Abstract

This paper studies the optimal dynamic provision of incentives in employment relationships with rents for the worker. In a model of relational contracts with limited liability, we show that the optimal relational contract generates definitive and joint implications on how job security, pay level, and the sensitivity of pay to performance change over time as the employment relationship progresses. We also show that for employment relationships with low surplus, the lack of commitment to long-term contract leads to both lower and more-volatile wages and productivity levels within firms and larger dispersion in wages and productivity across firms.

JEL Classifications: C61, C73, J33, L24

Keywords: Relational Contracts, Limited Liability, Efficiency Wages

* A previous version of this paper has circulated under the title "Relational Contracts, Limited Liability, and Employment Dynamics." We thank the Editor, Bruno Biais, and three anonymous referees for very detailed and helpful comments. We thank Guy Arie and Oscar Contreras for excellent research assistance. We especially thank Marina Halac for detailed comments. We also thank Ricardo Alonso, Attila Ambrus, Dan Bernhardt, Odilon Camara, George Deltas, Willie Fuchs, Bob Gibbons, Michihiro Kandori, Bentley MacLeod, Tony Marino, Niko Matouschek, John Matsusaka, Thomas Marriotti, Arijit Mukherjee, Kevin Murphy, Marco Ottaviani, Mike Powell, Peter Schnabl, Michael Song, Jano Zabojnik and seminar participants at McGill, Michigan State, NUS, Northwestern, University of Delaware, University of Hong Kong, UIUC, USC, the 7th International Industrial Organization Conference, the Society of Labor Economists Annual Meeting 2009, the North American Summer Meeting of the Econometric Society 2009, the 8th annual CSIO-IDEI joint workshop on Industrial Organization, and the 2009 International Symposium on Contemporary Labor Economics for helpful comments and discussions. All remaining errors are ours.
1 Introduction

A prominent line of research has emphasized the importance and consequences of moral hazard within firms; see Gibbons and Waldman (1999), Hart and Holmstrom (1987), and Prendergast (1999) for reviews. In employment relationships, the moral hazard problem has three salient features. First, employment relationships are typically ongoing, and many last a “lifetime.”\(^1\) The repeated interaction between the firm and the worker implies that the moral hazard problem has a dynamic aspect. Second, legal and institutional constraints restrict the degree to which a worker can be punished. In other words, workers are protected by a limited liability constraint.\(^2\) Finally, the costs of and difficulties in specifying and verifying performance often render formal contracts infeasible. Instead, the firms rely on "relational contracting" whereby workers are rewarded or punished at the firm’s discretion.

This paper develops a moral hazard model that reflects these three features. The first purpose of the paper is to explore the consequence of optimal relational contracts with limited liability on employment dynamics. Our model provides definitive and joint implications on how job security, pay level, and the sensitivity of pay to performance change over time as the employment relationship progresses. These patterns reveal the economic mechanism behind the firms’ ability to optimally extract rents from workers in a dynamic setting while continuing to offer work incentives.

The second purpose of this paper is to study how employment and productivity dynamics are affected by the firm’s ability to commit to a long-term contract. We show that when the surplus in the relationship is small, the firm’s lack of ability to commit implies that the relationship is more likely to be terminated. For some parameter ranges, the relationship survives with probability 1 under the optimal long-term contract yet it is terminated with probability 1 under the optimal long-term contract. In terms of long-run outcomes, we show that firms under relational contracts have lower and more-volatile productivity, and their workers also have lower and more-volatile wages. Finally, we show that lack of commitment leads to larger productivity and wage dispersion for ex ante identical relationships.

Specifically, we study a model of relational contracts with imperfect monitoring, which is an infinitely repeated principal-agent model where output is publicly observable but not contractible. The agent privately chooses to work or shirk, and by working the agent increases the

\(^1\)Hall (1982) finds that the average tenure of U.S. workers is eight years and that a quarter of the workforce holds what he calls "lifetime jobs," which are jobs lasting longer than 20 years. Beaudry and DiNardo (1991) also show that the labor market is better described by firms offering long-term contracts than by spot markets.

\(^2\)We use "limited liability" to describe any restriction that sets a lower bound on the worker’s pay. One example is minimum wage. Another example is the court’s unwillingness to enforce "liquidated damage" in contracts, i.e., provisions that call for larger penalties than the damage done; see Dickens et al. (1990) for a discussion. The limited liability constraint has been recognized as an important factor affecting incentives: formal models include Sappington (1983), Innes (1990), and Jewitt, Kadan, and Swinkels (2008).
probability of that the output being high. Unlike early models of dynamic moral hazard such as Lazear (1979), Shapiro-Stiglitz (1984), and Akerlof and Katz (1989), the output may be low even if the worker puts in effort. In addition, we model the worker’s limited liability constraint by requiring that the agent’s pay each period not fall below an exogenously given wage floor. This assumption sets us apart from the standard relational contracting model with imperfect public monitoring such as that of Levin (2003).

We follow Levin (2003) to define each relational contract as a Perfect Public Equilibrium (PPE) of the model and the optimal relational contract as the PPE that maximizes the principal’s payoff. We completely characterize the set of PPE payoffs using the method developed by Abreu, Pearce, Stacchetti (1990) and derive the implications of the optimal relational contract on employment dynamics. We also solve for the model when long-term contracts are feasible, and we compare the difference between the optimal long-term contract and the effect of the optimal relational contract on employment dynamics.

Our results show that limited liability drastically changes the nature and structure of the optimal relational contract. Without limited liability, Levin (2003) implies that the optimal relational contract in our setting is efficient and stationary, and that the firm can extract all of the surplus in the relationship. When limited liability is present, rents must be given to the worker to induce effort. In designing the optimal dynamic incentive structure, the firm faces a trade-off between surplus maximization and rent extraction. The optimal relational contract is no longer efficient nor is it stationary. But the structure of the optimal relational contract is surprisingly simple.

In particular, the employment relationship begins with a “probation phase,” during which the worker puts in effort and receives a constant wage equaling the wage floor. If the output history has been sufficiently favorable, the worker transitions into the “bonus phase,” during which he is incentivized with pay for performance. When the surplus in the relationship is sufficiently high, the optimal relational contract in the bonus phase can be implemented by a sequence of quasi-stationary contracts similar to those in Levin (2003): essentially, the worker receives a fixed bonus amount for each high output. If the output history has been sufficiently unfavorable, inefficiency occurs. Depending on the level of the wage floor, there are two types of inefficiency. If the wage floor is low, the inefficiency takes the form of temporary suspension of production: there are periods in which the worker is paid the wage floor but does not put in effort. If the wage floor is high, the inefficiency takes the form of permanent employee termination.

In the high wage floor case, our model gives joint predictions on turnover and pay dynamics.

---

3 For a definitive treatment of relational contracts with observable actions, see MacLeod and Malcomson (1988).
4 These properties also hold for the optimal long-term contract.
In terms of turnover, the model predicts that the turnover rate is initially increasing with respect to time on the job and is eventually decreasing to zero. In particular, the model can generate an inverse-U-shaped turnover rate as found in data; see, for example, Farber (1994). In terms of pay level, the model predicts that it is deferred in that there is a discrete jump in the average pay level after the agent transitions from the probation phase into the bonus phase. Moreover, this increase is associated with a discrete jump in the sensitivity of pay to performance.

This paper is mainly related to three strands of the literature. First, the employment dynamics in our model reflect features of optimal rent extraction under a dynamic setting. Since the worker has ex ante rents in the relationship, our model belongs to the class of efficiency wage models. Our model adds to classic efficiency wage models such as Shapiro and Stiglitz (1984) by featuring a) a stochastic production (on the equilibrium) and b) an explicit modelling of the limited liability constraint. The combination of these two features helps generate equilibrium turnover and allows us to make joint predictions on pay and turnover dynamics?

Second, our paper is related to a recent, vibrant literature on dynamic contracts with limited liability. In finance, in particular, several important papers have shown that many features of modern financial contracts can be understood through the lens of a dynamic principal-agent model in which the agent has private information about the cash flow and has a limited liability constraint; see, for example, Biais, Mariotti, Plantin, and Rochet (2007), DeMarzo and Fishman (2007), and DeMarzo and Sannikov (2006), and see Biais, Mariotti, and Rochet (2011) and Sannikov (forthcoming) for surveys. In addition to appearing in finance, the long-term contracting view has also been used by Clementi and Hopenhayn (2006) to study industrial organization and firm dynamics, by Myerson (2008) to study political economy, and by Lewis (2009) and Lewis and Ottaviani (2008) to study search behavior. Our paper can be considered as applying the optimal contracting view to labor economics, where we focus on how pay and turnover rates change over time.

Within this literature, two closely related papers are Biais, Mariotti, Plantin and Rochet (2007) (BMPR hereafter) and Zhu (forthcoming). Both papers study dynamic principal-agent models in which a risk-neutral agent is protected by limited liability. In BMPR, the optimal long-term contract generates dynamics very similar to ours: the relationship starts in a transition phase in which the agent receives the lowest possible wage and the bonus is paid out only when past performances are sufficiently good. Different from our model, the principal is more patient than the agent in BMPR, and they show that when the principal and the agent are both sufficiently patient, the relationship terminates with probability 1.5 We extend our model in

\footnote{In Biais, Mariotti, and Rochet (2011), they consider a simplified version of BMPR in which the principal and the agent share the same discount factors. The dynamics there is the same as ours in Section 4. We discuss in Section 5.1 when and why the optimal long-term and relational contracts can be the same and describe the differences between the two in Section 5.2.}
Section 6 to study the optimal contracts when the principal is more patient. A key new finding in our extension is that when the principal is significantly more patient than the agent, there are parameter ranges in which both the optimal relational and long-term contracts are essentially stationary. Moreover, we identify parameter ranges in which the optimal relational contract terminates with probability 1 yet the optimal long-term contract remains stationary.

Zhu (forthcoming) considers a continuous-time model in which the optimal contract can specify the agent to shirk (either as a reward for good performances or as a punishment for bad ones). Similar to Zhu, we also find that the agent can be suspended for production (corresponding to shirking). But unlike Zhu, suspension of effort occurs only when the agent is punished and not when he is rewarded. Another difference is that the level of wage plays an important role in our model in determining whether the agent is punished by suspension of effort or by termination. We show that there exists a threshold such that the agent is terminated if and only if the wage exceeds the threshold. Moreover, the threshold under the optimal relational contract is lower than that under the long-term contract. As a result, there are parameter ranges in which the agent is punished by termination under the relational contract yet he is punished by suspension of effort under the long-term contract.

Finally, our model adds to the growing literature that studies the dynamics of relational contracts. Classic models of relational contracts, such as Bull (1987), MacLeod and Malcomson (1988), and Levin (2003, first part), have focused on the conditions under which cooperation can be sustained, and the optimal relational contracts in these models are stationary. Several more recent papers have examined the evolution of the relationship when additional frictions are present. Halac (2012) studies a model in which the principal’s type is private information. In Yang (2012), the agent’s type is private information. Fuchs (2007) and Levin (2003, second part) both analyze the case in which the output is privately observed by the principal. In Chassang (2010), the efficient production function is unknown, and the agent’s moral hazard problem is linked to an experimentation problem.

In our model, the source of friction is the limited liability of the agent, so (monetary) transfer between the principal and the agent is constrained. A closely related paper on relational contracts with two-sided limited liability is Thomas and Worrall (2010). The main difference is that they have a partnership game, so outputs depend on the joint efforts of the two parties. More importantly, effort is perfectly observed in their model so the relationship does not terminate. They focus on the dynamics of the effort provision and show that effort may be overprovided at the earlier stage of the relationship. Another related paper is Padro i Miguel and Yared (2009), who, in a political-economy setting, study a similar repeated principal-agent model without commitment or transfer. In their study of the model without transfer between the players, they focus on how the surplus is destroyed in terms of the likelihood, duration, and intensity of the punishment.
The rest of the paper is organized as follows. We set up the model in Section 2. The PPE payoff set and the optimal relational contract are characterized in Section 3. Section 4 studies the empirical implications on employment dynamics, focusing on features that are shared by both the optimal long-term and relational contracts. In Section 5, we compare these two types of contracts in detail and examine the empirical implications of lack of commitment. Section 6 extends the main model to allow for different discount factors and continuous levels of effort. Section 7 concludes. The proofs of all the formal results are relegated to the Appendix.

2 Setup

There is one principal and one agent. Both are risk neutral, infinitely lived, and have a common discount factor $\delta$. Time is discrete and indexed by $t \in \{1, 2, ..., \infty\}$.

At the beginning of each period $t$, the principal decides whether to offer a contract to the agent: $d_t^P \in \{0, 1\}$. If a contract is offered, it specifies a legally enforceable wage $w_t \geq w$, where $w \in R$ is an exogenously given wage floor. When $w = 0$, the stage game is a standard model of moral hazard with limited liability.

In many relational contracting models, the contract also includes a discretionary bonus at the end of the period. The current setup is chosen to simplify notation and to facilitate the comparison to efficiency wages models. These two setups are equivalent in the sense that they give rise to the same set of equilibrium payoffs, and the results obtained here can be directly translated into a version with a discretionary bonus. In our setup, the discretionary bonus can be thought of as being postponed until the beginning of the next period and it becomes part of the wage offered. Specifically, for any wage $w_{t+1} > w$, it is equivalent that the principal pays out a bonus $(w_{t+1} - w)/\delta$ at the end of period $t$ and pays out $w$ at the beginning of period $t + 1$. In our discussion below, we will refer to wage payments above the wage floor as bonus payments.

If the principal offers a contract, the agent chooses $d_t^A \in \{0, 1\}$. If he accepts the offer, the principal pays out wage $w_t$. The agent then chooses effort $e_t \in \{0, 1\}$, and output $Y_t \in \{0, y\}$ is realized. If the agent works ($e_t = 1$), he incurs a cost of effort $c$, and $Y_t$ is equal to $y$ with probability $p \in (0, 1)$ and $0$ with probability $1 - p$. If the agent shirks ($e_t = 0$), no effort cost is incurred, and $Y_t$ is equal to $y$ with probability $q < p$.

We assume that if the principal does not make a contract offer ($d_t^P = 0$) or if the agent rejects an offer ($d_t^A = 0$), then the relationship is permanently terminated and the players receive their

\footnote{Malcomson and MacLeod (1998) provides a formal proof for the symmetric information case. Their proof can be adapted to the current case.}
outside options in every period thereafter. Let the agent’s normalized per-period outside option be $\underline{v}$ and the principal’s outside option be $\overline{v}$.

To make the analysis interesting, we assume that the value of the relationship exceeds the sum of outside options if and only the agent puts in effort:

$$py - c > \underline{v} + \overline{v} > qy.$$ 

In terms of the information structure, the effort of the agent is his private information. Outputs are publicly observable. In addition, we assume that a public randomization device is available.\footnote{The public randomization device is a commonly-made assumption in models of repeated games to convexify the equilibrium payoffs; see for example Mailath and Samuelson (2006), Section 3.4 for a discussion of its roles.} In particular, both the principal and the agent observe at the end of the period the realization of a random variable $x_t$ that is uniformly distributed between 0 and 1, and we also assume that $x_0$ exists at the very beginning of the game. We summarize the timing in Figure 1 below.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{timeline.png}
\caption{Timeline}
\end{figure}

Define $T := \min \{ t | d_t^P d_t^A = 0 \}$. At the beginning of any period $t$, the expected payoffs to the principal and the agent are given by

$$v_t = (1 - \delta) \left[ \sum_{\tau = t}^{T} \delta^{\tau - t} [y(q + (p - q)e_\tau) - w_\tau] + \sum_{\tau = T + 1}^{\infty} \delta^{\tau - t} \omega \right];$$

$$u_t = (1 - \delta) \left[ \sum_{\tau = t}^{T} \delta^{\tau - t} (w_\tau - ce_\tau) + \sum_{\tau = T + 1}^{\infty} \delta^{\tau - t} \omega \right].$$

Define $T := \min \{ t | d_t^P d_t^A = 0 \}$. At the beginning of any period $t$, the expected payoffs to the principal and the agent are given by
where we multiply throughout by \((1 - \delta)\) to express the payoffs as per period averages.

Our environment is an infinitely repeated game with imperfect public monitoring, and we follow the literature to use public perfect equilibrium (PPE) as the solution concept.\(^8\) A PPE is strategy profile such that a) players use strategies that depend on the public history, and b) the strategy profiles following any public history form a Nash Equilibrium.

Formally, we denote \(h_t = \{d_t^P, w_t, d_t^A, y_t, x_t\}\) as the public events that occur in period \(t\). Let \(h^t = \{h_n\}_{t=1}^{t-1}\) be a public history path at the beginning of period \(t\), and \(h^1 = \{x_0\}\). Let \(H^t = \{h^t\}\) be the set of public history paths till time \(t\), and define \(H = \cup_t H^t\) as the set of public histories.

In period \(t\), the principal’s actions are specified by

\[
\begin{align*}
D_t^P &: H^t \to \{0, 1\}; \\
W_t &: H^t \cup \{d_t^P\} \to [w, \infty).
\end{align*}
\]

where \(D_t^P\) is a function that determines whether to offer a contract, \(W_t\) is the function that determines the wage level. The principal’s public strategy, \(s^P\), is given by \(\{D_t^P, W_t\}_{t=1}^\infty\). The agent’s actions are specified by

\[
\begin{align*}
D_t^A &: H^t \cup \{d_t^P\} \cup \{w_t\} \to \{0, 1\}; \\
E_t &: H^t \cup \{d_t^P\} \cup \{w_t\} \to \{0, 1\},
\end{align*}
\]

where \(D_t^A\) is a function that determines whether to accept the contract and \(E_t\) is a function that determines the effort level. The agent’s public strategy, \(s^A\), is given by \(\{D_t^A, E_t\}_{t=1}^\infty\).

Let \(v(s^P, s^A|h_t)\) and \(u(s^P, s^A|h_t)\) be the principal’s and the agent’s expected payoffs following public history \(h_t\). A strategy profile \((s^P, s^A)\) is a PPE if and only if following any public history \(h_t\),

\[
\begin{align*}
s^P &\in \operatorname{arg\,max}_{s^P} v(s^P, s^A|h_t); \\
s^A &\in \operatorname{arg\,max}_{s^A} v(s^P, s^A|h_t).
\end{align*}
\]

We denote each PPE as a relational contract. The optimal relational contract is the PPE that maximizes the principal’s payoff at the beginning of the first period.

