The Endgame[*]

Anurag N. Banerjee
Durham Business School
University of Durham, U.K.

Sarit Markovich
Kellogg School of Management
Northwestern University, U.S.A.

Giulio Seccia[†]
Department of Economics
Nazarbayev University, KZ

October 1, 2018

Abstract

On December 1st, 2009 President Obama announced that the U.S. troops would have started leaving Afghanistan on July 2011. Rather than simply waiting "the U.S. troops out," the Taliban forces responded with a spike in attacks followed by a decline as the withdrawal date approached. These, at first, counter-intuitive phenomena, are addressed by studying a two-player, zero-sum game where the duration of the strategic interaction is either known or unknown (*i.e.,* the game can stop at any time with positive probability). We find that, conditional on the players' relative position, players' equilibrium strategies are non-stationary in a known duration game but they are stationary in the unknown duration case. Hence, introducing uncertainty, no matter how small, changes players' optimal behavior qualitatively and discontinuously: qualitatively because their behavior becomes stationary; discontinuously because the equilibrium is stationary only as far as the continuation probability is bounded away from 1.

Keywords: Stochastic games, information, conflict resolution.
JEL codes: C73, D74, D83

1. Introduction

In many instances, actors and observers recognize that knowing the exact length of a strategic interaction matters, independently of the length itself. By fixing the duration, the parties not only know that the game will end at a certain point in time, but they also know that it will *not* end before then. This is in contrast to the case where the game might be over at *any* point in time. This

[*]Giulio Seccia would like to thank Kyoto Institute for Economic Research (Kyoto University) for the hospitality during the academic year 2012-2013. We would like to thank José Apesteguia, Alessandro Lizzeri, Ignacio Palacios-Huerta and Levent Koçkesen for valuable comments. We benefited from conversations with Christoper Tuck of the British Military Staff College and U.S. Army Colonel Michael Musso.

[†]Corresponding author. Department of Economics, School of Humanities and Social Sciences, Nazarbayev University, Astana, Kazakhstan, Tel. +7 (7172) 70 57 41, giulio.seccia@nu.edu.kz.

paper shows that players' equilibrium behavior in a game with known fixed duration is qualitatively different than in a game with unknown duration.

An army involved in a foreign country intervention is a case in point. The duration of an armed conflict might be either uncertain or fixed. The uncertainty might be simply due to lack of information about how long it would take to resolve the conflict or lack of public or political support. Alternatively, the length of the involvement might be exogenously fixed, *e.g.,* by the budgetary decision of a political body.[1] Regardless of the reason for fixing the length of the involvement, the parties usually recognize that whether the duration is fixed or unknown affects their equilibrium behavior. The Iraq and Afghanistan wars are good examples to demonstrate this point. In both cases the American high command and politicians alike were very much aware of the implications of announcing a definite withdrawal date, as fixing the troop repatriation essentially fixes the conflict's length, thereby changing the nature of the game from unknown to known duration.[2] In anticipation of a subsequent change in both parties' strategies, the U.S. withdrawal announcement was either preceded by or made contemporary to a surge in troops deployment. Specifically, in the case of Iraq, in preparation of the agreement to hand over to the new Iraqi forces the control of the territory,[3] President Bush ordered a surge in troops in June, 15th 2007. In the case of Afghanistan, President Obama insisted that the announcement of both the troops surge (33,000 troops) as well as the beginning of the withdrawal (July 2011) would occur at the same time.[4] Indeed, both announcements were made during the same speech at West Point on December 1st, 2009 (White House (2009)).[5]

Woodward (2010) reports that President Obama had anticipated a surge in attacks following his West Point speech.[6] Consistent with these expectations, informed observers of the Afghanistan conflict have noticed a discontinuous change in the strategy of the Taliban army in response to Obama's announcement to fix the duration of the involvement of the U.S. forces. The two plots in

[1] In this paper we will not analyze the case where the duration is part of players' optimal choice.

[2] The setting of a date for troop withdrawal from Iraq was among the main points of Senator Obama's first presidential campaign: http://www.washingtonpost.com/wp-dyn/content/article/2007/01/30/AR2007013001586.html.

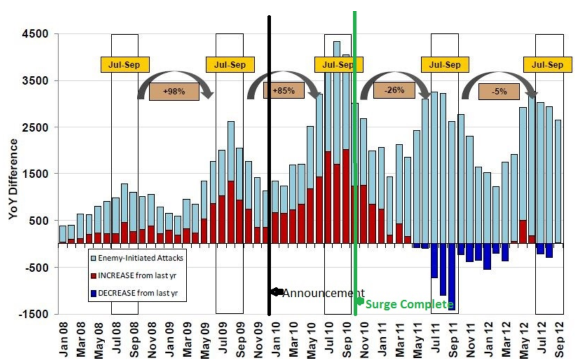
[3] This was later named the U.S. - Iraq Status of Forces Agreement which fixed the U.S. complete withdrawal to December 31, 2011. This date was later on postponed. For a timeline of the events see http://www.reuters.com/article/2011/12/15/us-iraq-usa-pullout-idUSTRE7BE0EL20111215.

[4] For an account of President Obama's decision of a surge and a withdrawal, see Baker (2009).

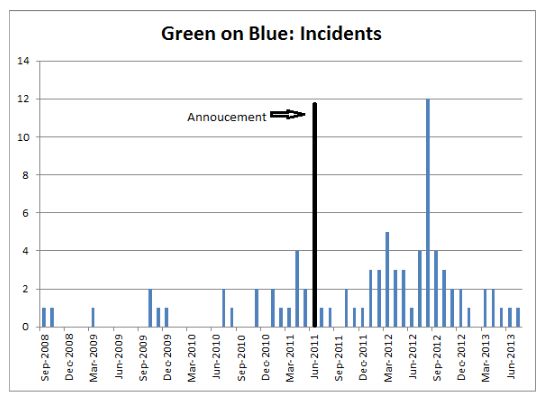
[5] This major surge was preceded by an increase in troops of minor entity in February 17, 2009 (17000 troops) and in March 27th, 2009 (4000 troops).

[6] "There is going to be tough, tough fighting in the spring and summer, he added. Anticipate a rise in casualties." (Woodward (2010), p.326). Thanks to Christopher Tuck for pointing this quote out.

Figure 1 provide some evidence for these claims. Figure 1(a), published by the NATO's Afghanistan Assessment Group, shows the "Enemy Initiated Attacks" (EIA) by Taliban forces across the period January 2008 - September 2012.[7] Abstracting from seasonality due to the Afghan winter, the figure shows a spike in attacks after the first announcement of troops withdrawal made in November 2009 (Afghanistan Assessment Group (2012)) followed by a gradual decrease in the number of incidents. Figure 1(b) shows the number of attacks on coalition forces by Afghan forces - the so-called "Green-on-Blue" attacks - for the period of September 2008 to June 2013 and includes the date of the second announcement made by the U.S. President (June 22nd, 2011) postponing the U.S. withdrawal to July 2014 along with a troops reduction starting in the following month. The data are consistent with Roggio and Lundquist (2012)'s claim that the number of "Green-on-Blue" attacks "[...] began spiking in 2011, just after President Barack Obama announced the plan to pull the surge forces, end combat operations in 2014, and shift security to Afghan forces. The Taliban also have claimed to have stepped up efforts at infiltrating the Afghan National Security Forces."[8]



(a) Year-to-year changes in EIA



(b) Green-on-Blue Attacks Afghanistan

Figure 1: Number of Attacks over Time

These reactions might at first appear counterintuitive. In particular, why did "announcing a timetable for a withdrawal" prompt a surge in attacks by the opponents' army rather than "merely send the Taliban underground until the Americans began to leave," as predicted by Senator McCain

[7] In the background (light blue) the total number of EIA. The red bars represent an increase of monthly EIA compared to the same month the year before; blue bars represent a decrease. The changes over three month periods are depicted at the top of the chart. Data Source: Afghan Mission Network (AMN) Combined Information Data Network Exchange (CIDNE) Database, as of 18 Sep 2012.

[8] This claim appears in *The Long War Journal,* among the most comprehensive, available data collection on the attacks conceded by the U.S. troops in the Afghan war. See also: Livingston and O'Hanlon (2017).

in his comment to the West Point speech (PBS (2009))? Similarly, why did President Obama announce a surge in troops concurrently with fixing the duration of the involvement? More generally, why does announcing the duration of the game result in such a discontinuous change in players' behavior?

Armed conflicts are inherently complex. Consequently, we do not attempt a comprehensive rationalization of such intricate events. Rather, we explain why and how knowing versus not knowing the duration of a strategic interaction affects players' equilibrium behavior and consequently the dynamics of the probability of the outcome. We start by modeling a two-player, zero-sum, dynamic game where at each point in time players' actions jointly determine the probability of scoring or conceding a point, or neither. The player that accumulates more points receives a positive, fixed payoff at the end of the game. Actions can be classified according to their governance as "attack" or "defense." We assume that all actions require the same level of effort but differ in their probability of determining the outcome. We study the game under two alternative settings: i) fixed known duration; and ii) unknown duration where at any time there is a strictly positive probability that the game ends.

We show that in both games the equilibrium is Markov perfect, unique, and in pure strategies. We find that in known duration games, players' equilibrium actions are non-stationary; specifically, monotonic over time and across players' relative positions. In unknown duration games, though still monotonic across players' relative positions, the equilibrium actions are stationary over time. The latter proof exploits the fact that the continuation probability in the unknown duration game plays the role of the discount factor. As such, though in our game the payoff is realized only when the game stops, it can be solved by using the standard tools of stochastic games with discounting. We further characterize the optimal strategies depending on players' relative position and on their relative advantage in attacking or defending, a concept we define in terms of relative probabilities of scoring or conceding a point. In both games, the equilibrium action is inversely related to difference in scoring. Intuitively, the more ahead a player is the more defensive he becomes, and vice versa. This holds for both games and is also intuitive as the leading player will try to keep the leading position by defending the score. The relative advantage in attacking or defending of a player plays an important role in determining the optimal equilibrium actions. In the known duration game, having an advantage in attacking determines the equilibrium dynamics overtime: the action will be decreasing overtime for the player with the relative advantage in attacking and increasing otherwise (equivalently, the player with a relative advantage in attacking starts with a high action that decreases as the endgame approaches). In the unknown duration case, though actions are stationary, the leading player always has an advantage in attacking and the player behind always an advantage in defending. A third characterization of interest is the role of time, in terms of its marginal value defined as the value of moving forward in the game, given the difference in scoring. In the known duration game, this is positive for the player with an advantage in attacking. In the unknown duration game, there is no getting closer to the endgame in a deterministic sense but only in probabilistic terms. This is possibly captured by a decrease in the continuation probability. We provide the conditions for such a decrease to play the same role as the marginal value of time in the known duration game. We show that under these conditions, the player with the advantage in attacking responds to an increase in the stopping probability as he would do in the known duration game when the end gets closer, *i.e.,* by decreasing the action level.

Not surprisingly, the equilibrium behavior of players' actions and the corresponding probability of scoring can be very different as the latter depends on the interaction between players' actions. In particular, we show that in known duration games, though actions are monotonic overtime, the probability of a player scoring a point can be monotonic or non-monotonic over time and across relative positions, depending on the functional form of the probability of scoring. In unknown duration games, the probability of scoring is always stationary as players' equilibrium actions are stationary. Finally, we study the dynamics of the probability of scoring over time and across players' relative positions for a given class of probability functions.

Our analysis is related to several strands in the literature. First, the literature on stochastic games that started with Shapley (1953). This literature has studied games with different payoff structures, mainly discounted, average, and final payoff games. Our game belongs to the latter. The payoff is quasi-binary as in González-Díaz and Palacios-Huerta (2016) and similar to the one studied by Lasso de la Vega and Volij (2018).[9] In the latter paper, however, the game stops when it reaches a predetermined absorbing state. In our case, the absorbing states in the known duration game are time dependent, *i.e.,* when there is not enough time for either player to equalize. In the unknown duration game, there are no absorbing states.[10]

[9] See also the binary games studied by Walker, Wooders and Amir (2011), Walker and Wooders (2001).

[10] The unknown duration game is essentially a discounted version of a quasi-binary game.

Second is the literature on bargaining games with deadlines (*e.g.,* Spier (1992); and Yildiz (2011)). We argue that knowing the duration of a game is not equivalent to having a deadline as the latter does not prevent the game from stopping beforehand. Consequently, we prefer to adopt the terminology of duration and clearly separate our analysis and results from the deadline effects found in the literature.

There is a large body of work on finite *vs.* infinitely repeated games (*e.g.,* Aumann and Shapley (1994); Rubinstein (1979); and Fudenberg and Maskin (1986)). The strategic situation we study, however, is different from the repeated game setup. Specifically, in our setting the players' final payoff depends on their actions in each period but are not the sum or the average of each period's outcomes. Furthermore, in contrast to the zero-sum nature of our game, the repeated games literature focuses on the incentives to coordinate. These differences explain the contrasting results in our paper and Dal Bó (2005) who studies, in an experimental set up, the equilibrium outcome in a finite *vs.* infinitely repeated prisoner's dilemma game with a random continuation rule. His finding that the probability of continuation matters for cooperation is driven by the higher expected punishment for deviators when the expected duration of the game is longer.[11]

Our model shares more the features of a sequential tournament, and specifically the case of tournaments where agents choose the level of risk.[12] This literature examines players' risk taking as a function of their position in the race. Modeling risk taking, Cabral (2003) finds that leaders choose the safe path while laggards the risky one. This view is also supported by González-Díaz and Palacios-Huerta (2016) in the context of chess tournaments. Hvide (2002) shows that if agents choose both the level of risk and the level of effort, it is possible to limit the risk level agents choose and induce higher effort. While it is easy to frame players' preference in our model in terms of risk and return, this requires additional assumptions on the probability function in order to link the action level to risk taking and would consequently limit the generality of our results.

