Introduction
Finance 467
Derivatives Markets II

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Course Content

- This course is broadly about
  - The economic and mathematical underpinnings of derivatives pricing
  - How modern financial institutions operate
  - The environment in which they operate
  - Three players: Market-makers, End-users, and Regulators
- Understanding modern finance
  - Introductory finance is about pricing linear payoffs; More general and important problem is pricing non-linear payoffs
  - Black and Scholes showed how to do this.
- Skills
  - modern analytical techniques (e.g. Monte Carlo valuation)
  - intuition about products, pricing, and hedging
  - basic computational skills

Administrative Details

- Office hours & getting in touch
- Texts & case packets
- Grading, problems sets, exams, & honor code
  - Note the three quizzes (week 3, week 6, and week 9)
- Your background and temperament
  - Derivatives Markets I or knowledge of binomial and Black-Scholes pricing
  - Analytical: much probability, and we will use calculus, although I will not expect you to repeat derivations
  - Excel skills
  - Willing suspension of disbelief
  - No guts, no glory (don't expect to understand everything the first time through)

Plan of the Course

- Foundations: lognormality, Monte Carlo, Itô processes, Itô's Lemma, Black-Scholes equation
- Interpretation of risk-neutral pricing and
- Exotic options as an application
- Fixed Income
- Value at Risk and credit risk
- This course will not turn you into a rocket scientist, but you will become acquainted with the language and concepts the rocket scientists use.

Review of the Black-Scholes Model

Forwards and Prepaid Forwards

- The **prepaid forward price** at time 0 for delivery at time T is

$$F^P_{0,T} = S_0 e^{-\delta T}$$

Intuition: You forego dividends, hence you reduce the payment by dividends not received. You would pay $S_0 e^{-\delta T}$ today.

- The **forward price** at time 0 for delivery at time T is

$$F_{0,T} = e^{rT} S_0 e^{-\delta T} = S_0 e^{(r-\delta)T}$$

In addition to not receiving dividends, you are also delaying payment for the stock, hence you pay the future value of the prepaid forward price, $e^{rT} S_0 e^{-\delta T}$.

- Example: Suppose $S = \$100$, $r = 0.08$, $T = 2$, $\delta = 0.05$. The prepaid forward price is

$$F^P_{0,T} = S_0 e^{-\delta T} = \$100 e^{-0.05\times 2} = \$90.48374$$

The forward price is

$$F_{0,T} = S_0 e^{(r-\delta)T} = \$100 e^{(0.08-0.05)T} = \$106.18365$$

The Black-Scholes Formula: I

- The Black-Scholes formula is the limit of the binomial model as $n \rightarrow \infty$

$$
\begin{aligned}
C(S, K, \sigma, r, T, \delta) &= S e^{-\delta T} N(d_1) - K e^{-r T} N(d_2) \\
d_1 &= \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

- This is calculated with the spreadsheet function (in *OptAll2.xls*:)

$$\text{BSCall}(S, K, \sigma, r, T, \delta)$$

- This is a standard way to write the Black-Scholes formula.
- Example: In addition to the assumptions before, suppose that $K = \$95$ and $\sigma = 0.30$:

$$\text{BSCall}(100, 95, 0.3, 0.08, 2, 0.05) = \$19.6501$$

- As before, let $F^P_{0,T}(S) = Se^{-\delta T}$ denote the prepaid forward price for the stock, and $F^P_{0,T}(K) = Ke^{-rT}$ denote the prepaid forward price for the strike. (The prepaid forward price is the amount you pay today if you are to receive the asset at time 0.)
- We can write the Black-Scholes formula as

$$C(F^P_{0,T}(S), F^P_{0,T}(K), \sigma, 0, T, 0) = F^P_{0,T}(S)N(d_1) - F^P_{0,T}(K)N(d_2)$$

$$d_1 = \frac{\ln(F^P_{0,T}(S)/F^P_{0,T}(K)) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

This corresponds to

$$\text{BSCall}(Se^{-\delta T}, Ke^{-rT}, \sigma, 0, T, 0) = \text{BSCall}(Se^{-\delta T}, Ke^{-rT}, \sigma\sqrt{T}, 0, 1, 0)$$

- The prepaid forward prices already incorporate dividends and interest. This is probably the simplest and (in my opinion) most intuitive[1] way to write the formula.