\(^8\)This model is a game of imperfect monitoring with "product structure", in that the output depends on the agent’s effort alone. It follows that our restriction to PPEs is without loss of generality; see Fudenberg and Levine (1994).
We solve for the optimal relational contract in the next section. To make the analysis interesting, we assume the following two conditions are satisfied. The first condition implies that the wage floor is sufficiently high so the limited liability condition matters:

\[ w > u - \frac{qc}{p - q}. \]

Otherwise, the principal is able to set sufficiently low wages to extract all of the surplus in the relationship, and in this case, the optimal relational contract is essentially stationary; see Levin (2003). The second condition guarantees a non-trivial relational contract exists:

\[ py - c - u - v > \frac{1 - p\delta}{\delta} \frac{c}{p - q} > w - c - u. \]  

(NT)

Specifically, this condition guarantees the following relational contract can be sustained. In each period, the agent puts in effort. If the output is low, the relationship is terminated. Otherwise, the agent is given a fixed bonus (to be paid out as the beginning of next period). The first inequality ensures that there exists a large enough bonus do induce the agent to put in effort. The second inequality ensures that the principal finds it incentive compatible to pay the bonus.

3 Analysis

In this section, we characterize the PPE payoff set by using the technique developed by Abreu, Pearce, and Stacchetti (1990). The first subsection gives a recursive representation of the PPE payoff set and shows that it is completely determined by the payoff frontier. The second subsection characterizes the payoff frontier and describes the optimal relational contract.

3.1 PPE Payoff Set

To give a recursive representation of the PPE payoffs, we start by establishing conditions each PPE payoff must satisfy. Denote the set of PPE payoffs by \( E \). For each payoff pair in \( E \), the recursive representation requires a specification of the equilibrium action taken by the players and the continuation payoffs associated with any publicly observable outcomes. In our setting, there is no loss of generality in assuming that if a publicly observable deviation occurs, the parties terminate their relationship, as this is the worst possible equilibrium that gives each party its minmax payoff. This implies that only the description of equilibrium actions and continuation payoffs are required.

Consider a PPE payoff pair \((u, v) \in E\). If \((u, v) = (u, v)\), then the payoff pair is sustained by termination. Otherwise, there are three possibilities: a) support with effort; b) support with suspension (no effort); c) support with randomization. When \((u, v)\) is supported with effort, there exists a wage payment \(w\) and continuation payoffs \((u_l, v_l, u_h, v_h)\) that satisfy the
conditions to be described below. Here, \( u_l \) and \( u_h \) are the agent’s continuation payoff following a high and low output and \( v_l \) and \( v_h \) are the principal’s payoff defined analogously. To support \((u, v)\), we need the following conditions to be satisfied. The limited liability condition of the agent is given by

\[ w \geq w; \quad \text{(LL)} \]

Moreover, to induce the agent to exert effort, the benefit of doing so must exceed the cost. The agent’s incentive constraint is given by

\[ \delta (p - q) (u_h - u_l) \geq (1 - \delta) c. \quad \text{(IC)} \]

In addition, the consistency of the PPE payoff decomposition requires that the parties’ payoffs are equal to the weighted sum of current and future payoffs. The promise-keeping constraints are therefore given by

\[ v = (1 - \delta) (py - w) + \delta (pv_h + (1 - p) v_l) \quad \text{(PK_P)} \]

for the principal and

\[ u = (1 - \delta) (w - c) + \delta (pu_h + (1 - p) u_l) \quad \text{(PK_A)} \]

for the agent. The last set of constraints are that we need the continuation payoffs to be PPE payoffs. These self-enforcing conditions are given by

\[ (u_l, v_l) \in E \quad \text{(SE_l)} \]

\[ (u_h, v_h) \in E \quad \text{(SE_h)} \]

Notice that these self-enforcing constraints correspond to the non-reneging constraints in models with bonus payments. Since the bonus takes the form of higher wage next period, the constraints that restrict the continuation payoffs to be in \( E \) correspond to the constraints that the principal is willing to pay out the bonus.

The second possibility for \((u, v) \in E\) is to support it with suspension. In this case, the payoff is supported with a wage payment \( w \) and continuation payoff \((u_s, v_s)\), where the subscript \( s \) denotes that the agent is suspended from working this period. To support \((u, v)\), we need \( w \geq w \) and \((u_s, v_s) \in E\) for the feasibility constraints. Moreover, it is clear that the agent would have no incentive to put in effort, so we do not need to consider the agent’s incentive constraint. Finally, for the promise-keeping constraints, we need

\[ u = (1 - \delta) w + \delta u_s; \]

\[ v = (1 - \delta) (py - w) + \delta v_s. \]
Third, \((u,v)\) can be supported with randomization. In this case, there exists two distinct PPE payoffs \((u_i,v_i) \in E, i = 1,2\) such that
\[
(u,v) = \alpha(u_1,v_1) + (1-\alpha)(u_2,v_2) \text{ for some } \alpha \in (0,1).
\]

To characterize \(E\), we need to determine, for each \((u,v) \in E\), whether it is supported by effort, by suspension, or by randomization. We then need to specify the corresponding actions and continuation payoffs, and in the case of randomization, how \((u,v)\) is randomized. The next lemma simplifies the analysis by showing that the PPE payoff set is completely determined by its frontier. Specifically, define the payoff frontier as
\[
f(u) \equiv \max\{v' : (u,v') \in E\}.
\]

In other words, \(f(u)\) is the principal’s highest PPE payoff when the agent’s payoff is given by \(u\). Also define \(\bar{\pi}\) as the largest PPE payoff of the agent:
\[
\bar{\pi} = \max\{u : (u,v) \in E\}.
\]

**Lemma 1.** The PPE payoff set \(E\) has the following properties: (i.) it is compact, (ii.) \(f\) is concave and payoff pair \((u,v)\) belongs to \(E\) if and only if \(u \in [\underline{u}, \bar{\pi}]\) and \(v \in [\underline{v}, f(u)]\).

Lemma 1 establishes several properties of the PPE payoff set \(E\). Part (i.) shows that \(E\) is compact, and this implies that both \(f\) and \(\bar{\pi}\) are well defined. The compactness of \(E\) follows essentially because in any equilibrium the actions are bounded. Specifically, the wage \(\omega\) is bounded because the principal will not find it incentive-compatible to make a payment that exceeds the future surplus of the relationship. Part (ii.) follows directly from the availability of the public randomization device. In particular, notice that for any \(f(u)\), the points below it can be sustained by randomizing among \((\underline{u},\underline{v})\), \((\bar{\pi},\underline{v})\), and \((u, f(u))\). Since the payoff frontier completely determines the PPE payoff set, we now turn to the next section, which characterizes the payoff frontier.

### 3.2 PPE Payoff Frontier

In this subsection, we start by establishing several properties of the PPE frontier. Lemma 2 shows that payoffs on the frontier are sequentially optimal in the sense that their continuation payoffs remain on the frontier. Lemma 3 shows that payoffs on the PPE frontier can be supported with effort as long as it is feasible to induce effort. Our main result of the section,
Proposition 1, then characterizes the PPE frontier and describes the associated optimal relational contract.

**Lemma 2:** For any payoff \((u, f(u))\) on the frontier, the followings hold.

(i.) If \((u, f(u))\) is supported with effort, the continuation payoffs satisfy \(v_h = f(u_h)\) and \(v_l = f(u_l)\).

(ii.) If \((u, f(u))\) is supported with suspension, the continuation payoff satisfies \(v_s = f(u_s)\).

Lemma 2 shows that payoffs on the frontier are sequentially optimal in the sense that their continuation payoffs remain on the frontier regardless of whether the agent is asked to put in effort. To see why this is true, note that the principal’s actions are publicly observable, so it is not necessary to punish her by moving below the PPE frontier. The reason for this sequential optimality is the same as similar results in Spear and Srivastava (1987) and the first part of Levin (2003) in which the actions of the principal are publicly observable. In contrast, when multiple parties have private information, joint punishments are necessary, see for example, Green and Porter (1984), Athey and Bagwell (2001), and the second part of Levin (2003).

Lemma 2 allows us to eliminate \(v_h, v_l,\) and \(v_s\) from our analysis since the principal’s continuation payoffs are completely determined by the agent’s continuation payoffs. The next lemma provides information on the agent’s effort and continuation payoffs. To simplify the exposition, we define

\[ k \equiv \frac{1 - \delta}{\delta} \frac{c}{p - q}, \]

which is the minimal difference between \(u_h\) and \(u_l\) necessary for effort. We also introduce the linear function \(L(u)\), defined by the following equation:

\[ u = (1 - \delta)(\underline{u} - c) + \delta[(1 - p)L(u) + p(L(u) + k)]. \]  

In other words, \(L(u)\) corresponds to the agent’s continuation payoff following a low output if he is paid \(\underline{u}\) and the agent’s IC is binding. Notice that to induce effort, the agent’s continuation payoff following a low output is at least \(L(u)\). Finally, we define

\[ u_0 = \inf\{u : (u, f(u)) \text{ can be supported with effort}\}. \]

**Lemma 3:** For any payoff \((u, f(u))\) on the frontier, the followings hold.

(i.) If \((u, f(u))\) is supported with effort, \((u, f(u))\) can be supported with continuation payoffs satisfying

\[ \delta (p - q) (u_h - u_l) = (1 - \delta) c. \]

(ii.) \((u, f(u))\) can be supported with effort for all \(u \geq u_0\), where \(u_0\) is given by \(L(u_0) = \underline{u}\).
Part (i.) follows from the concavity of the PPE frontier. Essentially, if the agent’s IC were not binding, one could always reduce $\eta$ and increase $\lambda$ in such a way that $u$ remains the same and all the relevant constraints continue to be satisfied. Since the PPE frontier is concave, however, such a change would make the principal better off. Part (ii.) shows that effort is weakly increasing in the agent’s payoff since all payoffs on the frontier can be sustained by effort as long as they exceed $u_0$. Since $L(u_0) = \underline{u}$ and $L$ is increasing, it implies that $L(u) < \underline{u}$ for all $u < u_0$. In other words, $u_0$ is the smallest payoff necessary for effort. Part (ii.) therefore implies that the PPE frontier can be supported by effort as long as it is feasible to induce effort.

Lemma 3 further simplifies the analysis by specifying the agent’s effort choice for $u \geq u_0$. Moreover, it shows that in this region $u_h$ is completely determined as long as $\lambda$ is specified. Given the simplifications from Lemma 2 and 3, we now turn to Proposition 1, which characterizes the PPE payoff frontier and describe the optimal relational contract. To simplify the exposition, we introduce the following cutoff discount factor:

$$\delta^* = \frac{1}{1 + ((p - q)(py - w) / c - p)}.$$  

As will be clear from our discussion below, $\delta^*$ is the smallest discount factor necessary for an efficient relational contract. For convenience, we will describe below the relationship as having high surplus when $\delta \geq \delta^*$ and having low surplus otherwise.

**PROPOSITION 1:** The PPE frontier $f(u)$ satisfies the following.

(i.) For $u \in (w, u_0)$,

$$f(u) = f(\underline{u}) + \frac{u - \underline{u}}{u_0 - \underline{u}}(f(u_0) - f(\underline{u})).$$

In addition, there exists $w^* < \underline{u}$ such that $f(u) = \underline{u}$ if and only if $w \geq w^*$.

(ii.) For $u \in [u_0, u_\epsilon]$, $(u, f(u))$ can be sustained with effort through $w = \underline{u}$, $\lambda = L(u)$, and $u_h = L(u) + k$.

$$f(u) = (1 - \delta)(py - w) + \delta[pf(L(u) + k) + (1 - p)f(L(u))],$$

(iii.) For $u \in (u_\epsilon, \overline{u}]$, $(u, f(u))$ can be sustained with effort through $w = \underline{u} + (u - u_\epsilon) / (1 - \delta)$, $\lambda = L(u_\epsilon)$ and $u_h = L(u_\epsilon) + k$.

$$f(u) = f(u_\epsilon) + u_\epsilon - u.$$  

In addition, $L(u_\epsilon) = u_\epsilon$ when $\delta \geq \delta^*$, and $L(u_\epsilon) = \overline{u} - k$ otherwise.

Proposition 1 shows that payoff frontier can be divided into three regions. Each region corresponds to a distinct phase of the employment relationship. Below, we discuss the equilibrium
play and the transitions of the continuation values associated with each of the regions. Figure 2 illustrates the PPE payoff frontier for the case where the relationship has high surplus (and the punishment takes the form of termination.

\[ f(u) = v(u) + k \]

\[ f(u) = v \]

**Figure 2: PPE Payoff Frontier with High Surplus and Termination**

The Left Region

In this region, the payoff frontier is obtained from a randomization between \((u, f(u))\) and \((u_0, f(u_0))\) according to the outcome of the public randomization device at the end of the previous period. For an agent with payoff \(u\), with probability \((u - u) / (u_0 - u)\) the randomization outcome gives the agent \(u_0\), and in this case the agent chooses effort. With probability \((u_0 - u) / (u_0 - u)\) the randomization outcome gives the agent \(u\). In this case, the agent does not exert effort (this period) and inefficiency occurs. Notice that the principal and the agent have incentive to renegotiate once the inefficiency occurs, but committing to such inefficiency improve the principal’s ex ante payoff.\(^9\)

Part (i) of the proposition shows that the inefficiency takes two possible forms. When the wage floor is above \(w^*\), \((u, f(u)) = (u, v)\), so the relationship terminates. When the wage floor is below \(w^*\), \((u, f(u))\) is supported with suspension. The agent does not put in effort this period, and his continuation payoff moves to \(u_\sigma(u) = [u - (1 - \delta)\bar{w}] / \delta\). For convenience of exposition, we refer to this type of inefficiency as temporary suspension of production (TSP). Notice that when TSP takes place, the agent’s current payoff within the relationship \(w\) is lower

\(^9\)While we do not consider the possibility of renegotiation, one way to avoid it is to allow the principal to replace the existing agent with a new one. This possibility is discussed in the conclusion section.
than his outside option \((u)\) since \(w^* < u\).

To see why the wage floor affects the form of inefficiency, notice that the wage floor affects how the surplus is divided in the relationship. When the wage floor is higher, the agent’s rent in the relationship is higher so termination becomes a more effective tool for inducing effort. Moreover, bigger rent for the agent means that the principal’s payoff in the relationship is smaller, so termination is less costly for her. Therefore, the principal would prefer termination over TSP when the wage floor is high.

**The Middle Region**

In the middle region, the agent puts in effort and receives \(w\). If the output is low, the agent’s continuation value moves to the left to \(u_l(u) = L(u)\). If the outputs have been consecutively low, the agent’s continuation payoff will eventually move to the left region and inefficiency will occur. When the output is high, the agent’s continuation moves to \(u_r(u) = L(u) + k\). Notice that when \(L(u) + k\) remains in the middle region, the agent’s wage next period will again be \(w\). In this case, the agent is motivated entirely through continuation payoffs instead of bonuses (in the form of higher wage next period).

The fact that the agent is always paid the lowest possible wage in the middle region reflects a general principle of dynamic incentive provision in jobs with rent: the principal should delay bonus payments as part of the reward as much as possible. To see why paying a wage exceeding \(w\) is suboptimal, note that when this happens, the principal can lower the wage payment and also lower the probability of punishing the agent when his continuation falls into the left region to keep the agent indifferent. Such change strictly benefits the principal because it keeps the agent’s payoff unchanged and at the same time increases the total surplus. The total surplus is increased because the change reduces the chance that inefficiency occurs in the future. In general, when a high output is realized, by substituting a bonus payment with a reduction in the probability of future punishment as part of the reward, the principal can increase her payoff by increasing the surplus of the relationship.

**The Right Region**

In this region, the PPE payoff frontier is a line segment with a slope of \(-1\), and as a result, the joint surplus is unchanged in this region. The right region is the region of bonus, and in general, there are a number of ways to support the PPE payoff frontier since there are some flexibility in arranging the timing of bonus payout. Proposition 1 shows one way to do so. In this case, an agent of payoff \(u\) receives payment \(w + (u - u_e)/(1 - \delta)\) and puts in effort. If the output is low, his continuation payoff moves to the left to \(u_e\). Otherwise, it moves to \(u_e + k\). Notice that \(L(u_e) < u_e\) when the relationship has low surplus \((\delta < \delta^*)\). In this case, inefficiency will occur in the future with probability 1 when a sufficiently long sequence of consecutive low outputs moves the agent’s continuation payoffs to the left region.
In contrast, $L(u_e) = u_e$ when the relationship has high surplus ($\delta \geq \delta^*$). Part (iii) then shows that the relational contract can become quasi-stationary (from the second period on) once the relationship moves into the right region. Specifically, the agent puts in effort and is guaranteed the wage floor in each period. He also receives a fixed bonus at the beginning of each period if the previous period’s output is high. Under this quasi-stationary contract, the amount of bonus for high output is invariant of time and history and the relationship is efficient in the sense that the agent always puts in effort.

The discussion above shows that there is a discontinuity in the long-run outcomes once the discount factor crosses $\delta^*$. The PPE frontier, however, is continuous in $\delta$. In particular, consider an agent who starts with a payoff $u_e$ and experiences $N$ consecutive low outputs. As $\delta$ approaches to $\delta^*$ from below, $L^N(u_e)$ also converges to $u_e$. This implies that for any large $N$, the equilibrium play under $\delta$ becomes very similar to that under $\delta^*$. Since the payoffs more than $N$ periods away are negligible given the discounting, the equilibrium payoff of the principal under $\delta$ becomes very similar to that under $\delta^*$. In other words, the payoff is contious at $\delta^*$.

Having described the properties of the PPE payoff, our next proposition completes our description of the optimal relational contract by providing information about where on the payoff frontier the optimal relational contract starts.

**Proposition 2:** When $\delta \geq \delta^*$, $f'(u_e) = -1$ so there exists $u < u_e$ such that

$$f(u) > f(u_e).$$

Proposition 2 shows that when the surplus of the relationship is sufficiently large the payoff frontier is differentiable at $u_e$ with a slope of $-1$. An immediate consequence of this is that the optimal relational contract is inefficient since Proposition 1 shows that the only way for the relationship to be efficient is to have $u \geq u_e$ and $\delta \geq \delta^*$. By choosing $u < u_e$, the principal sacrifices efficiency but can gain through extracting more rents out of the agent. Proposition 2 implies that the marginal cost of rent extraction to the principal starts at zero when the relationship has high surplus, so the principal always sacrifices some efficiency to extract more rent from the agent.

