Information Revelation in Relational Contracts

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Abstract

This paper shows that garbling signals intertemporally can increase the efficiency of relational contracting. Intertemporal garbling reduces the principal’s maximal reneging temptation by linking together the principal’s non-reneging constraints both across states and over time. To improve efficiency, an essential feature of the intertemporal garbling process is that past outputs have enduring effects on future signals.

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1 Introduction

This paper shows that partial and delayed revelation of information can sometimes increase the efficiency of relational contracting. Models of relational contracting are repeated principal-agent games with noncontractible side payments. They capture environments with high transaction cost of drafting and implementing formal contracts. The prevalence and importance of relational contracting have been emphasized both inside and outside the economics literature; see for example, Bull (1987), Baker, Gibbons, Murphy (1994, 2002), Chassang (2010), Fuchs (2007), Levin (2002, 2003), Macauley (1963), MacLeod (2007), MacLeod and Malcomson (1989, 1998), MacNeil (1978), and Malcomson (2008), and Rayo (2007).

It is well-recognized that relational contracting is harder to sustain when the underlying environment is more volatile. This is because to sustain a relational contract, the principal’s maximal reneging temptation must not exceed the amount of discounted future surplus of the relationship.\(^1\) A productive relationship can be prevented from starting when the reneging temptations are destructively high in some rare instances even if most of the time the principal’s reneging temptations is not of concern.

The main contribution of our paper is to show that, through a novel intertemporal signal-garbling process, the slackness of the non-reneging constraint in some states can be exploited to enhance the efficiency of relational contracting. Intertemporal garbling repartitions and smooths the principal’s reneging temptations both across states (information sets) and over time. This reduces the principal’s maximal reneging temptation.

A distinctive feature of the intertemporal garbling process considered here is that the initial, ungarbled signals have enduring effects on future, intertemporally-garbled signals. For example, suppose the initial signals are equal to the output levels in each period. One intertemporal garbling process with enduring effects is to garble the output levels to two signals: "Good" and "Bad". Signal “Good” is realized if the average past outputs exceeds a threshold; signal "Bad" is realized otherwise. In this example, since the average output level up till any time depends on past outputs, the output level at time \(t\) affects all garbled signals from time \(t\) on.

\(^1\)To simplify the exposition, we normalize the minimal reneging temptation to 0.
Specifically, the baseline model in this paper builds on Levin (2003), where the agent’s efforts are his private information but the outputs are publicly observed. The key innovation is that the public signal is not necessarily equal to the output in each period. Instead, the public signals are realized through a signal-generating function that maps the entire past history of outputs into probability distributions over the set of possible signals. To focus on the effect of information structure on efficiency, we study binary effort levels (effort or no effort) and binary output levels (high or low). An efficient equilibrium requires the agent to exert effort in each period.

We report three results. First, when players are restricted to use public strategies, then for any $p \in (0, 1)$, where $p$ is the probability of high output given effort, there exists an intertemporal garbling process (and an associated Perfect Public Equilibrium (PPE)) that improves efficiency over perfect observability, i.e., the signal is equal to the output in each period, as long as $p \neq \frac{1}{2}$. Second, when $p = \frac{1}{2}$, perfect observability is the optimal information structure. In other words, no information structure can sustain efficiency if it cannot be sustained under perfect observability.

Finally, our main result is that when the players can use private strategies, there exists an intertemporal garbling and an associated Perfect Bayesian Equilibrium (PBE) that improves efficiency for sufficiently small $p$. We focus on PBE instead of PPE, which is a more common equilibrium concept in this literature because the agent’s past effort, through affecting past outputs, influences the distribution of future signals under intertemporal garbling. This implies that the agent can utilize information of his past actions to his advantage in the future, and an appropriate equilibrium concept should allow the players to use private strategies.

Allowing for private strategies, however, brings in extra technical difficulties. The one-stage deviation-principle can no longer be used to verify that a strategy profile forms a PBE. Specifically, once the agent deviates from the equilibrium, he will have a different belief about the distribution of future signals from that of the principal (even if they share the same belief of future actions). This difference in beliefs implies that the agent may benefit from multi-stage deviations. Checking multi-stage deviations is in general difficult, and our technique of verifying that a strategy profile forms PBE may be of independent interest.

In terms of the relationship between information structure and efficiency, Kandori (1992) shows that garbling signals within periods weakly decreases efficiency in
repeated games with imperfect public monitoring. Applying Kandori’s analysis to our setting of relational contracting, one can show that garbling within periods also weakly decreases efficiency. When signals are garbled within-period, the agent is rewarded for low outputs with some probability in each period, leading to a weaker incentive for the agent to exert effort. Intertemporal garbling reduces this extra incentive cost because whenever an agent is rewarded for a low output, his continuation value suffers.

Abreu, Milgrom, and Pearce (1992) (AMP hereafter) in the context of repeated games and Fuchs (2007) in relational contracting with subjective evaluation show that efficiency can be increased by bundling signals across a fixed number periods. Such bundling of signals increases efficiency because it allows punishment (for low outputs) to be reused, and, thus, reduces the surplus destruction in the relationship. The AMP-Fuchs type of bundling does not help in our setting. Bundling increases the maximal bonus required to incentivize the agent since the bonus are paid out less frequently. In fact, any garbling process that allows the game to restart will not help. To increase the efficiency in our setting, it is essential that the bygones are never completely bygones, i.e., future signals are always affected by past outputs.

Kandori and Obara (2006) show that when signals do not have full support, then the use of private (mixed) strategies can give rise to more efficient equilibria. In our setting, signals have full support so there is no loss of generality in restricting to public strategies. Instead we show that efficiency can still be enhanced through persistent intertemporal signal garbling.

For the rest of the paper, we set up the model in Section 2. We present our main results in Section 3. Section 4 concludes.

2 Setup

Time is discrete and indexed by $t \in \{1, 2, ..., \infty\}$.

2.1 Players

There’s one principal and one agent. Both are risk neutral, infinitely lived, and have a common discount factor of $\delta$. The agent’s per period outside option is $u$; the
principal’s per period outside option is $\pi$.

2.2 Production

If the principal and the agent engage in production together in period $t$, the agent chooses effort $e_t \in \{0, 1\}$. The cost of effort is given by $c(0) = 0$ and $c(1) = c$. The output is binary: $Y_t \in \{0, y\}$. We assume that

$$\begin{align*}
\Pr\{Y_t = y | e_t = 1\} &= p; \\
\Pr\{Y_t = y | e_t = 0\} &= q,
\end{align*}$$

where $1 > p > q \geq 0$.

To make the analysis interesting, we assume that the relationship is valuable if and only if the agent puts in effort. In other words:

$$py - c > u + \pi > qy.$$

2.3 Information Structure

In each period $t$, the agent’s effort $e_t$ is his private information. In addition, the principal and the agent both observe a public signal $s_t$ after the output is realized. We assume that $s_t$ is generated by a signal generating function $S_t$ that maps the set of past outputs into the probability distribution on the set of possible signals $S$:

$$S_t : \prod_{j=1}^{t} Y_j \rightarrow \Delta S.$$ 

The signal generating function provides a framework for modelling information structures and below are some examples.