[1] Once you develop the appropriate intuition!

The Black-Scholes Formula: II (cont.)

- Example:

$$\text{BSCall}(100e^{-0.05\times 2}, 95e^{-0.08\times 2}, 0.3\times\sqrt{2}, 0, 1, 0) = \$19.6501$$

- In this version, the Black-Scholes formula has three inputs:

  1. The value today of what you will receive if you exercise the option
  2. The value today of what you will pay if you exercise the option
  3. The total uncertainty of $\ln(S_T/K)$

- Let $F_{0,T}(S) = Se^{r-\delta T}$ denote the forward price for the stock, and $F_{0,T}(K) = K$ denote the forward price for the strike.
- We can write the Black-Scholes formula as

$$C(F_{0,T}(S), F_{0,T}(K), \sigma, r, T, r) = e^{-rT} F_{0,T}(S) N(d_1) - e^{-rT} F_{0,T}(K) N(d_2)$$

$$d_1 = \frac{\ln(F_{0,T}(S)/F_{0,T}(K)) + (r - r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- This corresponds to

$$\text{BSCall}(F_{0,T}(S), F_{0,T}(K), \sigma, r, T, r)$$

- This version of the formula is called the **Black formula**. It is also written:

$$C(S, K, \sigma, r, T, \delta) \quad = \quad e^{-rT}\left[Se^{(r-\delta)T} N(d_1) - KN(d_2)\right]$$

The Black-Scholes Formula: III (cont.)

- We can rewrite this as

$$e^{-rT}\text{BSCall}(F_{0,T}, K, \sigma, 0, T, 0)$$

- Example:

$$e^{-0.08\times 2}\text{BSCall}(100e^{(0.08-0.05)\times 2}, 95, 0.30, 0, 2, 0)$$
$$= e^{-0.08\times 2}\text{BSCall}(106.18365, 95, 0.30, 0, 2, 0) = \$19.6501$$

Review of Option Greeks

- Delta (Δ): $\frac{\partial C}{\partial S}$
- Gamma (Γ): $\frac{\partial^2 C}{\partial S^2}$
- Vega: $\frac{\partial C}{\partial \sigma}$ (no Greek letter!)
- Theta (Θ): $\frac{\partial C}{\partial t}$ (be careful about t being *time*, not *time to expiration*)
- Rho (ρ): $\frac{\partial C}{\partial r}$
- Psi (ψ): $\frac{\partial C}{\partial \delta}$ (not in *OptAll.xls*).
- Elasticity (Ω): $\frac{S}{C} \times \frac{\partial C}{\partial S}$
- Note that Greek formulas can be different depending upon which version of the Black-Scholes formula you use! (Compare rho in the Black-Scholes and Black formulas.)

Useful Facts about Greeks

- For these Greeks: $\{\Delta, \Gamma, \text{vega}, \theta, \rho, \psi\}$, the Greek of a portfolio is the sum of the Greeks of the components
  - E.g., for a 40-45 call bull spread,

$$\Delta_{\text{spread}} = \Delta_{40} - \Delta_{45}$$

- The elasticity of a portfolio is the *weighted average* of the elasticities of the components:

$$\Omega_{\text{spread}} = \Omega_{40} \times \frac{C(40)}{C(40) - C(45)} - \Omega_{45} \times \frac{C(45)}{C(40) - C(45)}$$

Greeks, cont.

