

Valuing Outperformance Claims

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Suppose we have two assets for which the risk-neutral processes are

$$\frac{dS}{S} = (r - \delta_S)dt + \sigma_S dZ_1 \quad (1)$$

and

$$\frac{dQ}{Q} = (r - \delta_Q)dt + \sigma_Q dZ_2 \quad (2)$$

Suppose that $E(dZ_1 \times dZ_2) = \rho dt$, so that ρ is the correlation coefficient between S and Q .

Now suppose there is an option that at time T pays

$$V[S(T), Q(T), T] = \max[0, S(T) - kQ(T)] \quad (3)$$

How do we value this option?

I will present a solution that uses the techniques in McDonald (2003, Chapter 21). Before we discuss valuation, however, what if I present a solution and you don't believe the answer? *You can always compare the price you obtain using a formula with a price you obtain by using Monte Carlo valuation.* Since Monte Carlo is simple, you can easily check just about any pricing formula. I strongly urge you to do this. It is good practice with Monte Carlo valuation and it will increase your confidence that you understand how the formulas work.

You might be wondering why we can't simply view the difference $S(T) - kQ(T)$ as the value of a portfolio, and use the volatility of this portfolio as the volatility in the Black-Scholes formula. The reason this does not work is that *the volatility of this portfolio changes as $S(T)$ and $Q(T)$ change.* For example, as S increases relative to Q , the portfolio variance (calculated as $\text{Var}(S(T) - kQ(T))/(S(T) - kQ(T))$) will approach the volatility of S . As S approaches kQ , the portfolio volatility will generally approach infinity. A probability calculation using this portfolio volatility might be made to work, but not easily.

1 Pricing by Change of Numeraire

In this section we show how to value an outperformance option by computing the risk-neutral probability that $S(T) > kQ(T)$, and then performing a change of numeraire to get the value of a claim that pays shares instead of cash.

1.1 Digital Cash Outperformance Option

We begin by pricing a claim that pays \$1 if $S(T) > kQ(T)$. We can call this a digital cash outperformance option. Start with the observation that if S and Q follow equations (1) and (2), then by Itô's Lemma, we have

$$d\ln(S) = (r - \delta_S - 0.5\sigma_S^2)dt + \sigma_S dZ_1 \quad (4)$$

and

$$d\ln(Q) = (r - \delta_Q - 0.5\sigma_Q^2)dt + \sigma_Q dZ_2 \quad (5)$$

The price of the digital outperformance option is the discounted probability that $S(T) > kQ(T)$. We want to compute

$$\begin{aligned} \Pr(S(T) > kQ(T)) &= \Pr[\ln(S(T)) > \ln(k) + \ln(Q(T))] \\ &= \Pr[\ln(S(T)/Q(T)) > \ln(k)] \end{aligned}$$

The calculation of this probability hinges on the distribution of the difference $\ln(S(T)) - \ln(Q(T))$. The variance of this difference is

$$\nu^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\sigma_S\sigma_Q$$

We also have

$$\begin{aligned} E[\ln(S(T)) - \ln(Q(T))] &= \ln(S_0) + (r - \delta_S - 0.5\sigma_S^2)T - \ln(Q_0) - (r - \delta_Q - 0.5\sigma_Q^2)T \\ &= \ln(S_0/Q_0) + (\delta_Q - \delta_S + 0.5\sigma_Q^2 - 0.5\sigma_S^2)T \end{aligned}$$

Thus, we have

$$\ln(S(T)) - \ln(Q(T)) \sim \mathcal{N}[\ln(S_0/Q_0) + (\delta_Q - \delta_S + 0.5\sigma_Q^2 - 0.5\sigma_S^2)T, \nu^2 T] \quad (6)$$

Now we can compute the probability that $S(T) > kQ(T)$. We can treat the difference $\ln(S(T)) - \ln(Q(T))$ as being driven by a single normal random variable Z' . The probability calculation then works out as follows:

$$\begin{aligned} \Pr(S(T) > kQ(T)) &= \Pr[\ln(S(T)/Q(T)) > \ln(k)] \\ &= \Pr\left[\ln(S_0/Q_0) + (\delta_Q - \delta_S + 0.5\sigma_Q^2 - 0.5\sigma_S^2)T + \nu\sqrt{T}Z' > \ln(k)\right] \\ &= \Pr\left[\frac{\ln(S_0/Q_0) - \ln(k) + (\delta_Q - \delta_S + 0.5\sigma_Q^2 - 0.5\sigma_S^2)T}{\nu\sqrt{T}} < Z'\right] \\ &= N\left[\frac{\ln(S_0/kQ_0) + (\delta_Q - \delta_S + 0.5\sigma_Q^2 - 0.5\sigma_S^2)T}{\nu\sqrt{T}}\right] \end{aligned}$$