Example 1 (Perfect Observability of Outputs)

In a standard model of relational contracts with imperfect public monitoring, see for example Levin (2003), the outputs are observed each period. In this case, $S = \{0, y\}$, and in each period $t$ the signal $s_t = y_t$, the output in period $t$. More formally, the signal generating function is given by

$$\Pr(S_t(y_1, ..., y_t) = y_t) = 1, \text{ for all } \{y_1, ..., y_t\}.$$
Example 2: (T-period Revelation)

An information structure that has received considerable attention from economists is the T-period revelation, i.e., the outputs are perfectly revealed every \( T \) periods and no information revealed in between, see for example Abreu, Milgrom, Pearce (1991) and Fuchs (2007). In this case, \( S = \{0, y\}^T \cup \{N\} \), where \( N \) stands for no information. When \( t \neq nT \) for each \( n \in N \), the signal \( s_t = N \). When \( t = nT \), \( s_t = (y(n-1)T+1, \ldots, y_{nT}) \). More formally, when \( t \neq nT \), the signal distribution function is given by

\[
\Pr(S_t(y_1, \ldots, y_t) = N) = 1.
\]

When \( t = nT \),

\[
\Pr(S_t(y_1, \ldots, y_t) = (y(n-1)T+1, \ldots, y_{nT})) = 1, \text{ for all } \{y_1, \ldots, y_t\}.
\]

Example 3: (Partial Information Revelation)

In the two examples above, each output is known perfectly (eventually). In this example, only partial information about the past outputs is revealed. In particular, let the set of the signal be \( S = \{Success, Failure\} \). In period \( t \), the signal \( s_t = Success \) if more than half of the previous outcomes \( y = Y \), and \( s_t = Failure \) otherwise. More formally,

\[
\Pr(S_t(y_1, \ldots, y_t) = Success) = 1, \text{ if } \sum_{j=1}^{t} y_j > \frac{ty}{2};
\]

\[
\Pr(S_t(y_1, \ldots, y_t) = Failure) = 1, \text{ if } \sum_{j=1}^{t} y_j \leq \frac{ty}{2}.
\]

Under this signal generating function, the signal in any period depends on the entire past history of outputs. Therefore, each signal is affected by outputs in the long distant past. Conversely, each output affects signals in the arbitrarily far future.

2.4 Timing

The timing is as follows. At the beginning of period \( t \), the principal offers a contract that consists of a fixed wage \( w_t \). The agent chooses whether to accept or not: \( d_t \in \{0, 1\} \). If the agent rejects, both the principal and the agent receive their outside options. If the agent accepts, he chooses \( e_t \). The signal \( s_t \) is realized and the principal
pays out $W_t \geq w_t$. Just as in analysis of relational contracts with perfect revelation of outputs such as Levin (2002), this restriction to nonnegative bonus helps simplify the exposition without affecting the set of equilibrium payoffs sustainable by relational contracts.

Unlike the perfectly-revealed outputs case, however, the timing of the bonus does affect the set of equilibrium payoffs sustainable by relational contracts. The reason has to do with the possible multi-stage deviation by the agent that we will discuss in more detail in Subsection 3.3. Our timeline allows the principal to either pay the contingent bonus at the end of each period or postpone the bonus to the beginning of next period as part of the efficiency wage.\textsuperscript{2} In the former case, we have

$$W_t = w_t + b_t,$$

where $b_t$ is the contingent bonus. In the later case,

$$W_t = w_t;$$
$$w_{t+1} = w + \frac{b_t}{\delta},$$

where $w$ is the wage paid to the worker if the output is 0.

\section*{2.5 Strategy and Equilibrium Concept}

\subsection*{2.5.1 History}

We denote $h_t = \{w_t, d_t, s_t, W_t\}$ as public events that happen in period $t$. Denote $h^t = \{h_n\}_{n=0}^{t-1}$ as a public history path at the beginning of period $t$. $h^1 = \emptyset$. Let $H^t = \{h^t\}$ be the set of public history paths till time $t$. Finally, define $H = \cup_t H^t$ as the set of public histories. The principal only observes the public history. For the agent, at the beginning of period $t$, he also observes his past actions $e^t = \{e_j\}_{j=1}^{t-1}$. Denote $H^t_A = H^t \cup \{e^t\}$ as the set of the agent’s private history at the beginning of period $t$.

\subsection*{2.5.2 Strategy and Payoff}

In period $t$, the following functions capture the strategies of the players.

\textsuperscript{2}The principal can also do both, i.e., pay some of bonus at the end of a period and postpone some to the beginning of the following period.
The wage offer of the principal is given by
\[ w_t : H^{t-1} \to R. \]

The agent’s acceptance decision is given by
\[ d_t : H^{t-1}_A \times \{w_t\} \to \{0, 1\}, \]
where the second component in the cross-product denotes the set of wage offers.

The agent’s effort decision is given by
\[ e_t : H^{t-1}_A \times \{w_t\} \to \{0, 1\}. \]

The total compensation function is given by
\[ W_t : H^{t-1} \times D_t \times S_t \to R. \]

The pure strategy of the agent is given by
\[ s^A = \{d_t, e_t\}_{t=1}^\infty. \]
And the pure strategy of the principal is given by
\[ s^P = \{w_t, W_t\}_{t=1}^\infty. \]

We can also allow the principal and the agent to play mixed strategies. We restrict our attention to pure strategy partly to distinguish ourselves from the literature of repeated game with private monitoring, where mixed strategies play a crucial role.

Take a strategy profile \((s^A, s^P)\). The expected payoff of the agent following a private history \(h^t_A\) and \(w_t\) is given by
\[ U(h^t_A, w_t, s^A, s^P) = E[\sum_{\tau=t}^\infty \delta^{\tau-t} \{u + 1_{\{d_t=1\}} (-ce_\tau + W_\tau - w)\}|h^t_A, s^A, s^P]. \]

Similarly, we can define \(U(h^t_A, w_t, d_t, s^A, s^P)\), the expected payoff of the agent following his acceptance decision in period \(t\).
The expected payoff of the principal following a private history \( h_A^t \) (of the agent) is given by

\[
\pi(h_A^t, s^A, s^P) = E[\sum_{\tau=t}^{\infty} \delta^{\tau-t} \{ \pi + 1_{d_{\tau}=1}(y(p - q) + q) \} h_A^t, s^A, s^P].
\]

Since the principal does not observe the private history of the agent, we define \( \pi(h_A^t, s^A, s^P) = E_P[\pi(h_A^t, s^A, s^P) | h^t] \) as his expected payoff following public history \( h^t \). Here, the expectation is taken over all possible private histories of the agent \( (h_A^t) \) according to the principal’s belief \( (\mu^P) \) conditional on observing public history \( h^t \).

We also denote \( \pi(h_A^t, w_t, d_t, s_t, s^A, s^P) \) as the principal’s expected payoff in period \( t \) following the agent’s private history \( h_A^t \), the principal’s wage offer, the agent acceptance decision \( d_t \), and the signal \( s_t \). We define \( \Pi(h^t, w_t, d_t, s_t, s^A, s^P) \) similarly.

