Forward Prices with Discrete Proportional Dividends

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There has been some confusion (including my own) about how to compute the forward price when a stock pays a discrete proportional dividend. Logically there are three ways this can happen:

1. a stock dividend in which the firm grants δ additional shares for each share owned

2. a cash dividend in which the dividend payment is a fraction $\hat{\delta}$ of the *cum-dividend* price

3. a cash dividend in which the dividend payment is a fraction δ^* of the *ex-dividend* price.

I will derive formulas for each of these three cases.

1 Stock Dividend of δ

Suppose you have 1 share and the firm pays a stock dividend of δ at time t. How many shares will you have at time $T > t$? By definition, at time t you will receive δ shares for every share that you own, so you will have $1 + \delta$ shares at time T. This implies that the forward price is

$$F_{0,T} = S_0 e^{rT} \frac{1}{1+\delta} \tag{1}$$

To see this, consider Table 1, which lays out a cash-and-carry for this stock.

Position	Time 0 flow	Time t	Time T cash flow
Short forward	0		$F_{0,T} - S_T$
Buy $1/(1+\delta)$ shares	$-S_0/(1+\delta)$		$+S_T/(1+\delta)$
Dividend received		$+\delta S_t/(1+\delta)$	
Reinvest dividend		$-\delta S_t/(1+\delta)$	$+\delta S_T/(1+\delta)$
Borrow	$+S_0/(1+\delta)$		$-S_0 e^{rT}/(1+\delta)$
Total	0	0	$F_{0,T} - S_0 e^{rT}/(1+\delta)$

Table 1: Cash-and-carry when the stock pays a stock dividend of δ at time t.

2 Cash Dividend of $\hat{\delta}$ of the *Cum-Dividend* Value of the Stock

Here is the issue in this case: Suppose the stock pays a 5% dividend proportional to the cum-dividend value of the share. If the stock price is \$140 at the time of the dividend, the dividend is $0.05 \times \$140 = \$7.00$. The stock price will drop to \$133 when the dividend is paid, and the \$7 will buy $7/133 = 5.2632\%$ of a share—a fraction greater than the percentage stock dividend.

Here is how to work out the forward price in this case. If the dividend is $\hat{\delta}S_t$, the dividend can be used to purchase the fraction of a share

$$\frac{\hat{\delta}S_t}{S_t - \hat{\delta}S_t} \tag{2}$$

Thus, at time T the number of shares we will have is the initial share we started with plus this fractional share:

$$1 + \frac{\hat{\delta}S_t}{S_t - \hat{\delta}S_t} = \frac{S_t - \hat{\delta}S_t + \hat{\delta}S_t}{S_t - \hat{\delta}S_t} = \frac{1}{1-\hat{\delta}} \tag{3}$$

If we start with 1 share, we will end up with $1/(1-\hat{\delta})$ shares, so we tail the position by starting out with $1-\hat{\delta}$ shares in order to end up with 1 share. Thus, the forward price is

$$F_{0,T} = S_0 e^{rT}(1-\hat{\delta}) \tag{4}$$

To see this, consider Table 2.

Another way to think about why the forward price is scaled by $1-\hat{\delta}$ is that the present value of a dividend of $\hat{\delta}S_t$ is $\hat{\delta}S_0$, so the prepaid forward price is the stock price less the present value of dividends, or $S_0(1-\hat{\delta})$.

Position	Time 0 cash flow	Time t	Time T cash flow
Short forward	0		$F_{0,T} - S_T$
Buy $1 - \hat{\delta}$ shares	$-S_0(1 - \hat{\delta})$		$+S_T(1 - \hat{\delta})$
Dividend received		$+\hat{\delta}S_t$	
Reinvest dividend		$-\hat{\delta}S_t$	$+\hat{\delta}S_T$
Borrow	$+S_0(1 - \hat{\delta})$		$+S_0e^{rT}(1 - \hat{\delta})$
Total	0	0	$F_{0,T} - S_0e^{rT}(1 - \hat{\delta})$

Table 2: Cash-and-carry when the dividend is proportional to the cum-dividend stock price. The dividend, $\hat{\delta}S_t$, is paid at time t when the cum-dividend stock price is S_t, and reinvested in the stock.

3 Cash Dividend of δ^* of the *Ex-Dividend* Value of the Stock

Suppose we specify that the dividend is to be 5% of the ex-dividend value of the stock, and the *cum-dividend* stock price is \$140. Then, the dividend, D, is

$$D = 0.05 \times (\$140 - D)$$

which implies that $D = \$140 \times 0.05/(1 + 0.05) = \$6.67$. The dividend is 4.762% of the *cum-dividend* stock price.

Algebraically, if S_t is the cum-dividend price, we have

$$D = \delta^*(S_t - D)$$

or

$$D = \frac{\delta^*}{1 + \delta^*}S_t$$

This implies that the forward price, expressed in terms of the *ex-dividend* dividend yield, is

$$F_{0,T} = S_0e^{rT}\frac{1}{1 + \delta^*} \tag{5}$$

This is the same expression as in the case of a stock dividend of δ^*.

To see this, suppose you own x shares at time 0. Since the dividend yield δ^* is expressed as a percentage of the *ex-dividend* price, for each share you own, after reinvesting the dividend you will have $1 + \delta^*$ shares.

Table 3 depicts a cash-and-carry for this case.

Finally, it is important to emphasize that in the case of a proportional dividend (of any kind), the precise payment date of a dividend paid between

Position	Time 0 cash flow	Time t	Time T cash flow
Short forward	0		$F_{0,T} - S_T$
Buy $1/(1+\delta^*)$ shares	$-S_0/(1+\delta^*)$		$+S_T/(1+\delta^*)$
Dividend received		$+\delta^* S_t/(1+\delta^*)$	
Reinvest dividend		$-\delta^* S_t/(1+\delta^*)$	$+\delta^* S_T/(1+\delta^*)$
Borrow	$+S_0/(1+\delta^*)$		$-S_0 e^{rT}/(1+\delta^*)$
Total	0	0	$F_{0,T} - S_0 e^{rT}/(1+\delta^*)$

Table 3: Cash-and-carry when the stock pays a cash dividend at time t of $D = \delta^*(S_t - D)$, proportional to the ex-dividend value of the stock. The dividend is reinvested in the stock.

times 0 and T doesn't matter. Contrast this with the case of a fixed dividend, when the timing of the dividend does matter.