of s_N, s_K (subject to the constraint that $s_N + s_K = 1$) and ζ_{KN} determines the elasticities of the marginal product schedules. In effect, we are using the property that any constant returns to scale production function is locally approximated by a CES function.

Supplementary Note #2

Utility and Demand Theory

The objects of these notes are to add some additional information concerning utility and demand theory for the steady state of the growth model. The objective is to relate the elasticities of marginal utility used in the technical appendix and programs to more traditional demand concepts.

Consider an agent choosing consumption and labor supply (leisure demand) so as to maximize

$$u(c, L),$$

subject to a budget constraint

$$wL + c \leq w + a \equiv y^f,$$

where the time endowment is normalized to unity; a denotes other income; and y^f is "full income."

1. Results from Demand Theory

From standard demand analysis, we know that the Marshallian demands take the form $c = m_c(w, y^f)$ and $L = m_L(w, y^f)$. For these demands,
(i) share weighted income elasticities of demand must sum to unity;
(ii) share weighted price elasticities ϵ_L and ϵ_c satisfy an additional restriction:

$$\frac{wL}{y^f} + \frac{wL}{y^f}\epsilon_L + \frac{c}{y^f}\epsilon_c = \frac{w}{y^f}.$$

Given the shares, then, there are only two independent income and price elasticities. In addition, our growth restriction requires that

$$\frac{wD_1m_L}{L} + \frac{y^f D_2 m_L}{L} = 0,$$

i.e., the income elasticity and price elasticity of leisure demand sum to zero.

Taken together, then , in a steady state that has implicitly determined expenditure shares, there is only one degree of freedom.

2. Implications for Elasticities of Marginal Utility

With the type of preferences described in the main text, which we write as

$$u(c, L) = \frac{1}{1-\sigma}[cv(L) - 1]^{1-\sigma},$$

there is a marginal rate of substitution condition

$$D_2 u(c, L)/D_1 u(c, L) = [cDv(L)/v(L)] = w.$$

With a little algebra, this implies that

$$s_c[LDv(L)/v(L)][N/(1-N)] = s_N.$$

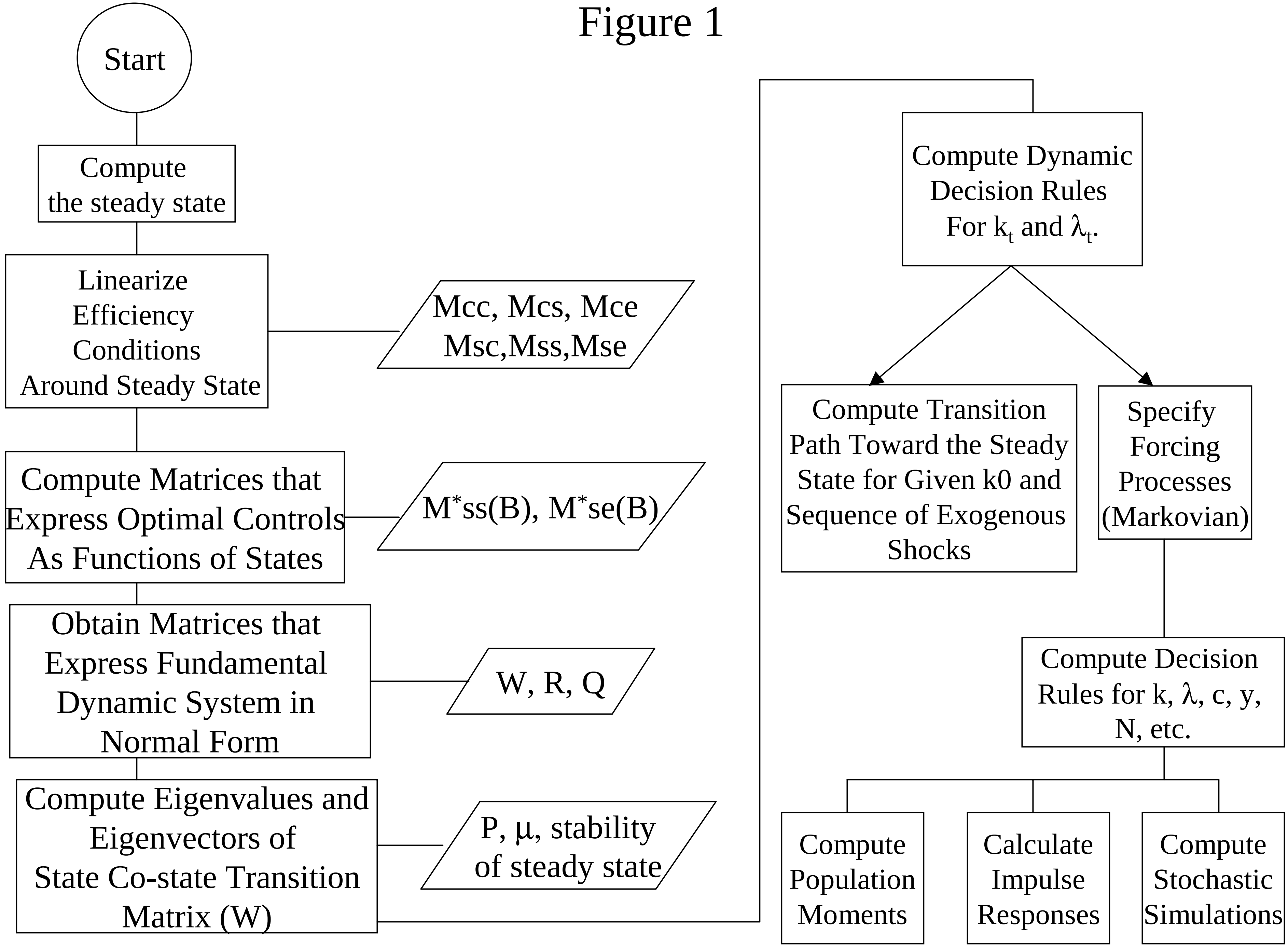
Hence, it follows that $[LDv(L)/v(L)]$. is pinned down given the steady state, not a free parameter.

With the specified preference, elasticities of marginal utility are given by

$$\begin{array}{ll} \xi_{cc} = -\sigma & \xi_{cL} = (1-\sigma)[LDv(L)/v(L)] \\ \xi_{Lc} = (1-\sigma) & \xi_{cL} = -\sigma[LDv(L)/v(L)] + [LD^2v(L)/Dv(L)]. \end{array}$$

These expressions contain two free parameters: σ, which controls intertemporal substitution/risk aversion and $[LD^2v(L)/Dv(L)]$. Log differentiation of the marginal rate of substitution condition indicates that σ does not enter, so that specification of $[LD^2v(L)/Dv(L)]$ is equivalent to specification of the single free parameter in the Marshallian demand system.

Figure 1



Start
Compute the steady state
Linearize Efficiency Conditions Around Steady State
Mcc, Mcs, Mce
Msc,Mss,Mse
Compute Matrices that Express Optimal Controls As Functions of States
M*ss(B), M*se(B)
Obtain Matrices that Express Fundamental Dynamic System in Normal Form
W, R, Q
Compute Eigenvalues and Eigenvectors of State Co-state Transition Matrix (W)
P, μ, stability of steady state
Compute Dynamic Decision Rules For k_t and λ_t.
Compute Transition Path Toward the Steady State for Given k0 and Sequence of Exogenous Shocks
Specify Forcing Processes (Markovian)
Compute Decision Rules for k, λ, c, y, N, etc.
Compute Population Moments
Calculate Impulse Responses
Compute Stochastic Simulations

Figure 2