We know that the discounted risk-neutral probability of an event (such as $S(T) > kQ(T)$) is the price of a claim that pays \$1 when the event occurs. Thus, the price of a claim that pays \$1 when $S(T) > kQ(T)$ (a digital cash outperformance option) is

$$e^{-rT} N\left[\frac{\ln(S_0/kQ_0) + (\delta_Q - \delta_S + 0.5\sigma_Q^2 - 0.5\sigma_S^2)T}{\nu\sqrt{T}}\right] \quad (7)$$

1.2 Digital Asset Outperformance Options

Now we wish to value claims that pay $S(T)$ and $Q(T)$ if $S(T) > kQ(T)$. In order to do this, we need to generalize the change of numeraire result (Proposition 21.1 in McDonald, 2003) to encompass the case of three random variables. Suppose that there is a third price, X , that follows the risk-neutral process

$$\frac{dX}{X} = (r - \delta_X)dt + \sigma_X dZ_3 \quad (8)$$

We will define the pairwise correlations between X , S , and Q as ρ_{XS} , ρ_{XQ} , and so forth.

Here is the generalization:

Proposition 1 *Let S , Q , and X follow geometric Brownian motion, as in equations (1), (2), and (8). Suppose a derivative claim has the value*

$$W(S_t, Q_t, \delta_S, \delta_Q, \dots, T - t)$$

Consider a claim that at time T pays

$$Y(X(T), S(T), Q(T)) = X(T)^b \times W(S(T), Q(T), \delta_S, \delta_Q, \dots, 0)$$

The time t value of this claim is given by

$$e^{-r(T-t)} F_{t,T}(X^b) W(S(T), Q(T), \delta_S - b\rho_{SX}\sigma_S\sigma_X, \delta_Q - b\rho_{QX}\sigma_Q\sigma_X, \dots, T - t) \quad (9)$$

This proposition tells us that, in addition to premultiplying by the prepaid forward price for X^b , we need only adjust the dividend yields of S and Q to account for the covariance of those variables with X .

Now we can value a claim that pays a share of S (or Q) if $S(T) > kQ(T)$. Proposition 1 tells us that we obtain the value by replacing δ_S with $\delta_S - \sigma_S^2$ and δ_Q with $\delta_Q - \rho\sigma_S\sigma_Q$. We also pre-multiply by the forward price for S . Notice that the value of the digital cash outperformance option contains the following expression:

$$\delta_Q - \delta_S + 0.5\sigma_Q^2 - 0.5\sigma_S^2$$

When we substitute for the dividend yields, we obtain

$$\begin{aligned} (\delta_Q - \rho\sigma_S\sigma_Q) - (\delta_S - \sigma_S^2) + 0.5\sigma_Q^2 - 0.5\sigma_S^2 &= \delta_Q - \delta_S + 0.5(\sigma_S^2 + \sigma_S^2 - 2\rho\sigma_S\sigma_Q) \\ &= \delta_Q - \delta_S + 0.5\nu^2 \end{aligned}$$

Thus, the time 0 value of a claim that pays $S(T)$ if $S(T) > kQ(T)$ is

$$S_0 e^{-\delta_S T} N \left[\frac{\ln(S_0/kQ_0) + (\delta_Q - \delta_S + 0.5\nu^2)T}{\nu\sqrt{T}} \right] \quad (10)$$

Similarly, suppose we have a claim that pays $Q(T)$ if $S(T) > kQ(T)$. Applying Proposition 1, we replace δ_S with $\delta_S - \rho\sigma_S\sigma_Q$ and δ_Q with $\delta_Q - \sigma_Q^2$. We obtain

$$\begin{aligned} (\delta_Q - \sigma_Q^2) - (\delta_S - \rho\sigma_S\sigma_Q) + 0.5\sigma_Q^2 - 0.5\sigma_S^2 &= \delta_Q - \delta_S - 0.5(\sigma_S^2 + \sigma_Q^2 - 2\rho\sigma_S\sigma_Q) \\ &= \delta_Q - \delta_S - 0.5\nu^2 \end{aligned}$$

The time 0 value of a claim that pays $Q(T)$ if $S(T) > kQ(T)$ is

$$Q_0 e^{-\delta_Q T} N \left[\frac{\ln(S_0/kQ_0) + (\delta_Q - \delta_S - 0.5\nu^2)T}{\nu\sqrt{T}} \right] \quad (11)$$

Notice that this has the same form as equation (10), except that instead of $+0.5\nu^2$ we have $-0.5\nu^2$.