- There are easy approximations for Greeks (e.g., Δ):

$$\Delta \approx \frac{C(S+\epsilon,\cdot) - C(S-\epsilon,\cdot)}{2\epsilon}$$

$$\Gamma \approx \frac{C(S+2\epsilon,\cdot) + C(S-2\epsilon,\cdot) - 2C(S,\cdot)}{4\epsilon^2}$$

- This approximation can be used to evaluate Greeks for any option for which you can evaluate a price.
- Example: What are the delta and vega of a knock-out call?

$$\Delta \approx \frac{\text{CallDownOut}(S+\epsilon,\cdot) - \text{CallDownOut}(S-\epsilon,\cdot)}{2\epsilon}$$

$$\text{Vega} \approx \frac{\text{CallDownOut}(\cdot,\sigma+\epsilon,\cdot) - \text{CallDownOut}(\cdot,\sigma-\epsilon,\cdot)}{2\epsilon} \times \frac{1}{100}$$

- Why divide by 100?

Greeks, cont.

- What is the gamma of a down-and-out call?
- Gamma is the change in delta, hence,

$$\Gamma \approx \frac{\Delta_{S+\epsilon} - \Delta_{S-\epsilon}}{2\epsilon}$$

Letting F denote the down-and-out call price function, we have

$$\Gamma = \frac{\left(\frac{F(S+2\epsilon,\cdot)-F(S,\cdot)}{2\epsilon}\right) - \left(\frac{F(S\epsilon,\cdot)-F(S-2\epsilon,\cdot)}{2\epsilon}\right)}{2\epsilon}$$

$$= \frac{F(S+2\epsilon,\cdot) + F(S-2\epsilon,\cdot) - 2F(S,\cdot)}{4\epsilon^2}$$

VBA

- Programming language built into Microsoft Office with which you can create
  - custom functions (provide inputs, compute a number, just like Excel functions)
  - subroutines (perform a series of steps)
- Loops, conditional execution ("if" statement), etc...
- Matlab (for example) is far faster and more powerful
- Optional VBA session in two weeks before class
- VBA is *extremely* limited; Matlab is a more powerful but expensive alternative for serious computation. There are free, open-source alternatives to Matlab (e.g., FreeMat and Octave) that I have not evaluated.
  - For cross-platform use, OpenOffice has a version of basic identical to VB except for the application interface (calling built-in functions, writing to cells, etc.)

VBA: Computing Implied Volatility

- Here is an algorithm for computing implied volatility. We start with an observed option price $\hat{C}$, and we guess an initial volatility, σ_0. Repeat the following until $|x| < \epsilon$:

  1. Compute option price $C_1(\sigma_0)$.
  2. The number of volatility units in error is
  $$x = (C_1(\sigma_0) - \hat{C})/\text{vega}(\sigma_0)$$
  3. Guess new σ: $\sigma_1 = \sigma_0 - x$, until $\sigma_0 = \sigma_1$.
  4. Set $\sigma_0 = \sigma_1$, go back to step 1.

- *Very* fast convergence (for reasonable inputs, price is approximately linear in volatility)
- See *optall.xls*

Vega and Gamma of a Down-and-out Call

The following code computes the vega of a down-and-out call:

```
Function docallvega(s, k, v, r, t, d, h)
    z = 0.00001
    x1 = CallDownOut(s, k, v + z, r, t, d, h)
    x0 = CallDownOut(s, k, v - z, r, t, d, h)
    docallvega = (x1 - x0) / (2 * z)
End Function
```

The following code computes the gamma of a down-and-out call:

```
Function docallgamma(s, k, v, r, t, d, h)
    z = 0.00001
    x2 = CallDownOut(s + 2 * z, k, v, r, t, d, h)
    x1 = CallDownOut(s, k, v, r, t, d, h)
    x0 = CallDownOut(s - 2 * z, k, v, r, t, d, h)
    docallgamma = ((x2 - x1) - (x1 - x0)) / (4 * z 2)
End Function
```


Delta-Hedging and Pricing

- Delta-hedged exposure, all funds borrowed. I.e., if we buy a \$100 share of stock, we can borrow \$100 to do so.
  - This ignores the margin that would be required in practice
- $S = 40$, $K = 40$, $r = 8\%$, $\sigma = 30\%$, $\delta = 0$, $t = 0.25$
- $\Delta = 58.24$ for option on 100 shares