### 2.5.3 Equilibrium Concept

When outputs are publicly observed, the standard equilibrium concept is Perfect Public Equilibrium (PPE). A strategy profile forms a PPE if the strategies of the players strategy only depend on the public history. Moreover, following any public history, the strategies of the players form a Nash Equilibrium. The advantage of PPE is that the set of PPE payoffs is stationary following any history, and this allows for a recursive formulation in characterizing the PPE payoffs. This recursive formulation keeps the analysis tractable. In Section 3, we show that if the players are restricted to use public strategies, then intertemporal garbling can help improve efficiency for all \( p \neq \frac{1}{2} \) when the discount factor approaches 1.

It is well-known, however, that the restriction to public strategies is not without loss of generality. One reason is it prevents the players from exploiting the efficiency gain of using mixed strategies. In the current setting, PPE is restrictive even if all players use pure strategies. When future signals depend on past outputs, the agent can (and should) use his private actions in the past to form expectation of the distribution of future signals. Therefore, we think the more appropriate equilibrium
A Perfect Bayesian Equilibrium (PBE) in this model consists of the strategy of the principal ($s^P$), the strategy of the agent ($s^A$), the belief of the principal ($\mu^P$), and the belief of the agent $\mu^A$ such that

- following any history $\{h^t_A, w_t\}$ and $\{h^t_A, w_t, d_t\}$,

$$U(h^t_A, w_t, s^A, s^P) \geq U(h^t_A, w_t, s^A, s^P);$$
$$U(h^t_A, w_t, d_t, s^A, s^P) \geq U(h^t_A, w_t, d_t, s^A, s^P);$$

- following any history $h^t$ and $\{h^t, w_t, d_t, s_t\}$,

$$\Pi(h^t, s^A, s^P) \geq \Pi(h^t, s^A, s^P);$$
$$\Pi(h^t, w_t, d_t, s_t, s^A, s^P) \geq \Pi(h^t, w_t, d_t, s_t, s^A, s^P).$$

- the beliefs are consistent with $\sigma^*$ and are updated with Bayes rule whenever possible.

In addition, PBE requires specifying the beliefs of the players. In this game, since only the agent has private information, the belief of the agent is degenerate. The principal knows the public history, and his belief of the agent’s private history is again degenerate whenever the action of the agent is consistent with the equilibrium play. When the agent’s action does not conform to the equilibrium play, we assume the principal believes that the agent has never put in effort in the past.

While PBE is the more appropriate concept, it is difficult to establish that a strategy profile forms a PBE. The difficulty arises because there is in general no recursive formulation when the agent’s strategy can depend on his private actions. In particular, if the agent deviates from the equilibrium action, his belief about the distribution of future signals becomes different from that of the principal (even if they share the same belief about the future actions). The difference in beliefs results from the intertemporal garbling of signals: future signals are affected by past outputs which in turn are affected by the agent’s past actions. So if the agent deviates from the equilibrium action, he forms a different belief about the future from the principal,
and this difference in belief implies that the set of PBE payoffs cannot be formulated recursively, making it difficult to check a strategy profile is a PBE.

3 Analysis

In this section, we study how the information structure affects the efficiency of the relational contract. In Section 3.1, we review the necessary and sufficient condition to sustain an efficient relational contract when outputs are perfectly observed. In Section 3.2, we present the main idea of how intertemporal garbling can in general help smooth bonus payments and increase efficiency, except when $p = \frac{1}{2}$. Section 3.3 formally construct an information structure (and an associated PBE) that obtains the efficient outcome when it is impossible to do so when outputs are perfectly observed.

3.1 Review: Perfect Observability of Outputs

Suppose signals are perfectly informative of the outputs, i.e., $s_t = y_t$ for all $t$. In this setting, Levin (2003) shows that the optimal relational contract is (constrained) efficient and can be implemented by a sequence of stationary contracts. Applying Levin (2003) to our setting, the stationary contract can be characterized by a base wage $w$ and a performance bonus $b > 0$ for high output.

To induce the effort from the agent, the bonus has to be big enough such that

$$b \geq \frac{c}{p - q}. \quad (1)$$

In addition, the principal can capture the entire surplus of the relationship by setting the base wage so that the agent’s payoff inside the relationship is equal to his outside option:

$$w - c + pb = u.$$

Finally, since the bonus is non-contractible, for the principal not to renege on the bonus, it must be that

$$b \leq \frac{\delta(py - c - u - \pi)}{1 - \delta}, \quad (2)$$

where $\frac{\delta(py - c - u - \pi)}{1 - \delta}$ is the discounted expected future surplus that is completely captured by the principal.
Combining equations (1) and (2), we see that a relational contract can induce effort if and only if
\[
\frac{c}{p-q} \leq \frac{\delta(py-c-u-\pi)}{1-\delta}.
\] (3)
In other words, the incentive cost should be smaller than the discounted expected future surplus. Inequality (3) implies that the sustainability of the relational contract is about the extremes. In other words, controlling for the value of the maximal reneging temptation \(\frac{c}{p-q}\) and the expected per period surplus in the relationship \((py-c-u-\pi)\), the set of discount factors \(\delta\) that allow for efficiency is completely determined.

### 3.2 Intertemporal Signal Garbling

When outputs are observed perfectly, the principal’s non-reneging constraints only bind in states where the outputs are high and are slack otherwise. When the probability of high outputs \(p\) decreases, the reneging temptations become more concentrated and are larger in size \(\frac{c}{p-q}\) when they are realized. This makes the efficient relational contract harder to sustain (controlling for surplus, i.e., keeping \(py\) and \(c\) the same).

Also when \(p\) is small, the reneging temptations are slack most of the time.

The basic idea of intertemporal garbling is that, by making the signals less informative of the outputs, we can repartitions the reneging temptations across different states (high or low outputs) and over time. In other words, intertemporal garbling re-partitions the information sets to smooth the bonus payments and thus reduces the maximal reneging temptation. Specifically, when the signals are garbled intertemporally, a high output increases both the current and future payoffs of the agent. This allows the principal to reduce the current bonus (compared to the case of perfect observability of outputs) while maintaining incentive. The difficulty is to do this every period without causing the agent’s future payoff to explode following some equilibrium play path.

To implement this idea, consider the following signal generating process for \(p < \frac{1}{2}\). The case of \(p > \frac{1}{2}\) is its mirror image and will be described after Theorem 1. There are two public signals: good and bad. Following any public history, there are \(n > \frac{1-p}{1-2p}\) "secret states," with lower states being more favorable to the agent. For an agent in state \(k \geq 2\), if the output is high, then the public signal is good, and the agent transitions into secret state \(k-1\) with probability \(\frac{k}{n-1}\) and stays in secret state \(k\) with
probability $\frac{n-k}{n-1}$. If the output is low, then the signal is bad, and the agent transitions into secret state $k - 1$ with probability $\frac{(n+1-k)p}{(n-1)(1-p)}$ and stays in secret state $k$ with probability $1 - \frac{(n+1-k)p}{(n-1)(1-p)}$. Note that the players do not know which secret state the agent is.

For the agent in secret state 1, if the output is high, then the signal is good, and the agent stays in secret state 1. If the output is low, then with probability $\frac{pn}{(1-p)(n-1)}$, the signal is good, and the agent transitions into secret state $n$. With probability $1 - \frac{pn}{(1-p)(n-1)}$, the signal is bad, and the agent is in secret state 1.