1.3 Outperformance Option

We can now easily value an option that pays $\max[0, S(T) - kQ(T)]$. The value is the difference between the expressions in (10) and (11). You can verify that we can compute the option price using the Black-Scholes formula:

$$\begin{aligned} &\text{BSCall}(S_t, Q_t, \nu, \delta_Q, T - t, \delta_S) \\ &= S_0 e^{-\delta_S T} N \left[\frac{\ln(S_0/kQ_0) + (\delta_Q - \delta_S + 0.5\nu^2)T}{\nu\sqrt{T}} \right] \\ &\quad - Q_0 e^{-\delta_Q T} N \left[\frac{\ln(S_0/kQ_0) + (\delta_Q - \delta_S - 0.5\nu^2)T}{\nu\sqrt{T}} \right] \quad (12) \end{aligned}$$

This is the Black-Scholes formula, except that the strike price is replaced by $kQ(T)$, the risk-free rate is replaced by δ_Q and the volatility is replaced by ν .

2 Proof of Proposition 1

Suppose that we have a derivative that is a function of time and N prices, S_1, S_2, \dots, S_N . The dividend yield on each asset is δ_i and the covariance between the i^{th} and j^{th} price is σ_{ij} . The price of this claim satisfies the Black-Scholes equation:

$$V_t + \sum_{i=1}^N (r - \delta_i) S_i V_i + 0.5 \times \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij} S_i S_j V_{ij} = rV \quad (13)$$

Suppose there is another claim that at time T pays

$$Y(X, S_1, \dots, S_N, T) = X(T)^b V[S_1(T), \dots, S_N(T), 0] \quad (14)$$

By the multivariate Black-Scholes equation, the value of the claim $Y(t)$ must satisfy

$$Y_t + (r - \delta_X)XY_X + \sum_{i=1}^N (r - \delta_i)S_iY_i + 0.5\sigma_X^2 X^2 Y_{XX} + \sum_{i=1}^N \sigma_{X_i} X S_i Y_{X_i} + 0.5 \times \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij} S_i S_j Y_{ij} = rY \quad (15)$$

where σ_{X_i} is the covariance between the returns on X and the i^{th} asset.

We guess that the price for Y has the form

$$Y(S_1, \dots, S_N, X, t) = Ae^{-(r-\delta^*)(T-t)} X^b W(S_1, \dots, S_N, T-t) \quad (16)$$

This solution, equation (16), can satisfy the boundary condition, equation (14), as long as, at time T (when $t = T$), we have

$$W(S_1, \dots, S_N, 0) = V(S_1, \dots, S_N, 0)$$

The guessed solution, (16), must satisfy a specific form of the Black-Scholes equation. We can evaluate the derivatives in equation (15) using equation (16). This yields:

$$-(r - \delta^*)Y + Y/W W_t + (r - \delta_X)bY + \sum_{i=1}^N (r - \delta_i)S_i W_i Y/W + 0.5\sigma_X^2 b(b-1)Y + \sum_{i=1}^N \sigma_{X_i} S_i b W_i Y/W + 0.5 \times \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij} S_i S_j W_{ij} Y/W = rY$$

Divide by Y and rewrite this equation to obtain

$$(r - \delta_X)b - (r - \delta^*) + 0.5\sigma_X^2 b(b-1) + \frac{1}{W} \left[W_t + \sum_{i=1}^N (r - \delta_i + b\sigma_{X_i})S_i W_i + 0.5 \times \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij} S_i S_j W_{ij} - rW \right] = 0 \quad (17)$$

The partial differential equation in square brackets is the same as equation (13), except that δ_i is replaced with $\delta_i - b\sigma_{X_i}$. We know that the boundary condition, equation (14), is satisfied if $V = W$ at time T . Thus, a solution is to choose W to equal V , except that δ_i is replaced by $\delta_i - b\sigma_{X_i}$.

With this replacement, from equation (??), the term in square brackets equals 0. Equation (17) becomes

$$\delta^* - r + b(r - \delta_X) + \frac{1}{2}\sigma_X^2 b(b-1) = 0$$

This equation is satisfied if $\delta^* = r - b(r - \delta_X) - \frac{1}{2}\sigma_X^2 b(b-1)$. The parameter A then equals one.

References

McDonald, R. L., 2003, *Derivatives Markets*, Addison Wesley, Boston, MA.