The trade-off between efficiency and rent extraction also exists in static moral-hazard model with limited liability. But rent extraction in a dynamic setting leads to qualitative differences. Suppose the wage floor is sufficiently high for the inefficiency to take the form of termination. In stationary relational contracts, in each period the agent earns a base wage, receives a performance bonus for high output (at the beginning of next period), and faces a fixed probability of termination for low output. The principal may extract more rent from the worker by increasing
this fixed probability of termination. However, termination in such a stationary way is costly for the principal: the entire relationship may end after period 1. In fact, termination with a fixed probability can impose such a high cost to the principal that for some parameters the optimal stationary relational contracts use no termination. In other words, the optimal stationary relational contract may actually be efficient.

In contrast, Proposition 2 implies that the marginal cost of using termination starts at 0 in the optimal relational contract (without the stationarity restriction). Therefore, termination always occurs with positive probability and the optimal relational contract is inefficient. Moreover, it is non-stationary: the termination probability differs across periods and is path-dependent. For example, Proposition 1 implies that the optimal relational contract starts in the middle region and the termination probability can be zero following period 1.

Taking the standard carrot-and-stick metaphor, rent extraction in a stationary way requires in each period giving out a carrot for high output and using a stick (with some probability) for low output. Both are costly for the principal. Optimal dynamic rent extraction implies that neither the carrot nor the stick should be used immediately. Instead, the relationship begins in the middle region, with the principal simply holding the stick. The principal raises the stick higher (i.e., increases the probability of future termination) following each high output and lowers it otherwise. Following a sufficiently bad output history, the stick is raised high enough (the agent’s continuation value falls in the left region) and the principal will use the stick with positive probability. Following a sufficiently good output history, the stick is held low enough (the agent’s continuation value falls in the right region) and carrots are given out for high outputs. When the surplus in the relationship is high, the principal will eventually put down the stick following a sufficiently favorable output history. After that, the agent is incentivized exclusively with carrots.

4 Employment Dynamics

This section explores the empirical implications from the previous section. Section 4.1 describes the predictions on employment dynamics. Subsection 4.2 relates our predictions to empirical findings and alternative theories on employment dynamics.

To study turnover dynamics, we restrict our attention to the high wage floor case \((w \geq w^*)\) so that punishment takes the form of termination. In addition, we assume in this section that the relationship has high surplus \((\delta \geq \delta^*)\). As will be clear in the next section, the employment dynamics from the relational and long-term contract are identical when the surplus is high. In other words, predictions on patterns of employment dynamics in this section are related in general to models of dynamic moral hazard problem with limited liability, and not
specific that the principal has commitment problem. The difference between commitment and non-commitment will be examined in detail in the next section, where we also study the cases of $w < w^*$ and $\delta < \delta^*$.

4.1 Predictions on Employment Dynamics

The optimal relational contract gives a specific prediction on the pattern of employment dynamics. In particular, the worker starts the employment relationship in a “probation phase” in which he receives $w$ regardless of his performance, and depending on his performance, he either transitions into the bonus phase or is terminated. In the bonus phase, the worker’s pay depends on his performance. If the relationship has a high surplus, then once the worker transitions into the bonus phase, the continuation optimal relational contract can be implemented by a sequence of quasi-stationary contracts and the worker is no longer terminated. Proposition 3 characterizes the employment structure formally.

**Definition 1 (Histories)** We define three types of histories:

(i.) Probation phase ($H_1$): For $h^t \in H_1$, the agent’s pay is equal to $w$ and the agent exerts effort, i.e.,

\[
\begin{align*}
w(h^t) & = w \\
e(h^t) & = 1,
\end{align*}
\]

(ii.) Termination ($H_2$): For $h^t \in H_2$, players each receive their outside option, i.e.,

\[
\begin{align*}
u_t(h^t|h^t) & \in H_2 = w; \\
v_t(h^t|h^t) & \in H_2 = v.
\end{align*}
\]

(iii.) Bonus phase ($H_3$): When $h^t \in H_3$, the optimal relational contract can be implemented in the following way:

\[
w(h^{t+1}) = \begin{cases} 
    w & \text{if } y_t = 0; \\
    \frac{w + c/(\delta(p - q))}{w + c/(\delta(p - q))} & \text{if } y_t = y.
\end{cases}
\]

*In other words, the optimal relational contract in $H_3$ may be quasi-stationary.*

---

10In this section we focus on the case of $\delta \geq \delta^*$. If $\delta < \delta^*$, then when $h^t \in H_3$, the agent’s wage would be given by

\[
w_t = w + \max\{u_t - u_e, 0\},
\]

where $u_e$ is determined by Proposition 1, and $u_t$ is the agent’s continuation value.
PROPOSITION 3: Suppose \( \delta \geq \delta^* \) and \( w > w^* \). In the optimal relational contract, the set of histories can be partitioned into \( H = H_1 \cup H_2 \cup H_3 \), such that

(i.) Employment starts with the probation phase \( (H_1) \).

(ii.) With positive probability, the relationship is terminated, after which both the principal and the agent receive their outside options \((u, v)\):

\[
\Pr(h^t \in H_2) > 0 \text{ for some } t.
\]

(iii.) With positive probability, the agent transitions into the bonus phase \( (H_3) \):

\[
\Pr(h^t \in H_3) > 0 \text{ for some } t.
\]

(iv.) In the long run, the agent stays in the probation phase with probability 0:

\[
\lim_{t \to \infty} \Pr(h^t \in H_1) = 0.
\]

The three phases of the employment dynamics correspond to the three regions in the PPE payoff set. Proposition 3 implies that the optimal relational contract starts in the middle region, which corresponds to the probation phase. While in the probation phase, the agent’s continuation values vary according to the outputs. If the output is high, the agent’s continuation value moves to the right. Otherwise, it moves to the left. When the continuation value moves across the right threshold \((u_e)\), there are two possibilities. If the surplus of the relationship is high, the agent receives permanent employment and the continuation optimal relational contract can be implemented by a sequence of quasi-stationary contracts in which the worker receives a bonus for each high output. If the surplus is not high enough, the agent still receives a bonus for each high output as long as he stays in the right region but after a large enough number of low outputs, he will fall back to the probation phase. When the continuation value crosses the left threshold \((u_0)\), termination occurs with positive probability. This employment pattern leads to predictions on turnover, pay level, and the sensitivity of pay to performance, which we discuss below.

**Turnover Dynamics**

In terms of turnover dynamics, Proposition 3 has two predictions. First, the turnover rates are initially increasing with time on the job. This low turnover rate at the beginning of the employment relationship is a feature of the optimal relational contract. As mentioned in Section 3, to optimally extract rents from the worker, the principal does not terminate the agent immediately for a low output but instead raises the agent’s termination probability in the future. This explains why the turnover rate is initially low.
Second, the turnover rates eventually drop down to zero. This follows because workers who have not received permanent employment always face some risk of termination (possibly in the future). As time passes by, these workers will either be terminated or receive permanent employment. It follows that for workers who have remained on the job, the fraction of them who have received permanent employment increases over time, and the turnover rate eventually moves towards zero.

Combining these two predictions, the model suggests that the turnover rate may be inverse-U-shaped with job tenure. We should note, however, the turnover rate is not necessarily inverse-U-shaped. Due to the discreteness of the time periods, the turnover rate may not be very smooth and can have multiple peaks. Nevertheless, the following provides an example where the turnover rate is a degenerate inverse-U: the turnover rates are zero both before and after a fixed date, and it is positive on that date.

COROLLARY 1: If \( \delta \geq \delta^* \) and \( p \leq q + \delta(u - w)/(1 - \delta - \delta q)\), there exists \( T^* \) such that the turnover rate is 0 for \( t < T^* \) and is again 0 for \( t > T^* + 1 \). Generically, turnover happens only in \( T^* \).

The turnover process in Corollary 1 follows an "up-or-out" rule: the worker is terminated at a fixed date if he does not receive tenure by then. Note that the condition in Corollary 1 is more likely to be satisfied when the probability of success \( (p) \) is small. In this regard, Corollary 1 appears to fit some segment of the academic labor market. A few small-probability home-run publications bring an assistant professor to tenure, and failure to do so before a fixed date leads to termination.

**Pay Dynamics**

In terms of pay level, Proposition 3 predicts that pay is deferred and the average pay is immediately higher once the worker transitions out of the probation case. The logic is straightforward: during the probation phase, the firm rewards for high output with future job security instead of bonus, and this results in lower average pay. For each individual worker, the model predicts deferred pay between phases of employment and not within. At the aggregate level, since the durations of the probation phases are in general stochastic, the model predicts that the average pay rises with time on the job. Moreover, it predicts that the increase in the average pay eventually goes to zero as most workers will have transitioned out of the probation phase.

In terms of the sensitivity of pay to performance, the model predicts that the (short-term) pay is insensitive to performance during the probation phase and responds to performance only in the bonus phase. This happens because the rewards for good performance in a worker’s earlier career take the form of increased future job security as opposed to bonus payments. Just as in pay level, the increase in pay sensitivity occurs between, and not within, phases. But at
the aggregate level, the average pay sensitivity increases with time on the job as more workers receive permanent employment over time.

4.2 Empirical Findings and Related Theories

In terms of turnover dynamics, the highlight of our model is that it can generate an inverse-U-shaped turnover rate, which fits well with empirical findings. Farber (1994) uses NLSY data and finds that the monthly hazard rate of job ending is not monotone decreasing. Rather, it increases to a maximum at three months and declines thereafter. The inverse-U-shaped pattern is typically explained by the matching model of Jovanovic (1979). In the Jovanovic model, a worker learns about the match quality with his employer over time, and the worker leaves for a new employer if the expected match quality falls below a (time-dependent) threshold. Turnover in Jovanovic’s model is efficient and, thus, the model does not distinguish voluntary and involuntary turnovers.

In contrast, the separation in our model is ex post inefficient, and since the worker has rents in the job, turnover in our model is better characterized as involuntary turnovers. In the CEO labor market, the distinction between quits and dismissals have been emphasized in the empirical literature. Brookman and Thistle (2009) find that the inverse-U-shaped pattern holds for both forced and unforced CEO turnovers. However, Farber (1994) does not distinguish between voluntary and involuntary turnovers, and it will be useful to study whether the inverse-U-shape pattern also folds for involuntary turnovers in the labor market in general.

Terminations also occur in two notable models of relational contracts. In Fuchs (2007), the source of termination is two-sided moral hazard. In his model, the output is the principal’s private information. When the principal reports a low output, it could either be that the output is low (which calls for punishing the agent) or that the principal has lied (which calls for punishing the principal). The inability to identify which player has led to the bad outcome implies that surplus destruction is necessary and termination will occur. It is also possible that the termination probability starts out at zero in Fuchs (2007) especially when the surplus in the relationship is high. However, Fuchs (2007) is a model of repeated game with private monitoring, and there are great technical difficulties in solving such games. The turnover pattern associated with the optimal relational contract remains unknown. In Yang (2012), the source of termination is worker heterogeneity. In his model, high-ability workers always produce high outputs in equilibrium. A low output, therefore, is perfectly indicative of low ability. Yang (2012) shows that there exists a type of pooling equilibrium in which the worker’s ability is gradually revealed. A worker is immediately terminated following a low output. In Yang

\[ \text{The worker strictly prefers working for the firm than his outside option when } w > u. \text{ When } w < u, \text{ it is less clear whether the nature of turnover is voluntary or involuntary in this model.} \]
(2012), the turnover rate is monotone decreasing and depends on the proportion of low-ability workers.

In terms of pay level, our model predicts, broadly, that the average pay is increasing with job tenure, which is a well-documented empirical finding; see for example Rubinstein and Weiss (2006) for a recent survey. The universe of theories that generates this prediction is vast and includes (at least) firm-specific human capital, selection, screening, and learning and insurance. Most relatedly, it is well-known since the seminal works by Lazear (1979, 1981) that agency theory can generate upward-sloping wage profiles. In Lazear’s models, the upward-sloping wage profiles help prevent the worker from shirking close to the retirement age, and workers in his models do not have rents in the employment relationship. In contrast, the upward-sloping wage profile arises in our model as a feature of optimal rent extraction. One distinctive prediction of our model is a discrete increase in the expected pay following the probation phase.\(^{12}\)

In terms of pay sensitivity, the model predicts that pay becomes more sensitive to performance over time. The empirical evidence on this issue is broadly, though not exclusively, supportive. Hashimoto (1979) finds that the bonus to wage ratio is increasing with experience in Japanese firms. Gibbons and Murphy (1992) show that the pays of older CEOs are more sensitive to stock-market performance. Gompers and Lerner (1999) document that the sensitivity of pay to performance is smaller for newer venture capitalists. Misra, Coughlan and Narasimhan (2005) find that the salary to total compensation ratio is decreasing with salesperson seniority. However, Khan and Sherer (1990) find that bonuses are more sensitive to performance for less senior managers.\(^{13}\)

The increasing sensitivity of pay to performance is often explained by career-concern models; see for example Gibbons and Murphy (1992) and Gompers and Lerner (1999). In these models, pay for performance and the outside market are substitutes in incentive provision. Over time, the worker’s ability becomes better known to the market and the incentive from the outside market weakens. To maintain proper incentives for the worker, pay becomes more sensitive to performance. In our model, in contrast, pay for performance and future job security are substitutes in incentive provision. Over time, pay becomes more sensitive to performance to substitute for the weakened job-security incentive. We think these two mechanisms complement each other and both can be relevant in practice.

Recently, Garrett and Pavan (2009, 2010) show that pay can also become more sensitive to performance over time in environments in which outputs depend both on the agent’s effort and

\(^{12}\)Models of screening can also generate a discrete jump in wages following the probation phase. Unlike our model, however, such discrete jumps are not always necessary.

\(^{13}\)Unlike the rest of the authors cited above, Kahn and Sherer (1990) base their findings on a single firm. Workers in this firm have considerable job security: the annual discharge rate is 0.5%, one-tenth of the industry average.
ability and the agent’s ability is his private information and changes over time. Using a dynamic mechanism approach, they show that the optimal long-term contracts can be implemented by a sequence of (path-dependent) linear contracts and the average slopes of the linear contracts increase over time. Similar to our paper, the increased sensitivity of pay to performance also reflects dynamic rent extraction. Unlike our paper, there is no limited liability constraint and the agent’s rent comes from better information about his ability. In particular, efforts are distorted downwards to induce truth-telling from the agent. More interestingly, the distortions become smaller over time because abilities in the further future are less correlated with the current ability, and, thus, the effort distortions in the future are less effective in inducing truth-telling today. Garrett and Pavan (2010) allow the principal to replace the current agent and restart the relationship with a new agent drawn from a pool of infinitely many potential agents. In this case, the principal fires agents of lower ability. In addition, the optimal retention policy becomes more lenient over time, and there is excessive retention in the long run.

In all of the models of cited above, less is known about how turnover and pay dynamics are affected by exogenous variables. One unique feature of our model is its ability to predict how turnover and pay dynamics should change with respect to the firm’s ability to commit, which we discuss next.

5 The Effects of Non-Commitment

In this section, we study how the lack of commitment from the principal affects employment and firm dynamics by comparing optimal relational contracts with optimal long-term contracts. The first subsection shows that the optimal long-term contract and the optimal relational contract are identical when the relationship has a high surplus. The second subsection examines the differences between the two contract types when the relationship has a low surplus. The third subsection discusses the empirical implications of non-commitment associated with the low-surplus case.

5.1 High Surplus

In this subsection, we show that the ability to commit to a long-term contract is unnecessary when the relationship has a high surplus ($\delta \geq \delta^*$). A high-enough discount factor is always a necessary condition for the optimal relational and long-term contracts to be the same. But, in general, it is not sufficient. Our discussion therefore focuses on the condition that guarantees that the optimal relational and long-term contracts will be identical. To facilitate our discussion, denote $f_R(u|\delta)$ and $f_{LT}(u|\delta)$ as the PPE frontiers of the optimal relational and long-term contracts, respectively.
PROPOSITION 4: \( f_R(u|\delta) = f_{LT}(u|\delta) \) for all \( u \) if and only if \( \delta \geq \delta^* \). Therefore, the optimal relational and long-term contracts are identical if and only if \( \delta \geq \delta^* \).

The reason for Proposition 4 is as follows. First, fundamentally the logic of backloading applies to both the optimal relational contract and the long-term contract. In particular, when choosing between using a bonus and a continuation payoff to reward the agent, the principal will delay paying the bonus as long as possible. Relational contracts can be thought of as long-term contracts with the extra non-reneging constraints. If all of the non-reneging constraints are satisfied in the relational contract, then the two contract types are identical. As the discount factor increases, the future surplus goes up and helps relax the non-reneging constraints. For sufficiently high discount factors, all of the non-reneging constraints in our model are satisfied, so the optimal relational and long-term contracts are identical.

When we look beyond our model, it is no longer true that all of the non-reneging constraints can be satisfied even if the discount factor is arbitrarily close to 1. As a result, the optimal relational and long-term contracts are in general different in a broader class of models (see, for example, Harris and Holmstrom (1982) and Thomas and Worrall (1988). The persistence of the difference under a high discount factor typically occurs in models where the optimal long-term contract is non-stationary. Specifically, the optimal long-term and relational contracts are different if, along the equilibrium play, the principal’s continuation payoffs can reach or fall below his outside option. In this case, for any fixed discount factor \( \delta \), there will be histories after which the principal’s payoff will be sufficiently small (or zero or negative) that he will prefer to renege.

The key condition for the optimal relational and long-term contracts to be identical is that the principal’s continuation payoffs with positive reneging temptations must be strictly bounded away from his outside option. When these continuation payoffs are strictly above the principal’s outside option, the gain from keeping the promise becomes infinitely more valuable relative to the gain from reneging as the discount factor goes to 1. This condition therefore implies that, following all histories, the non-reneging constraints are satisfied when the principal is sufficiently patient, so the optimal relational and long-term contracts are equal.