The literature on the act of sabotage where players expend some of their resources for "the act of raising rivals' costs" represents another strand of existing work to which our paper is related.[13] In the Spanish soccer context, for example, Garicano and Palacios-Huerta (2014) find that increasing the number of points awarded for a win resulted in an increase in the amount of sabotage effort

[11] See also the literature on repeated trust games, *e.g.,* Engle-Warnick and Slonim (2004).

[12] This is in contrast to the literature on tournaments where agents choose effort levels. See for example, Lazear and Rosen (1981) and Nalebuff and Stiglitz (1983), among many others.

[13] See Chowdhury and Gurtler (2015) for definition and survey of the literature.

undertaken by teams, measured by the number of fouls, yellow cards and red cards. Allowing players to choose how to allocate their effort between acts that increase one's own probability of success and acts of sabotage adds complexity to the game. Given that, as far as we are aware, this is the first paper to study behavior under known and unknown duration games, we leave the question on the effect of sabotage for further research.

A particularly interesting link is to the literature on demand for suspense (*i.e.,* Eli, Frankel and Kamenica (2015)). We find that games of known duration display swings in optimal actions between attack and defense and hence such swings would lead to higher suspense in known duration games relative to unknown duration games.

2. The game

Consider a game played by two players, $i = A, B$, over time $t = 1, 2, \ldots$ At each t, each player chooses an action taking values in the closed unit interval $\mathbf{I} = [0, 1]$. Denote by $a \in \mathbf{I}$ and $b \in \mathbf{I}$ player A and B's actions, respectively. We interpret higher values of the action as offensive play (or attack) and lower values as defensive play (or defense). At each t, players' actions jointly determine the probability distribution of the random variable X_t, *i.e.,* $\Pr(X_t = x|a, b) = p_x(a, b)$. Specifically, x takes the value of 1 if player A scores a point, 0 if neither players scores a point and -1 if player B scores a point. The function p_x takes values in the interior of the unit interval, is twice continuously differentiable in both players' actions, and is defined as:

$$p_x : \mathbf{I} \times \mathbf{I} \to Int(\mathbf{I}), \;\; x = -1, 1,$$

and $p_0 = 1 - p_1 - p_{-1}$. All actions require the same level of effort and yield the same direct costs or disutility. Equivalently, we may assume that they are costless.[14] In addition, we assume the following:

Assumption 1. *1. In the interior of the action set, the scoring probability of each player is increasing in both actions a and b,* i.e.*, for $(a, b) \in Int(\mathbf{I} \times \mathbf{I})$:*

$$\partial_a p_x(a, b) > 0 \;\; and \;\; \partial_b p_x(a, b) > 0, \;\; x = -1, 1.$$

[14]Defending typically requires high effort, so there is no direct relationship between the action level and effort level in our model, *e.g.,* action $a = 0$ does not mean inaction.

2. At the boundaries of the action set, the marginal probability of scoring by players A and B are given by:

$$\lim_{a\to 1} \partial_a p_1(a,b) = \lim_{a\to 0} \partial_a p_{-1}(a,b) = 0,\ \ b \in \mathbf{I};$$

$$\lim_{b\to 0} \partial_b p_1(a,b) = \lim_{b\to 1} \partial_b p_{-1}(a,b) = 0,\ \ a \in \mathbf{I}.$$

3. In the interior on the action set, the probability of scoring is concave in each player's own action and convex in the opponent's action, i.e.*, for $(a,b) \in Int(\mathbf{I}\times\mathbf{I})$:*

$$\partial_a^2 p_1(a,b) \ < 0 \ \ and \ \ \partial_b^2 p_{-1}(a,b) < 0;$$

$$\partial_a^2 p_{-1}(a,b) > 0 \ \ and \ \ \ \partial_b^2 p_1(a,b) > 0.$$

Assumption 1.1 implies that a player choosing a higher action increases both his own and the opponent's probability of scoring; the latter representing an implicit cost of attacking. This assumption is meant to capture circumstances where a more offensive action increases the probability of scoring but weakens the defense level. This trade-off between offense and defense is typical in conflictual situations where one can identify actions of attack or defense. In armed conflicts, for example, an offensive action increases both the probability of inflicting casualties to the enemy and of suffering casualties. In sports games like soccer, playing in attack implies an increasing probability of scoring as well as conceding a goal by counter-attack. Assumption 1.2 provides sufficient conditions for obtaining an equilibrium in the interior of the action space and implies that there is neither harm nor benefit in slowing down the attack a bit when the player is attacking at the maximum level. Assumption 1.3 implies that the marginal probability of scoring decreases in the player's own action level and increases in the opponent's action level as well as guaranteeing concavity of the players' objective function.

While the game is on, there is no payoff. When the game ends, player A's payoff is 1 if the difference in points scored is positive, 0 if the difference in points scored is nil and -1 if it is negative, *i.e.,* player A receives a payoff given by:

$$V(d) = I_{\mathbb{Z}_+}(d) - I_{\mathbb{Z}_-}(d), \tag{1}$$

where I denotes the indicator function, $\mathbb{Z}_+$ (resp. $\mathbb{Z}_-$) the set of non-negative (resp. non-positive) integers, and d the difference in the points scored at time T. Player B receives a value $-V(d)$.[15] The stopping time is governed by two alternative exogenous rules. The game either lasts T periods and does not stop earlier, or stops at t with probability π. We will refer to the former as the *known duration game* or *fixed stopping time* and to the latter as the *unknown duration game* or *random stopping time*. Players know the rule and choose their optimal strategy accordingly.

We proceed by analyzing the game under the two alternative rules in Sections 2.1 and 2.2. In Section 3, we characterize the dynamics of the probability of a scoring across the two types of games. All proofs are in the Appendix.

2.1 Known duration: fixed stopping time

Suppose that both players know that the game will last until time $T+1$, *i.e.,* they will take their last actions at T and at $t = T+1$ they will receive their payoff depending on the difference in points scored at T+1. Let D_t be the difference in the number of points from player A's perspective at time t. Thus the process $D_{t+1} = D_t + X_{t+1},\ t = 1, \dots, T$ is a Markov process with transition probability $\Pr(D_{t+1} = d + x | D_t = d, a, b) = p_x(a, b)$ for all $d \in \mathbb{Z}$, the state space of the process. Without loss of generality let $D_1 = 0$.[16]

At any $t = 1, \dots, T$, let the $2 \times (T - t + 1)$ matrix $(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}) = ((a_\tau, b_\tau) : \tau = t, ..., T)$, denote the current and future action profiles of player A and B, respectively. Using the Markovian property of the transition probability function, we iteratively define the transition probability from t to T as:

$$\Pr\left(D_{T+1} = d |\ D_t = d', \boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}\right) = \sum_{x=-1}^{1} p_x(a_t, b_t) \Pr(D_{T+1} = d |\ D_{t+1} = d' + x; \boldsymbol{a}_{\geq t+1}, \boldsymbol{b}_{\geq t+1}),$$

where $d, d' \in \mathbb{Z}$. Players' payoff at time $t = 1$ and state 0 can be written as:

$$\begin{aligned} U_1^A(\boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1}; 0) &= \mathbb{E}_0^{(\boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1})} V \\ &= \sum_{d \in \mathbb{Z}} \Pr(D_{T+1} = d | D_1 = 0; \boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1}) V(d) = -U_1^B(\boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1}; 0). \end{aligned}$$

Using iterative expectations we can write player A's expected payoff as:

[15] These values are for simplicity. The results hold for any increasing function in d that is symmetric around zero.
[16] Notice that the transition probability does not depend on the state.

$$U_1^A(\boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1}; 0) = \sum_{x=-1}^{1} p_x\,(a_1, b_1)\,\mathbb{E}_x^{(\boldsymbol{a}_{\geq 2}, \boldsymbol{b}_{\geq 2})} V.$$

Notice that $\mathbb{E}_x^{(\boldsymbol{a}_{\geq 2}, \boldsymbol{b}_{\geq 2})} V$ is independent of (a_1, b_1) and is only a function of the future actions profile $(\boldsymbol{a}_{\geq 2}, \boldsymbol{b}_{\geq 2})$ and the state d.[17] Consistently, letting $U_2^A(\boldsymbol{a}_{\geq 2}, \boldsymbol{b}_{\geq 2}; d) = \mathbb{E}_d^{(\boldsymbol{a}_{\geq 2}, \boldsymbol{b}_{\geq 2})} V$, we can write $U_1^A(\boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1}; 0) = \sum_{x=-1}^{1} p_x\,(a_1, b_1)\,U_2^A(\boldsymbol{a}_{\geq 2}, \boldsymbol{b}_{\geq 2}; x)$. Similarly, we can iteratively define at any t and d, $U_t^A(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d) = \mathbb{E}_d^{(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t})} V = -U_t^B(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d)$. Being the game zero-sum at each t, we can analyze the game from player A's perspective only and drop the upper indexes identifying the players' identity. The payoff at any $t = 1, \ldots, T$ can be written as:

$$U_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d) = \sum_{x=-1}^{1} p_x\,(a_t, b_t)\,U_{t+1}(\boldsymbol{a}_{\geq t+1}, \boldsymbol{b}_{\geq t+1}; d + x), \tag{2}$$

and, though actions at $T + 1$ are not taken, it is convenient to denote the payoff at $T + 1$ to be given by $U_{T+1}(d) = V(d)$.

In order to allow for more general (behavior) strategies, define the profile $\boldsymbol{\sigma}^i = (\sigma_1^i, \ldots, \sigma_\tau^i, \ldots, \sigma_T^i)$, $i = A, B$, where $\sigma_\tau^i : \mathbb{Z} \to \mathscr{P}(\mathbf{I})$ is a map from the state into a probability distribution on the action space $\mathbf{I}$. Letting $\boldsymbol{\sigma}_{\geq t}^i = (\sigma_\tau^i : \tau = t, ..., T)$, player $i = A, B$ strategy from t onward then the expected payoff at $t = 1, \ldots, T$ can be recursively defined as:

$$\mathbb{E}_{(\boldsymbol{\sigma}_{\geq t}^A, \boldsymbol{\sigma}_{\geq t}^B)} U_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d) = \int \sum_{x=-1}^{1} p_x(a, b) \mathbb{E}_{(\boldsymbol{\sigma}_{\geq t+1}^A, \boldsymbol{\sigma}_{\geq t+1}^B)} U_{t+1}(\boldsymbol{a}_{\geq t+1}, \boldsymbol{b}_{\geq t+1}; d + x) d\left(\sigma_t^A \otimes \sigma_t^B\right).$$

Being the expected payoff history independent, we can restrict ourselves to Markov strategies and define an equilibrium of the game as follows:

Definition 1. *A strategy* $(\boldsymbol{\sigma}^{A*}, \boldsymbol{\sigma}^{B*})$ *is a* ***Markov perfect equilibrium*** *for the known duration game if for any* $t = 1, \ldots, T$ *and state* d*:*

$$\mathbb{E}_{(\boldsymbol{\sigma}_{\geq t}^{A*}, \boldsymbol{\sigma}_{\geq t}^{B*})} U_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d) = \sup_{\boldsymbol{\sigma}_{\geq t}^A} \inf_{\boldsymbol{\sigma}_{\geq t}^B} \mathbb{E}_{(\boldsymbol{\sigma}_{\geq t}^A, \boldsymbol{\sigma}_{\geq t}^B)} U_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d). \tag{3}$$

For each t, let $\overline{d}_t = T - t + 1$. The values $\overline{d}_t$ and $-\overline{d}_t$ are the minimum differences in the number of

[17] The game satisfies the sufficiency conditions of Maskin and Tirole (2001, p. 204) for a game to be Markov: by construction, actions (a_1, b_1) do not restrict future actions (condition (i)) and the final payoff function V is also unaffected (condition (ii)).

points necessary for either player A (for $d = \overline{d}_t$) or player B (for $d = -\overline{d}_t$) to ensure victory at time t. We define $\overline{d}_t$ and $-\overline{d}_t$ the *absorbing states* corresponding to time t. Reaching an absorbing state implies that there is not enough time for the lagging player to catch up or win. At an absorbing state, the value of the game for all successive stages is fixed at either 1, if player A is winning or -1, if player B is winning.

If $(\boldsymbol{\sigma}^{A*}, \boldsymbol{\sigma}^{B*})$ is a sequence of degenerate probability distributions, then the equilibrium is in *pure strategies* and will be denoted by $(\boldsymbol{a}^*, \boldsymbol{b}^*) = ((a^*_\tau(d), b^*_\tau(d)) : \tau = 1, \dots, t, \dots, T)$.