A key feature of this transition function is that it helps maintain stationarity. In particular, if the $n$ states are equally likely initially and the agent puts in effort, then in the next period these states are again equally likely regardless of whether the signal is good or bad and the probability that a good signal is announced each period is given by

$$p + \frac{1}{n} \left( (1-p) \frac{pn}{(1-p)(n-1)} \right) = \frac{n p}{n-1} = \rho.$$

The stationarity is not essential for the idea of intertemporal garbling, but it helps the analysis by ensuring that the principal’s maximal reneging temptation is simply the bonus amount.

The following figure depicts the signal generating process and the transition of states for $n = 3$:

![Figure 1: Intertemporal Signal Garbling w/ $n = 3$](image)
Note that the signal generating process above re-partitions the information sets across time: if the signals are only garbled within each period, then the signals become less informative of effort and this exacerbates the incentive problem. Such insight is due to Kandori (1992) who formalizes it in the context of repeated games without transfers. Also note that intertemporal garbling will not help in our environment if it is conducted across a fixed number of periods because once the last period of the fixed number of periods is reached, the agent cannot be rewarded with a higher continuation payoff anymore. This helps explain why garbling using AMP’s T-period revelation will not improve efficiency in the current setting.

Now suppose the agent is paid a base wage $w$ each period, puts in effort, and receives a bonus whenever a good signal is realized. The transition function above allows us to write down the value of each state.

For $k \geq 2$, we have

$$v_k = w - c + p\{b + \delta[(k-1)yv_{k-1} + (1-(k-1)y)v_k]\}$$

$$+ \delta(1-p)[(n+1-k)zv_{k-1} + (1-(n+1-k)z)v_k],$$

where $y = \frac{1}{n-1}$ and $z = \frac{p}{(n-1)(1-p)}$.

For $k = 1$,

$$v_1 = w - c + p[b + \delta v_1]$$

$$+ (1-p)\left[\frac{pn}{(1-p)(n-1)}(b + \delta v_n) + (1 - \frac{pn}{(1-p)(n-1)})\delta v_1\right],$$

or alternatively,

$$v_1 = \frac{pb - c + \rho b + \delta \rho v_n}{1 - \delta(1 - \rho)},$$

where $\rho = \frac{pn}{n-1}$. Note that we have $n$ equations and $n$ unknowns, so the value of each of the states ($v_k, k \in \{1, \ldots, n\}$) can be calculated.

When the states are equally likely, the expected payoff of the agent is given by

$$EV = \frac{\sum_{i=1}^{n} v_i}{n} = \frac{w - c + \rho b}{1 - \delta}.$$
low output respectively.\textsuperscript{3} Note that the difference in $EV_H$ and $EV_L$ is proportional to $b$, and define
\[ K(\delta, n) = \frac{EV_H - EV_L}{b}, \]
as a measure of the effectiveness of payment smoothing.

Now suppose the players use public strategies.\textsuperscript{4} In particular, consider strategies characterized by the following two-state automaton. On the equilibrium path, a) the principal always offers the same stationary contract with a base wage $w$ and a bonus $b$ and pays out the bonus for good signal, and b) the agent always accepts and puts in effort. Off the equilibrium path (if either party publicly deviates), the principal and the agent take their outside options.\textsuperscript{5} In addition, let the principal set the base wage $w = u + c - \frac{m}{n-1}b$ so that she captures the entire surplus of the relationship.

With this strategy profile, the agent is willing to put in effort if and only if
\[ \frac{c}{p - q} \leq EV_H - EV_L = K(\delta, n)b. \]
The principal will not renege the bonus if and only if
\[ b \leq \frac{\delta(py - c - u - \pi)}{1 - \delta}. \]
Combining these two conditions together, we see that this strategy profile can be a PPE if
\[ \frac{c}{(p - q)K(\delta, n)} \leq \frac{\delta(py - c - u - \pi)}{1 - \delta}. \]
Recall that the necessary condition to sustain an efficient relational contract with

\textsuperscript{3}In particular, we have $EV_H = w - c + b + \delta \sum_{i=1}^{\frac{m}{n-1}} v_i$.

\textsuperscript{4}As mentioned in Section 2, the restriction to public strategy is not without loss of generality. Nevertheless, such restriction simplifies the analysis and helps illustrate why intertemporal garbling helps improve efficiency. The restriction to public strategies is lifted in the next section where, focusing on the case of small $p$, we show that intertemporal garbling can also help improve efficiency when the players can use private strategy.

\textsuperscript{5}The public deviation of principal include offering a different contract and not paying bonus for a good signal. The public deviation of the agent includes not accepting the contract. More formally, if a public deviation has ever occurred, then the principal offers a base wage $w = u - 1$ and will not pay out the bonus. The agent rejects all contracts with base wage $w < u$ and does not put in effort.
perfect observability of outputs is given by

\[ \frac{c}{(p - q)} \leq \frac{\delta(py - c - u - \pi)}{1 - \delta}. \]

Therefore, if \( K(\delta, n) > 1 \), then the signal generating process under the \( n \)-state transition rule above helps sustain efficiency for a larger range of discount factors. Theorem 1 shows that as the discount factor approaches 1, \( K(\delta, n) > 1 \) for sufficiently large \( n \).

**Theorem 1:** Let \( p < \frac{1}{2} \). For \( n > \frac{1}{1-2p} \), under the \( n \)-state transition rule,

\[ \lim_{\delta \to 1} K(\delta, n) = 1 + \frac{(1 - 2p)n - 1}{2(1 - p)n(n - 1)}. \]

Theorem 1 implies that if the signals can be intertemporally garbled, then as the players become more patient, the efficiency of the game can be improved if the players use public strategies and \( p < \frac{1}{2} \). As discussed above, the source of the gain comes from linking the principal’s reneging constraints across states and over time. In particular, a high output not only leads to a good signal today but also moves the agent into more favorable states, making future good signals more likely.

While intertemporal garbling helps bonus smoothing, it is not ex ante clear that it helps efficiency. Once the signals are garbled, the agent can sometimes receive a bonus even if the output is low. (In our \( n \)-state construction, this can happen when the agent is in state 1). Paying out bonus for low output reduces the agent’s incentive to work, and this makes the relational contract harder to sustain. In addition, intertemporal garbling implies that part of the reward paid to the agent is delayed to the future. When the bonus is delayed and the agent is impatient, the total expected bonus increases, and this makes the principal’s non-reneging constraint harder to satisfy. As the discount factor goes to 1, however, the damage from delayed bonus disappears and Theorem 1 shows that intertemporal garbling can help.

In Theorem 1, the success probability \( p \) is less than \( \frac{1}{2} \). When \( p \in (\frac{1}{2}, 1) \), we can construct a signal generating process that is the mirror image of the signal generating process in the \( p < \frac{1}{2} \) case. In essence, we turn the graph in Figure 1 upside down: we again have two signals, good and bad, and each signal (information set) contains \( n \) states. But the states are ranked in the reverse order from the \( p < \frac{1}{2} \) case. In addition, the high output corresponds to the low output in the \( p < \frac{1}{2} \) case, and the
good signal here corresponds to the bad signal there. Under the new information structure, the agent pays the principal a bonus if and only if the signal is good. Using this transition function, we can establish the following corollary:

**Corollary 1:** Let \( p > \frac{1}{2} \). For \( n > \frac{1}{2p-1} \), under the reversed \( n \)–state transition function,

\[
\lim_{\delta \to 1} K(\delta, n) = (1 + \frac{(2p - 1)n - 1}{2pn(n - 1)}).
\]

Theorem 1 and Corollary 1 show that intertemporal garbling can help increase efficiency for all \( p \in (0, 1) \) except at \( p = \frac{1}{2} \). When \( p = \frac{1}{2} \), perfect observability of output is in fact the optimal information structure. In other words, if no efficient relational contracts can be sustained when the outputs are perfectly observed when \( p = \frac{1}{2} \), no information structure can obtain efficiency. Note that Theorem 2 is a general result: it holds regardless of whether the players use private strategies or not.