Notice that this condition is clearly satisfied in models where the optimal long-term contracts are stationary; see, for example, MacLeod and Malcomson (1989), Baker, Murphy and Gibbons (1994) and Levin (2003). Therefore, the optimal relational and long-term contracts are equal for sufficiently patient players. In contrast, this condition is violated in models where the principal’s continuation payoff can reach or fall below his outside option; see, for example, Harris and Holmstrom (1982) and Thomas and Worrall (1988). As a result, the optimal relational and long-term contracts are always different in these models. In particular, optimal long-term contracts are...
contracts in these models feature non-decreasing wages, while wages can decrease in optimal relational contracts.

5.2 Low Surplus

In this subsection, we show that there are important differences between the optimal relational contract and the optimal long-term contract when the relationship has a low surplus. First, the lack of commitment makes the relationship less likely to survive in the long run. In particular, under some range of parameter values, the optimal relational contract always terminates while the optimal long-term contract never terminates. Second, the lack of commitment generates more diversity in productivity and employment dynamics in the long run. The relationship is quasi-stationary under the optimal long-term contract but cycles under the optimal relational contract.

Recall that in the optimal relational contract, there exists a $w^*$ such that termination is used at $w$ when the wage floor is above $w^*$ and TSP is used otherwise. To facilitate our discussion, we now use $w^*_R$ to denote this threshold and let $w^*_LT$ be its counterpart under the long-term contract.

**Proposition 5:** When $\delta < \delta^*$, we have $w^*_R < w^*_LT$ and the following holds.

(i.) When $w > w^*_LT$, the relationship survives with positive probability under the optimal long-term contract but survives with probability 0 under the optimal relational contract.

(ii.) When $w \in [w^*_R, w^*_LT]$, the relationship survives with probability 1 under the optimal long-term contract but survives with probability 0 under the optimal relational contract.

Proposition 5 demonstrates the two ways that non-commitment reduces the survival probability. Part (i.) reflects a common reason why a lack of commitment leads to termination. When the surplus is low, the bonus under the relational contract is constrained. As a result, the principal must also always use the threat of termination as part of the incentive to induce effort. This implies the agent never receives tenure and a sufficiently long sequence of failures leads to termination. In contrast, when the principal can commit, the bonus is no longer constrained. As a result, the bonus is sufficiently backloaded that by the time the first bonus is paid out, the agent has been guaranteed tenure.

More interestingly, part (ii.) shows that for $w \in [w^*_LT, w^*_R]$, relational contracting lowers the survival probability by affecting the actions the agent takes. For wage floors within this interval, the optimal long-term contract uses TSP to destroy surplus so the relationship never terminates. In contrast, the optimal relational contract uses termination to destroy surplus so the relationship terminates with probability 1. To see why the surplus is destroyed differently, recall that a lower wage floor raises the cost of termination because the principal’s value in the
relationship is higher. $w^*_R$ is the wage-floor level at which the principal is indifferent between using TSP and termination under the long-term contract. At $w^*_R$, however, the principal’s value under relational contracting is smaller, making termination more effective over TSP. This implies that there is a range of wage floors below $w^*_R$ in which the principal prefers termination to TSP under the optimal relational contract while the opposite is true for the long-term contract. As a consequence, for wage floors between $w^*_L$ and $w^*_R$, relationships terminate with probability 1 under the optimal relational contract but they never terminate under the optimal long-term contract. Figure 3 illustrates the punishment choices under the optimal relational contract and the optimal long-term contract.

![Figure 3: Comparison of Punishment Path](image)

In addition to survival probability, lack of commitment also has important implications on productivity and employment dynamics in the long run.

**PROPOSITION 6:** When $\delta < \delta^*$, the following holds.

(i.) Under the optimal long-term contract, the surviving relationship reaches the quasi-stationary phase ($u \geq u_e$) with probability 1 in the long run. The expected output is equal to $py$ and the worker’s expected wage is equal to $\bar{w} + (1 - \delta) pc/(\delta (p - q))$.

(ii.) Under the optimal relational contract, the surviving relationship cycles in the long run. The expected output is strictly below $py$ and the worker’s expected wage is smaller than $\bar{w} + (1 - \delta) pc/(\delta (p - q))$.

Part (i.) describes the long-run wage and productivity dynamics of relationships governed by the optimal long-term contract. It shows that all of the surviving relationships in the long run are quasi-stationary. The reason is that, conditional on survival, the relationship must have reached the right region with probability 1. Once the relationship falls into this region, it never leaves and the optimal long-term contract can be implemented in an essentially stationary way.

Part (ii.) describes the long-run dynamics of the relationship governed by the optimal relational contract. Such a relationship never becomes quasi-stationary. Instead, the agent’s pay
fluctuates between the wage floor (when the agent’s payoff is not in the right region) and higher wage levels (when it is in the right region). Moreover, the expected productivity fluctuates between $py$ (when the agent puts in effort) and $qy$ (when TSP is used). These fluctuations occur because the PPE frontier under relational contracting does not have an absorbing region when the surplus is low. As a result, a sufficiently long streak of low outputs will lead to the suspension of production. The differences in wages and productivity in parts (i.) and (ii.) suggest that there are empirically distinguishable differences between relationships governed by long-term contracts and those governed by relational contracts. We turn to these in the next subsection.

5.3 Empirical Implications

In this subsection, we explore the empirical implications of non-commitment resulting from the low-surplus case. The first part examines the long-run differences as described in Proposition 6; the second part focuses on the transitional dynamics involving wage growth and turnover.

5.3.1 Long-Run Differences

Proposition 6 points to several implications of non-commitment for productivity and employment dynamics in the long run. First, the lack of commitment leads to lower productivity for the firm and lower wages for the worker. Second, the lack of commitment leads to larger productivity and wage dispersion across ex ante identical firms. Third, the lack of commitment leads to higher volatility in productivity and wages. Notice that the second and third predictions follow because all surviving relationships under the long-term contract are essentially stationary and the relationships under relational contracts cycle.

The predicted differences in the previous subsection shed light on a number of findings on productivity and wages across firms. Many studies, for example, have examined the differences in employment outcomes in formal and informal sectors. According to Maloney (1999), "In developing countries roughly 40 percent of urban workers are not protected by labor legislation and work in small, informal firms." Since labor laws do not protect workers in the informal sector, employment relationships there are better characterized by relational contracts.

In terms of wage levels, it is perhaps not surprising that most papers on the topic find that workers in the informal sector have lower wages (Mazumdar 1981, Heckman and Hotz, 1986, Roberts 1989, Pradhan and van Soest 1995, Tansel 1999, and Gong and van Soest 2001). Of course, many reasons can be given for why these workers have lower wages, but we should note that the studies cited above have controlled for worker and firm characteristics. One paper that finds no wage differences is Pratap and Quintin (2006), which uses a semi-parametric approach to examine data from Argentina. Interestingly, the authors report that wages increase with
To the extent that the non-commitment issue is particularly relevant for smaller firms in the informal sector, our model suggests running a wage regression on the interaction between firm size and the formality of the sector, and it will be interesting to see whether the coefficient on this variable is negative.

Some of the above studies on wage levels have also examined wage dispersion between informal and formal sectors. For example, Pradhan and van Soest (1995) find smaller residual wage variance for Bolivian workers in the formal sector, and Tansel (1999) reports a similar finding for workers in Turkey. In a recent study, Meghir et al. (2012) also find that a larger dispersion of wages exists for Brazilian workers in the informal sectors. Their study, however, does not have detailed control of worker characteristics, but they do find that larger wage dispersion persists in each submarket when they control for sex and regions.

Our theory also suggests that productivity is lower for firms in the informal sector and both wages and productivity are more volatile in these firms, after controlling for worker and firm characteristics. While we cannot find studies at the firm level, there are some indirect and suggestive findings at the aggregate level. For example, to the extent that good institutions help enforce contracts, non-commitment becomes less of an issue in countries with stronger institutions. Along this line, Rigobon and Rodrik (2005) find a strong positive correlation between the rule of law and incomes across countries. Hsieh and Klenow (2009) show that there is larger TFP dispersion in India and China relative to the U.S. Acemoglu et al. (2003) report higher macroeconomic volatility in countries with weaker institutions. It will be interesting to see whether these correlations also hold at the firm level.

5.3.2 Differences in Transitory Dynamics

Last but not least, in addition to the long-run outcomes, the lack of commitment also affects how employment dynamics evolve over time. In particular, with commitment, a worker’s expected pay and pay sensitivity are (weakly) increasing with time on the job: they jump up once the worker transitions into the tenure case and they do not drop again. Without commitment, however, expected pay and pay sensitivity are not always increasing with time on the job. After the worker transitions into the bonus phase, a sufficiently poor output sequence will bring him out of it, resulting in lower expected pay and pay sensitivity.

Our discussion on the impact of commitment on pay and job security sheds new light on the empirical findings that relate firm characteristics with employment dynamics. For example, Krueger (1991) examines the differences in employment dynamics between company-owned and

\footnote{There is a large literature reporting the positive correlation between wages and firm size; see Oi and Idson (1999) for a review.}
franchisee-owned fast food restaurants. For lower-level managers, the starting pay is about the same at company-owned and franchisee-owned restaurants, but the pay rises more rapidly at company-owned restaurants. The steeper earning profile contributes to a 9% higher pay on average at company-owned restaurants. In addition, the tenure of these managers is half a year longer at company-owned restaurants.

Krueger (1991) argues that the pay difference arises because company-owned restaurants cannot monitor their employees as well and have to rely on paying efficiency wages. While monitoring costs are clearly relevant, this explanation alone does not explain the difference in turnover rates or the higher wage growth. Our discussion suggests that the difference in commitment power can provide a unified explanation for the empirical findings. Since the companies are larger in size than the franchisees, they are better able to commit to future promises.\textsuperscript{15} Our model therefore suggests that, first, company-owned stores can better backload wage payments, leading to steeper earning profiles. Moreover, higher commitment power enables company-owned stores to better use future job security as part of the reward, leading to better job security.

Krueger (1991) also reports that for full-time crew workers, the pay difference is less than 2% and there is no statistical difference in job tenure. The jobs of the crew workers are more routine and close to our model with \( p = 1 \). When \( p = 1 \), our model implies that there is no equilibrium turnover and the relationship is efficient (when it can be sustained). In this case, commitment does not affect employment dynamics.\textsuperscript{16}

In our explanation above, we have implicitly assumed that the wage floors at company-owned and franchisee-owned restaurants are the same. This assumption appears empirically plausible since the starting pay, which is a proxy for the wage floor, is about the same in both types of restaurants. For some restaurant workers, this wage floor is just the minimum wage, but not all workers have minimum wage as their starting pay. For these workers, it is desirable to have a theory for what determines their wage floor, which is currently left as exogenous. Without a theory, the reason why company-owned restaurants do not reduce the starting pay for their workers, who have more rents than workers in franchisee-owned restaurants remains an empirical puzzle. Nevertheless, to the extent that the starting pay can be viewed as the wage floor, our model sheds light on why pay and job security evolve differently between company-owned and franchisee-owned restaurants.

\textsuperscript{15} Ji and Weil (2009) find that franchisees care less about the brand reputation.

\textsuperscript{16} The previous version of this paper has a detailed analysis on the \( p = 1 \) case, and it is available upon request.
6 Extensions

In this section, we consider two extensions of the main model. First, we examine the case in which the principal is more patient. Second, we explore the case with continuous effort level.

6.1 More Patient Principal

In this subsection, we characterize the optimal relational contract when the principal is more patient than the agent. For simplicity, we focus on parameter cases in which it is very costly for the agent not to put in effort, so punishment will be carried out in the form of termination rather than TSP. In this case, we show that the optimal relational contract takes one of two forms. In the first case, the optimal relational contract is quasi-stationary, and this can occur when the principal is considerably more patient than the agent. In the second case, the optimal relational contract is similar to that in the main model with the exception that the right region is not absorbing, so the relationship terminates with probability 1. We also explore the role of commitment, and the main results in the previous section are robust to this extension. Specifically, we identify the parameter range under which the optimal long-term contract never terminates yet the optimal relational contract terminates with probability 1.

To carry out our analysis, we let the agent’s discount factor to remain at $\delta$ and let the principal’s discount factor be $\rho > \delta$. We start by describing the PPE payoff frontier.

**Lemma A1:** There exist $u_0 \leq u_e$ such that the PPE frontier $f(u)$ is given by the following.

(i) For $u \in [u, u_0)$,

$$f(u) = u + \frac{u - u}{u_0 - u} (f(u_0) - u).$$

(ii) For $u \in [u_0, u_e]$,

$$f(u) = (1 - \rho)(p y - w) + \rho [p f(L(u) + k) + (1 - p) f(L(u))].$$

(iii) For $u \in (u_e, \bar{u}]$,

$$f(u_e) + \frac{1 - \rho}{1 - \delta} (u_e - u).$$

Lemma A1 implies that the PPE frontier under a more patient principal shares many features with that in the main model. In particular, the PPE frontier can again be categorized into three regions. In the left region, the payoff frontier is sustained by randomization, implying that termination occurs in this region. In the right region, the payoff frontier is a straight line, implying that the bonus (in the form of wage higher than $w$) is paid out here. In the middle region, the payoff frontier satisfies the same functional equation as the one in the main model.
adjusting for the principal’s discount factor, implying that the agent receives \( w \) in this region. These similarities with the main model is a reflection of the same idea that the principal benefits from backloading bonuses because it reduces surplus destruction.

Despite the similarities with the main model, Lemma A1 also indicates that there are several differences. First, unlike the main model, part (i.) shows that the slope of the PPE frontier in the right region is strictly larger than -1, reflecting the fact that the principal is more patient. This smaller slope has implications on how optimal relational contract is implemented. In particular, when the principal and the agent are equally patient, the timing of payment is irrelevant. Therefore, there is some flexibility in how to implement optimal relational contract in the right region in the main model. When the principal is more patient, however, there is strict gain in paying the agent earlier, and this implies that optimal relational contract is uniquely implemented in the right region.

Second, and more importantly, the boundaries separating the three regions \((u_0\) and \(u_e\)) are different when the principal is more patient. In the main model, \(u_0\) is strictly smaller than \(u_e\), so the middle region always has interior points. When the principal is more patient, in contrast, the middle region can collapse to a single point. A related difference is that \(u_e = L(u_e)\) in the main model when the relationship has high enough surplus. When the principal is more patient, however, we can have \(u_e < L(u_e)\) so that the right region fails to be absorbing for arbitrarily high surplus.

The reason for these differences is that when the principal is more patient than the agent, there is mutual gain in conducting intertemporal trade by paying the less patient agent earlier and letting the more patient principal to consume later. As a result, the principal has an incentive to frontload the payment to the agent. Recall that in the main model, the optimal relational contract is driven by the principal’s incentive to backload the payment. Therefore, when the principal is more patient than the agent, the optimal relational contract must balance the incentive to backload with that to frontload. The next proposition captures the impact of frontloading incentive on the structure and the long-run dynamics of the optimal relational contract. To simplify the exposition, we define the following condition.

**Definition 2 (Condition A)**

\[
p_y - w - v - \frac{\rho}{\delta} p - q \geq \max\left\{ \frac{1 - \rho}{\delta} c \frac{1 - \rho}{1 - \delta \rho(1 - p) - \delta} \left( w + \frac{qc}{p - q} - u \right) \right\}
\]

Notice that Condition A is a combination of two inequalities. The first inequality (that the left hand side is bigger than \((1 - \rho) c / (\delta (p - q))\)) corresponds to the high-surplus condition in
the main model. It is satisfied when the surplus is sufficiently high. The second inequality is a
technical condition that measures the cost of termination, and it essentially gives a lower bound
to the slope of the line segment that links \((u, v)\) and \((u_e, f(u_e))\).

**PROPOSITION A1:** The optimal relational contract takes one of the two forms.

(i.) If \(\delta < \rho(1 - p)\) and Condition A is satisfied, the optimal relational contract is quasi-
stationary. Specifically, the relationship starts at \(u_e = \frac{\varphi}{\rho - \eta}\). The agent always puts in effort.
Along the equilibrium path, \(w(u) = \varphi + (u - u_e) / (1 - \delta)\), \(u_l(u) = u_e\), and \(u_h(u) = u_e + k\).

(ii.) If \(\delta \geq \rho(1 - p)\) or Condition A fails, \(L(u_e) < u_e\), and the relationship terminates with
probability 1. The optimal relational contract starts in \([u_0, u_e]\) and takes the following form:

(a): For \(u \in [u_0, u_0]\), the relationship is terminated with probability \((u_0 - u) / (u_0 - u)\)
and goes to \((u_0, f(u_0))\) with probability \((u - u) / (u_0 - u)\).

(b): For \(u \in [u_0, u_e]\), the agent puts in effort, \(w(u) = \varphi\), \(u_l(u) = L(u)\), and \(u_h(u) =
L(u) + k\).

(c): For \(u \in (u_e, \bar{w}]\), the agent puts in effort, \(w + (u - u_e) / (1 - \delta), u_l(u) = L(u_e)\), and \(u_h(u) = L(u_e) + k\).

Proposition A1 describes the two possible forms that the optimal relational contract can take,
and it provides a necessary and sufficient condition that determines which form arises. Part
(i.) shows that, unlike the main model, the optimal relational contract can be quasi-stationary.
Notice that the condition for quasi-stationarity requires the principal to be considerably more
patient than the agent \((\delta < (1 - p)\rho)\). In this case, the incentive to frontload is so important
that it completely dominates the incentive to backload. The agent is rewarded as early as possible,
and consequently, the optimal relational contract is quasi-stationary.

Part (ii.) shows that the optimal relational contract is similar to that in the main model
when it is not quasi-stationary. The relationship starts in a probation phase, in which the agent
receives a wage equal to the wage floor. In the middle region, the agent’s continuation moves to
the left to \(L(u)\) when the output is low. Otherwise, it moves to \(L(u) + k\). Once the relationship
moves into the right region, the agent starts to receive bonus. And once the relationship moves
into the left region, the agent risks termination.