Lemma 1. *The known duration game has a Markov perfect equilibrium in pure strategies. Specifically: 1. if $|d| < \overline{d}_t - 1$ the equilibrium actions are unique and in the interior,* i.e., $(a^*_t(d), b^*_t(d)) \in Int(\mathbf{I}\times\mathbf{I})$*; 2. if $d = \overline{d}_t - 1$ then $(a^*_t(d), b^*_t(d)) = (0, 1)$ and if $d = -(\overline{d}_t - 1)$ then $(a^*_t(d), b^*_t(d)) = (1, 0)$. 3. If $|d| \geq \overline{d}_t$, the equilibrium actions are indeterminate.*

The *value* of the known duration game at $t = 1, \dots, T-1$ and d can be written as:

$$V_t(d) = \max_a \min_b U_t((a, \boldsymbol{a}^*_{\geq t+1}), (b, \boldsymbol{b}^*_{\geq t+1}); d) = \sum_{x=-1}^{1} p_x(a^*_t, b^*_t) V_{t+1}(d + x), \tag{4}$$

and at T this is given by:

$$V_T(d) = \max_a \min_b U_T(a_{\geq T}, b_{\geq T}; d) = \sum_{x=-1}^{1} p_x(a^*_T, b^*_T) V(d + x). \tag{5}$$

Part 1 of the lemma states the existence and uniqueness of an interior equilibrium when the state of the game is away from the boundaries. Uniqueness also implies that there exists a function $\beta : Int(\mathbf{I}) \to Int(\mathbf{I})$ such that $\beta(a^*) = b^*$ with $\partial_a \beta(a)|_{a=a^*} < 0$. Part 2 identifies the behavior "one point away" from an absorbing state. For an intuition of players' behavior in this case consider the game at T and $d = 1$, *i.e.,* players have only one period left to play and player A is ahead by one point. Player A can choose a relatively offensive action in order to try to increase the probability of finishing the game with two points ahead. At the same time, this will increase the probability of conceding a point and hence ending the game in a tie. Alternatively, player A could choose a more defensive strategy, for example, set $a = 0$ and maximize the probability of ending the game at $d = 1$. Since $V(2) = V(1) = 1$, and since setting $a = 0$ minimizes the probability of conceding a point, the latter strategy is optimal. Similarly, player B can either set $b = 1$ and maximize the probability of scoring (along with increasing the probability of conceding a point) or set $b < 1$ and increase the probability of ending the game at $d = 1$. Since $V(0) < V(1) = V(2)$ the first strategy dominates for player B. Finally, Part 3 of the lemma refers to the case where the game has reached an absorbing state. Not surprisingly, in this case the actions are indeterminate as the value of the game cannot be changed while all action levels bear the same cost.

For the remaining part of this section we turn to the characterization of the *interior equilibrium* of a known duration game, *i.e,* at any time t and state d such that $|d| < \overline{d}_t - 1$. To this end we define player A's *relative elasticity of scoring* for a given action pair (a, b) as the following ratio:[18]

$$\epsilon^A(a,b) = \frac{\partial_a p_1(a,b)/p_1(a,b)}{\partial_a p_{-1}(a,b)/p_{-1}(a,b)}.$$

The relative elasticity of scoring for player B, $\epsilon^B(a, b)$, is defined in a similar way.

The variable $\epsilon^A(a, b)$ represents the player's odds of scoring relative to conceding a point. If $\epsilon^A(a, b) > 1$ then an increase in the action by player A, *i.e.,* becoming more offensive, improves the player's relative odds of scoring a point as compared to conceding one. Similarly, $\epsilon^A(a, b) < 1$ implies that decreasing the level of A's action, *i.e.,* becoming more defensive, improves the player's relative odds of preventing player B from scoring compared to A's odds of scoring. Accordingly, we say that at (a, b) player A has the *relative advantage in attacking (defending)* if $\epsilon^A(a, b) > 1$ ($\epsilon^A(a, b) < 1$). Similarly for player B.

Denote by $p^*_{x,t}(d)$ the equilibrium probability of $X = x$ for the given equilibrium actions $(a^*_t(d), b^*_t(d))$, *i.e.,* $p^*_{x,t}(d) = p_{x,t}(a^*_t(d), b^*_t(d))$. Similarly, let $\epsilon^{A*}_t(d) = \epsilon^A(a^*_t(d), b^*_t(d))$.

Lemma 2. *At equilibrium, the relative elasticity of scoring equals the ratio of the expected losses from conceding a point to the expected gains from scoring one,* i.e.,

$$\epsilon^{A*}_t(d) = \frac{p^*_{-1,t}(d)\,[V_{t+1}(d) - V_{t+1}(d-1)]}{p^*_{1,t}(d)[V_{t+1}(d+1) - V_{t+1}(d)]}. \tag{6}$$

Hence, if the expected losses of conceding a point, relative to the gains from scoring one, are high, in probabilistic terms, the relative gains to the relative losses from increasing the action must also be high.

[18] Accordingly, one could define the term $\partial_a p_1(a,b)\frac{a}{p_1(a,b)}$ as player A's elasticity of scoring and $\partial_a p_{-1}(a,b)\frac{a}{p_{-1}(a,b)}$ as the elasticity of conceding a point. Notice that the elasticity is defined for any (a, b) as the probability takes values in the interior of $\mathbf{I}$ only.

By equation (6), it is easy to show that, at an interior solution, $\epsilon_t^{A*}(d) = [\epsilon_t^{B*}(d)]^{-1}$ and hence that at any given point on the equilibrium trajectory only one player can have the relative advantage in attacking and only one the relative advantage in defending.

Before proceeding to the next result, let us denote by $\mathcal{A}_+(b) = \{a : \epsilon^A(a,b) > 1\}$ the set of player A's actions such that, given action b, player A has the relative advantage in attacking. Similarly let $\mathcal{A}^*_{+,t}(d) = \mathcal{A}_+(b^*_t(d)) = \{a : \epsilon^A(a, b^*_t(d)) > 1\}$. Finally let $\mathcal{A}_-(b) = \{a : \epsilon^A(a,b) < 1\}$ and $\mathcal{A}^*_{-,t}(d)$ accordingly.[19]

Lemma 3. *1. At any t, the value function is monotonically increasing in d,* i.e., $V_t(d+1) > V_t(d)$. *2. if $a^*_t(d) \in \mathcal{A}^*_{+,t}(d)$ then given d, the value function is monotonically increasing in t,* i.e., $V_{t+1}(d) > V_t(d)$. *The opposite holds for $a^*_t(d) \in \mathcal{A}^*_{-,t}(d)$.*

Part 1 of the lemma implies that, in the interior, the marginal value of scoring is always positive. Part 2 states that "shortening the game", *i.e.,* getting one period closer to the end, has positive marginal value for the player with the relative advantage in attacking. Vice versa for the opponent. The claim follows from rearranging equation (5) to obtain:

$$\overbrace{V_{t+1}(d) - V_t(d)}^{\text{marginal value of time}} = \overbrace{p^*_{-1,t}(d)[V_{t+1}(d) - V_{t+1}(d-1)]}^{\text{expected losses}} - \overbrace{p^*_{1,t}(d)[V_{t+1}(d+1) - V_{t+1}(d)]}^{\text{expected gains}}.$$

The equation shows that the marginal value of time is positive if the expected losses from playing an additional time period are greater than the expected gains. By equation (6) this holds for the player with the relative advantage in attacking. Lemma 3 leads to the following proposition:

Proposition 1. *1. Player A's equilibrium action decreases in d at any given t,* i.e., $a^*_t(d+1) < a^*_t(d)$; *2. If $a^*_t(d) \in \mathcal{A}^*_{+,t}(d)$, player A's equilibrium action decreases in t at any given d,* i.e., $a^*_{t+1}(d) < a^*_t(d)$. *The opposite holds if $a^*_t(d) \in \mathcal{A}^*_{-,t}(d)$.*

Player B's equilibrium action behaves symmetrically.

The result is driven by the assumption that attacking increases the probability of conceding a point. In particular, the proposition follows from the fact that the action is inversely related to

[19] Notice that, apart from degenerate cases, $\mathcal{A}^*_{+,t}(d) \cup \mathcal{A}^*_{-,t}(d)$ is non empty. *E.g.,* the case $\epsilon_t(d) = 1$ for all t can occur when $d = 0$ and when players are symmetric, *i.e.*, the functional forms of p_1 and p_{-1} are symmetric. Clearly a degenerate case.

the value of the game. Across d, since the value of the game increases in d, the action decreases at the same time. The leading player, *i.e.,* the player with higher value of the game, will be more complacent and conversely the losing player more aggressive. Over time, however, this holds only if the expected losses are greater than the expected gains. The player with the relative advantage in attacking chooses a high action at the beginning or, equivalently, reduces his action as the end of the game becomes closer.[20] Vice versa for the other player.

Proposition 1 offers a suggestive interpretation for why Senator McCain's prediction of the Taliban army waiting the U.S. troops out following President Obama's announcement did not materialize. Figure 1.a in the introduction shows a behavior that is consistent with Proposition 1.2. The announcement prompted a spike in Enemy Initiated Attacks followed by a gradual decrease in subsequent years. Indeed, the announcement represented the beginning of a known duration game. Moreover, if it is reasonable to assume the Taliban had an advantage in attacking in the period right after the announcement then such a response is consistent with the optimal strategy of the player with advantage in attacking in a known duration overtime. This view is supported by Obama's comment on the necessity of breaking the Taliban's momentum with a surge in troops.[21]

2.2 Unknown duration: random stopping time

Suppose now that the players do not know the exact duration. Let the duration T of the game be a random variable where at each time t the players assign a probability $\pi = \Pr(T > t|T \geq t)$ that the game will continue. Equivalently, assume that at each stage there is a probability $1 - \pi = \Pr(T = t|T \geq t)$ that the game might not continue to the next stage. We assume π to be strictly positive, otherwise the game would stop immediately, and less than 1, otherwise it would never end. Notice that there are four possible states next period: the game stops with probability $(1 - \pi)$, or it continues with probability $\pi p_x(a, b)$ where $x = -1, 0, 1$. The transition probability $\Pr(D_{t+1} = d + x|D_t = d, a, b) = \Pr(D_{t+1} = d + x|D_t = d, T > t, a, b) \Pr(T > t|T \geq t) = \pi p_x(a, b)$ for all $d \in \mathbb{Z}$.

For each t, denote the sequence of present and future actions by $(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}) = ((a_\tau, b_\tau) : \tau = t, ..., \infty)$.

[20]The result is consistent with the empirical observation in the soccer context by Garicano and Palacios-Huerta (2014) where they observe that "[...] when a team is ahead it deploys a strategy aiming at conserving the score relative to the possibility of scoring more goals."

[21]See Woodward (2010), p.329, "[...] 'We have to break the momentum of the Taliban' [Obama] said."

The payoff function at time $t = 1$ can be defined as:

$$\widetilde{U}_1^A(\boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1}; 0) = \mathbb{E}_0^{(\boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1})} V = -\widetilde{U}_1^B(\boldsymbol{a}_{\geq 1}, \boldsymbol{b}_{\geq 1}; 0).$$

As in the known duration game, we can drop the upper indexes and analyze the game from player A's perspective. At any t, using iterated expectations, the payoff can be written iteratively as:

$$\begin{aligned}\widetilde{U}_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d) &= \mathbb{E}_d^{(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t})} V \\ &= (1-\pi)\, V(d) + \pi \sum_{x=-1}^{1} p_x\,(a_t, b_t)\, \widetilde{U}_{t+1}(\boldsymbol{a}_{\geq t+1}, \boldsymbol{b}_{\geq t+1}; d + x).\end{aligned}$$

Similarly to the known duration game, let $\boldsymbol{\sigma}^i = (\sigma_1^i, \ldots, \sigma_\tau^i, \ldots \sigma_\infty)$ denote the (behavior) strategy for $i = A, B$, where σ_τ^i is defined as in Section 2.1. The expected payoff is recursively defined as:

$$\begin{aligned}\mathbb{E}_{(\boldsymbol{\sigma}_{\geq t}^A, \boldsymbol{\sigma}_{\geq t}^B)} \widetilde{U}_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d) &= (1-\pi)\, V(d) \\ &\quad + \pi \int \sum_{x=-1}^{1} p_x\,(a, b)\, \mathbb{E}_{(\boldsymbol{\sigma}_{\geq t+1}^A, \boldsymbol{\sigma}_{\geq t+1}^B)} \widetilde{U}_{t+1}(\boldsymbol{a}_{\geq t+1}, \boldsymbol{b}_{\geq t+1}; d + x) d\left(\sigma_t^A \otimes \sigma_t^B\right),\end{aligned}$$

where $\boldsymbol{\sigma}_{\geq t}^i = (\boldsymbol{\sigma}_\tau^i : \tau = t, ..., \infty)$ are the present and future strategies of player $i = A, B$. In this case as well we can restrict ourselves to Markov strategies and define the equilibrium as follows:

Definition 2. *A strategy* $(\widetilde{\boldsymbol{\sigma}}^A, \widetilde{\boldsymbol{\sigma}}^B)$ *is a* ***Markov perfect equilibrium*** *for the unknown duration game if at any* $t = 1, \ldots \infty$ *and state* d*:*

$$\mathbb{E}_{(\widetilde{\boldsymbol{\sigma}}_{\geq t}^A, \widetilde{\boldsymbol{\sigma}}_{\geq t}^B)} \widetilde{U}_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d) = \sup_{\boldsymbol{\sigma}_{\geq t}^A} \inf_{\boldsymbol{\sigma}_{\geq t}^B} \mathbb{E}_{(\boldsymbol{\sigma}_{\geq t}^A, \boldsymbol{\sigma}_{\geq t}^B)} \widetilde{U}_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d). \tag{7}$$

The equilibrium in pure strategies for this game will be denoted by $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}) = ((\tilde{a}_\tau(d), \tilde{b}_\tau(d)) : \tau = 1, ..., \infty)$. The equilibrium is said to stationary if the strategies are time independent, *i.e.,* $(\tilde{a}_\tau(d), \tilde{b}_\tau(d) = (\tilde{a}(d), \tilde{b}(d))$ for all τ's. The following proposition states that there exists a unique equilibrium in pure strategies for the game of unknown duration and provides a characterization of the equilibrium actions.

