Delta-Hedging and Pricing a Quanto

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Appendix A to Chapter 21 shows that if the price of a derivative depends on multiple state variables, it must solve the following partial differential equation:

$$V_t + \sum_{i=1}^{n}(r-\delta_i)S_iV_{S_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{i,j}S_iS_jV_{S_iS_j} = rV \tag{1}$$

This is a generalization of the Black-Scholes equation when the stock price is the only state variable. In addition to satisfying equation (1), the price of a claim must satisfy the appropriate boundary condition.

Consider a quanto prepaid forward contract on a stock index denominated in a foreign currency, such as the Nikkei quanto contract discussed in Chapter 22. The contract could pay, for example, \$5 times the price of the Nikkei index. If $Q(t)$ is the time t price of the Nikkei index, one way to write this payoff at time T is

$$\$5 \times Q(T) \tag{2}$$

If we let $Y(t) = x(t)Q(t)$, another way to write this payoff is

$$\$5 \times \frac{x(T)Q(T)}{x(T)} = \$5\frac{Y(T)}{x(T)} \tag{3}$$

The reason for writing the payoff in this fashion is that $Y(T) = x(T)Q(T)$ represents the price of the currency-translated index, which is an asset that dollar-denominated investors can actually hold, unlike the Nikkei index denominated solely in yen.

We can write risk-neutral processes for $x(t)$ and $Y(t)$ as

$$\begin{aligned}\frac{dx(t)}{x(t)} =& (r-r_f)dt + sdW \\ \frac{dY(t)}{Y(t)} =& (r-\delta)dt + \sigma dZ + sdW\end{aligned}$$

The process for Y reflects the fact that the currency translated index has both currency and index risk. We assume that $E(dWdZ) = \rho dt$ and the the price of the quanto contract is $V(x_t, Y_t, t)$. With this assumption, equation (1) becomes

$$\frac{1}{2}\left[(dx)^2 V_{xx} + (dY)^2 V_{YY} + 2dx\, dY V_{xY}\right] + (r-\delta)YV_Y + (r-r_f)xV_x + V_t = rV \quad (4)$$

Using the multiplication rules from Chapter 20, we can write $(dx)^2 = s^2 dt$, $(dY)^2 = (s^2 + \sigma^2 + 2\rho s\sigma)dt$, and $dx\, dY = (s^2 + \rho s\sigma)dt$. Equation (4) then becomes

$$\frac{1}{2}\left[s^2 x^2 V_{xx} + (s^2 + \sigma^2 + 2\rho s\sigma)Y^2 V_{YY} + 2(s^2 + \rho s\sigma)xYV_{xY}\right] + (r-\delta)YV_Y + (r-r_f)xV_x + V_t = rV \quad (5)$$

The price of the quanto prepaid forward must satisfy equation (5), subject to the constraint

$$V(x(T), Y(T)) = \bar{x}\frac{Y(T)}{x(T)} \quad (6)$$

where $\bar{x}$ is the fixed exchange rate. We will let $\bar{x} = \$1$ for simplicity.

The solution to equation (5) subject to equation (6) is

$$V(x(t), Y(t)) = \frac{Y(t)}{x(t)}\, e^{(r_f - \delta - \rho s\sigma - r)(T-t)} \quad (7)$$

It is also possible to write $Q(t)$ instead of $Y(t)/x(t)$, but writing the solution as in equation (7) helps us understand how to hedge a quanto.

Now consider a market-maker who sells a quanto prepaid forward and wishes to delta-hedge. The delta with respect to the exchange rate, $x(t)$, is

$$\begin{aligned}\frac{\partial V}{\partial x(t)} &= -\frac{Y(t)}{x(t)^2}\, e^{(r_f - \delta - \rho s\sigma - r)(T-t)} \\ &= -\frac{1}{x(t)} V(x(t), Y(t))\end{aligned}$$

The delta with respect to the currency-translated index, $Y(t)$, is

$$\begin{aligned}\frac{\partial V}{\partial Y(t)} &= \frac{1}{x(t)}\, e^{(r_f - \delta - \rho s\sigma - r)(T-t)} \\ &= \frac{1}{Y(t)} V(x(t), Y(t))\end{aligned}$$

Example 0.1 *Assume the same inputs as in the Nikkei quanto example in Chapter 22. The quanto prepaid forward price is*

$$\$200e^{.04-.02-0.2\times 0.15\times 0.10-0.08} = \$187.789$$

The deltas with respect to x *and* Y *are*

$$\Delta_x = -\frac{1}{x(t)}V(x(t), Y(t)) = -\frac{1}{0.01}187.789 = ¥18,789$$
$$\Delta_Y = \frac{1}{Y(t)}V(x(t), Y(t)) = \frac{1}{200}187.789 = 0.939$$

Thus, we short ¥18,789 in the spot market and go long 0.939 units of the currency-translated index.

If the contract is a forward contract rather than a prepaid forward contract, then the forward price is $e^{r(T-t)}V(x(t), Y(t))$*, and we have* $\Delta_x = e^{0.08}¥18,789 = -¥20,342.91$ *and* $\Delta_Y = e^{0.08}0.939 = 1.0171$ *units of the currency-translated index. The number of currency units we short is the same as the quanto forward price.*