While the second form of the optimal relational contract is similar to that in the main model,
there is one key difference. In particular, the agent never receives tenure since the right region
is not absorbing. As a result, the relationship terminates with probability 1 in the long run
since there will always be an arbitrarily long sequence of consecutive low outputs that drives the
relationship to the left region. Notice that the reason for the right region to fail to be absorbing
is because of the incentive to frontload as described above. Relative to the main model, there
is gain in rewarding the agent with monetary payment earlier. Consequently, less job security
is offered in this case.
We end this subsection by exploring the role of commitment and compare the optimal relational contracts with the optimal long-term contracts when the principal is more patient.

**PROPOSITION A2:** The following holds.

(i) There exists a $\rho^*$ such that the optimal relational contract and the optimal long-term contract are identical if and only if $\rho \geq \rho^*$.

(ii) If $\delta < \rho (1 - p)$ and

$$\frac{1 - \rho}{\delta} \frac{c}{p - q} > py - w - \frac{c}{p - q} \frac{\rho}{1 - \delta} \frac{\rho p}{\rho (1 - p) - \delta} \left( \frac{w}{p - q} + \frac{qc}{p - q} - u \right),$$

the optimal long-term contract is quasi-stationary and the optimal relational contract terminates with probability 1.

Proposition A2 shows that the results on the role of commitment in Section is robust to this extension. Part (i.) shows that when the principal is sufficiently patient, commitment from the principal is not needed: the optimal relational contract and the optimal long-term contract are identical. The reason for this result is identical to that in Proposition 4: when the principal is sufficiently patient, she values the future more and is thus less likely to renege. As her discount factor increases, all of the non-reneging constraints are satisfied by the optimal long-term contracts.

Part (ii.) shows that the lack of commitment makes the relationship more likely to terminate. The stark contrast in the survival probability results from the following situation. On the one hand, the principal is considerably more patient than the agent and the incentive to frontload the payment completely dominates the incentive to backload ($\delta < \rho (1 - p)$ and the second weak inequality). This implies that the optimal long-term contract is quasi-stationary. On the other hand, the surplus of the relationship within this parameter range is not sufficiently high (the first weak inequality). This implies that, under the relational contract, the agent must also be punished for a low output through a lower continuation payoff. In the long run, an arbitrarily long sequence of low outputs arises with probability 1, so the relationship terminates with probability 1.

### 6.2 Continuous Effort

In this subsection, we enrich the main model by allowing the agent’s effort level be continuous. Many important features of the optimal relational contract derived in the main model are preserved under this extension. In particular, the relationship always starts in the probation phase.

---

17 Notice that there will be parameter values in which the condition in part (ii.) is satisfied. To see this, notice that the first and the third term does not contain $y$ so that we can choose the value of $y$ properly as along as the first term is larger than the third one. In addition, $u$ only occurs in the third term, so it can be chosen to make the third term small.
in which the wage is equal to the wage floor, and the agent receives a bonus (in the form of wage above the wage floor) if and only if his continuation payoff is above a threshold. Importantly, we show that termination can still occur as part of the optimal relational contract when the effort level is continuous.

To carry out the analysis, we assume that the agent can choose effort \( e \in [0, 1] \). When effort \( e \) is chosen, the high output is realized with probability \( p(e) \), where \( p'(e) > 0 \) and \( p''(e) \leq 0 \). In addition, the agent incurs a cost of \( c(e) \), where \( c(e) \) is increasing and convex with and \( c(0) = 0 \). Just as in the main model, we assume that

\[
p(0)y < u + v < p(e^{FB})y - c(e^{FB}),
\]

so the relationship is valuable only when a sufficiently high level of effort is chosen. The next proposition describes the PPE payoff frontier.

**Proposition A3:** There exist \( u_0 \leq u_e \) such that the PPE payoff frontier \( f(u) \) is given by the following.

(i.) For \( u \in [u, u_0) \),

\[
f(u) = f(u) + \frac{u - u}{u_0 - u}(f(u_0) - f(u)).
\]

(ii.) For \( u \in [u_0, u_e] \),

\[
f(u) = (1 - \delta)(p(e)y - w) + \delta[p(e)f(u_h(u)) + (1 - p(e))f(u_l(u))]
\]

when \( (u, f(u)) \) can be supported with pure action. In this case, \( e(u) > 0 \), \( w(u) = w \) and when \( c'(e) \) exists,

\[
u_l(u) = \frac{1}{\delta} \left(u - (1 - \delta)(w - c(e) + \frac{p(e)c'(e)}{p'(e)})\right),
\]

\[
u_h(u) = u_l(u) + \frac{1 - \delta}{\delta} \frac{c'(e)}{p'(e)}.
\]

(iii.) For \( u \in (u_e, \overline{u}] \),

\[
f(u) = f(u_e) + (u_e - u).
\]

In this case, we can choose \( e(u) = c(u_e), w = w + (u - u_e) / (1 - \delta) \), \( u_l(u) = u_e \), and \( u_h(u) = u_e + (1 - \delta) c'_-(u_e) / (\delta p'(e(u_e))). \)

Proposition A3 implies that the PPE payoff frontier, and therefore, the optimal relational contract with continuous effort shares many features with the main model. In particular, the PPE payoff frontier again can be divided into three regions, where randomization occurs in the
left region and the bonus is paid out only in the right region. Moreover, the optimal relational contract starts in the middle region, so the employment relationship starts in a probation phase with $w = \underline{w}$.

Despite the similarities, there are clear differences. First, we can no longer guarantee that randomization is not needed in the middle region (although any outcome from the randomization still leads to $w = \underline{w}$, and therefore, the relationship always starts in a probation phase). Second, even when the payoff frontier can be sustained by pure actions, there is typically no explicit characterization of the effort level. There appears to be no general rule for the effort dynamics: it is not clear, for example, whether the agent’s effort is monotone in his continuation payoff.

While a full characterization of the employment dynamics is difficult for a general effort cost function, we are able to show that termination can still occur under the optimal relational contract for some cost functions. For example, let $p(e) = pe$ and consider the following piecewise linear cost function:

$$c(e) = \begin{cases} 
    c_0 e & e \in [0, a] \\
    c_0 a + c_1 (e - a) & e \in (a, 1]
\end{cases},$$

where $c_1 > c_0 > 0$. We assume that $c_1 < py$ so higher level of effort increases joint surplus.

**COROLLARY A1**: If $\underline{w} \geq u$, there exists an $a^*$ such that the relationship terminates with positive probability for all $a \leq a^*$.

We make it clear in the proof of Corollary A1 that when $a$ is sufficiently small, the optimal relational contract becomes equivalent to the one described in the main model. There exists $u_0(a)$ such that $e = 1$ will be chosen if $u \geq u_0(a)$ and otherwise the relationship terminates with some probability. While the cost functions in Corollary A1 are special, the result for termination is more general. Essentially, a sufficient condition for our result is that as the agent’s effort goes to zero, the probability of the output being high also goes to zero.

To see this, notice that when the agent’s payoff goes to his outside option, the level of feasible effort goes to zero. In this case, the relationship may prefer to terminate for two reasons. First, for low enough effort level, the sum of payoffs through termination exceeds the joint payoff within the relationship for the current period. Second, and more importantly, when a low enough effort level is chosen, the probability of a low output becomes very high under the condition above. This means that the agent’s continuation payoff is likely to drop, so the effort level is likely to be low in the next period as well. Therefore, when the agent’s payoff is sufficiently low, not only the effort is low today, but also the relationship is likely to remain in low effort levels for long periods of time in the future. As a result, termination becomes a better option.
7 Conclusion

This paper studies a model of relational contracts with limited liability. The optimal relational contract generates several implications on employment dynamics. In particular, the employment relationship starts with a probation phase in which the worker’s pay is equal to the wage floor. In terms of pay dynamics, there is a discrete jump in both the average pay level and pay sensitivity to performance after the worker transitions into the bonus phase. In terms of turnover dynamics, the turnover rate is initially increasing and is eventually decreasing.

We also study the role of commitment by comparing the optimal relational contract with the optimal long-term contract. When the surplus is high, these two are identical. When the surplus is low, significant differences arise. In terms of survival probability, the lack of commitment makes the termination of the relationship more likely. In terms of long-run dynamics, all surviving firms reach the first best and are stationary under long-term contracts. Under relational contracts, in contrast, all surviving firms cycle in their productivity and wages. This not only leads to lower productivity and wage levels but also renders both more volatile.

The current model studies a single principal-agent relationship. A natural extension would be to allow for multiple firms and workers. One consequence of this extension would be that the optimal relational contract could become renegotiation-proof. In particular, when the firm can easily find a replacement in the case of agent termination, the left region in the PPE payoff becomes a flat line instead of an upward-sloping one. When the agent’s continuation payoff falls into the left region, the principal is indifferent between keeping the current agent and finding a new one, and, hence, no renegotiation takes place. In general, allowing the firm to replace the worker does not affect the basic structure of employment dynamics. However, replacement does affect the turnover rate and the action the agent takes, and further research on this is needed.

Another consequence of the extension would be that there could be multiple self-fulfilling equilibrium turnover patterns. The reason is that turnover patterns affect the outside options, which in turn affect the surplus in the relationship, which affects the turnover pattern. In particular, consider an economy in which vacant firms and unemployed workers match randomly. When it becomes easier to form new employment relationships, the surplus in the existing relationship is lowered due to the higher outside options. Our analysis implies that the relationships are more likely to dissolve when the surplus is lower. This increases the number of vacant firms and unemployed workers, making it even easier to form new employment relationships. Such multiplicity may help shed light on the large cross-country differences in employment patterns.
References


8 Appendix

Proof of Lemma 1: Part (i.): Note that \( u \) and \( v \) are the agent’s and the principal’s minmax payoffs. It is then immediate that in any PPE the wages are bounded. As a result, we can restrict the agent’s and the principal’s actions to compact sets. Standard arguments then imply that the PPE payoff set \( E \) is compact so that

\[
 f(u) = \max \{ v, (u,v) \in E \}.
\]

Part (ii.): the concavity of \( f \) follows immediately from the availability of the public randomization device. Next, we show that \((\pi,\nu)\) is in the PPE payoff set. To do this, notice that \((\pi, f(\pi)) \in E\) by part (i.), and it suffices to show that \( f(\pi) = \nu \). Now suppose to the contrary that \( f(\pi) > \nu \). Since \((\pi, f(\pi))\) is an extremal point, it must be sustained by pure actions. First suppose that it is supported with effort in period 1, and the associated wage is \( w \) and the associated continuation payoffs be \((u_1, u_h, v_l, v_h)\). Now consider an alternative strategy profile with the same first-period continuation payoffs \((u_1, u_h, v_l, v_h)\) but in which first-period wage is given by \( \tilde{w} = w + \varepsilon \). It follows from the promise keeping constraints PK\(P\) and PK\(A\) that under this alternative strategy profile the payoffs are given by \((\pi + (1 - \delta) \varepsilon, f(\pi) - (1 - \delta) \varepsilon)\). When \( f(\pi) > \nu \), it can be checked that this alternative strategy profile satisfies all the constraints in Section 3.1. and therefore constitutes a PPE. Since \( \tilde{u} > \pi \) this contradicts the definition of \( \pi \). Next, if \((\pi, f(\pi))\) is supported with suspension, identical argument (with the same variation \( \tilde{w} = w + \varepsilon \)) can be carried out, and this proves that \( f(\pi) = \nu \), and thus, \((\pi, \nu)\) is in the PPE payoff set. It then follows that, given the public randomization device, any payoff on the line segment between \((u, \nu)\) and \((\pi, \nu)\) can be supported as a PPE payoff. In other words, \((u, v)\) is a PPE payoff for any \( u \in [u, \pi] \). Finally, the randomization between \((u, \nu)\) and \((u, f(u))\) allows us to obtain any payoff \((u, v')\) for all \( v' \in [\nu, f(u)] \). ⊓⊔

Proof of Lemma 2: Part (i.): Let \((u, f(u))\) be associated with first-period wage \( w \) and the continuation payoffs \((u_1, u_h, v_l, v_h)\). Suppose to the contrary of the claim that \( v_h < f(u_h) \). Now consider an alternative strategy profile with the same \( w \) but in which the continuation payoffs are given by \((u_1, u_h, v_l, \tilde{v}_h)\), where \( \tilde{v}_h = v_h + \varepsilon \) and where \( \varepsilon > 0 \) is small enough such that \( v_h + \varepsilon \leq f(u_h) \). It follows from the promise keeping constraints PK\(P\) and PK\(A\) that under this alternative strategy profile the payoffs are given by \( \tilde{u} = u + \delta p \varepsilon > f(\pi) \). It can be checked that this alternative strategy profile satisfies all the constraints in Section 3.1 and therefore constitutes a PPE. Since \( \tilde{u} > f(u) \) this contradicts the definition of \( f(u) \) from Lemma 1. Thus it must be that \( v_h = f(u_h) \). The argument for \( v_l = f(u_l) \) is identical, and similarly, so is the proof for \( v_s = f(u_s) \) in part (ii.). ⊓⊔

Proof of Lemma 3: Part (i.): Let \((u, f(u))\) be associated with first-period wage \( w \) and the continuation payoffs \((u_1, u_h, f(u_1), f(u_h))\). Suppose that for this PPE the IC is slack, that is,
\( \delta (p - q)(u_h - u_l) > (1 - \delta) c. \) Now consider an alternative strategy profile with the same first-period wage \( w \) but in which the continuation payoffs are given by \((\tilde{\pi}, \tilde{\gamma}, f(\tilde{\pi}), f(\tilde{\gamma}))\), where \( \tilde{\pi} = u_s + \rho \varepsilon \) and \( \tilde{\gamma} = u_h - (1 - p) \varepsilon \) for \( \varepsilon > 0 \). It follows from the promise keeping constraints \( \text{PK}_P \) and \( \text{PK}_A \) that under this strategy profile the payoffs are given by \( \tilde{\pi} = u \) and

\[
\tilde{\pi} = (1 - \delta) (py - w) + \delta ((1 - p) f(\tilde{\pi}) + pf(\tilde{\gamma})).
\]

From the concavity of \( f \) it then follows that

\[
\tilde{\pi} \geq (1 - \delta) (py - w) + \delta ((1 - p) f(\pi) + pf(\gamma)) = f(\pi).
\]

It can be checked that for sufficiently small \( \varepsilon \) this alternative strategy profile satisfies all the constraints in Section 3.1. and therefore constitutes a PPE. Since \( \tilde{\pi} \geq f(\pi) \) this implies that for any PPE with payoffs \((\pi, u(\pi))\) for which IC is not binding there exists another PPE for which IC is binding and which gives the parties weakly larger payoffs.

Part (ii.): Consider a PPE payoff \((u, v)\) that is supported with suspension. In this case, the agent’s promise-keeping condition then implies \( u_s(\pi) = (u - (1 - \delta) w)/\delta \). Define

\[
g(\pi) \equiv (1 - \delta) (qy - w) + \delta f(u_s(\pi)),
\]

which is the maximum PPE payoff that gives the agent \( u \) and requires suspension in period 1. In Step 1 of the proof, we show that if \( g(\pi) = f(\pi) \) for some \( \pi \), then \( g(\pi') = f(\pi') \) for all \( \pi' \in [\underline{\pi}, \overline{\pi}] \) and that \( f \) is a straight line in \([\underline{\pi}, \overline{\pi}]\). In Step 2, we show that there exists \( \pi^* \geq \pi_0 \) such that \((\pi, f(\pi))\) can be supported with effort for all \( \pi \geq \pi^* \) and \( f(\pi) \) is a straight line in \([\underline{\pi}, \pi^*]\). In Step 3, we prove \( \pi^* = \pi_0 \).

For Step 1, there are two cases to be considered. First, \( \pi \leq \pi_0 \) and we argue below that it is possible. In this case, \( u_s(\pi') \leq \pi' \) for all \( \pi' \leq \pi \) since the \( \text{PK}_A \) implies that \( u_s(\pi') = (\pi' - (1 - \delta) w)/\delta \). It follows that the right derivative of \( g \) satisfies

\[
g'_+(\pi') = f'_+(u_s(\pi')) \geq f'_+(\pi'),
\]

where the inequality follows because \( f \) is concave. This implies that

\[
g(\pi) = g(\pi) + \int_{\pi}^{\pi_0} g'_+(x) dx \geq f(\pi) + \int_{\pi}^{\pi_0} f'_+(x) dx = f(\pi),
\]

where the integral is in the sense of Lebesgue. By the definition of \( f \), we must also have \( f(\pi) \geq g(\pi) \), and therefore, \( f(\pi) = g(\pi) \). By the definition of \( g \), however,

\[
g(\pi) = (1 - \delta) (qy - w) + \delta f(\pi) = (1 - \delta) (qy - w) + \delta g(\pi),
\]

42
implying \( g(w) = qy - w = f(w) \). But this is impossible because \( qy - w \leq qy - u < v \).

Next, we come to the second case that \( u > w \). In this case, notice that we have \( u_s(u') > u' \) for all \( u' \in [w, u] \). As a result,

\[
g'_+ (u') = f'_+ (u_s(u')) \leq f'_+ (u') \quad \text{for all} \quad u' \in [w, u].
\]

Now if \( w \geq u \), then there is a contradiction (in \( g(u) = f(u) \)) because

\[
g(u) = g(w) + \int_w^u g'_+(x)dx < f(w) + \int_w^u f'_+(x)dx = f(u),
\]

where the strict inequality follows because we established that \( g(w) < f(w) \) in the previous case. This implies that we must have \( w < u \). In this case,

\[
f(u) = g(u) = g(w) + \int_w^u g'_+(x)dx = f(u).
\]

Therefore, for this chain of inequality to be satisfied, we must have a) \( g(u) = f(u) \) and b) \( f'_+(u') = g'_+(u') = f'_+(u_s(u')) \) almost everywhere. Since \( u_s(u') > u' \) for all \( u' \in [w, u] \), b) implies that \( f'_+(x) \) is a constant for \( x \in [u', u_s(u')] \) for all \( u' \in [w, u] \). The concavity of \( f \) then implies that \( f \) is a straight line in \([w, u]\). Moreover, by replacing \( u \) with any \( u' < u \) in the chain of inequality above, we have that \( f(u') = g(u') \) for all \( u' \in [w, u] \). This proves Step 1.