Proposition 2. *The unknown duration game has a unique Markov perfect equilibrium. Specifically: 1. The equilibrium is in pure strategies, stationary, and in the interior. 2. The value of the* *game is increasing in d and player A's equilibrium actions are decreasing in d. Player B behaves symmetrically.*

The value of an unknown duration game at t and d can be recursively written as:

$$W_t(d) = \max_a \min_b \tilde{U}_t((a, \tilde{\boldsymbol{a}}_{\geq t+1}), (b, \tilde{\boldsymbol{b}}_{\geq t+1}); d) = (1-\pi)V(d) + \pi \sum_{x=-1}^{1} p_x(\tilde{a}, \tilde{b}) W_{t+1}(d+x). \quad (8)$$

Proposition 2 has several implications. First, and most importantly, the stationarity of the solution implies that removing the certainty about the duration of the game changes players' optimal behavior qualitatively and discontinuously: qualitatively because their behavior becomes stationary; discontinuously because this property holds only as long as $0 < \pi < 1$.[22] The key observation in the proof is that we can apply the contraction mapping theorem for bounded continuous functions to show that the problem is time independent. The role played by the continuation probability is similar to the one played by time discounting in standard dynamic programming. Intuitively, stationarity follows from the fact that at any t, across d's, players know that next period they will face exactly the same problem so, given d, today and tomorrow choices are the same.[23]

As in the case of known duration, it is possible to decompose the value of the game in terms of expected losses and gains. Interestingly, in the unknown duration case, the *marginal value of stopping the game* plays the same role as the marginal value of time in known duration games. At equilibrium, the marginal value of stopping the game must equal the expected losses from continuing. This follows from equation (8) and using the fact that the value as well as the probability functions are t-independent:

$$\overbrace{(1-\pi)\left(V(d) - W(d)\right)}^{\text{marginal value of stopping}} = \pi(\overbrace{\tilde{p}_{-1}(d)[W(d) - W(d-1)]}^{\text{expected losses}} - \overbrace{\tilde{p}_1(d)[W(d+1) - W(d)]}^{\text{expected gains}}), \quad (9)$$

where $\tilde{p}_x(d),\ x = -1, 1$ denotes the equilibrium probability at d. In the unknown duration game the continuation probability weighs both sides of the equations and hence plays a role that is obviously absent in the case of known duration. Moreover, since $|W(d)| < 1$, the marginal value of stopping is always positive for the leading player.

[22] Walker, Wooders and Amir (2011) analyze stationary equilibria in unknown duration games. Their analysis focuses on binary Markov games.

[23] The distinction in players' equilibrium behaviour between unknown and known duration games is very much related to the well-known problem in the analysis of unit roots of AR(1) processes encountered in time series.

In unknown duration games the characterization of the relative advantage of attacking and defending is relatively simple and is related to the marginal value of stopping. In fact, plugging the first order condition with respect to player A's action,

$$\frac{\partial_a \tilde{p}_{-1}(d)}{\partial_a \tilde{p}_1(d)}[W(d) - W(d-1)] = W(d+1) - W(d),$$

into (9) obtain:

$$\begin{aligned}(1-\pi)(W(d) - V(d)) &= \pi \tilde{p}_{-1}(d)\left[\frac{\tilde{p}_1(d)}{\tilde{p}_{-1}(d)}\frac{\partial_a \tilde{p}_{-1}(d)}{\partial_a \tilde{p}_1(d)} - 1\right][W(d) - W(d-1)] \\ &= \pi \tilde{p}_{-1}(d)\left[\frac{1}{\tilde{\epsilon}^A(d)} - 1\right][W(d) - W(d-1)], \qquad (10)\end{aligned}$$

where $\tilde{\epsilon}^A(d)$ is player A's relative elasticity of scoring defined in the equivalent way as $\epsilon_t^A(d)$ in known duration games. If $d > 0$, then $\tilde{\epsilon}^A(d) > 1$ and player A must have the relative advantage in attacking (the opposite holds for $d < 0$). That is, in unknown duration games, the leading player always has the relative advantage in attacking, while the player that is behind always has the relative advantage in defending.

The effect of a change in the continuation probability π on the optimal equilibrium action is less easily determined at this level of generality. We can however identify the forces at play. From equation (10), it is easy to see the effect of an increase in the duration π on the value of the game and hence on the equilibrium actions. Taking the derivative with respect to π and using the first order condition with respect to the action of player A in problem (8) obtain:

$$\begin{aligned}\partial_\pi W(d) = &\frac{1}{1-\pi}\overbrace{[\sum_x p_x(d)W(d+x) - V(d)]}^{\text{negative for } d \geq 1} \\ &+ \frac{\pi}{1-\pi}\,\partial_\pi[\overbrace{p_1(d)(W(d+1) - W(d))}^{\text{expected gains}} - \overbrace{p_{-1}(d)(W(d) - W(d-1))}^{\text{expected losses}}]. \qquad (11)\end{aligned}$$

The first order effect of an increase in the continuation probability π is detrimental to the leading player. Independently of the magnitude of the other terms, this will dominate for π low enough. The second order effects involve the marginal continuation values only. While we are unable to sign the total change, the equation shows how demanding the condition on the cross derivatives is for an increase in the continuation probability to be favored by a winning player. In fact, for the right hand side to be positive, one of the (weighted) cross derivative has to compensate the other (weighted) cross derivative plus the first order effect. If that stringent condition is not met or if π is low enough, the winning player will increase his action as a consequence of an increase in π (and vice versa).

The latter observation brings an additional interesting parallel between the two games. Specifically, a decrease in the continuation probability has a similar effect on players' behavior as getting closer to the endgame in known duration games. In probabilistic terms, an increase in the stopping probability $1 - \pi$ brings the game closer to an end. If the second order effect of the cross derivative is not high enough or if π is low enough, the player with an advantage in attacking will increase his defence. This is the same as in the known duration game where the player with an advantage in attacking has a positive marginal value of time and decreases the action level as the endgame approaches.

3 Characterizing the probability of scoring

The previous sections have provided a characterization of players' equilibrium actions. However, empirically, in many instances only the actions' *outcomes* or *consequences* are observable (*i.e.,* the realization of X) and not the actions themselves. In the case of armed conflicts, for example, records of the attacks (actions) of the armies involved are rarely available and only data on casualties (outcomes) might be recorded. Similarly, in sport events, players' points are usually recorded, rather than their actions. In the soccer game, for instance, until recently only goals were recorded and not teams' actions. By observing the outcome x across different d's and over time t one may recover the probability of the outcomes. In this section, we show that the results obtained thus far on the equilibrium actions provide testable empirical hypotheses on the behavior of the probability of scoring *without further restrictions beyond Assumption 1.* Moreover, we show that in known duration games monotonicity of the equilibrium actions does not necessarily translate into monotonicity of the probability of success. In Section 3.1, we study a specific functional form for the probability which allows us to better characterize its possible trajectories.

Let us look at changes in the equilibrium probability in known duration games, $p^*_{x,t}(d)$, over t and across d. For unknown duration games the probability function at equilibrium is obviously stationary, an immediate consequence of the stationarity of the equilibrium strategies in such games. Recall that according to Lemma 1.1, player B's equilibrium action can be written as $b_t^*(d) = \beta(a_t^*(d))$. This implies that the equilibrium probability can be written as a function of $a_t^*(d)$ only, *i.e.,* abusing notation $p_x(a_t^*(d))$. In the interior solution of a known duration game, changes to the probability of scoring due to changes in t and d can be computed as follows:

$$p_{x,t}^*(d+1) - p_{x,t}^*(d) \equiv \Delta_d p_{x,t}^*(d) \approx \frac{dp_x(a^*)}{da^*} \Delta_d a_t^*(d), \tag{12}$$

$$p_{x,t+1}^*(d) - p_{x,t}^*(d) \equiv \Delta_t p_{x,t}^*(d) \approx \frac{dp_x(a^*)}{da^*} \ \Delta_t a_t^*(d), \tag{13}$$

where Δ_d and Δ_t denote the partial difference with respect to d and t (the approximation is due to the discreteness of d and t). Notice that equations (12) and (13) differ only in the terms Δ_d and Δ_t. Since these are monotonic by Proposition 1, non-monotonicities of the probability function are driven uniquely by the non-monotonicity of $\frac{dp_x(a^*)}{da^*}$. The sign of the latter can be determined as:

$$\overbrace{\frac{dp_x(a^*)}{da^*}}^{?} = \overbrace{\partial_a p_x(a^*, \beta(a^*))}^{+} + \overbrace{\partial_b p_x(a^*, \beta(a^*))}^{+} \ \overbrace{\beta'(a^*)}^{-}. \tag{14}$$

The sign of the left hand side is determined by the relative magnitude of the partials $\partial_a p_x(a^*, \beta(a^*))$ and $\partial_b p_x(a^*, \beta(a^*))$ (positive by Assumption 1) and the absolute value of the term $\beta'(a^*)$ (negative by Lemma 1.1). By equation (14) it follows that the equilibrium probability of scoring is such that:

$$\frac{dp_x(a^*)}{da^*} \geq 0 \ \text{ if and only if } \ \frac{\partial_a p_x(a^*, \beta(a^*))}{\partial_b p_x(a^*, \beta(a^*))} \geq -\beta'(a^*). \tag{15}$$

The following result is an immediate corollary of Proposition 1 along with equations (12)-(15).

Proposition 3. *In known duration games, $p_{x,t}^*(d)$ is stationary in t and d if and only if the equality in equation* (15) *is satisfied for all possible a^*.*

The proposition has important implications. Namely, since a turning point in the probability of scoring is determined by the common term $\frac{dp_x(a^*)}{da^*}$, then the probability of scoring has a turning point in t if and only if it has a turning point in d. In order to gain a better understanding of how changes in t and d translate into changes in the probability $p_{x,t}^*(d)$ via changes in players' actions, one needs more information, or impose further restrictions, on the actual form of the probability function itself. To this end, we study a fairly nonrestrictive, yet conveniently simple, class of functions that will help in computing the projection of $p^*_{x,t}(d)$ on d and t and hence identify how the probability of observing a player scoring might evolve across differences in the number of accumulated points and over time.

3.1 An example

Let us consider the exponentially wrapped log-convex functions:

$$p_1(a,b) = \exp(C_a a - f(a) + f(b)), \tag{16}$$

$$p_{-1}(a,b) = \exp(C_b b - f(b) + f(a)), \tag{17}$$

where C_a and $C_b \in \mathbb{R}_+,\ 0 \le f' \le \min\{C_a, C_b\}$ with $f'' > (max(C_a, C_b) - f')^2$, satisfying Assumption 1. All parameters are such that $p_1(a,b) + p_{-1}(a,b) < 1$ for any (a,b).[24] The terms C_a and C_b do not have an intrinsic behavioral meaning but simply allow for interesting asymmetries between the two players.[25] The given functional form allows for the following explicit derivation of $\beta(a^*)$:[26]

$$\beta(a^*) = [f']^{-1}\left[\frac{C_b}{C_a}[C_a - f'(a^*)]\right]. \tag{18}$$

Two facts play a role in the next sections. First, the function $\beta(a^*)$ is a function of f' and hence $\beta'(a^*)$ is a function of f''. It follows that changes in the slope of $p_1(a^*)$ are determined by changes in f'' and hence by the third derivative of f. Second, using (18), it is easy to see that the ratio of $\partial_a p_1(a^*, \beta(a^*))$ and $\partial_b p_1(a^*, \beta(a^*))$ is C_a/C_b. Thus, changes in $dp_1(a^*)$ are determined by whether $-\beta'(a^*)$ lies above or below C_a/C_b. We can now turn to the characterization of the trajectory of $p^*_{x,t}(d)$, first across d and then over t. Notice that equation (15) and Lemma 2 imply that $\frac{dp_{-1}(a^*)}{da^*} \ge 0$ if and only if $\frac{dp_1(a^*)}{da^*} \ge 0$. Hence, $p^*_{1,t}(d)$ and $p^*_{-1,t}(d)$ have the same behavior across d and over t. Therefore, we can concentrate on characterizing the behavior of $p^*_{1,t}(d)$ only.

[24] Alternatively, we could pre-multiply the two functions by a constant small enough to satisfy the inequality.

[25] Under this specification, if $C_a = C_b$ then the players are symmetric, *i.e.,* for any action pair (a,b), $p_1(a,b) = p_{-1}(b,a)$ if and only if $C_a = C_b$. We consider the latter a degenerate case in our setting (see also footnote 19 for the relative elasticity with symmetric players). This differs from the analysis in the literature which analyzes similar games, *e.g.,* Palomino, Rigotti and Rustichini (1998)

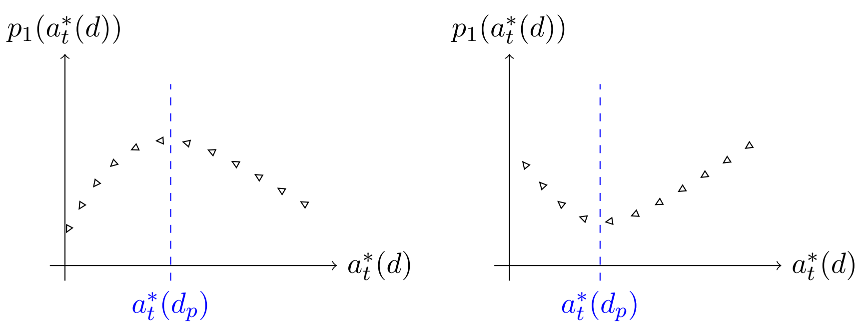
[26] For the derivation of $\beta(a^*)$ see Lemma A1 in the Appendix.