**Theorem 2:** When \( p = 1/2 \), the optimal information structure is given by \( s_t = Y_t \) for all \( t \).

To see why intertemporal garbling does not help when \( p = \frac{1}{2} \), consider the benchmark case of imperfect public monitoring in Subsection 3.1. Suppose inequality (3) holds so there’s an efficient relational contract. In this case, one can construct a stationary equilibrium in which the agent receives a bonus of \( \frac{c}{p-q} \) each time for a high output. Since high output happens with probability \( p \), the average reneging temptation of the principal is \( \frac{pc}{p-q} \). Alternatively, we can adjust the base wage properly and have the agent pay back a bonus of \( \frac{c}{p-q} \) each time for a low output. Since low output occurs with probability \( 1 - p \), the average reneging temptation is \( \frac{(1-p)c}{p-q} \). One can consider more general bonus schemes, but it can be shown that, to induce effort, a necessary condition is that the **average reneging temptation** is at least

\[
\min\{p, 1-p\} \cdot \frac{c}{p-q},
\]

regardless of the information structure.

Now consider a stationary contract that pays out a bonus \( B \) with frequency \( \rho \). To minimize the reneging temptation, we would like to minimize \( B \). For this contract to induce effort, however, the discussion above implies that the average reneging temptation should be at least \( \frac{c}{p-q} \min\{p, 1-p\} \). In other words, we would like to examine the following program:
\[ \min_{B, \rho} B \]
\[ \text{s.t. } B \min \{ \rho, 1 - \rho \} \geq \frac{c}{p - q} \min \{ p, 1 - p \}. \]

It is easy to see that \( B \) is minimized when \( \rho^* = \frac{1}{2} \), and we have \( B^* = \frac{2c}{p - q} \min \{ p, 1 - p \} \). Therefore,
\[ \frac{B^*}{c/(p - q)} = 2 \min \{ p, 1 - p \}. \]
This expression suggests that intertemporal garbling is more effective when the success probability is uneven. Moreover, when \( p = \frac{1}{2} \), \( B^* = -\frac{c}{p - q} \) and intertemporal garbling does not help. The discussion above provides an intuition for Theorem 2, but the actual proof of the theorem relies on a slightly different logic and also deals with contracts that are not stationary. It shows if intertemporal garbling could help, the variance of values within each information set would diverge to infinity.

### 3.3 Perfect Bayesian Equilibrium

In the previous subsection, we showed that when the probability of success is not equal to \( \frac{1}{2} \), intertemporal garbling can help increase efficiency when the players use public strategies. However, the restriction to public strategies is not particularly appropriate when the signals are intertemporally garbled. Since past outputs affect future signals, the agent can have a different belief about the distribution of future signals from that of the principal when he deviates (even if he follows the equilibrium play in the future). Therefore, the agent’s private history matters for future play, and a more appropriate equilibrium concept should allow the agent’s strategy to depend on his private history.

In this subsection, we show that intertemporal garbling can also improve efficiency when Perfect Bayesian Equilibrium (PBE) is used as the equilibrium concept. While the basic idea of why intertemporal garbling helps remains the same, it is significantly harder to check that a strategy profile forms a PBE: when the agent no longer holds the same belief as the principal after a deviation, there is typically no recursive structure in the game and all relevant multi-stage deviations need to be checked to
ensure that the strategy profiles form a PBE. To keep the analysis tractable, we let \( q = 0 \) in this subsection, so that if the agent does not put in effort, the output must be low.\(^6\)

Consider the following signal generating process. Let the signals be either *good* or *bad*. Within each information set, there are two secret states: *up* or *down*. If the output is high, then regardless of the state, the signal is *good* \((g)\) and the agent will be in the *up* state (within the good signal) next period. If the output is low, then if the agent is in the *up* state, the signal is good and the agent will be in the *down* state next period. If the agent is in the *down* state, then the signal is *bad* \((\sim g)\) and at the same time the state is moved to *up* with probability \( \rho^* \) and to *down* with probability \( 1 - \rho^* \).

The probability \( \rho^* \) satisfies

\[
\rho^* = \frac{p}{p + (1 - p)\rho^*}
\]

to maintain stationarity. In particular, if with probability \( \rho^* \) the agent is in the *up* state and he puts in effort, then it can be checked that in the next period the agent will again be in the *up* state with probability \( \rho^* \) regardless of which signal is realized. The following figure illustrates how different outputs and previous states lead to different signals and states:

\(^6\)The game remains one of imperfect monitoring as long as \( p < 1 \) because when the output is low, the principal is unable to infer whether it is due to lack of effort or bad luck.
This signal generating process is similar in spirit to the two-state case in Subsection 3.2. However, this process is not a special case of that in Subsection 3.2 because the two states here are not of equal probability. Given the generating function (with $\rho^*$ as the initial probability of the up-state), consider the following strategies:

The principal offers

$$w_1 = w,$$

in period 1. If the agent has always accepted the contract, the principal offers for $t > 1$

$$w_t = \begin{cases} 
  w & \text{if } s_{t-1} = g, \\
  w + \frac{B}{\delta} & \text{if } s_{t-1} = g.
\end{cases}$$

If the agent has ever rejected the principal’s offer, the principal offers

$$w_t = u - 1.$$

In addition, no discretionary bonus is given out so

$$W_t = w_t \text{ for all } t.$$

The principal always believes that the probability of up-state is $\rho^*$. 
The agent accepts the principal’s contract offer if

\[ w_t > u \]

or if the principal has never deviated. The agent puts in effort if the principal has never deviated and the probability of the up-state satisfies \( \rho \leq \rho^* \). The agent calculates the probability of up-state using the Bayes rule according to his past history of efforts.\(^7\)

**Theorem 3**: Let \( w = c + (1 - \delta)u - \frac{p}{\rho'} B \) and \( B = \frac{c}{A(p, \delta) p} \), where \( A(p, \delta) = \left( \frac{1}{\rho^*} - \frac{\rho^*}{p} + \delta \frac{(1 - \rho^* - \delta(1 - \rho^*) \rho^*)}{1 - \rho^* \delta^2 (1 - \rho^*)} \right) \). The strategy and belief above form a PBE if and only if

\[ B \leq \frac{\delta}{1 - \delta} (py - c - u - \pi). \]

In addition,

\[ \lim_{p \to 0} A(p, \delta) = 1 + \delta. \]

Theorem 3 implies that an efficient PBE exists if

\[ \frac{c}{p} \leq A(p, \delta) \frac{\delta (py - c - u - \pi)}{1 - \delta}. \]

Recall that when the outputs are publicly observed each period, the necessary condition for an efficient relational contract is given by

\[ \frac{c}{p} \leq \delta \frac{(py - c - u - \pi)}{1 - \delta}. \]

When \( p \) goes to 0, Theorem 3 states that \( A(p, \delta) \) goes to \( 1 + \delta \). Since \( A(p, \delta) \) is continuous in \( p \), intertemporal garbling helps increase efficiency for small enough \( p \). For \( p \) close to 0, the signal generating process above cuts the surplus required for efficiency by a factor of \( \frac{1}{1 + \delta} \).