In Step 2, we show that there exists \( u^* \geq u_0 \) such that \((u, f(u))\) can be supported with effort for all \( u \geq u^* \) and \( f(u) \) is a straight line in \([u, u^*] \). To see this, notice that there are three possibilities with respect the relationship between \( f \) and \( g \). First, \( f(u) < g(u) \) for all \( u \). Second, \( f(u) = g(u) \) and \( f(u) < g(u) \) for all \( u > u^* \). Third, \( f(u) = g(u) \) for some \( u > u_0 \). We claim that in all three cases, there exists a nontrivial line segment with \((u, f(u))\) as its left end point. For the first two cases, the reason is that \((u, f(u))\) must be sustained by randomization for all \( u \in (u, u_0) \) since in this region \( g(u) < f(u) \) (so it is not supported with suspension) and \( L(u) < u \) (so it cannot be supported with effort). It follows that \( f \) is a straight line in \([u, u_0] \). In the third case, Step 1 implies that \( f \) is a straight line in \([u, u] \) for some \( u > u \). Let \( u^* \) be the agent’s payoff associated with the right end point of this line segment. Notice that \( u_s(u^*) > u^* \), so \((u^*, f(u^*))\) cannot be supported with suspension by the argument in Step 1. Moreover, since \((u^*, f(u^*))\) is an extremal point, it cannot be supported with randomization. Therefore \((u^*, f(u^*))\) must be supported with effort. Combining all three cases, we see that there exists \( u^* \geq u_0 \) such that a) \( f(u) \) is a straight line in \([u, u^*] \), b) \((u^*, f(u^*))\) is supported with effort, and c) \( g(u) < f(u) \) for all \( u \geq u^* \) so the payoff frontier cannot be supported with suspension for \( u \geq u^* \).

To finish Step 2, it remains to be shown that for each \( u \geq u^* \), \((u, f(u))\) can be supported with effort. To see this, suppose to the contrary that \((u, f(u))\) is not sustained by effort for some \( u > u^* \). The argument above then implies that \((u, f(u))\) must be sustained by ran-
domination (since it cannot be sustained by suspension.) Then there exists a \( \rho \in (0,1) \), a \( \tilde{u}_1 < u \), and a \( \tilde{u}_2 > u \) such that (i.) \((\tilde{u}_1, f(\tilde{u}_1))\) and \((\tilde{u}_2, f(\tilde{u}_2))\) are sustained by pure actions, (ii.) \((u, f(u)) = \rho(\tilde{u}_1, f(\tilde{u}_1)) + (1 - \rho)(\tilde{u}_2, f(\tilde{u}_2))\). Since \( u > u^* \) and \( f \) is concave, it must be that \( \tilde{u}_1 \geq u^* \). It follows that both \((\tilde{u}_1, f(\tilde{u}_1))\) and \((\tilde{u}_2, f(\tilde{u}_2))\) are supported with effort. Now suppose \((\tilde{u}_i, f(\tilde{u}_i)), i = 1, 2\) are associated with first-period wages \( w_i \) and continuation payoffs \((u_i, u_{hi}, v_i, v_{hi})\). Consider an alternative strategy profile with first-period wage \( \hat{w} \) and continuation payoffs \((\hat{u}_i, \hat{u}_{hi}, \hat{v}_i, \hat{v}_{hi})\), where \( \hat{w} = \rho w_1 + (1 - \rho)w_2 \) and where \( \hat{u}_i, \hat{u}_{hi}, \hat{v}_i, \hat{v}_{hi} \) are defined analogously. It follows from the promise keeping constraints \( PK_P \) and \( PK_A \) that under this alternative strategy profile the payoffs are given by \( \hat{u} = \rho \tilde{u}_1 + (1 - \rho)\tilde{u}_2 = u \) and \( \hat{v} = \rho(\tilde{u}_1, f(\tilde{u}_1)) + (1 - \rho)(\tilde{u}_2, f(\tilde{u}_2)) = f(u) \). Moreover, it can be checked that the new strategy profile satisfies all of the constraints in Section 3.1, so it is a PPE. This implies that \((u, f(u))\) can be supported with effort. This finishes Step 2.

Finally, we prove that \( u^* = u_0 \). Suppose to the contrary that \( u^* > u_0 \). It suffices to show that \((u, f(u))\) can be supported with effort for some \( u < u^* \). By the definition of \( u^* \), \( f \) is a straight line in \([u, u^*]\). Denote its slope as \( s \). In addition, Step 2 implies that \((u^*, f(u^*))\) is supported with effort. Let \( w \) be its period 1 wage and \((u_l, u_{hl}, f(u_l), f(u_{hl}))\) be its associated continuation payoffs. Notice that \( u_l > u \) when \( u^* > u_0 \). Now consider an alternative strategy profile with the same first-period wage \( w \) but in which the continuation payoffs are given by \((\tilde{u}_l, \tilde{u}_{hl}, f(\tilde{u}_l), f(\tilde{u}_{hl}))\), where \( \tilde{u}_l = u_l - \epsilon \) and \( \tilde{u}_{hl} = u_{hl} - \epsilon \) for \( \epsilon > 0 \). It follows from the promise keeping constraint \( PK_A \) that under this strategy profile the agent’s payoff is given by \( \hat{u} = u^* - \delta \epsilon \). In addition, the promise keeping constraint \( PK_P \) implies that under this strategy profile the principal’s payoff is given by

\[
\hat{v} = (1 - \delta)(py - w) + \delta((1 - p)f(\tilde{u}_l) + pf(\tilde{u}_{hl}))
= f(u^*) + \delta((1 - p)(f(\tilde{u}_l) - f(u_l)) + p(f(\tilde{u}_{hl}) - f(u_{hl})))
\geq f(u^*) - s\delta \epsilon,
\]

where the inequality follows because \( f \) is concave so its slope is bounded above by \( s \). Moreover, the only constraint that new strategy profile tightens is \( SE_1 \). But since \( u_l > u \), \( SE_4 \) remains satisfied for for small enough \( \epsilon \). In other words, the new strategy profile satisfies all of the constraints in Section 3.1, and it is a PPE. Since \( \hat{u} < u^* \), \( f(\hat{u}) = f(u^*) - s\delta \epsilon \) by assumption. This implies that \( \hat{v} \geq f(\hat{u}) \), so \((\hat{u}, f(\hat{u}))\) can be supported with effort. This finishes the proof.

**Proof of Proposition 1:** For expositional convenience, we start with part (iii.). To prove part (iii.), we first show that \( f'_+(u) \geq -1 \), and \( f'_+(u) = -1 \) when \( w(u) > w \). To do this, notice that given the concavity of \( f \), it suffices to show the above for all \( u \in (u^*, \pi) \), where \( u^* \) is defined in Lemma 3. Now consider a payoff pair \((u, f(u))\) on the frontier with \( u \in (u^*, \pi) \). Lemma 3 implies that \((u, f(u))\) is supported with effort. Let \( w \) be its period 1 wage and \((u_l, u_{hl}, f(u_l), f(u_{hl}))\)
be its associated continuation payoffs. Now consider an alternative strategy profile with the first-period wage \( w + \varepsilon \) and the same continuation payoffs \((u_l, u_h, f(u_l), f(u_h))\). It follows from the promise keeping constraint \( PK_A \) that under this strategy profile the agent’s payoff is given by \( \hat{u} = u + (1 - \delta)\varepsilon \). In addition, the promise keeping constraint \( PK \) implies that under this strategy profile the principal’s payoff is given by \( \hat{v} = f(u) - (1 - \delta)\varepsilon \). Moreover, for small enough \( \varepsilon > 0 \), \( \hat{v} > v \), so the new strategy profile is a PPE. By the definition of \( f \), it then follows that \( f(\hat{u}) \geq \hat{v} \), or equivalently

\[
 f(u + (1 - \delta)\varepsilon) \geq f(u) - (1 - \delta)\varepsilon.
\]

Sending \( \varepsilon \) to 0, we obtain that \( f'_+(u) \geq -1 \). Next, suppose \( w > w \). Then we can consider the same variation as above with \( \varepsilon < 0 \). In this case, sending \( \varepsilon \) to 0 leads to \( f'_-(u) \leq -1 \). It then follows that

\[
-1 \geq f'_-(u) \geq f'_+(u) \geq -1,
\]

and therefore, \( f'(u) = -1 \). Now define \( u_e = \inf\{u : w(u) > w\} \). It follows that \( f(u) = f(u_e) + u_e - u \) for all \( u \geq u_e \), and \( w(u) = w \) for all \( u \in [u_0, u_e] \).

To determine \( u_e \), there are two possibilities. First, when \( \delta \geq \delta^* \), it can be checked that the following quasi-stationary contract can be supported as a PPE: in every period, the agent gives the agent a normalized payoff of \( \pi - (w + qc/(p - q)) \). Moreover, the joint-surplus of this relationship is \( \pi - c \), so it obtains the first-best. It then follows that \( u_e \leq w + c/\delta(p - q) \), \( f(u) = \pi - c - u \) for \( u \geq u_e \). Now suppose to the contrary that \( u_e < w + c/\delta(p - q) \). Notice that

\[
L \left( w + \frac{qc}{p - q} \right) = w + \frac{qc}{p - q},
\]

so \( L(u_e) < u_e \). It then follows that \( L(u_e) + f(L(u_e)) < \pi - c \) by the definition of \( u_e \). But this implies that

\[
\begin{align*}
    u_e + f(u_e) &= (1 - \delta)(\pi - c) + \delta[p(L(u_e) + k) + f(L(u_e) + k)] + (1 - p)(L(u_e) + f(L(u_e))) \\
    &< \pi - c,
\end{align*}
\]

contradicting \( u_e + f(u_e) = \pi - c \). Therefore, \( u_e = L(u_e) = w + qc/(p - q) \) when \( \delta \geq \delta^* \).

Second, when \( \delta < \delta^* \), it is clear that \( u_e < w + qc/(p - q) \) (because otherwise we reach first best by the above, contradicting \( \delta < \delta^* \)). It then follows that \( L(u_e) < u_e \). Recall that \((u_e, f(u_e))\) is supported with effort with first-period wage \( w \). Denote its associated continuation payoff as \((u_l, u_h, f(u_l), f(u_h))\), where we may assume \( u_l = L(u_e) \) and \( u_h = L(u_e) + k \) by part
(i.) of Lemma 3. Now suppose to the contrary that $u_h < \overline{\pi}$. Now consider an alternative strategy profile with the same first-period wage $w$ but in which the continuation payoffs are given by $(\tilde{u}_t, \tilde{u}_h, f(\tilde{u}_t), f(\tilde{u}_h))$, where $\tilde{u}_t = u_t + \varepsilon$ and $\tilde{u}_h = u_h + \varepsilon$ for $\varepsilon > 0$. It follows from the promise keeping constraint $PK_A$ that under this strategy profile the agent's payoff is given by $\tilde{u} = u_e + \delta \varepsilon$. In addition, the promise keeping constraint $PK_P$ implies that under this strategy profile the principal's payoff is given by

$$ \tilde{v} = (1 - \delta)(py - w) + \delta((1 - p)f(\tilde{u}_t) + pf(\tilde{u}_h))$$

$$ = f(u_e) + \delta((1 - p)(f(\tilde{u}_t) - f(u_t)) + p(f(\tilde{u}_h) - f(u_h)))$$

$$ > f(u_e) - \delta \varepsilon,$$

where the strict inequality follows because $f(\tilde{u}_t) - f(u_t) > -\varepsilon$ since $u_t < u_e$. Moreover, the only constraint that new strategy profile tightens is $SE_h$. But since $u_h < \overline{\pi}$, $SE_h$ remains satisfied for for small enough $\varepsilon$. In other words, the new strategy profile satisfies all of the constraints in Section 3.1, and it is a PPE. But this is impossible because the above implies that $\tilde{v} > f(u_e) - \delta \varepsilon = f(\tilde{u})$ contradicting the definition of $f$. This finishes the proof that $L(u_e) + k = \overline{\pi}$ for $\delta < \delta^*$.

Part (ii.): take $u \in [u_0, u_e]$. By Lemma 3 (part (ii)), $(u, f(u))$ is supported with effort. In this case, suppose $(u, f(u))$ is associated with first-period wage $w$ and continuation payoff $(u_t, u_h, f(u_t), f(u_h))$. By part (iii.) above, we have $w = w$. Finally, the promise keeping constraint $PK_P$ implies that $f(u) = (1 - \delta)(py - w) + \delta[pf(L(u) + k) + (1 - p)f(L(u))]$.

Before moving to the left region (part (i.)), we show that $u_e < \overline{\pi}$. To see this, notice that if $\delta \geq \delta^*$, $f(u_e) - \overline{w} \geq k$. This implies that $f(u_e) > \overline{u}$, and therefore, $u_e < \overline{\pi}$ by Lemma 1. Now consider $\delta < \delta^*$. In this case, suppose to the contrary that $u_e = \overline{\pi}$. By Condition (NT) (at the end of Section 2), we have $L(\overline{\pi}) > \overline{w}$. Part (ii.) then implies that

$$ f'(u_e) = (1 - p)f'_-(L(u_e)) + pf'_-(u_e).$$

Since $f$ is concave and $L(u_e) = u_e - k < u_e$, this implies that $f'_-(u_e) = f'_-(L(u_e))$, so $f$ is a straight line between $[u_e - k, u_e]$. Let the slope of this line segment be $s$. Notice that $s < 0$, and as a result, $L(u_e) > u_0$. It then follows that $L(u_e)$ is again in the middle region, and as a result, we have

$$ f'_-(L(u_e)) = (1 - p)f'_-(L(u_e)) + pf'_-(L(u_e) + k),$$

or alternatively, $f'_-(L^2(u_e)) = s$. Repeating the same argument, we have $f'_-(L^N(u_e)) = s$ for all $N > 0$. It then follows that we must have $L(u_e) = u_e$. But this contradicts $\delta < \delta^*$.

Finally, we prove part (i.). Notice that the randomization part follows directly from part (ii.) of Lemma 3. It remains to prove that there exists $w^* < \overline{u}$ such that $f(w^*) = \overline{u}$ if and only if $\overline{w} \geq w^*$. Also recall from the proof of part (ii.) of Lemma 3 that $g(u) < f(u)$ for all
$u$ when $w \geq u$. This implies that $w^* < u$. Therefore, if suffices to show that if $f(u, w) > v$ then $f(u, w - \varepsilon) > v$ for $\varepsilon > 0$, where we use $f(u, w)$ to indicate the dependence of $f$ on $w$. In addition, we denote $u_s(|w|)$ as the agent’s continuation payoff (following suspension) when the wage floor is $w$. By PKA, we have $u_s(|w| - \varepsilon) = u_s(|w|) + (1 - \delta)\varepsilon/\delta$. Now suppose $(1 - \delta)(qy - w) + \delta f(u_s(|w|)) > v$. It then follows that

$$
(1 - \delta)(qy - (w - \varepsilon)) + \delta f(u_s(|w| - \varepsilon), w - \varepsilon) \\
\geq (1 - \delta)(qy - (w - \varepsilon)) + \delta f(u_s(|w|) + 1 - \frac{\delta}{\delta} \varepsilon, w) \\
\geq (1 - \delta)(qy - w) + \delta f(u_s(|w|), w) \\
\geq v.
$$

The first inequality follows from the fact that $f$ is weakly decreasing in $w$. The second inequality follows because $f(u, w)$ has a left derivative (with respect to $u$) weakly bigger than $-1$ for all $u$. This completes the proof. ■

**Proof of Proposition 2:** Notice that Proposition 1 implies that $(u_e, f(u_e))$ is supported with effort with associated first-period wage $w$ and continuation payoff $(u_e, u_e + k, f(u_e), f(u_e + k))$. Now consider an alternative strategy profile that is also associated first-period wage $w$ but with continuation payoff $(u_e - \varepsilon, u_e + k - \varepsilon, f(u_e - \varepsilon), f(u_e + k - \varepsilon))$. The promise keeping constraints PKA and PKP imply that this new strategy profile gives the agent payoff $\tilde{u} = u_e - \delta\varepsilon$ and the principal a payoff

$$
\tilde{v} = (1 - \delta)(py - w) + \delta[pf(u_e + k - \varepsilon) + (1 - p)f(u_e - \varepsilon)].
$$

Moreover, for small enough $\varepsilon > 0$, it can be checked that this strategy satisfies all the constraints in Section 3.1 and is a PPE. In this case, the definition of $f$ implies that

$$
\tilde{v} = f(u_e) + \delta[p\varepsilon + (1 - p)(f(u_e + k - \varepsilon) - f(u_e))] \leq f(\tilde{u}) = f(u_e - \delta\varepsilon).
$$

Sending $\varepsilon$ to 0, the inequality above implies that

$$
f'_-(u_e) \leq -1.
$$

Since $f'_+(u_e) = -1$ and $f$ is concave, this implies that $f'(u_e) = -1$, and as a consequence, there exists $u < u_e$ such that $f(u) > f(u_e)$. ■

**Proof of Proposition 3:** From Propositions 1 and 2, we know that the game starts in $H_1$ and this directly gives (i). Since $(\underline{w}, \underline{v})$ is the unique PPE payoff in which the agent’s payoff is $\underline{u}$ when $w \geq w^*$ and it is supported by players taking their outside options, once the agent’s
payoff reaches $\underline{u}$, he stays there forever. The wage formula in history $H_3$ follows directly from
Proposition 1.

Next, we show that $L(u^*) < u^*$ and $L(u^*) + k > u^*$, where $u^*$ is the agent’s payoff associated
with the optimal relational contract.\footnote{If there are multiple agent’s payoffs that maximize the principal’s payoff, take $u^*$ as the smallest one.} Given $L(u_e) = u_e$, and since $u^* < u_e$, we immediately
have $L(u^*) < u^*$. To see that $L(u^*) + k > u^*$, note that $f_-'(u^*) \geq 0$ and $f_+(u^*) \leq 0$. For small
$\varepsilon > 0$, we have

$$f(u^* + \varepsilon) - f(u^*) = \delta[p(f(u^* + \varepsilon) + k) - f(L(u^*) + k)] + (1 - p)(f(L(u^* + \varepsilon)) - f(L(u^*))).$$

Sending $\varepsilon$ to 0, we have

$$f_+(u^*) = pf_+(L(u^*)) + (1 - p)f_+(L(u^*) + k)$$

If $L(u^*) + k \leq u^*$, then $L(u^*) < u^*$, so $f_+(L(u^*)) > 0$. Now consider two cases. In case 1,
$L(u^*) + k < u^*$, and the left hand side is strictly positive, violating the condition $f_+(u^*) \leq 0$.
In case 2, $L(u^*) + k = u^*$, and we have $f_+(u^*) = f_+(L(u^*)) > 0$, again violating the condition
$f_+(u^*) \leq 0$.