3.2 The probability of scoring across d

The following results characterize $\Delta_d p^*_{1,t}(d)$ for the given functional form.

Proposition 4. *Irrespective of the rules governing the endgame, if $f''' > 0$ ($f''' < 0$) then, given t, $p_1(a^*_t(d))$ is inverted-U shaped (U shaped) across d.*

Figure 2 describes the trajectory of the probability of scoring by player A as a function of his optimal action, given the time of play. The arrows identify the direction of the trajectory as d increases. Fix a given t. Let a_p denote the value where (15) holds with equality and let d_p be its projection on d so that a_p is identified by $a^*_t(d_p)$ in the figure. That is, at a_p a change in $a^*_t(d)$ is exactly compensated by an opposite change in $\beta(a_p)$ with the reaction given by $\beta'(a_p) = -\frac{\partial_a p_1}{\partial_b p_1}$ (for the functional form given in equations (16) and (17) this is equal to $-\frac{C_a}{C_b}$). If d increases from d_p to d_p+1 (since the arrows in the figure point to the left, this corresponds to moving leftward of a_p), by Proposition 1.1 player A's action decreases by an amount, say δ, to $a_p - \delta = a^*_t(d_p+1) < a^*_t(d_p) = a_p$ and player B's optimal action moves to $\beta(a^*_t(d_p+1)) > \beta(a^*_t(d_p))$. If $-\beta'(a_p) < -\beta'(a_p - \delta)$ then $p^*_{1,t}(d_p+1) < p^*_{1,t}(d_p)$ and hence $p^*_{1,t}(d)$ is decreasing (Figure 2(a)).[27] The trajectory is increasing otherwise (Figure 2(b)). Since $\beta'(a_p) - \beta'(a_p - \delta) \approx \delta\beta''(a_p)$, then $-\beta'(a_p) < -\beta'(a_p - \delta)$ if and only if $\beta''(a_p) > 0$ (holding for $f''' < 0$). The same argument applies for decreases in d (moving rightward to a_p). Since for the given functional form a_p is unique (Lemma A2), the behavior of $p_1(a^*_t(d))$ is monotonic thereafter.



(a) $f''' > 0$ (b) $f''' < 0$

Figure 2: Trajectory of the probability of scoring for increasing d, given t

[27] See Lemma A2.

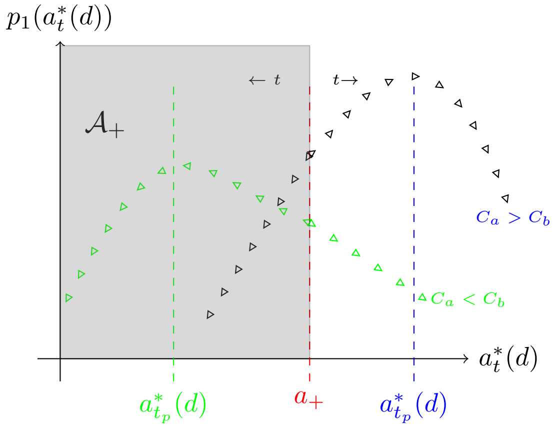
3.3 The probability of scoring over t

As discussed above, in known duration games, the probability of scoring is non-stationary. In this section, we show that for the functional forms in (16) and (17) the dynamics of the probability of scoring depends on the relative position of the two following values along with the shape of the function f. Similarly to the dynamics over d, fix now the state d. The first value is the turning point $a^*_{t_p}(d)$, given by the projection of a_p –the value where (15) holds with equality on t– where the function $p_1(a^*_t(d))$ is locally concave (convex) if $f''' > 0$ ($f''' < 0$). The second value, denoted by a_+, determines the position of the sets over which player A has a relative advantage in attacking and defending. For the given functional form, this is given by $a_+ = [f']^{-1}(C_a/2)$.[28] The point a_+ partitions player A's action space into two time and state independent connected subsets that can now be written as $\mathcal{A}^*_+$ and $\mathcal{A}^*_-$. The action value $a^*_t(d) \in \mathcal{A}^*_+$ if and only if $a^*_t(d) < a_+$. Moreover, the following proposition states that the relative position of $a^*_{t_p}(d)$ with respect to a_+ depends on the relative magnitude of C_a and C_b.

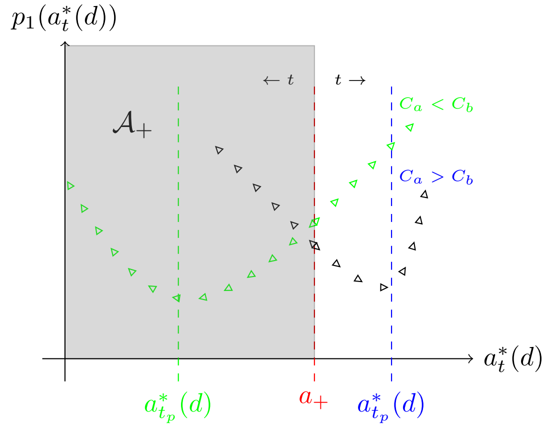
Proposition 5. *Suppose the duration of the game is known. 1. Let $f''' > 0$. Then, given d, $p_1(a^*_t(d))$ is inverted-U shaped over t on $\mathcal{A}^*_+$ if and only if $C_b > C_a$ or on $\mathcal{A}^*_-$ if and only if $C_b < C_a$. $p_1(a^*_t(d))$ is monotonically decreasing in the complementary sets.*
2. Let $f''' < 0$. Then, given d, $p_1(a^*_t(d))$ is U-shaped over t on $\mathcal{A}^*_+$ if and only if $C_b > C_a$ or on $\mathcal{A}^*_-$ if and only if $C_b < C_a$. $p_1(a^*_t(d))$ is monotonically increasing in the complementary sets.

Figure 3 shows the four qualitatively, non-degenerate[29] configurations of $p_1(a^*_t(d))$ that can occur according to Proposition 5. Graphs (a) and (b) present the cases for $f''' > 0$ and $f''' < 0$, respectively. The light/green and dark/blue arrows trace the dynamics corresponding to $C_b > C_a$ and $C_b < C_a$, respectively. The shaded area represents the set $\mathcal{A}_+$. The arrows to the left (right) of a_+ show the dynamics over time when the player has a relative advantage in attacking (defending).
Consider, for example, the light/green path in plot (a). This represents the case where $f''' > 0$ and $C_a < C_b$. Since $f''' > 0$, $p_1(a^*(d))$ is inverted-U shaped with a turning point at $a^*_{t_p}(\cdot)$. $C_a < C_b$ then implies that $a_p < a_+$, *i.e.,* a_p belongs to the set of equilibrium actions where the player has a relative advantage in attacking. The mechanism explaining the inverted-U shape of the dynamics over t is the same as for the trajectory over d represented in Figure 2. In this case, player A's

[28] See Lemma A3 in the appendix.
[29] The degenerate configurations are for $f''' = 0$ and $C_a = C_b$. The graphs are available from the authors.



(a) $f''' > 0$



(b) $f''' < 0$

Figure 3: Dynamics of the equilibrium probability given d.

optimal equilibrium action decreases over t if $a_t^*(d) \in [0; a_+]$. For $a_t^*(d) > a_+$, the optimal action increases overtime and the equilibrium probability $p_1(a_t^*(d))$ decreases with the action, resulting in a decreasing probability of scoring. Notice that since the optimal action moves always away from a_+ (*i.e.,* a_+ is a repeller), given d, the optimal action will never cross a_+ overtime and players will not switch relative advantage (unless there is a change in d). If, however, the difference in scoring increases, $a_t^*(d)$ moves faster away from a_+ if $a_t^*(d) \in \mathcal{A}_+^*$ and it is pushed back towards a_+ if $a_t^*(d) \in \mathcal{A}_-^*$. If the increase in d is sufficiently large, then $a_t^*(d)$ crosses a_+ into $\mathcal{A}_+^*$. So, for the case plotted in the green/light path of (a), changes in relative advantage for player A can occur only from defense to attack for sufficiently large increases in d and from attack to defense for sufficiently high decreases in d. The same logic applies to other paths. Finally, it is possible to compute $p_{-1}(a_t^*(d))$ in a similar way.

Notice that equation (12) applies, *mutatis mutandis,* to the dynamics of the probability of scoring across d's and hence the characterization of $\Delta_d \tilde{p}_{x,t}(d)$ is the same as $\Delta_d p_{x,t}^*(d)$. The dynamics over t is trivial as $\tilde{p}_{x,t}(d)$ is stationary.

3.4 Discussion

Figure 4 plots the casualties (outcomes) suffered by the allied forces between February 2008 and August 2013. The graph is constructed from the data available in Roggio and Ludquist (2012).

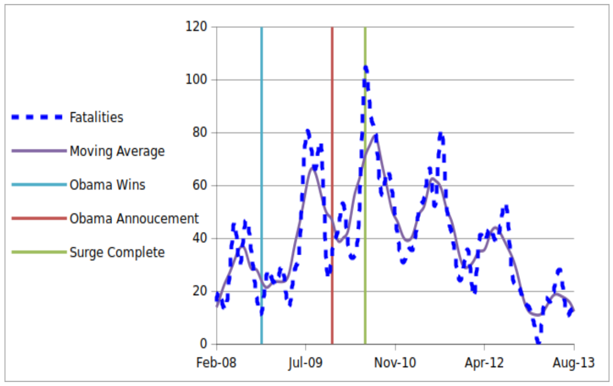


Figure 4: Casualties in the Afghanistan war

This period includes one year before and four years after the U.S. election and the withdrawal announcement. Interestingly, the trajectory of the fatalities is non-monotonic and specifically follows an inverted-U shape overtime. As noted before, Obama's comment on the need to break the Taliban's momentum suggests that it is reasonable to assume that the Taliban had an advantage in attacking in the period right after the announcements, and the U.S. army needed an increase in action, *i.e.*, an increase in troops. In that case, the pattern in Figure 4 is indeed consistent with the optimal strategy of a player with relative advantage in attacking as the game gets closer to an end.

Though suggestive, the war represents one point observation so it is difficult to justify any claim of its statistical relevance beyond a case study. In contrast, sport tournaments allow for a better representation of the probability of scoring in our model. The game of soccer is a case in point. Unlike the Afghanistan war which is a single sample observation, we have data on multiple games in various leagues and therefore we can estimate the probability of success. In the case of the first 90 minutes of each soccer match (the regular time), the game is of known duration as players know that the game will not end before the 90th minute. We collect our data set from the primary league matches starting with the 1995-1996 season and ending with the 2003-2004 season for England, Germany, Ireland, Italy, Scotland and Spain. For each match, we recorded the total number of goals scored and how far into the game each goal was scored.[30] We compute the probability of scoring as the average number of goals scored at time t over all matches, which is equivalent to $p_1(a^*(t))$ in the model. Figure 5 plots the value of the scoring probability over time. Once again the

[30]The data for the analysis were compiled from individual games box scores, largely obtained from Soccerbot.com, an online site reporting results and standings for soccer leagues around the world.

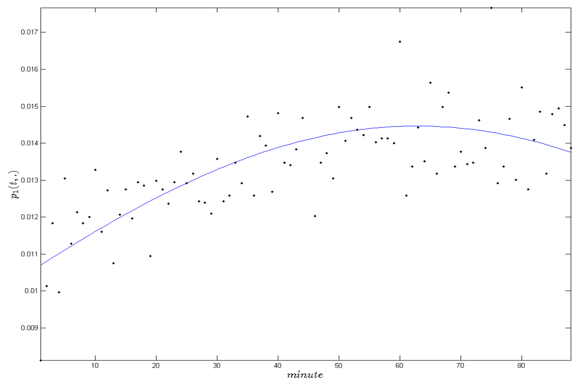


Figure 5: The probability of scoring in soccer

dynamics is non monotonic and with an inverted-U shape. Here as well, the behavior is consistent with an optimal strategy where the player with relative advantage in attacking starts with a high action and gradually decreases it as the game approaches the end of the game. Note that the player with relative advantage in attacking may change as players' relative position, in terms of differences in the number of points, changes over time.

4. Concluding remarks

The 2008 U.S. presidential campaign, where the setting of a withdrawal date from Afghanistan was central to the debate, provides evidence of policymakers' awareness of the potential implications of disclosing the duration of a conflict, both as a response to public opinion pressure and as a strategic commitment. A better understanding of these issues would also help in the management of international peace-keeping missions, especially when considering the optimal allocation of troops across multiple fronts.

Though the game we study is zero-sum the basic point of the model is not specific to this class of games. However, zero-sum games make the characterization of the dynamics much simpler and the results easier to interpret. Similarly for the payoff structure where the outcome is either -1, 0 or 1. We believe the latter not to be essential and the results to hold for more general, bounded and increasing functions of the difference in scoring. That case, however, would deliver only interior equilibria as there would be no absorbing states. Hence, the type of game *per se* does not seem to be crucial as long as the mechanism at the root of the results holds, *i.e.,* the different solution procedures adopted in order to identify the equilibrium: in the known duration case, one proceeds backward in solving the game; in the unknown duration game, the conditions imply the contraction mapping theorem must hold.