To see why \( 1 + \delta \) is the proportion of gain as \( p \) goes to 0, let \( \beta \) be the benefit of being in the up state instead of the down state. When the probability of up-state is

\[ \beta \]

\(^7\)If the contract is not accepted in a period, no signal is generated in that period. And future signals are generated as if that period does not exist.
\( \rho^* \), the agent’s benefit of exerting effort is given by

\[
p(1 - \rho^*) B + \delta p \left[ 1 - (1 - \rho^*) \rho^* \right] \beta,
\]

where \( p(1 - \rho^*) \) is the additional probability of a good signal and \( p \left[ 1 - (1 - \rho^*) \rho^* \right] \) is the additional probability of being in the up-state. When \( p \) goes to zero, \( \rho^* \) also goes to zero. More importantly, as \( p \) goes to zero, the probability of receiving a bonus goes to 0. However, once the agent is in the up state, he receives a bonus with probability one. In other words, As \( p \) goes to zero, being in the up state increases the probability of getting a bonus from almost zero to one. Therefore, \( \beta \) goes to \( B \) as \( p \) goes to zero.

It follows that as \( p \to 0 \),

\[
p(1 - \rho^*) B + \delta p \left[ 1 - (1 - \rho^*) \rho^* \right] \beta
\approx
p(1 - \rho^*) B + \delta p \left[ 1 - (1 - \rho^*) \rho^* \right] B
\approx
(1 + \delta) B.
\]

This explain why the benefit of exerting effort can reach \( (1 + \delta) pB \), as compared to \( pB \) when the signal is not garbled. In other words, with intertemporal signal garbling, the same amount of bonus can provide a stronger incentive to put in effort.

The formal proof of the theorem is complicated by the fact that the signal garbling process destroys the common knowledge between the principal and the agent. To see this, consider an agent’s deviation. If the agent deviates, then he will privately know the actual probability that he is in the up state. As a result, following such reporting rule, the agent’s belief can be different from the principal’s belief since the agent’s belief depends on the entire past history of private actions. With the breakdown of common knowledge, the one-stage deviation principle cannot be used to check the equilibrium. To show that the agent puts in effort in each period with this signal garbling process, it remains to prove that any arbitrary multi-stage deviation is unprofitable, which is difficult in general.

Our construction bypasses this difficulty by restricting the number of secret states to 2. In doing so, the probability of the up-state becomes a sufficient statistic of the agent’s future payoff. This allows for a recursive formulation of the agent’s value function in which the probability of the up-state is the state variable. Moreover, our construction allows us to calculate the value function explicitly. The value function
is piecewise linear in the probability of the up-state and has a kink at $\rho^*$. 

The value function is constructed as if the bonus were paid at the end of each period (and the agent never takes the outside option). In the actual equilibrium, the bonus is paid out at the beginning of the next period (as a part of the enlarged base wage). Unlike the case of imperfect public monitoring case such as in Levin (2003), the difference in timing turns out to matter here in preventing multi-stage deviations. Had the bonus been paid out at the end of the period, the agent might find it profitable to shirk for one period and once he received a bonus despite shirking, he would Bayesian update that he is in the down state. In that case, he would strictly prefer to take his outside option, which exceeds the value function (see the graph above). By paying out the bonus in the form of a higher base salary in the next period, the principal essentially makes the bonus payout contingent on the agent’s acceptance of next period’s contract as well. And this helps prevents the deviating agent from exiting the relationship upon learning that he is in the down state.

4 Conclusion

This paper shows that intertemporally garbling of signals can sometimes help relational contracting. Through repartitioning and smoothing the reneging temptations both across states and over time, intertemporal garbling increases the efficiency by reducing the principal’s maximal reneging temptation. To improve efficiency, an
essential feature of the intertemporal garbling process considered here is that past outputs have enduring effects on future signals.

Our theoretical investigation has implications on how to better sustain relational contracting in practice. For example, since the full revelation of information is in general suboptimal, intermediaries can help relational contracting by controlling information flows. By reducing the transparency of the relationship, an intermediary can sometimes increase its efficiency. In addition, our analysis suggests that gain from using an intermediary is larger in relationships in which the temptation to renege is small in all but a few rare instances. Obviously, introducing an intermediary creates a host of other issues. Further research in this area is needed.

References


Appendix

Proof of Theorem 1. The proof proceeds by explicitly calculating the value of $EV_H - EV_L$. Note that

$$EV_H = w - c + b + \delta \frac{\sum_{i=1}^{n-1} v_i}{n-1}$$

$$= w - c + b + \delta \frac{nEV - v_n}{n-1}.$$  

and

$$EV = w - c + \rho b + \delta EV.$$  

Therefore,

$$\frac{EV_H - EV_L}{EV_H - EV} = \frac{1}{1 - p}$$

$$= \frac{1}{1 - p} [(1 - \rho)b + \delta(\frac{nEV - v_n}{n-1} - EV)]$$

$$= b + \frac{1}{(1 - p)(n - 1)} [\delta[EV - v_n] - \rho b].$$  

To calculate $EV - v_n$, define $s_{k+1} = v_{k+1} - v_k$. Given

$$v_k = w - c + p\{b + \delta[(k - 1)yv_{k-1} + (1 - (k - 1)y)v_k]\}$$

$$+ \delta(1 - p)[(n + 1 - k)zv_{k-1} + (1 - (n + 1 - k)z)v_k],$$

and

$$v_{k+1} = w - c + p\{b + \delta[(k_yv_k + (1 - k_y)v_{k+1}]$$

$$+ \delta(1 - p)[(n - k)zv_k + (1 - (n - k)z)v_{k+1}],$$
we have

$$s_{k+1} = v_{k+1} - v_k$$

$$= p\delta[kyv_k + (1 - ky)v_{k+1} - (k - 1)yv_{k-1} - (1 - (k - 1)y)v_k]$$

$$+ \delta(1 - p)[(n - k)zv_k + (1 - (n - k)z)v_{k+1} - (n + 1 - k)zv_{k-1} - (1 - (n + 1 - k)z)v_k]$$

$$= [k(1 - p)(v_k - v_{k-1}) + (1 - ky)(v_{k+1} - v_k)]$$

$$+ \delta(1 - p)[(1 - (n - k)z)(v_{k+1} - v_k) + (n + 1 - k)z(v_k - v_{k-1})]$$

$$= p\delta[(k - 1)ys_k + (1 - ky)s_{k+1}] + \delta(1 - p)[(1 - (n - k)z)s_{k+1} + (n + 1 - k)zs_k]$$

$$= \delta\{[(p(k - 1)y + (1 - p)(n + 1 - k)z)s_k + [p(1 - ky) + (1 - p)(1 - (n - k)z)]s_{k+1}]}.$$

This implies that

$$s_{k+1} = \delta[\rho s_k + (1 - \rho)s_{k+1}], \text{ for } k \geq 2,$$

and, thus, for $k \geq 2$,

$$s_k = (1 + \frac{1 - \delta}{\delta \rho})s_{k+1} = C(\delta)s_{k+1},$$

where $C(\delta) = (1 + \frac{1 - \delta}{\delta \rho})$ is independent of $k$.

