

Notes on Doubling Strategies

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You would think that with a fair game, repeating the game would not affect the expected outcome. In particular, if you are gambling, repeating the gamble many times should not raise your odds of leaving the casino with more money than you came with.

Under fairly plausible and realistic conditions, this instinct is correct. However, it turns out that if you can play forever and have access to unlimited funds, then a fair gamble will have a positive expected value.¹

The results from following various gambling strategies has been extensively analyzed. (After all, formal statistical analysis was created to study gambling!) The *optional sampling theorem* is the basic result in the analysis of repeated gambling such as coin flips (e.g. Karlin and Taylor, 1975, pp. 253–278). Essentially, this theorem says that

1. if you play a game where you are certain to not play forever, and
2. if the expected value of the largest amount money you have at risk is finite

then the expected value of playing the game repeatedly is zero. This description suggests that there is room for a different result when you can play forever, or if there is no limit on the possible size of your bets. A simple example shows that if you have a limited number of plays of a game, the expected value of following a doubling strategy is zero. If you can play forever, however, the expected value can be positive.

Consider a fair wager that is a coin flip. We will follow what is known as a *doubling strategy*. We will bet \$1 initially, and will double the bet if we lose, continuing this way until we win.

Finite Bets

First, suppose that we can bet no more than N times. We will stop at the first win. The expected value of this bet is zero. You can see this by direct computation of the expected net cash winnings. We first pay \$1. If we flip heads, we win \$2 and stop, while if we flip tails, we invest \$2 in the next bet.

¹This statement does not mean that a risk-averse investor will be willing to play such a game. See for example Omberg (1989).

If we then flip heads, we win \$4 and stop, while if we flip tails, we invest \$4. Continuing in this way, we get

$$-1 + 0.5 \times (2 - 2) + 0.5^2 \times (4 - 4) + \dots + 0.5^N \times (2^N) = -1 + 1 = 0 \quad (1)$$

Because we stop on the last wager, there is no subsequent investment of 2^N dollars. The possible win of 2^N on the last flip, with probability 0.5^N , exactly offsets our initial investment.

There is a slightly different way to reach the same answer. This second approach will be helpful when we consider the case of an infinite number of flips. In this second approach, at each flip we will compute the net winnings, conditional on flipping a heads at that point. If we flip heads the first flip, we win $2 - 1 = 1$. If we flip heads the second flip, we win

$$4 - (1 + 2) = 1$$

If we flip heads on the third flip, we win

$$8 - (1 + 2 + 4) = 1$$

Note that if n is finite, we have

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1 \quad (2)$$

Thus, on the n^{th} flip, we obtain

$$2^n - \sum_{i=0}^{n-1} 2^i = 2^n - (2^n - 1) = 1$$

This demonstrates that on each round, if we flip a head, our *net* winning is \$1.

Suppose we cannot flip more than N times. If we win on the N^{th} flip, we receive \$1. If we lose on the N^{th} flip, from equation (2) we lose $2^N - 1$. Thus, the expected value of N flips is

$$1 \times 0.5 + 1 \times 0.5^2 + 1 \times 0.5^3 + \dots + 1 \times 0.5^N - (2^N - 1) \times 0.5^N \quad (3)$$

To evaluate this expression, note that²

$$1 \times 0.5 + 1 \times 0.5^2 + 1 \times 0.5^3 + \dots + 1 \times 0.5^N = 1 - 0.5^N$$

Thus, equation (3) becomes

$$1 - 0.5^N - (2^N - 1) \times 0.5^N = 0$$

²You can verify this by letting $g = 0.5$ in the formula

$$\sum_{i=1}^N g^i = \frac{g - g^{N+1}}{1 - g}$$

This formula obtains for $g \neq 1$ and finite N . It holds for $N = \infty$ if $g < 1$.

Infinite Bets

Now what happens if we can bet forever? In this case there is no point at which we simply lose our accumulated bets; there is always another flip. Thus, on each flip there is 0.5 probability of receiving a net winning of \$1. Expected winnings are³

$$1 \times 0.5 + 1 \times 0.5^2 + 1 \times 0.5^3 + \dots = 1$$

We have a positive expected value. Infinity plays a strange role here!

References

- Karlin, S. and Taylor, H. M., 1975, *A First Course in Stochastic Processes*, Academic Press, New York, 2nd edition.
- Omberg, E., 1989, "The Expected Utility of the Doubling Strategy," *Journal of Finance*, 44(2), 515–524.

³You may be wondering why we don't simply set $N = \infty$ in equation (3). The reason is that we obtained this equation by using equation (2), which is not defined for $N = \infty$.