Now given $L(u^*) < u^*$, consider the history in which $n$ consecutive low outputs occurred.
Notice that $L(u) - u$ is increasing in $u$, so $L(u) - u < 0$ for all $u < u^*$. Moreover, the step
size of moving to the left, $|L(u) - u|$, is larger for smaller $u$. It then follows immediately that
for sufficiently large $n$, the agent’s payoff goes into the left region, and this proves part (ii.).
Similarly, given that $L(u^*) + k > u^*$, a sequence of consecutive high outputs will move the agent’s
continuation payoff into the right region since $L(u) + k - u$ is increasing in $u$ (so that the step
size of moving to the right is increasing after each high output). This proves part (iii.). Part
(iv.) follows because there is no absorbing states other than $u = \underline{u}$ and $u \in [u_e, \overline{u}]$.

Proof of Corollary 1: The condition $\delta \geq \delta^*$ implies efficient outcome is attainable and the
condition $p \leq q + \delta(u - \underline{u})/(1 - \delta)$ implies that $u + k \geq u_e$, or for all $u \in (u_0, u_e)$,
$L(u) + k \geq u_e$. In this case, we can explicitly solve the PPE payoff frontiers. In particular,
there exists a sequence of $\{u_n\}_{n=1}^\infty$ with $u_n = u_0 + \delta(1 - \delta^n)(u_0 - u)/ (1 - \delta)$ such that for each
$u \in [u_n, u_{n+1}]$, we have $f'(u) = s_{n+1}$, where $s_n = s_0 - (1 + s_0)(p + (1 - (1 - p)^{n+1}))$ and

$$s_0 = \frac{(1 - \delta)(py - w) + \delta((1 - p)y + p(py - c - (u + k))) - w}{(1 - \delta)(w - c) + \delta(u + pk) - w}.$$ 

Now there are two cases to consider. In the first case (the generic case), there’s a unique $u_n$
that maximizes $f(u)$. In this case, if the outputs in the first $n + 2$ periods are all bad, then the
agent’s continuation payoff moves to $u_{n+1}$ in period $t$ and goes down to $\underline{u}$ in period $t = n + 2$,
leading to termination of the relationship. However, if any of the output in the first \(n+2\) periods is positive, the agent’s payoff is raised beyond \(u_e\) and receives permanent employment.

In the second case (the non-generic case), there exists \(n\) such that \(f(u)\) is maximized in \([u_{n-1}, u_n]\). In this case, if no positive outcome has been generated, the agent’s continuation will be in \([u_{n-t}, u_{n+1-t}]\) in time \(t\). And the agent will be terminated either in time \(t = n + 1\) or \(t = n + 2\). The proof is completed by setting \(T^* = n + 1\). ■

**Proof of Proposition 4:** Solving the optimal long-term contract follows the same steps of solving the optimal relational contract except that the self-enforcing conditions (SE\(_l\)) and (SE\(_h\)) are not imposed on the optimal relational contract. Removing these constraints may only potentially improve efficiency by allowing the principal’s payoff to fall below \(\bar{u}\). Define \(\tilde{u}\) by \(L(\tilde{u}) = \tilde{u}\). When \(\delta \geq \delta^*\), following any path in which the agent’s payoff is raised above \(\tilde{u}\), the continuation equilibrium is efficient and can be implemented by a continuation equilibrium in which the players receive \((\tilde{u}, py - c - \tilde{u})\) if the outcome in the previous period was bad and \((\tilde{u} + k, py - c - (\tilde{u} + k))\) if the output was good, where

\[
py - c - \tilde{u} > py - c - (\tilde{u} + k) \geq \bar{u}.
\]

Therefore, allowing \(v\) to fall below \(\bar{u}\) does not expand the equilibrium payoff set.

On the other hand, when \(\delta < \delta^*\), the efficient outcome in which \(u + v = py - c\) is not attainable in a relational contract. Therefore, imposing constraints SE\(_l\) and SE\(_h\) cause \(f_R(u|\delta) < f_{LT}(u|\delta)\) for \(u \geq u_e\). On the other hand, since \(f_R(u|\delta) \leq f_{LT}(u|\delta)\) for all \(u \in [\underline{u}, \bar{u}]\) and \(u^*_R(u) > u_e\) for sufficiently large number \((n)\) of good outcomes, so \(f_R(u|\delta) < f_{LT}(u|\delta)\) for all \(u \in [u_0, u_e]\). And \(f_R(u_0|\delta) < f_{LT}(u_0|\delta)\) further implies \(f_R(u|\delta) < f_{LT}(u|\delta)\) for all \(u \in (\underline{u}, u_0)\). ■

**Proof of Proposition 5:** First, we prove that \(w^*_R < w^*_{LT}\). Recall from the proof of Proposition 4 that if \(\delta < \delta^*\), then \(f_{LT}(u) > f_R(u)\) for all \(u \in (\underline{u}, \bar{u})\). By definition of \(w^*_{LT}, w^*_R\), and the fact that \(f_{LT}(u_s(\underline{u})) > f_R(u_s(\underline{u})),
\]

\[
(1 - \delta) (qy - w^*_R) f_R(u_s(\underline{u})) = v = (1 - \delta) (qy - w^*_{LT}) f_{LT}(u_s(\underline{u}))
\]

\[
> (1 - \delta) (qy - w^*_{LT}) f_R(u_s(\underline{u})).
\]

It follows that \(w^*_R < w^*_{LT}\).

Part (i.): When \(\underline{w} > w^*_R\) \((> w^*_{LT})\), both under optimal relational contract and optimal long-term contract, when \(u\) falls below \(u_0\), the relationship is terminated with a positive probability. Under the optimal long-term contract, the agent still gets tenure when \(u\) is raised above \(u_e\), which happens with a positive probability. However, under the optimal relational contract, the agent never gets tenure, the relationship is eventually terminated.

Part (ii.): When \(\underline{w} \in [w^*_R, w^*_{LT}]\), following the same logic as in Part (ii), the optimal relational contract terminates with probability one. By contrast, since the optimal long-term contract is
not punished with termination, it survives with probability one. ■

Proof of Proposition 6: Part (i.): First, recall that under the optimal relational contract, in the right region of the equilibrium payoff frontier, full efficiency is achieved in the bonus phase and when the relationship reaches the bonus phase, \( E(Y_t) = py \). Also, the efficient outcome is supported by a quasi-stationary contract in which the agent receives a base wage of \( w \) and a bonus of \( \frac{1-\delta}{\delta} \frac{c}{p-q} \) for previous period’s good high output with probability \( p \). In the long run, the agent is either terminated or in the absorbing bonus phase, and obviously every surviving relationship in the long run is in the quasi-stationary bonus phase.

Part (ii.): In the case that \( w > w^*_R \), the optimal relational contract survives with probability 0 according to Proposition 5. In the case that \( w \leq w^*_R \), bad outcomes are punished by temporary suspension of effort instead of termination. Therefore, the relationship survives with probability one. Therefore, any relationship surviving in the long run must be one with \( w \leq w^*_R \). However, since the bonus phase is not absorbing, the relationship cycle among temporary suspension of effort, the probation phase, and the bonus phase. Since temporary suspension of effort occurs with a positive probability, \( E(Y_t) < py \). Since the relationship stays in the bonus phase with probability less than one, \( E(w_t) < w + \frac{1-\delta}{\delta} \frac{pc}{p-q} \). ■

50
9 Appendix A (For Online Publication)

Proof of Lemma A1: We prove part (i.) first. We show that there exists $u_0$ such that $f$ is sustained by randomization if and only if $u \in (u, u_0)$. To see this, again let $u_h(u)$ be the continuation payoff following a high output and $u_l(u)$ the continuation following a failure. To induce effort, the incentive compatibility constraint of the agent requires

$$u_h(u) - u_l(u) \geq k.$$ 

This implies that if the worker puts in effort, his payoff satisfies

$$u = (1 - \delta)(w - c) + \delta(u_l(u) + p(u_h(u) - u_l(u)))$$

$$\geq (1 - \delta)(w - c) + \delta(u + pk).$$

Therefore, for $u \in (u, (1 - \delta)(w - c) + \delta(u + pk))$, effort cannot be provided, so the payoff frontier is sustained by randomization. It follows that for $u \in [u, (1 - \delta)(w - c) + \delta(u + pk)]$, the payoff frontier is a line segment where the left point is $(u, v)$, i.e., $f(u) = v + s(u - u)$ for some $s > 0$. Now denote the right end point of the line segment as $(u_0, f(u_0))$, i.e.,

$$u_0 \equiv \max\{u : f(u) = v + s(u - u)\}.$$ 

By the definition of $u_0$, we have $f(u) = f(u) + \frac{u - u}{u_0 - u}(f(u_0) - v)$ for all $u \in [u, u_0]$.

Next, we show that randomization is not needed for all payoff pairs $(u, f(u))$ when $u > u_0$. To see this, suppose that

$$(u, f(u)) = \alpha(u_1, f(u_1)) + (1 - \alpha)(u_2, f(u_2))$$

where $\alpha \in (0, 1)$, and $(u_i, f(u_i))$ are two extremal points on the frontier for $i = 1, 2$ with $u_1 < u_2$. Notice we must have $u_1 > u$ because a linear combination using $(u, v)$ and $(u_2, f(u_2))$ is strictly dominated by the corresponding linear combination using $(u_0, f(u_0))$ and $(u_2, f(u_2))$. Let $(w_i, u_i, u_h)$ be the associated wages and continuation payoffs for $i = 1, 2$. Now consider a payoff pair sustained by the following. Specifically, the principal pays out $w = \alpha w_1 + (1 - \alpha)w_2$ in the first period (and the worker puts in effort), and the continuation payoffs following a high and low output be $(u_1, f(u_1))$ and $(u_h, f(u_h))$, where $u_1 = \alpha u_1 + (1 - \alpha)u_2$ and $u_h = \alpha u_h + (1 - \alpha)u_h$. It is clear that this payoff pair is a PPE payoff and it gives the agent a payoff of $u$. Moreover, it gives the principal a payoff of

$$(1 - \rho)(py - w) + \rho(pf(u_h) + (1 - p)f(u_1))$$

$$\geq \alpha f(u_1) + (1 - \alpha)f(u_2).$$

51
by the concavity of $f$. This shows that the payoff frontier can be sustained by pure strategies for all $u \geq u_0$.

We prove part (iii.) next. To see this, we first show that $f'_+ (u) \geq -(1 - \rho) / (1 - \delta)$ for all $u$. Notice that for $u < u_0$, this is clearly satisfied since $f'(u) > 0$ for $u < u_0$ by (i). For $u \in [u_0, \bar{u}]$, part (i.) above shows that $(u, f(u))$ can be sustained by pure actions. Moreover, it is clear that $f(u) > \bar{u}$. Now suppose $(u, f(u))$ is sustained by a wage $w$ in the first period and the continuation payoffs $(u_i, f(u_i))$ for $i = l, h$. Consider a new payoff pair sustained by first-period wage $w + \varepsilon$ and the same continuation payoffs. For small enough positive $\varepsilon$, this new set of actions and continuation payoffs satisfy all of the constraints and constitutes a PPE. Moreover, this PPE payoff is given by $(u + (1 - \delta) \varepsilon, f(u) - (1 - \rho) \varepsilon)$. By the definition of $f$, we have

$$f(u + (1 - \delta) \varepsilon) \geq f(u) - (1 - \rho) \varepsilon.$$ 

Sending $\varepsilon$ to 0, we obtain $f'_+ (u) \geq -(1 - \rho) / (1 - \delta)$. Now define $u_\varepsilon = \inf \{ u, f'(u) = -\frac{1 - \rho}{1 - \delta} \}$. This proves (iii).

Finally, we prove part (ii.). In particular, suppose $(u, f(u))$ is sustained by $(w, u_l, u_h)$, we need to show that $w = \underline{w}$, $u_l = L(u)$ and $u_h = L(u) + k$ for all $u \leq u_\varepsilon$. We first show that $w(u) = \underline{w}$. To see this, suppose to the contrary that $w > \underline{w}$. Then the same argument as in the proof of part (iii.) shows that $(u + (1 - \delta) \varepsilon, f(u) - (1 - \rho) \varepsilon)$ is again a PPE payoff for small negative $\varepsilon$, and therefore, $f'_- (u) \leq -(1 - \rho) / (1 - \delta)$. Since $f'_- (u) \geq f'_+ (u)$, we must have $f' (u) = -(1 - \rho) / (1 - \delta)$. This contradicts the definition of $u_\varepsilon$, and therefore, we must have $w(u) = \underline{w}$ for all $u \leq u_\varepsilon$. Next, we show $u_l = L(u)$ and $u_h = L(u) + k$. Suppose the contrary. Then the promise-keeping constraint for the agent and the incentive compatibility constraint for the agent must imply that $u_l < L(u)$ and $u_h > L(u) + k$. Now consider a new payoff pair sustained by $w = \underline{w}$, $u'_l = L(u)$ and $u'_h = L(u) + k$. For small enough $\varepsilon > 0$, this new set of action and continuation satisfies all constraints and supports a PPE payoff. Moreover, the new PPE gives the agent a payoff of $u$ and the principal a payoff of

$$(1 - \rho) (py - \underline{w}) + \rho ((1 - p) f(L(u)) + pf(L(u) + k))$$ 

$$\geq (1 - \rho) (py - \underline{w}) + \rho ((1 - p) f(u_l) + pf(u_h)),$$

where the inequality follows because $f$ is concave, $u_l < L(u) < L(u) + k < u_h$, and

$$(1 - p) u_l + pu_h = (1 - p) L(u) + p (L(u) + k).$$

By the definition of $f$, we must then have

$$f(u) = (1 - \rho) (py - \underline{w}) + \rho ((1 - p) f(L(u)) + pf(L(u) + k)).$$

This proves part (ii.).
Notice that for $f$ to be the PPE payoff frontier, we should also check that

$$ f(u) \geq \max_{w \geq u, u_s \in [u, \bar{u}]} (1 - \rho) (qy - w) + \rho f(u_s), \text{ for all } u \in [u, \bar{u}] $$

so that it will not be optimal for the agent to shirk. Since $f' \geq - (1 - \rho) / (1 - \delta)$, the above condition is equivalent to

$$ f(u) \geq \max_{w \geq u, u_s \in [u, \bar{u}]} (1 - \rho) (qy - w) + \rho f\left(\frac{u - (1 - \delta)}{\delta}\right), \text{ for all } u \in [u, \bar{u}] \text{ such that } \frac{u - (1 - \delta)}{\delta} \in [u, \bar{u}]. $$

While the inequality above cannot be directly mapped into an inequality that involves exogenous conditions only, a sufficient condition for it to hold is that

$$ (1 - \rho) (qy - w) + \rho (py - c - w) \leq v. $$

In this case, expected payoff from suspending the agent is so costly for the principal that termination is preferred.

Next, we prove Proposition A2. Notice that to show Proposition A2, it suffices to prove the following lemma.

**Lemma A2:** The followings hold.

(i.) $u_t(u_e) \leq u_e.$

(ii.) $u_t(u_e) = u_e$ if and only if $\delta < \rho(1 - p)$

$$ py - w - \frac{\rho p c}{\delta (p - q)} \geq \max\left\{ \frac{1 - \rho}{\delta} \frac{c}{p - q}, \frac{1 - \rho}{1 - \delta \rho(1 - p) - \delta} \left( w + \frac{qc}{p - q} - u \right) \right\}. \text{ (Condition A) }$$

**Proof of Lemma A2:** Part (i). First note that $w(u_e) = w$ and $u_t(u_e) - u_l(u_e) = k$. Therefore, $L(u_e) \leq u_e$ is equivalent to $u_t(u_e) \leq u_e$. Suppose to the contrary that $u_t(u_e) > u_e$. Consider an alternative strategy profile sustained by first-period wage $w$ and continuation payoffs $u_t(u_e) - \varepsilon$ and $u_h(u_e) - \varepsilon$. For small enough $\varepsilon > 0$, the first-period action and continuation payoffs satisfy all constraints and support a PPE payoff. Moreover, the PPE gives the agent a payoff of $u_e - \rho \varepsilon$ and the principal a payoff of

$$ (1 - \rho) (py - w) + \rho ((1 - p) f(u_l(u_e)) - \varepsilon) + pf(u_h(u_e)) = (1 - \rho) (py - w) + \rho ((1 - p) f(u_l(u_e)) + pf(u_h(u_e))) + \frac{1 - \rho}{1 - \delta \rho} \rho \varepsilon \leq f(u_e - \rho \varepsilon), $$

where the last inequality follows the definition of $f$. 

53
But this implies that
\[
f(u_e) - f(u_e - \rho \varepsilon) \leq -\frac{1 - \rho}{1 - \delta} \rho \varepsilon,
\]
so the slope between \(u_e - \rho \varepsilon\) and \(u_e\) is weakly smaller than \(-(1 - \rho)/(1 - \delta)\). By Lemma A1, this implies that the slope between \(u_e - \rho \varepsilon\) and \(u_e\) must be equal to \(-(1 - \rho)/(1 - \delta)\). But this contradicts the definition of \(u_e\). This proves part (i).