	Purchased	Written
Call Price	2.7804	-2.7804
Delta	0.5824	-0.5824
Gamma	0.0652	-0.0652
Theta	-0.0173	0.0173

Profit Details

Day 1, $S_1 = \$40.50$

Gain on 58.24 shares	$58.24 \times (40.50 - 40)$	=	29.12
Gain on written option	$278.04 - 306.21$	=	−28.17
Interest	$-(e^{0.08/365} - 1) \times 2051.56$	=	−0.45
Overnight Profit			**0.50**

Day 2, $S_2 = \$39.25$

Gain on 61.42 shares	$61.42 \times (39.25 - 40.50)$	=	−76.78
Gain on written option	$306.21 - 232.82$	=	73.39
Interest	$-(e^{0.08/365} - 1) \times 2181.31$	=	−0.48
Overnight Profit			**−3.87**

Profit Over 4 Days

Day	0	1	2	3	4
Stock	40.00	40.50	39.25	38.75	40.00
Call	278.04	306.21	232.82	205.46	271.04
Delta	0.5824	0.6142	0.5311	0.4956	0.5806
Investment	2,051.58	2,181.31	1,851.65	1,715.12	2,051.35
Interest		(0.45)	(0.48)	(0.41)	(0.38)
Cap Gain		0.95	(3.39)	0.81	(3.62)
Daily Profit		0.50	(3.87)	0.40	(4.00)

Interpretation of Profit

- Profit is cash surplus after re-hedging.
- In going from day 1 to day 2, we
  - buy $61.42 - 58.24 = 3.18$ additional shares, at a cost of $3.18 \times 40.50 = \$128.79$
  - Because we have additional shares and the market value of the option has changed, we can borrow an additional

$$(40.50 \times 61.42 - 40 \times 58.24) - (306.21 - 278.04) = 0.95$$

  - Pay interest of \$0.45
  - We're left with \$0.50. This can be pocketed.

Why Isn't Delta-Hedging Perfect?

- Delta changes with S; gamma is not zero.
- Over the interval 40 to 40.75, for example,

$$\Delta_{\text{Average}} = 0.5 \times (\Delta_{40} + \Delta_{40.75})$$

- We can approximate:

$$\Delta_{40.75} = \Delta_{40} + 0.75 \times \Gamma = 0.5824 + 0.75 \times 0.0652 = 0.6313$$

- Thus,

$$\begin{aligned}\Delta_{\text{Average}} &= 0.5 \times [\Delta_{40} + (\Delta_{40} + 0.75 \times \Gamma_{40})] \\ &= \Delta_{40} + 0.5 \times \Gamma_{40} \times 0.75 = 0.6069\end{aligned}$$