In addition, note

$$v_n = w - c + p(b + \delta v_{n-1})$$

$$+ \delta(1 - p)[\frac{p}{(n - 1)(1 - p)}v_{n-1} + (1 - \frac{p}{(n - 1)(1 - p)})v_n].$$

Since $v_{n-1} = v_n - s_n$, we have

$$s_n = \frac{w - c + pb - (1 - \delta)v_n}{\delta \rho}.$$

Using $v_k = v_{k+1} - s_{k+1}$, it can be seen that

$$\sum_{i=2}^{n} v_i = (n - 1)v_n - (n - 2)s_n - ... - s_3$$

$$= (n - 1)v_n - K(\delta)s_n,$$

where it can be shown that

$$K(\delta) = \frac{C^{n-1} - 1 - (C - 1)(n - 1)}{(C - 1)^2}.$$
To see the above, note that

\[ s_3 + 2s_4 + \ldots + (n-2)s_n = s_n(C^{n-3} + 2C^{n-4} + \ldots + (n-3)C + (n-2)). \]

Let

\[ K = C^{n-3} + 2C^{n-4} + \ldots + (n-3)C + (n-2), \]

then

\[ CK = C^{n-2} + 2C^{n-3} + \ldots + (n-3)C^2 + (n-2)C, \]

so

\[ (C-1)K = C^{n-2} + C^{n-3} + \ldots + C + 1 - (n-1) \]

\[ = \frac{C^{n-1} - 1}{C - 1} - (n-1), \]

so

\[ K = \frac{C^{n-1} - 1 - (n-1)(C-1)}{(C-1)^2} \]

Therefore,

\[ nEV = \sum_{i=1}^{n} v_i \]

\[ = v_1 + (n-1)v_n - K(\delta)s_n \]

\[ = \frac{w - c + (p + \rho)b + \delta\rho v_n}{1 - \delta(1-\rho)} + (n-1)v_n + K(\delta)\frac{(1-\delta)v_n - (w - c + pb)}{\delta\rho}. \]

This implies two things. First,

\[ n(EV - v_n) = \frac{w - c + (p + \rho)b - (1-\delta)v_n}{1 - \delta(1-\rho)} + K(\delta)\frac{(1-\delta)v_n - (w - c + pb)}{\delta\rho}. \]

Note that the \( w - c \) term will cancel out, so we might just assume that they’re equal to 0, the equation above simplifies to

\[ n(EV - v_n) = \frac{(p + \rho)b - (1-\delta)v_n}{1 - \delta(1-\rho)} + K(\delta)\frac{(1-\delta)v_n - pb}{\delta\rho}. \]
This can be rewritten as

\[ v_n(n - \frac{(1 - \delta)}{1 - \delta(1 - \rho)} + K(\delta)\frac{(1 - \delta)}{\delta \rho}) = nEV - \frac{(p + \rho)b}{1 - \delta(1 - \rho)} + K(\delta)\frac{pb}{\delta \rho}, \]

and, thus,

\[ v_n = \frac{n \cdot \frac{pb}{1 - \delta} - \frac{(p + \rho)b}{1 - \delta(1 - \rho)} + K(\delta)\frac{pb}{\delta \rho}}{n - \frac{(1 - \delta)}{1 - \delta(1 - \rho)} + K(\delta)\frac{(1 - \delta)}{\delta \rho}}, \]

Now let \( \delta \) go to 1, note that \( C = 1 + \frac{1 - \delta}{\delta \rho} \equiv 1 + \varepsilon \), so we have \( \varepsilon \to 0 \) as \( \delta \to 1 \), and thus \( C^M \simeq 1 + M\varepsilon \). This implies that

\[ \lim_{\delta \to 1} K(\delta) = \lim_{\delta \to 1} \frac{1 + C + C^2 + \ldots + C^{n-2} - (n - 1)}{C - 1} \]

\[ \simeq \frac{\sum_{i=1}^{n-2} M\varepsilon}{\varepsilon} = \frac{(n - 1)(n - 2)}{2}. \]

Therefore, as \( \delta \to 1 \),

\[ \lim_{\delta \to 1} v_n(1 - \delta) = \lim_{\delta \to 1} \frac{n \rho b - \frac{(1 - \delta)(p + \rho)b}{1 - \delta(1 - \rho)} + (1 - \delta)K(\delta)\frac{pb}{\delta \rho}}{n - \frac{(1 - \delta)}{1 - \delta(1 - \rho)} + K(\delta)\frac{(1 - \delta)}{\delta \rho}} \]

\[ = \rho b. \]

In addition, as \( \delta \to 1 \),

\[ \lim_{\delta \to 1} n(EV - v_n) = \lim_{\delta \to 1} \frac{(p + \rho)b - (1 - \delta)v_n}{1 - \delta(1 - \rho)} + \lim_{\delta \to 1} K(\delta)\frac{(1 - \delta)v_n - pb}{\delta \rho} \]

\[ = \frac{pb}{\rho} + (n - 1)(n - 2)\frac{(\rho - p)}{\rho}b \]

\[ = \frac{(n - 1)}{n} + \frac{(n - 1)(n - 2)}{2n}b \]

\[ = \frac{(n - 1)}{2}b. \]
Finally,

\[
\lim_{\delta \to 1} (EV_H - EV_L) = b + \frac{1}{(1-p)(n-1)}[\lim_{\delta \to 1} [EV - v_n] - pb]
\]

\[
= b + \frac{1}{(1-p)(n-1)}\left(\frac{n-1}{2n} - p\right)b
\]

\[
= (1 + \frac{(1-2p)n-1}{2(1-p)n(n-1)})b.
\]

\[\blacksquare\]

**Proof of Theorem 2.** First recall that when \(s_t = Y_t\) for all \(t\), the necessary and sufficient condition for sustaining cooperation is given by equation (3):

\[
\frac{c}{p-q} \leq \frac{\delta}{1-\delta}(py - c - u - \pi) \equiv S,
\]

where without confusion in this proof, \(S\) denotes the surplus of the relationship (when the agent puts in effort each period.) We want to show that if the inequality above fails, it is impossible to construct an equilibrium in which the agent puts in effort. In particular, using standard argument as in Fuchs (2007), it suffices to show that there does not exist an equilibrium in which the agent always puts in effort (unless the relationship is terminated).

Consider an arbitrary information partition process. Pick one information set \((h^t)\). Use \(x\) to denote the possible states within the information set. One interpretation of \(x\) is some output realizations \(y^t\) that falls into \(h^t\).

Let \(V(x)\) be the agent’s continuation payoff in state \(x\) after \(e_{t+1}\) is put in but before \(y_{t+1}\) is realized and \(W_{t+1}\) is paid out. Let \(V(x_i)\) be the agent’s continuation payoff in state \(x\) after \(e_{t+1}\) is put in, \(y_{t+1}\) is realized but before \(W_{t+1}\) is paid out. Within each state \(x\), we have \(x_i \in \{x_y, x_0\}\), where \(x_y\) denotes that \(Y_t = y\) is realized following \(x\), and \(x_0\) denotes that \(Y_t = 0\) is realized.