Part (ii): We first show that if \(u_l(u_e) = u_e\), it must be the case that \(\delta < \rho (1 - p)\) and Condition A is satisfied. Notice that when \(u_l(u_e) = u_e\), we have \(u_h(u_e) = u_e + k\). As a result, \(u_e = w + qc/(p - q)\) and \(f(u_e) = py - w - pc/(p - q)\delta\). Condition A is then equivalent to
\[
f(u_e) - v \geq \frac{1 - \rho}{1 - \delta} \max\{\frac{\rho p}{\rho (1 - p) - \delta} (u_e - u), k\}.
\]

Notice that \(f(w) = v\) and \(w \geq u_h(u_e) = u_e + k\). It follows that
\[
f(u_e) - v = f(u_e) - f(w) = \frac{1 - \rho}{1 - \delta} (w - u_e) \geq \frac{1 - \rho}{1 - \delta} k.
\]

It then remains to show that \(\delta < \rho (1 - p)\) and \(f(u_e) - v \geq \frac{1 - \rho}{1 - \delta} \frac{\rho p}{\rho (1 - p) - \delta} (u_e - u)\). Consider a payoff sustained with the same actions and the continuation payoffs being \(u_l = u_e - \varepsilon\), \(u_h = u_e + k - \varepsilon\), \(f(u) = f(u_e - \varepsilon)\), \(f(u_h) = f(u_e + k - \varepsilon)\). This alternative profile generates payoff of \(u_e - \delta \varepsilon\) for the agent, and gives the principal a payoff of
\[
v = f(u_e) - \rho ((1 - p)(f(u_e - \varepsilon) - f(u_e)) + p (f(u_e + k - \varepsilon) - f(u_e + k))).
\]

By the definition of \(f\), we have \(v \leq f(u_e - \delta \varepsilon)\), or equivalently,
\[
f(u_e) - f(u_e - \delta \varepsilon) \\
\geq \rho ((1 - p)(f(u_e - \varepsilon) - f(u_e)) + p (f(u_e + k - \varepsilon) - f(u_e + k))).
\]

Sending \(\varepsilon\) to zero and noticing that \(f'(u) = \frac{1 - \rho}{1 - \delta}\) for \(u > u_e\), we obtain.
\[
(\delta - \rho (1 - p)) f'_-(u_e) \leq -\rho p \frac{1 - \rho}{1 - \delta}.
\]

Since the right hand side is negative, it is clear that we cannot have \(\delta - \rho (1 - p) = 0\). Now if \(\delta - \rho (1 - p) > 0\), this implies that
\[
f'_-(u_e) \leq -\frac{\rho p}{\delta - \rho (1 - p)} \frac{1 - \rho}{1 - \delta} < -\frac{1 - \rho}{1 - \delta},
\]
contradicting the concavity of \(f\). This proves that if \(u_l(u_e) = u_e\), we must have \(\delta - \rho (1 - p) < 0\).
Notice that when $\delta < \rho(1 - p)$, the argument above implies that

$$f'(u_e) \geq \frac{1 - \rho}{1 - \delta} \frac{\rho p}{\rho (1 - p) - \delta}.$$  

Moreover, since for all $u \in [u, u_e]$, the payoff frontier $f$ is weakly dominated by the randomization between $(u, v)$ and $(u_e, f(u_e))$. It follows that $f'_-(u_e) \leq (f(u_e) - v) / (u_e - u)$. Therefore,

$$\frac{f(u_e) - v}{u_e - u} \geq \frac{1 - \rho}{1 - \delta} \frac{\rho p}{\rho (1 - p) - \delta},$$

so

$$f(u_e) - v \geq \frac{1 - \rho}{1 - \delta} \frac{\rho p}{\rho (1 - p) - \delta} (u_e - u).$$

This proves the "if" part.

Next, we show that if $\delta < \rho(1 - p)$ and Condition A is satisfied, then $u_t(u_e) = u_e$. To see this, define $f (u)$ as follows.

$$f(u) = \begin{cases} v + \frac{u - u}{u_{\tilde{u}}} (f(u_e) - v) & \text{for } u \in [u, u_e] \\ f(u_e) + \frac{1 - \rho}{1 - \delta} (u_e - u) & \text{for } u \in [u_e, \tilde{u}] \end{cases},$$

where $u_e = \tilde{u} + \frac{q^c}{p - q}$, $f(u_e) = p y - \tilde{w} - p \frac{c}{p - q} \tilde{r}$, and $	ilde{w} = u_e + (1 - \delta) (f(u_e) - v) / (1 - \rho)$.

It is clear that under $f$, we have $u_t(u_e) = u_e$. We will show below that $f$ is the payoff frontier of the long-term contracts, and therefore, the payoff frontier of the relational contracts, and this finishes the proof.

Notice that the payoff frontier of the long-term contract must satisfy the following functional equation:

$$f(u) = \max\{f_1(u), f_2(u)\},$$

where

$$f_1(u) = \max_{u_1, u_2, \alpha \in [0, 1]}, \quad \alpha u_1 + (1 - \alpha) u_2 = u$$

and

$$f_2(u) = \max_{w \geq \omega, u_1, u_h} (1 - \rho) (p y - w) + \rho ((1 - p) f (u) + pf(u_h))$$

such that

$$(1 - \delta) (w - c) + \delta ((1 - p) u_t + pu_{h}) = u; \quad u_h - u_t \geq k.$$  

Moreover, define the operator $Tf \equiv \max\{f_1(u), f_2(u)\}$, where $T$ is defined on bounded functions with support in $[u, M]$. It can be checked that $T$ is a, monotone, non-expansive mapping, and,
thus, has a unique fixed point. Therefore, it suffices to show that $f$ constructed above satisfies $Tf = f$.

It is easy to check that $f(u) = \max\{f_1(u), f_2(u)\}$ for $u \geq u_e$. For $u < u_e$, we have $f(u) = f_1(u)$ by construction. Therefore, the proof is complete once we show that $f(u) \geq f_2(u)$ for $u < u_e$. While $f_2$ is cast as the solution to a maximization problem, argument from argument in the proof of Lemma A1 immediately that the maximizers must satisfy $w = \bar{w}$, $u_l = L(u)$ and $u_h = L(u) + k$ for $u \leq u_e$. This implies that for $u \leq u_e$, we have

$$f_2(u) = (1 - \rho)(py - w) + \rho((1 - p)f(L(u)) + pf(L(u) + k)).$$

Now notice that $f(u_e) = f_2(u_e)$ by construction. Moreover, define $s \equiv \frac{f(u_e) - u}{u_e - \bar{u}}$ as the slope of $f$ between $\bar{u}$ and $u_e$, then

$$f'_2(u) = \frac{\rho}{\delta}((1 - p)f'(L(u)) + pf'(L(u) + k)) \geq \frac{\rho}{\delta}(1 - p)s + p\left(-\frac{1 - \rho}{1 - \delta}\right) \geq s,$$

where the last inequality follows because it is equivalent to $(\rho(1 - p) - \delta)s \geq \rho p(1 - \rho)/(1 - \delta)$. Since $f(u_e) = f_2(u_e)$ and $f'_2(u) \geq f'(u)$ for all $u \leq u_e$, this implies that $f_2(u) \leq f(u)$ for all $u \leq u_e$. It follows that the constructed $f$ is the unique solution to $f = Tf$. From the construction of $f$, it is clear that $u_l(u_e) = u_e$. Therefore, we have $u_l(u_e) = u_e$ if and only if $\delta < \rho(1 - p)$ and Condition A holds.

**Proof of Proposition A2:** Part (i.) follows from exactly the same step as in Proposition 4 and is omitted. For part (ii.), notice that on the one hand, if $\delta < \rho(1 - p)$ and

$$py - w - v - p\frac{c}{p - q}\frac{\rho}{\rho p - (1 - p) - \delta} \geq \frac{1 - \rho}{1 - \delta}(\frac{\rho}{\rho p - (1 - p) - \delta} - \frac{\rho}{\delta} + \frac{qc}{p - q} - \bar{u}),$$

we can use the same argument as that in the proof of Lemma A2 to show that the payoff frontier under the long-term contract is given by

$$f_{LT}(u) = \begin{cases} f_{LT}(u_{e_{LT}}) - v & \text{for } u \in [\bar{u}, u_{e_{LT}}] \\ f_{LT}(u_{e_{LT}}) + \frac{1 - \rho}{1 - \delta} (u_{e_{LT}} - u) & \text{for } u \in [u_{e_{LT}}, \bar{u}] \end{cases},$$

where $u_{e_{LT}} = \frac{w + qc}{p - q}$, $f_{LT}(u_{e_{LT}}) = py - w - p\frac{c}{p - q}\frac{\rho}{\rho p - (1 - p) - \delta}$, and $\bar{u} = u_{e_{LT}} + (1 - \delta)(f_{LT}(u_{e_{LT}}) - v)/(1 - \rho)$. It follows that $u_{e_{LT}} = L(u_{e_{LT}})$, and as a result, the optimal long-term contract is stationary. On the other hand, when

$$\frac{1 - \rho}{\delta} \frac{c}{p - q} > py - w - v - p\frac{c}{p - q}\frac{\rho}{\delta},$$

56
the condition implies that stationary relational contract is impossible. As a result, \( L(u_{eR}) < u_{eR} \), and the relationship terminates with probability 1.

**Proof of Proposition A3:** For expository convenience, we assume that \( c \) is differentiable. When \( c'(e) \) does not exist, the argument can be adapted through the use of left-derivative. We prove part (i) first. Define \( u'_0 \) as the smallest payoff of the agent in which the payoff frontier requires a positive effort, i.e.,

\[
u'_0 = \inf\{ u : e(u) > 0 \}.
\]

then the payoff frontier is a straight line between \( (u, f(u)) \) and \( (u'_0, f(u'_0)) \). Define \( u_0 \) as the right end point of the line segment, and then part (i) follows.

Next, we prove part (iii). Notice that the same argument as in Proposition 1 shows that \( f'(u) \geq -1 \) for all \( u \). Moreover, the same argument as in Proposition 1 shows that if \( w(u) > w \), then \( f'(u) = -1 \). Define \( u_e \) as the smallest payoff of the agent such that \( f'_+(u) = -1 \), i.e.,

\[
u_e = \inf\{ u : f'_+(u) = -1 \}.
\]

It is then clear that \( f(u) = f(u_e) + (u_e - u) \) for all \( u \geq u_e \), and the stated effort and and continuation payoffs support the payoff frontier.

Part (ii): When \( u \) is in the middle region. The definition of \( u_e \) implies that the agent’s wage is given by \( w \). In addition, the agent chooses effort \( e \) to maximize

\[
(1 - \delta)(w - c(e)) + \delta[p(e)u_h(u) + (1 - p(e))u_l(u)],
\]

and the first order condition with respect to \( e \) gives that

\[
(1 - \delta)c'(e) = \delta p'(e)(u_h(u) - u_l(u)),
\]

and, thus,

\[
u_h(u) = u_l(u) + \frac{1 - \delta c'(e)}{\delta p'(e)}.
\]

Finally, the promise-keeping condition of the agent’s utility gives that

\[
u = (1 - \delta)(w - c(e)) + \delta[p(e)u_h(u) + (1 - p(e))u_l(u)] = (1 - \delta)(w - c(e)) + \delta[u_l(u) + p(e)\frac{1 - \delta c'(e)}{\delta p'(e)}],
\]

which implies that

\[
u_l(u) = \frac{1}{\delta} \left( u - (1 - \delta)(w - c(e)) + \frac{p(e)c'(e)}{p'(e)} \right).
\]

This completes the proof.
Proof of Corollary A1: We show that when $\alpha$ is small enough, the PPE payoff frontier is essentially identical to that in the main model in the sense that if $e > 0$ is chosen then $e = 1$. To do this, we take two steps. First, we show that if $e > 0$ is chosen, then either $e = \alpha$ or $e = 1$ is chosen. Second, we show that for sufficiently small $\alpha$, only $e = 1$ is chosen.

In Step 1, we first show that if $e \in (0, \alpha]$ is chosen, it is dominated by the choice of $e = \alpha$. Suppose $e \in (0, \alpha]$ is chosen, then the agent’s IC is given by

$$u_h(e) - u_I(e) = \frac{1 - \delta c_0}{\delta} \text{ for all } e \in (0, \alpha].$$

The promise-keeping condition of the agent then implies that as long as $e > 0$, we have

$$u = (1 - \delta) (w - c_0 e) + \delta (u_I(e) + pe (u_h(e) - u_I(e)))
= (1 - \delta) (w - c_0 e) + \delta \left( u_I(e) + e \frac{1 - \delta}{\delta} c_0 \right).$$

Solving the equation above, we have

$$u_I(e) = \left( u - (1 - \delta) \frac{w}{\delta} \right)/\delta.$$

Notice that $u_I(e)$ is independent of $e$, and the IC constraint then immediately implies that $u_h(e)$ is also independent of $e$. Moreover, the expression for $u_I(e)$ implies that the smallest $u$ to support effort $e \in (0, \alpha]$ is given by $u_0(e) \equiv (1 - \delta) w + \delta u_I$, which is also independent of $e$.

Next, notice that if $u$ is sustained by effort $e > 0$, the joint surplus $(u + f(u))$ is given by

$$u + (1 - \delta) (p y - c_0 e) + \delta (u_I(e) + pe (u_h(e) - u_I(e)))
= (1 - \delta) (w - c_0 e) + \delta (u_I(e) + f(u_I(e) + f(u_h(e) - u_I(e)))) \geq u_I + f(u_I).$$

This term is increasing in $e$ since $p y - c_0 > 0$ and $u_h + f(u_h) \geq u_I + f(u_I)$. The second inequality follows because $f'(u) \geq -1$ by Proposition A3. It follows that if $e > 0$ is chosen, it is strictly dominated by the choice of $e = \alpha$.

Next, we show that if $e \in (\alpha, 1]$ is chosen, it is dominated by the choice of $e = 1$. In this case, then the agent’s IC is given by

$$u_h(e) - u_I(e) = \frac{1 - \delta c_1}{\delta} \text{ for all } e \in (\alpha, 1].$$

The promise-keeping condition then implies that

$$u = (1 - \delta) (w - c_0 a - c_1 (e - a)) + \delta (u_I(e) + pe (u_h(e) - u_I(e)))
= (1 - \delta) (w + (c_1 - c_0) a) + \delta u_I(e),$$

and therefore,

$$u_I(e) = \frac{1}{\delta} \left( u - (1 - \delta) (w + (c_1 - c_0) a) \right).$$
Notice again that \( u_l(e) \) is independent of \( e \) for all \( e \in (a, 1] \), and therefore, so is \( u_h(e) \). Moreover, the smallest \( u \) to support effort \( e \in (a, 1] \) is given by

\[
    u_0(e) = (1 - \delta)(\overline{w} - c_0a - c_1(e - a)) + \delta \left( \frac{u}{\delta} + \frac{1 - \delta c_1}{p} e \right) = (1 - \delta)(\overline{w} + (c_1 - c_0)a) + \delta u,
\]

which is independent of \( e \).

Next, notice that similar to the case of \( e \in (0, a] \), if \( u \) is sustained by effort \( e \in (a, 1] \), the joint surplus is given by

\[
    (1 - \delta)(py - c_0a - c_1(e - a)) + \delta (u_l + f(u_l) + (pe)(u_h + f(u_h) - u_l - f(u_l))).
\]

This term is again increasing in \( e \) since \( py - c_1 > 0 \) and \( u_h + f(u_h) \geq u_l + f(u_l) \). It follows that if \( e \in (a, 1] \) is chosen, it is strictly dominated by the choice of \( e = 1 \). This finishes Step 1.

Step 2: We now prove that for small enough \( a \), only \( e = 1 \) is chosen. First note that \( e = 0 \) will not be chosen. The proof of this is identical to that in Lemma 3 (where we showed that \( g(u) < f(u) \) when \( \overline{w} \geq \underline{w} \) ) and is omitted here. Next, we show that \( e = a \) will not be chosen for small enough \( a \). Notice that if \( e = a \) is chosen, the joint surplus is equal to

\[
    S(a) = (1 - \delta)(pay - c_0a) + \delta (u_l(a) + f(u_l(a)) + (pa)(u_h(a) + f(u_h(a)) - u_l(a) - f(u_l(a)))).
\]

There are two cases to consider. In the first case, \( u \geq (1 - \delta)(\overline{w} + (c_1 - c_0)a) + \delta u \). In this case, both \( e = 1 \) and \( e = a \) are feasible. The joint surplus from choosing \( e = 1 \) is given by

\[
    S(1) = (1 - \delta)(py - c_0a - c_1(1 - a)) + \delta (u_l(1) + f(u_l(1)) + p(u_h(1) + f(u_h(1)) - u_l(1) - f(u_l(1)))).
\]

Notice that as \( a \) goes to zero,

\[
    \lim_{a \to 0} S(a) = \delta \left( \frac{1}{\delta} (u - (1 - \delta) \overline{w}) + f \left( \frac{1}{\delta} (u - (1 - \delta) \overline{w}) \right) \right).
\]

In contrast,

\[
    \lim_{a \to 0} S(1) = (1 - \delta)(py - c_1) + \delta \left( \frac{1}{\delta} (u - (1 - \delta) \overline{w}) + f \left( \frac{1}{\delta} (u - (1 - \delta) \overline{w}) \right) \right) + \left( \frac{1 - \delta c_1}{p} + f \left( \frac{1}{\delta} (u - (1 - \delta) \overline{w}) + \frac{1 - \delta c_1}{p} \right) - f \left( \frac{1}{\delta} (u - (1 - \delta) \overline{w}) \right) \right).
\]
Notice that \( f' \geq -1 \), so

\[
\left( \frac{1 - \delta c_1}{\delta} + f \left( \frac{1}{\delta} (u - (1 - \delta) w) + \frac{1 - \delta c_1}{\delta} p \right) - f \left( \frac{1}{\delta} (u - (1 - \delta) w) \right) \right) \geq 0.
\]

This implies that

\[
\lim_{\alpha \to 0} S(1) - S(a) \geq (1 - \delta) (py - c_1),
\]

and as a result, there exists \( a_1^* \) such that for all \( a \leq a_1^* \), \( S(1) > S(0) \). In other words, \( e = a \) cannot be chosen for all \( a \leq a_1^* \) here.

In the second case, \( u < (1 - \delta) (w + (c_1 - c_0)a) + \delta w \). In this case, \( e = 1 \) cannot be sustained. However, \( u_i(u) \to \underline{u} \) as \( a \) goes to 0. Therefore,

\[
\lim_{a \to 0} S(a) = \delta (\underline{u} + f(\underline{u})) = \delta (\underline{u} + \underline{v}) < \underline{u} + \underline{v},
\]

which is a contradiction because the joint surplus cannot be smaller than the sum of the outside options. It follows that there exists \( a_2^* \) such that for \( a \leq a_2^* \), \( f(u) \) cannot be sustained with \( e = a \) for \( u < (1 - \delta) (w + (c_1 - c_0)a_2^*) + \delta w \).

Combining the two cases by having \( a^* = \min\{a_1^*, a_2^*\} \), then only \( e = 1 \) can be chosen for all \( a < a^* \), and the corollary is proved. \( \blacksquare \)