- We can predict the change in the option price by

$$C(40.75) = C(40) + \Delta_{\text{Average}} \times 0.75 = 3.2355$$

Delta-Gamma Approximations

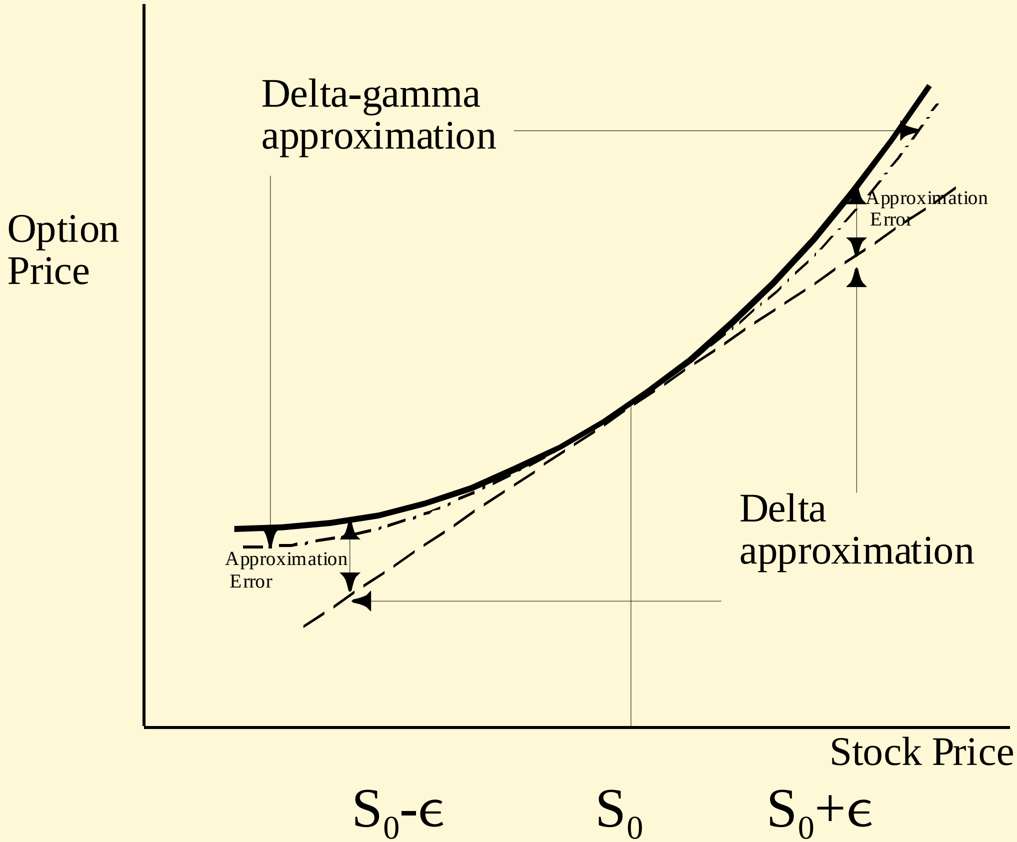
- $\epsilon = S_1 - S_0$ (definition of ϵ)
- $\Delta_1 = \Delta_0 + \epsilon\Gamma$ (definition of Γ)
- $\Delta_{\text{Average}} = \Delta_0 + 0.5 \times \Gamma_0 \times \epsilon$
- Delta-gamma approximation of new option price:

$$\begin{aligned} C(1) &= C(0) + \Delta_{\text{Average}} \times \epsilon \\ &= C(0) + \epsilon \times (\Delta_0 + 0.5 \times \Gamma_0 \times \epsilon) \end{aligned}$$

- Thus,

$$\boxed{C(1) \approx C(0) + \epsilon \times \Delta_0 + 0.5 \times \Gamma \times \epsilon^2}$$

Graphical Interpretation



Accounting for Time

- Stock price moves occur over time, hence we need to account for Θ:

$$\boxed{C(S_{t+h}, t+h) \approx C(S_t, t) + \epsilon\Delta_0 + 0.5 \times \Gamma\epsilon^2 + \Theta h}$$

- Given that the option moves like this, what is market-maker profit?

Market-Maker Profit

- Buy Δ shares and sell one option
- Profit for a delta-hedged option writer is

$$\Delta(S_{t+h} - S_t) - [C(S_{t+h}) - C(S_t)] - rh[\Delta S_t - C(S_t)]$$

- Rewrite using the approximation formula for $C(S_{t+h})$:

$$-\left(\frac{1}{2}\epsilon_h^2\Gamma + \Theta h + rh\left[\Delta S_t - C(S_t)\right]\right)$$

Three Profit Drivers

- **Gamma** $-\frac{1}{2}\epsilon_h^2\Gamma$ Larger stock price moves lower profit
- **Theta** $-\Theta h$ Option value decreases as time goes by
- **Interest Cost** $-rh\left[\Delta S_0 - C(S_0)\right]$ Carry cost of the position
- Theta and interest are predictable; *how big is the stock price move?*