Although we believe zero-sum is not essential, what is essential is that the payoff of the game is realized at the end of the interaction so that there is no time discounting between stages of the game. For instance, our framework would not fit situations where players consume resources during the game and where there is a trade-off between present and future consumption.

Indeed, there are many strategic interactions where the final outcome is the one that matters. Beyond the example of armed conflicts, sport competitions are a case in point (with random *vs.* non-random stopping). In a companion paper, Banerjee, Markovich and Seccia (2017), we analyze the effect of the change in rule regarding the added time in soccer, introduced by FIFA in 1998. According to the new rule the referee was required to announce the length of the added time changing the endgame of the match from unknown to known duration. Another sports example is chess, where players play either in blitz competition or other timing rules. Businesses may face known and unknown duration games as well. For example, in research and development competitions when companies announce the launch of a product beforehand *vs.* an open ended outcome. The same holds when a company depends on the launch of a product by its supplier who can choose to announce or keep secret the launch date. Alternatively, investors at times face a limited or unlimited period of time before they need to show returns. For example, venture capital firms, at times, receive funds that are released in installments and thus need to show returns on investment before the next financial installment is released.

In our theoretical model, neither the game's duration nor the communication of the duration is part of the players' strategies, as both are taken to be exogenous. An interesting extension of the model would consider the case where agents can unilaterally fix the duration and then decide whether to release this information or keep it private.[31]

[31]Notice that having abstracted away from this case does not detract from the interest of the analysis as in many situations the duration of the game is not part of the players actions' set. For example, in the case of armed conflicts or peace missions, budgetary and political considerations often determine the length of the involvement, which is only then communicated to the actors on the field. In case of the UN peace missions The Fifth Committee (Administrative and Budgetary) sets the Peacekeeping Budget each year from July to June. However, the committee reviews and adjusts the budget throughout the year. Since peace missions vary in number and duration, contributions to the Peacekeeping Budget fluctuate widely from year to year" (Global Policy Forum (2014)).

References

Afghanistan Assessment Group (2012), International Security Assistance Force in Afghanistan Monthly Data - Trends through February 2012. http://www.isaf.nato.int/images/20120320_isaf_ monthly_data.pdf.

Amir, R. (1996), Continuous Stochastic Games of Capital Accumulation with Convex Transitions, *Games and Economic Behaviour,* 15, 111-131.

Aumann, R.J. and L.S. Shapley (1994), *Essays in Game Theory,* Springer New York.

Baker, P. (2009), How Obama came to plan for "surge" in Afghanistan, *New York Times,* 5/12/2009.

Banerjee, A.N., S. Markovich and G. Seccia (2017), The endgame in soccer: a natural experiment. Mimeo.

Cabral, L.M.B. (2003), R&D competition when firms choose variance, *Journal of Economics and Management Strategy*, 12(1), 139-150.

Chowdhury, S.M. and O. Gurtler (2015), Sabotage in contests: a survey, *Public Choice*, 164(1-2), 135-155.

Dal Bó, P. (2005), Cooperation under the shadow of the future: experimental evidence from infinitely repeated games, *American Economic Review*, 95(5), 1591-1604.

Eli, J., A. Frankel and E. Kamenica (2015), Suspense and surprise, *Journal of Political Economy*, 123(1), 215-260.

Engle-Warnick, J. and R. Slonim (2004), The evolution of strategies in a repeated trust game. *Journal of Economic Behavior and Organization,* 55(4), 553-573.

Fundeberg, D. and E. Maskin (1986), The folk theorem in repeated games with discounting and with incomplete information, *Econometrica*, 54(3), 533-554.

Garicano, L. and I. Palacios-Huerta (2014), Sabotage in tournaments: making the beautiful game a bit less beautiful, in Palacios-Huerta (2014).

González-Díaz, J. and I. Palacios-Huerta (2016), Cognitive performance in competitive environments: evidence from a natural experiment, *Journal of Public Economics,* 139, 40-52.

Global Policy Forum (2014), Tables and charts on UN Peacekeeping Operations Budget, http://www.global policy.org/un-finance/tables-and-charts-on-un-finance/the-un-peacekeeping-operations-budget.html.

Hvide, H.K. (2002), Tournament rewards and risk taking, *Journal of Labor Economics,* 20(4), 945-976.

Lasso de la Vega, C. and O. Volij (2018), The value of a draw, *Economic Theory,* 1-22.

Lazear, E.P. and S. Rosen (1981), Rank-order tournaments as optimum labor contracts, *Journal of Political Economy*, 89(5), 841-864.

Livingston, I.S. and M. O'Hanlon (2017), Afghanistan Index, https://www.brookings.edu/wp-content/uploads

/2016/07/21csi_20170525_afghanistan_index.pdf

Maskin, E. and J. Tirole (2001), Markov Perfect Equilibrium, I: Observable Actions, *Journal of Economic Theory*, 100, 191-219.

Nalebuff, B.J. and J.E. Stiglitz (1983), Prices and incentives: towards a general theory of compensation and competition, *Bell Journal of Economics*, 14(1), 21-43.

Palacios-Huerta, I. (2014), *Beautiful Game Theory,* Princeton University Press.

PBS (2009), Obama outlines plan for Afghanistan troop surge, https://www.pbs.org/newshour/nation/military-july-dec09-obamapeech_12-01.

Palomino, F.A., L. Rigotti and A. Rustichini (1998), Skill, strategy and passion: an empirical analysis of soccer. *CentER Discussion Paper*, 1998-129.

Roggio, B. and L. Lundquist (2012), Green-on-blue attacks in Afghanistan: the data, *The Long War Journal,* http://www.longwarjournal.org/archives/2012/08/green-on-blue_attack.php.

Rosen, J.B. (1965), Existence and uniqueness of equilibrium points for concave N-person games, *Econometrica*, 33(3), 520-534.

Rubinstein, A. (1979), Equilibrium in supergames with the overtaking criterion, *Journal of Economic Theory,* 21(1), 1-9.

Shapley, L.S. (1953), Stochastic Games, PNAS, 39, 1095-1100.

Spier, K.E. (1992), The Dynamics of pretrial negotiation, *The Review of Economic Studies,* 59(1) 93-108.

Stokey, L.N., R.E. Lucas and E.C. Prescott (1989), *Recursive Methods in Economic Dynamics*, Harvard University Press.

Sundaram, R.K. (1989), Perfect equilibrium in non-randomized strategies in a class of symmetric dynamic games, *Journal of Economic Theory*, 47, 153-177.

Walker, M. and J. Wooders (2001), Minimax Play at Wimbledon, *American Economic Review*, 91(5), 1521-1538.

Walker, M., J. Wooders and R. Amir (2011), Equilibrium play in matches: binary Markov games, *Games and Economic Behaviour,* 71, 487-502.

White House, Office of Press Secretary (2009), *Remarks by the president in address to the nation on the way forward in Afghanistan and Pakistan,* http://www.obamawhitehouse.archive.gov/the-press-office/remarks-president-address-nation-way-forward-afghanistan-and-pakistan.

Woodward, B. (2010), *Obama's Wars*, Simon & Schuster.

Yildiz, M. (2011), Bargaining with optimism, *Annual Review of Economics,* 3, 399-420.

Appendix

Proof of Lemma 1: We prove only parts 1 and 2 of the lemma as part 3 is obvious. We start by proving that a fix point to the problem in (3) exists and is unique at T and at any non-absorbing state. We then show that the argument can be applied backward.

At T, the non-absorbing states are only for $d = -1, 0, 1$. Given Assumption 1.3 and recalling that the game is zero-sum, computing the second derivatives of the payoff functions at T obtain:

$$\partial_a^2 U_T(a, b; d) = \partial_a^2 p_1(a, b)(V(d+1) - V(d)) - \partial_a^2 p_{-1}(a, b)(V(d) - V(d-1)) < 0; \tag{A1}$$

$$-\partial_b^2 U_T(a, b; d) = -\partial_b^2 p_1(a, b)(V(d+1) - V(d)) + \partial_b^2 p_{-1}(a, b)(V(d) - V(d-1)) < 0. \tag{A2}$$

Being the action space compact and the payoff functions continuous and concave, by Kakutani's fixed point theorem (as in Theorem 1 Rosen (1965)) a pure strategy equilibrium $(a_T^*(d), b_T^*(d))$ exists with associated value function $V_T(d)$ defined as in equation (5) for all d's.

To show uniqueness, we appeal to the sufficient condition of Theorem 6 in Rosen (1965). Consider the Jacobian matrix of the slope vectors of $U_T(a, b; d)$ and $-U_T(a, b; d)$ given by:

$$H_T(a, b; d) = \begin{pmatrix} \partial_a^2 U_T(a, b; d) & -\partial_{ab}^2 U_T(a, b; d) \\ \partial_{ba}^2 U_T(a, b; d) & -\partial_b^2 U_T(a, b; d) \end{pmatrix}.$$

For any vector $\mathbf{y} = (y_1, y_2)^T$, by Assumption 1.3 obtain:

$$\mathbf{y}^T [H_T(a, b; d) + H_T(a, b; d)^T] \mathbf{y} = 2[\partial_a^2 U_T(a, b; d) y_1^2 - \partial_b^2 U_T(a, b; , d) y_2^2] < 0$$

and hence the matrix $[H_T(a, b; d) + H_T(a, b; d)^T]$ is negative semi-definite satisfying the sufficient condition for the value $(a_T^*(d), b_T^*(d))$ to be the unique solution to the problem in (3) at T and $d = -1, 0, 1$.

In order to prove part 1, let $d = 0$ (the only state satisfying the condition $|d| < \overline{d}_T - 1$) so that $V(d+1) > V(d) > V(d-1)$. Recall that by Assumption 1.2, $\partial_a p_1(1, b) = \partial_a p_{-1}(0, b) = 0$ and hence obtain:

$$\partial_a U_T(0, b; 0) = \partial_a p_1(0, b)(V(1) - V(0)) - \partial_a p_{-1}(0, b)(V(0) - V(-1)) = \partial_a p_1(0, b) > 0;$$
$$\partial_a U_T(1, b; 0) = \partial_a p_1(1, b)(V(1) - V(0)) - \partial_a p_{-1}(1, b)(V(0) - V(-1)) = -\partial_a p_{-1}(1, b) < 0.$$

Also since $\partial_b U_T(a, 1; 0) > 0$ and $\partial_b U_T(a, 0; 0) < 0$, the unique solution $(a_T^*(0), b_T^*(0))$ such that $\partial_a U_T(a_T^*(0), b_T^*(0); 0) = \partial_b U_T(a_T^*(0), b_T^*(0); 0) = 0$ must belong to the interior of $Int(\mathbf{I} \times \mathbf{I})$.

In order to prove part 2, consider the cases where players are one point away from an absorbing state. For T these are $d = 1$ and $d = -1$. If $d = 1$ then $V(2) = V(1) > V(0)$ and $U_T(a, b; 1) = 1 - p_{-1}(a, b)$ and $-U_T(a, b; 1) = -1 + p_{-1}(a, b)$. Thus for any given b, the optimal action for player A is $a_T^*(1) = 0$ and, for any given a, the optimal action for player B is $b_T^*(1) = 1$. Similarly, for $d = -1$, obtain $(a_T^*(-1), b_T^*(-1)) = (1, 0)$.

Proceeding backward, given the solution at T, the payoff function in (4) at time $T - 1$ and $|d| \leq 2$ is given by:

$$U_{T-1}((a, \boldsymbol{a}_{\geq T}^*), (b, \boldsymbol{b}_{\geq T}^*); d) = \sum_{x=-1}^{1} p_x(a, b) V_T(d + x). \tag{A3}$$

The proof that a maxmin of (A3) exists and has the same properties described for T goes through in the same way, provided we check that the second derivatives are negative. The calculation is the same as for equations (A1) and (A2) and the inequalities hold provided $V_T(d)$ is increasing in d. To see that this is the case, consider the first order condition of the problem in (5):

$$V_T(d) - V_T(d - 1) = \frac{\partial_a p_1(a, b)}{\partial_a p_{-1}(a, b)} [V_T(d + 1) - V_T(d)]. \tag{A4}$$

By Assumption 1.1, $\partial_a p_x > 0$ for $x = -1, 1$. It follows that $V_T(d) - V_T(d-1)$ and $V_T(d+1) - V_T(d)$ must have the same sign and hence that $V_T(d)$ is a monotone function in d. Since at the absorbing states $V_T(-\bar{d}_T) = -1$ and $V_T(\bar{d}_T) = 1$, it follows that $V_T(d)$ must be increasing in d. This proves that a solution $(a_{T-1}^*(d), b_{T-1}^*(d))$ to the problem (3) at time $T - 1$ and state d for $|d| \leq 2$ exists and hence the corresponding value $V_{T-1}(d)$ also exists.

The same argument can now be applied to $T - 2$ and $|d| \leq 3$ and then to all t's and d's such $|d| \leq \bar{d}_t - 1$. □

Proof of Lemma 2: From the first order condition of (5) computing the derivative at $a^* = a_t^*(d)$

and $b^* = b_t^*(d)$, recalling that the game is zero-sum and using $p_0 = 1 - p_1 - p_{-1}$ obtain:

$$\partial_a p_1(a^*, b^*)[V_{t+1}(d+1) - V_{t+1}(d)] = \partial_a p_{-1}(a^*, b^*)[V_{t+1}(d) - V_{t+1}(d-1)]. \tag{A5}$$

Rearranging (A5) and multiplying both sides by $\frac{p^*_{-1,t}(d)}{p^*_{1,t}(d)}$ obtain equation (6). The elasticity $\epsilon_t^{B^*}(d)$ can be obtained by rearranging the first order condition of (5) with respect to b in a similar way. □

Proof of Lemma 3. Part 1 of the lemma follows from equation (A4) holding at each time t and state d and hence is a corollary of Lemma 1. Part 2 is proved in the main text. □

Proof of Proposition 1: We first show in a static setting that an exogenous increase of the value of the game reduces the equilibrium action of player A. We then appropriately reinterpret this result in our setting to derive the changes in the optimal action due to changes in d and t. Consider the following static problem:

$$u(a, b; \mathbf{V}) = \min_b \max_a \sum_{x=-1}^{1} p_x(a, b) V_x, \tag{A6}$$

where a and $b \in Int(\mathbf{I})$ are two actions chosen by two players as in our dynamic game, $\mathbf{V} = (V_x : x = -1, 0, 1)$ with $V_1 > V_0 > V_{-1}$ represents a vector of parameters and the function $p_x(a, b),\ x = -1, 0, 1$ satisfies Assumption 1. Letting $\alpha(b, \mathbf{V})$ and $\beta(a, \mathbf{V})$ denote the reaction functions of player A and B, respectively, the solutions to the problem (A6) are given by $a^*(\mathbf{V}) = \alpha\left(b^*(\mathbf{V}), \mathbf{V}\right)$ and $b^*(\mathbf{V}) = \beta\left(a^*(\mathbf{V}), \mathbf{V}\right)$. Implicitly differentiating with respect to V_0 obtain:

$$\frac{da^*(\mathbf{V})}{dV_0} = \partial_b \alpha\left(b, \mathbf{V}\right) \frac{db^*(\mathbf{V})}{dV_0} + \partial_{V_0} \alpha\left(b, \mathbf{V}\right),$$

$$\frac{db^*(\mathbf{V})}{dV_0} = \partial_a \beta\left(a, \mathbf{V}\right) \frac{da^*(\mathbf{V})}{dV_0} + \partial_{V_0} \beta\left(a, \mathbf{V}\right).$$

Solving for $\frac{da^*(\mathbf{V})}{dV_0}$ obtain:

$$\frac{da^*(\mathbf{V})}{dV_0} = \frac{\partial_b \alpha\left(b, \mathbf{V}\right) \partial_{V_0} \beta\left(a, \mathbf{V}\right) + \partial_{V_0} \alpha\left(b, \mathbf{V}\right)}{1 - \partial_b \alpha\left(b, \mathbf{V}\right) \partial_a \beta\left(a, \mathbf{V}\right)}. \tag{A7}$$

Let us first sign the denominator of (A7). From the first order conditions of (A6) with respect to

a, *i.e.*, $\partial_a u(a, b; \mathbf{V}) = 0$ obtain:

$$\partial_a p_1(a, b)(V_1 - V_0) = \partial_a p_{-1}(a, b)(V_0 - V_{-1}), \tag{A8}$$

substituting for the reaction function $\alpha(b, \mathbf{V})$ in (A8) obtain:

$$\partial_a p_1(\alpha(b, \mathbf{V}), b)(V_1 - V_0) = \partial_a p_{-1}(\alpha(b, \mathbf{V}), b)(V_0 - V_{-1}). \tag{A9}$$

Differentiating (A9) with respect to b obtain:

$$\begin{aligned}
&\partial_a^2 p_1(\alpha(b, \mathbf{V}), b)\partial_b \alpha(b, \mathbf{V}) + \partial_{ab} p_1(\alpha(b, \mathbf{V}), b)(V_1 - V_0) = \\
&\qquad \partial_a^2 p_{-1}(\alpha(b, \mathbf{V}), b)\partial_b \alpha(b, \mathbf{V}) + \partial_{ab} p_{-1}(\alpha(b, \mathbf{V}), b)(V_0 - V_{-1}),
\end{aligned}$$

and solving for $\partial_b \alpha(b, \mathbf{V})$ obtain:

$$\partial_b \alpha(b, \mathbf{V}) = -\frac{\partial_{ab}^2 p_1(\alpha(b, \mathbf{V}), b)(V_1 - V_0) - \partial_{ab}^2 p_{-1}(\alpha(b, \mathbf{V}), b)(V_0 - V_{-1})}{\partial_a^2 p_1(\alpha(b, \mathbf{V}), b)(V_1 - V_0) - \partial_a^2 p_{-1}(\alpha(b, \mathbf{V}), b)(V_0 - V_{-1})} = -\frac{\partial_{ab}^2 u(a, b; \mathbf{V})}{\partial_a^2 u(a, b; \mathbf{V})}. \tag{A10}$$

Similarly, from the first order condition of (A6) with respect to b, *i.e.*, $\partial_b u(a, b; \mathbf{V}) = 0$, obtain:

$$\partial_b p_1(a, b)(V_1 - V_0) = \partial_b p_{-1}(a, b)(V_0 - V_{-1}), \tag{A11}$$

and using the reaction function $\beta(a, \mathbf{V})$ obtain:

$$\partial_b p_1(a, \beta(a, \mathbf{V}))(V_1 - V_0) = \partial_b p_{-1}(a, \beta(a, \mathbf{V}))(V_0 - V_{-1}). \tag{A12}$$

Differentiating (A12) with respect to a we obtain:

$$\begin{aligned}
\partial_a \beta(a, \mathbf{V}) &= -\frac{\partial_{ab}^2 p_1(a, \beta(a, \mathbf{V}))(V_1 - V_0) - \partial_{ab}^2 p_{-1}(a, \beta(a, \mathbf{V}))(V_0 - V_{-1})}{\partial_b^2 p_1(a, \beta(a, \mathbf{V}))(V_1 - V_0) - \partial_b^2 p_{-1}(a, \beta(a, \mathbf{V}))(V_0 - V_{-1})} \\
&= -\frac{\partial_{ab}^2 u(a, \beta(a, \mathbf{V}))}{\partial_b^2 u(a, \beta(a, \mathbf{V}))}.
\end{aligned} \tag{A13}$$

Since $\partial_b \alpha(b, \mathbf{V})$ and $\partial_a \beta(a, \mathbf{V})$ computed in (A10) and (A13) have opposite signs, *i.e:* $\partial_b \alpha(b, \mathbf{V})\partial_a \beta(a, \mathbf{V}) < 0$, it follows that the denominator of (A7) must be positive.

In order to sign the numerator of (A7), differentiate (A9) and (A12) with respect to V_0 and solving for $\partial_{V_0}\alpha(b,\mathbf{V})$ and $\partial_{V_0}\beta(a,\mathbf{V})$ respectively obtain:

$$\partial_{V_0}\alpha(b,\mathbf{V}) = \frac{\partial_a p_1(\alpha(b,\mathbf{V}),b) + \partial_a p_{-1}(\alpha(b,\mathbf{V}),b)}{\partial_a^2 u(\alpha(b,\mathbf{V}),b)} < 0, \tag{A14}$$

and

$$\partial_{V_0}\beta(a,\mathbf{V}) = \frac{\partial_b p_1(a,\beta(a,\mathbf{V})) + \partial_b p_{-1}(a,\beta(a,\mathbf{V}))}{\partial_b^2 u(a,\beta(a,\mathbf{V}))} > 0, \tag{A15}$$

where the signs follow from Assumption 1.1 and 1.3.

Now plugging $\alpha(b,\mathbf{V})$ in (A11) and differentiating with respect to V_0 obtain:

$$\begin{aligned}&\partial_{ab}^2 p_1(\alpha(b,\mathbf{V}),b)\partial_{V_0}\alpha(b,\mathbf{V})(V_1 - V_0) - \partial_b p_1(\alpha(b,\mathbf{V}),b)\\ &\qquad = \partial_{ab}^2 p_{-1}(\alpha(b,\mathbf{V}),b)\partial_{V_0}\alpha(b,\mathbf{V})(V_0 - V_{-1}) + \partial_b p_{-1}(\alpha(b,\mathbf{V}),b).\end{aligned}$$

Using the fact that $\partial_{ab}^2 u(a,b;\mathbf{V}) = \partial_{ab}^2 p_1(\alpha(b,\mathbf{V}),b)(V_1 - V_0) - \partial_{ab}^2 p_{-1}(\alpha(b,\mathbf{V}),b)(V_0 - V_1)$, obtain:

$$\partial_{ab}^2 u(a,b;\mathbf{V})\ \partial_{V_0}\alpha(b,\mathbf{V}) = \partial_b p_1(\alpha(b,\mathbf{V}),b) + \partial_b p_{-1}(\alpha(b,\mathbf{V}),b).$$

Using (A14) obtain:

$$\frac{\partial_{ab}^2 u(a,b;\mathbf{V})}{\partial_a^2 u(a,b;\mathbf{V})} = \frac{\partial_b p_1(\alpha(b,\mathbf{V}),b) + \partial_b p_{-1}(\alpha(b,\mathbf{V}),b)}{\partial_a p_1(\alpha(b,\mathbf{V}),b) + \partial_a p_{-1}(\alpha(b,\mathbf{V}),b)}.$$

Multiplying both sides by $\partial_{V_0}\beta\,(a,\mathbf{V})$ in (A15) obtain:

$$-\frac{\partial_{ab}^2 u(a,b;\mathbf{V})}{\partial_b^2 u(a,b;\mathbf{V})}\partial_{V_0}\beta\,(a,\mathbf{V}) = -\frac{\left(\partial_b p_1(\alpha(b,\mathbf{V}),b) + \partial_b p_{-1}(\alpha(b,\mathbf{V}),b)\right)^2}{\left(\partial_a p_1(\alpha(b,\mathbf{V}),b) + \partial_a p_{-1}(\alpha(b,\mathbf{V}),b)\right)\partial_b^2 u(a,b;\mathbf{V})}. \tag{A16}$$

Using (A10) finally obtain: $\partial_b\alpha\,(b,\mathbf{V})\,\partial_{V_0}\beta\,(a,\mathbf{V}) < 0$. Therefore, the numerator of (A7) is negative and hence $\frac{da^*(\mathbf{V})}{dV_0}$ is negative. The proof of $\frac{db^*(\mathbf{V})}{dV_0} > 0$ is similar.

The formulation of the static problem above is convenient as now we can prove the two parts of the proposition by appropriately reinterpreting a^* and b^* as $a_t^*(d)$ and $b_t^*(d)$, respectively, and V_0 as $V_{t+1}(d)$ in problem (5).

Part 1 of the proposition follows from Lemma 3.1 stating that the value of the game $V_t(d)$ is increasing in the differences in points d.

Part 2 of the proposition follows from Lemma 3.2, stating that if $a^* \in \mathcal{A}_{+,t}(d)$ then the value of the game is increasing in t, *i.e.,* $V_{t+1}(d) > V_t(d)$. Hence, if player A has an advantage in attacking, his optimal actions will decrease in t.

Being a zero-sum game, the behaviour of player B's optimal action is symmetric. □

Proof of Proposition 2: *Part 1*: The proof proceeds in three steps. Step 1 proves that player A's best response to any given behavior strategy of player B is stationary, and vice versa. The standard dynamic programming arguments can be applied in this case as well, though the setup here is different as the payoff is realized only when the game ends as opposed to being the sum of discounted payoffs at every t. Step 2 proves that, given B's strategy, the value of the game is strictly increasing in d. Similarly, holding fixed player A's strategy, the value of the game is strictly decreasing in d. This implies that player A's payoff function is concave and player B's is convex. Finally, Step 3 appealing again to Rosen (1965) proves that a pure strategy equilibrium exists (Theorem 1) and is unique (Theorem 6).

Step 1: *Player A's best response to any strategy of player B is stationary, and vice versa.*

Let $\mathbf{\Sigma}_\infty$ be the set of all (behavior) strategies $\boldsymbol{\sigma}$ as defined in Section 2.2. Also let $\Upsilon = \{w : \mathbf{\Sigma}_\infty \times \mathbb{Z} \to (-1,1)\}$ be the Banach space of the functions $w(\boldsymbol{\sigma}; d)$ that are bounded and continuous in $\boldsymbol{\sigma}$ with respect to $\|\cdot\|_\infty$ and are endowed with the uniform norm $\|\cdot\|_\infty$.