Stock Price Moves

- If annual volatility is σ, volatility over period h is $\sigma\sqrt{h}$.
- Binomial model: $S_{t+h} = S_t e^{rh \pm \sigma\sqrt{h}} \approx S_t(1 \pm \sigma\sqrt{h}) \implies$ the stock moves one standard deviation ($S_{t+h} - S_t = S_t\sigma\sqrt{h}$).
  - You can verify that for small h, $\sigma\sqrt{h}$ is much larger than rh
  - This is why all different binomial models (CRR, Jarrow-Rudd, Forward model) have the same limiting behavior
- Economic assumption: market-maker breaks even with one standard deviation move
- This implies $\epsilon^2 = \sigma^2 h S^2$, hence

$$\boxed{\frac{1}{2}\sigma^2 h S^2 \Gamma + rh\Delta S_t + \Theta h = rhC(S_t)}$$

Profit Over 4 Days, 1 σ moves

Stock evolves according to $S_{t+h} = S_t e^{rh \pm \sigma\sqrt{h}}$

Day	0	1	2	3	4
Stock	40.000	40.642	40.018	39.403	38.797
Call	278.04	315.00	275.57	239.29	206.14
Option Delta	0.5824	0.6232	0.5827	0.5408	0.4980
Investment	2,051.58	2,217.66	2,056.08	1,891.60	1,725.95
Interest		(0.45)	(0.49)	(0.45)	(0.41)
Capital Gain		0.43	0.51	0.46	0.42
Daily Profit		(0.02)	0.02	0.01	0.00

The Black-Scholes Equation

- The preceding argument holds for any kind of option which has a payoff at time t based on the stock price at time t
- The Black-Scholes equation is a relationship among the option Greeks delta, gamma, and theta
- The Black-Scholes equation must be satisfied if an option pricing formula is correct (the formula must also give right price at expiration or exercise)
- Residual risk from imperfect delta-hedging must be absorbed by market-maker capital

What are the Gains to Frequent Hedge Rebalancing?

- From the gain/loss equation for a delta-hedged option writer:

$$-\left(\frac{1}{2}\epsilon_h^2\Gamma + \Theta h + rh\left[\Delta S_0 - C(S_0)\right]\right)$$

$$= -\left(\frac{1}{2}[\epsilon_h^2 - \sigma^2 S^2 h]\Gamma + \frac{1}{2}\sigma^2 S^2 h\Gamma + \Theta h + rh\left[\Delta S_0 - C(S_0)\right]\right)$$

$$= -\frac{1}{2}\sigma^2 S^2 h[\frac{\epsilon_h^2}{\sigma^2 S^2 h} - 1]\Gamma$$

If S is normally distributed for small moves, $\xi^2 = \epsilon_h^2/\sigma^2 S^2 h$ has the chi-squared distribution with variance 2.

- Thus, if we're delta-hedged, the return is

$$R_{h,i} = \frac{1}{2}S^2\sigma^2\Gamma(1-\xi_i^2)h$$

with

$$\text{Var}(R_{h,i}) = \frac{1}{4} \times 2 \times \left(S^2\sigma^2\Gamma h\right)^2 = \frac{1}{2}\left(S^2\sigma^2\Gamma h\right)^2$$

Gains to Frequent Hedging, cont.

- Compare hedging once vs. n times over an interval of length h. The variance of the sum of n independent hedged returns is

$$\text{Var}\left(\sum_{i=1}^{n} R_{\frac{h}{n},i}\right) = \sum_{i=1}^{n} \frac{1}{2}\left(S^2\sigma^2\Gamma\frac{h}{n}\right)^2$$
$$= \frac{1}{n} \times \text{Var}(R_{h,1})$$

- Hedging hourly instead of daily, for example, reduces variance by a factor of 24, standard deviation by a factor of $\sqrt{24} = 4.90$.

Other Issues

- Since delta-hedging can fail, ultimately risk must be borne by capital markets
- Experience during 1987 crash illustrates problems with delta-hedging and the need for capital.
- We'll see later that *hedging is not necessary for the Black-Scholes formula to give the "right" price*. It is also a fair price.
  - However, when hedging is not possible, pricing depends upon risk premia.