Fix any (not necessarily Markov) behavior strategy for player B, $\overline{\boldsymbol{\sigma}}^B \in \mathbf{\Sigma}_\infty$ and consider player A's optimization problem $W_t^A(\overline{\boldsymbol{\sigma}}^B; d) = \sup_{\boldsymbol{\sigma}^A_{\geq t}} \mathbb{E}_{(\boldsymbol{\sigma}^A_{\geq t}, \overline{\boldsymbol{\sigma}}^B_{\geq t})} \widetilde{U}_t(\boldsymbol{a}_{\geq t}, \boldsymbol{b}_{\geq t}; d)$. Writing recursively obtain:

$$W_t^A(\overline{\boldsymbol{\sigma}}^B; d) = (1-\pi)V(d) + \pi \sup_{\sigma_t^A \in \mathscr{P}(\mathbf{I})} \int \sum_{x=-1}^{1} p_x(a,b) W_{t+1}^A(\overline{\boldsymbol{\sigma}}^B; d+x) d(\sigma_t^A \otimes \overline{\sigma}_t^B). \tag{A17}$$

By the Theorem of the Maximum, $W_t^A(\overline{\boldsymbol{\sigma}}^B; d)$ is bounded and continuous in $\overline{\boldsymbol{\sigma}}^B$ with respect to the uniform norm $\|\cdot\|_\infty$ and hence $W_t^A(\overline{\boldsymbol{\sigma}}^B; d) \in \Upsilon$. Define the functional Φ on Υ such that:

$$\Phi\left(W_{t+1}^A(\overline{\boldsymbol{\sigma}}^B; d)\right) = (1-\pi)V(d) + \pi \sup_{\sigma_t^A \in \mathscr{P}(\mathbf{I})} \int \sum_{x=-1}^{1} p_x(a,b) W_{t+1}^A(\overline{\boldsymbol{\sigma}}^B; d+x) d(\sigma_t^A \otimes \overline{\sigma}_t^B).$$

Then,

$$\left\|\Phi\left(W_{t+1}^A(\overline{\boldsymbol{\sigma}}^B; d)\right) - \Phi\left(W_{t+2}^A(\overline{\boldsymbol{\sigma}}^B; d)\right)\right\|_\infty \leq \pi \left\|W_{t+1}^A(\overline{\boldsymbol{\sigma}}^B; d) - W_{t+2}^A(\overline{\boldsymbol{\sigma}}^B; d)\right\|_\infty.$$

Writing recursively,

$$\left\| \Phi\left(W^A_{t+1}(\overline{\boldsymbol{\sigma}}^B; d)\right) - \Phi\left(W^A_{t+k}(\overline{\boldsymbol{\sigma}}^B; d)\right)\right\|_\infty \leq \pi^k \left\| W^A_{t+1}(\overline{\boldsymbol{\sigma}}^B; d) - W^A_{t+k}(\overline{\boldsymbol{\sigma}}^B; d)\right\|_\infty, \tag{A18}$$

is a convergent Cauchy sequence. Being $0 < \pi < 1$ this converges to a fixed point such that[32]

$$W^A(\overline{\boldsymbol{\sigma}}^B; d) = (1-\pi)V(d) + \pi \sup_{\sigma^A \in \mathscr{P}(\mathbf{I})} \int \sum_{x=-1}^{1} p_x(a,b) W^A(\overline{\boldsymbol{\sigma}}^B; d+x) d\left(\sigma^A \otimes \overline{\sigma}^B_t\right). \tag{A19}$$

Fixing $\overline{\boldsymbol{\sigma}}^A \in \mathbf{\Sigma}_\infty$ a similar argument shows that the value function $W^B(\overline{\boldsymbol{\sigma}}^A, d)$ and player B's best response are also stationary. Notice also that a stationary Markov strategy is optimal among all possible strategies, including those that depend on the past history. That is, each player cannot do any better by using strategies that are not Markov.[33]

Step 2: Player A's (B's) payoff function is concave (convex).

From Step 1, at state d player A's payoff function at any t for a given $\overline{\boldsymbol{\sigma}}^B$ can be written as:

$$(1-\pi)V(d) + \pi \sum_{x=-1}^{1} p_x(a,b) W^A(\overline{\boldsymbol{\sigma}}^B; d+x). \tag{A20}$$

The first order condition of the maximization of (A20) is given by:

$$W^A(\overline{\boldsymbol{\sigma}}^B; d+1) - W^A(\overline{\boldsymbol{\sigma}}^B; d) = \frac{\partial_a p_1(a,b)}{\partial_a p_{-1}(a,b)}[W^A(\overline{\boldsymbol{\sigma}}^B; d) - W^A(\overline{\boldsymbol{\sigma}}^B; d-1)].$$

By Assumption 1.1, it follows that $W^A(\overline{\boldsymbol{\sigma}}^B; d)$ is monotonic in d. Moreover, since $\lim_{d\to-\infty} W^A(\overline{\boldsymbol{\sigma}}^B; d) < 0$ and $\lim_{d\to+\infty} W^A(\overline{\boldsymbol{\sigma}}^B; d) > 0$, $W^A(\overline{\boldsymbol{\sigma}}^B; d)$ is strictly increasing in d. Under Assumption 1.3, the second derivative with respect to player A's action is given by:

$$\partial^2_a p_1(a,b)[W^A(\overline{\boldsymbol{\sigma}}^B; d+1) - W^A(\overline{\boldsymbol{\sigma}}^B; d)] - \partial^2_a p_{-1}(a,b)[W^A(\overline{\boldsymbol{\sigma}}^B; d) - W^A(\overline{\boldsymbol{\sigma}}^B; d-1)] < 0, \tag{A21}$$

proving that the payoff function of player A is concave.

A similar argument shows that the payoff function of player B is convex.

Step 3: There exists a unique, stationary Markov Perfect Equilibrium. The solution lies in the

[32] See Theorems 4.2, 4.3, Stokey, Lucas and Prescott (1989).

[33] This has been observed by Amir (1992) p. 116 and Sundaram (1989) p. 161 among others.

interior for all d's.

The action space is compact and the payoff functions are continuous and concave. By Kakutani's fixed point theorem (as in Theorem 1 of Rosen (1965)) it follows that for all d's there is a pure strategy equilibrium $(\tilde{a}(d), \tilde{b}(d))$ with associated value function $W(d)$.

Note that, since for all d's $-1 < W(d) < 1$, the equilibrium $(\tilde{a}(d), \tilde{b}(d))$ is in the interior (the proof is similar to the one for Lemma 1). As in case of Lemma 1, it is possible to verify that the sufficient conditions of Theorem 6 in Rosen (1965) apply and hence that the equilibrium is unique.

Part 2: Proceeding along the lines of the proof of Proposition 1 it is easy to show that $\frac{d\tilde{a}(d)}{dW(d)} < 0$ and $\frac{d\tilde{b}(d)}{dW(d)} > 0$. Since $W(d)$ is increasing in d the result follows. □

Lemma A1. *Player B's equilibrium action is given by* $\beta(a^*) = [f'^{-1}\left[C_b[1 - C_a^{-1} f^{'}(a^*)]\right]$.

Proof of Lemma A1: From the equilibrium condition on the relative elasticities it is easy to show that $\epsilon^{A*} = [\epsilon^{B*}]^{-1}$. For the functional forms specified in (16) and (17) obtain:

$$\frac{C_a - f^{'}(a^*)}{f^{'}(a^*)} = \frac{f^{'}(\beta(a^*))}{C_b - f^{'}(\beta(a^*))} \;\Rightarrow\; f^{'}(\beta(a^*)) = \frac{C_b}{C_a}\left[C_a - f^{'}(a^*))\right], \tag{A22}$$

Lemma A2. *There is a unique turning point* a_p *such that* $p_1(a^*)$ *is locally concave (convex) at* a_p *if and only if* $f''' > 0$ *(*$f''' < 0$*).*

Proof of Lemma A2: Changes in $p_1(a^*, \beta(a^*))$ with respect to action a^* are obtained by solving:

$$\frac{dp_1(a^*)}{da^*} = \partial_a p_1(a^*, \beta(a^*)) + \partial_b p_1(a^*, \beta(a^*))\beta^{'}(a^*),$$

where $\partial_a p_1(a, b) = (C_a - f'(a))p_1(a, b) > 0$ and $\partial_b p_1(a, b) = f'(b)p_1(a, b) > 0$. Imposing the equilibrium condition in (A22) it is straightforward to show that:

$$\frac{dp_1(a^*)}{da^*} = \left(C_a - f^{'}(a^*)\right)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)p_1(a^*),$$

where $p_1(a^*, \beta(a^*)) = p_1(a^*)$. Note that there exists a turning point a_p of $p_1(a_p)$ such that $\frac{dp_1(a^*)}{da^*} = 0$ if and only if $\beta^{'}(a^*) = -\frac{C_a}{C_b}$. The first derivative of $\beta(a^*)$ can also be computed from (A22) obtaining: $\beta^{'}(a^*) = -\frac{C_b}{C_a}\frac{f^{''}(a^*)}{f^{''}(\beta(a^*))}$. Equating the two values gives:

$$[\frac{C_b}{C_a}]^2 \frac{f''(a_p)}{f''(\beta(a_p))} = 1, \tag{A23}$$

where $\beta(a_p)$ can be obtained by Lemma A1. Since $f^{''}$ is strictly monotonic the turning point a_p of $p_1(a_p)$ is unique.

Let us now show that the function $p_1(a^*)$ is locally concave at a_p. Taking the second derivative of $p_1(a^*) = p_1(a^*, \beta(a^*))$ obtain:

$$
\begin{aligned}
\frac{d^2 p_1(a^*)}{d^2 a^*} &= \frac{d^2 p_1(a^*, \beta(a^*))}{d^2 a^*} = d\left[\left(C_a - f^{'}(a^*)\right)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)p_1(a^*)\right] \\
&= -f^{''}(a^*)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)p_1(a^*) + \frac{C_b}{C_a}\beta^{''}(a^*)\left(C_a - f^{'}(a^*)\right)p_1(a^*) \\
&+ \left(C_a - f^{'}(a^*)\right)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)\frac{dp_1(a^*)}{da^*} \\
&= \left[-f^{''}(a^*)\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right) + \frac{C_b}{C_a}\beta^{''}(a^*)\left(C_a - f^{'}(a^*)\right)\right]p_1(a^*) \\
&\qquad + \left(C_a - f^{'}(a^*)\right)^2\left(1 + \frac{C_b}{C_a}\beta^{'}(a^*)\right)^2 p_1(a^*).
\end{aligned}
$$

By equation (A23) at $a^* = a_p$ it follows that: $d^2 p_1(a_p) = \beta^{''}(a_p)\left(C_a - f^{'}(a_p)\right)p_1^*(a_p)$. The latter is negative if and only if $\beta^{''}(a_p) < 0$. Computing $\beta^{''}(a_p)$ obtain:

$$
\beta^{''}(a_p) = -\frac{C_b}{C_a}\frac{f^{'''}(a_p)f^{''}(\beta(a_p)) - \beta^{'}(a_p)f^{'''}(\beta(a_p))f^{''}(a_p)}{\left[f^{''}(\beta(a_p))\right]^2}.
$$

Then $\beta^{''}(a_p) < 0$ if and only $f^{'''} > 0$ that is the necessary and sufficient condition for the local concavity of the equilibrium $p_1(a^*)$ at a_p. $\square$

Proof of Proposition 4: For both types of games, the point d_p is the projection of a_p on d. Given t this can be computed by using (12). $\square$

Lemma A3. *There exists a unique action $a_+ = [f^{'}]^{-1}(C_a/2)$, such that $\mathcal{A}_+(b^*) = \{a : a < a_+\}$.*

Proof of Lemma A3: From (16) and (17) obtain: $\epsilon^A(a, b^*) = \frac{C_a - f^{'}(a)}{f^{'}(a)}$. Therefore, $\mathcal{A}_+(b^*) = \left\{a : \epsilon^A(a, b^*) > 1\right\} = \{a : \frac{C_a - f^{'}(a)}{f^{'}(a)} > 1\}$. Note that a_+ is unique since $\frac{\partial \epsilon^A(a,b^*)}{\partial a} = -\frac{C_a f^{''}(a)}{[f^{'}(a)]^2} < 0$ for all a's. $\square$

Proof of Proposition 5: We first show that $a_p \in \mathcal{A}_+^*$ if and only if $C_b > C_a$. Notice that, at equilibrium, from (A23) $\left[\frac{C_b}{C_a}\right]^2 f^{''}(a_p) = f^{''}(\beta(a_p))$. If $\frac{C_b}{C_a} < 1$ then $f^{''}(a_p) > f^{''}(\beta(a_p))$ and since $f^{''} > 0$ it follows that $a_p > \beta(a_p)$.

Using (A22) obtain: $f^{'}(\beta(a_p)) - f^{'}(a_p) = \frac{C_b}{C_a}\left[C_a - f^{'}(a_p)\right] - f^{'}(a_p)$. Being $f^{''} > 0$:

$$
\begin{aligned}
0 &< f^{'}\left(\beta\left(a_p\right)\right) - f^{'}\left(a_p\right) = \frac{C_b}{C_a}\left[C_a - f^{'}\left(a_p\right)\right] - f^{'}\left(a_p\right) \\
0 &< \frac{C_b}{C_a}\left[C_a - f^{'}\left(a_p\right)\right] - \frac{C_b}{C_a} f^{'}\left(a_p\right) < \frac{C_b}{C_a}[C_a - 2f^{'}(a_p)] < [C_a - 2f^{'}(a_p)]
\end{aligned}
$$

implying that $f^{'}(a_p) < \frac{C_a}{2} = f^{'}(a_+)$. Thus $a_+ > a_p$ and hence $a_p \in \mathcal{A}_+^*$. From Lemma A2, if $f^{'''} > 0$ then $\frac{dp_1(a^*)}{da^*} < 0$ for $a < a_p$ $\left(\frac{dp_1(a^*)}{da^*}\right) > 0$ for $a > a_p$. Since $\Delta_t a_t(d) > 0$ for $a \in \mathcal{A}_+^*$ it follows that $\Delta_t p_{1,t}(a_t^*(d)) < 0$ if $a_t^*(d) < a_p = a_{t_p}^*(d)$ (and $\Delta p_{1,t}(a^*(d)) > 0$ if $a_t^*(d) > a_p = a_{t_p}^*(d)$), where t_p is a projection of a_p given d that can be computed using (13).

On the complementary set $\mathcal{A}_-^*$ the function is monotonically decreasing overtime as $\frac{dp_1(a^*)}{da} > 0$ but $\Delta_t a_t(d) < 0$. A similar argument proves that if $C_b < C_a$ then $a_p \in \mathcal{A}_-^*$. $\square$