Note that

\[
V(x) = V(x) + p(V(x_y) - V(x)) + (1-p)(V(x_0) - V(x)).
\]

And since the output \(Y_t\) is independent of the past state, we have \(Cov(V(x_i) - V(x), V(x)) = 0\).
To induce effort, we need

\[ E_x[V(x_y) - V(x_0)] = E_x[V(x_y) - V(x)] - E_x[V(x_0) - V(x)] \geq \frac{c}{p-q}. \]

This helps give a lower bound for \( Var(V(x_i)) \). In particular,

\[
Var(V(x_i)) = Var(V(x)) + Var(V(x_i) - V(x))
\]

\[
= Var(V(x)) + E_y[Var(V(x_i) - V(x))|Y]
\]

\[
+Var(E_x[V(x_i) - V(x)|Y])
\]

\[
\geq Var(V(x)) + E_y[Var(V(x_i) - V(x))|Y]
\]

\[
+ p(1-p)(\frac{c}{p-q})^2
\]

\[
\geq Var(V(x)) + p(1-p)(\frac{c}{p-q})^2,
\]

where the first line follows because \( Cov(V(x_i) - V(x), V(x)) = 0 \), the second line uses the variance decomposition formula, the third line follows because \( E_x[V(x_i) - V(x)|Y] \) is a binary value (\( Y \in \{0, y\} \)) such that with probability \( p \) its value is \( E_x[V(x_y) - V(x)] \) and with probability \( 1-p \) its value is \( E_x[V(x_0) - V(x)] \), and \( E_x[V(x_y) - V(x)] - E_x[V(x_0) - V(x)] \geq \frac{c}{p-q} \). Now let’s provide an upper bound for \( Var(V(x_i)) \). Suppose a public signal \( s(x_i) \) will be sent out after state \( x_i \). Let \( b(s) \) be the bonus paid out to the agent (at the end of the period) following signal \( s \). This allows us to write

\[
V(x_i) = b(s(x_i)) + \delta V_{s(x_i)}(x_i),
\]

where \( V_{s(x_i)}(x_i) \) is the continuation payoff of \( x_i \), which goes to the information set by signal \( s(x_i) \).

Note that for the principal to be willing to pay the bonus, we must have

\[
\max_s \{b_s + \delta E_{x_i}[V_s(x_i)|s]\} - \min_s \{b_s + \delta E_{x_i}[V_s(x_i)|s]\} \leq S.
\]

Because otherwise the expected payoff of the principal following some signal will be below his outside option.

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Decomposing the variance on the signals, we have

\[
\text{Var}(V(x_i)) = \text{Var}(E[b_s + \delta V_s(x)|s]) + E[\text{Var}(b_s + \delta V_s(x_i)|s)] \\
\leq \frac{1}{4}S^2 + \delta^2 E[\text{Var}(V_s(x)|s)].
\]

Now combining the upper and lower bound for \(\text{Var}(V(x_i))\), we get that

\[
\frac{1}{4}S^2 + \delta^2 E[\text{Var}(V_s(x_i)|s)] \geq \text{Var}(V(x)) + p(1 - p)\left(\frac{c}{p-q}\right)^2,
\]
or equivalently,

\[
E[\text{Var}(V_s(x_i)|s)] \geq \frac{1}{\delta^2}(\text{Var}(V(x)) + p(1 - p)\left(\frac{c}{p-q}\right)^2 - \frac{1}{4}S^2).
\]

When \(p = \frac{1}{2}\), and \(\frac{c}{p-q} > S\), the inequality above implies that

\[
E[\text{Var}(V_s(x_i)|s)] > \frac{1}{\delta^2}(\text{Var}(V(x))).
\]

In particular, there will be one information set (associated with a signal) whose variance exceeds \(\frac{1}{\delta^2}(\text{Var}(V(x)))\). Now we can perform the same argument on this new information set, and we can construct a sequence of information set whose variance approaches infinity. This leads to a contradiction. ■

**Proof of Theorem 3.** It can be checked that given the choice of the base wage, the principal captures the entire surplus. So when \(B \leq \frac{\delta}{1-\delta}(py - c - u - \psi)\), the principal will not renege. The key is to check that the agent will not deviate. To simplify the exposition, we set \(u = 0\) here. It follows that

\[
w = c - \frac{p}{\rho^*}B.
\]

Given the strategy of the principal, the agent’s payoff is completely determined by his probability of being in the up-state. Let \(V(\rho)\) denote the agent’s value function if the previous signal is bad, i.e. the agent will receive \(w\) if he accepts the contract today.

Recognizing that the agent can choose between exerting and not exerting effort
given every $\rho$, the value function has the following recursive representation:

$$V(\rho) = \max \{w - c + (p + (1 - p)\rho)(B + \delta V\left(\frac{p}{p + (1 - p)\rho}\right)) + (1 - (p + (1 - p)\rho))\delta V(\rho^*), w + \rho (B + \delta V(0)) + (1 - \rho)\delta V(\rho^*)\}.$$ 

In the above expression, we implicitly assume that the agent does not choose his outside option. This will be verified.

Now note that the operator on the right hand side satisfies the Blackwell Sufficiency Conditions, so it is a contraction mapping. Therefore, there is a unique value function $V$ that satisfies this equation.

To check the strategy profile forms a PBE, we take the following steps. First, we give an explicit expression for $V$. Second, given the value function, we show that, if the agent accepts the contract, his optimal response is to put in effort if $\rho \leq \rho^*$ and not put in effort otherwise. Third, we check that the agent will accept the contract.

For the value function, it takes the following form:

$$V(\rho) = \begin{cases} (B + \delta V(0))(\rho - \rho^*) & \text{for } \rho > \rho^* \\ (1 - p)(B - \delta \rho^*(B + \delta V(0)))(\rho - \rho^*) & \text{for } \rho \leq \rho^*, \end{cases}$$

where

$$V(0) = \frac{p(1 - \frac{1}{\rho^*} + \delta (1 - \rho^*))}{1 - p\delta^2(1 - \rho^*)} B. \quad (4)$$

To see that this is the value function (given the agent’s equilibrium strategy), we first note that the choice of $\rho^*$ and $w$ insures that $V(\rho^*) = 0$. In addition, it can be checked that the slope of $V$ for $\rho > \rho^*$ is a constant $(B + \delta V(0))$. In other words, $V$ is linear for $\rho > \rho^*$ although it remain to be checked that

$$V(\rho^*_+) \equiv \lim_{\rho \to \rho^*_+} V(\rho) = V(\rho^*).$$

Next, the linearity of $V$ for $\rho > \rho^*$ implies that $V$ is linear for $\rho < \rho^*$ as well, since
for \( \rho < \rho^* \),

\[
V(\rho) = w - c + (p + (1 - p)\rho)(B + \delta V(\frac{p}{p + (1 - p)\rho})) + (1 - (p + (1 - p)\rho))\delta V(\rho^*)
\]

\[
= w - c + (p + (1 - p)\rho)\left[ B + \delta\left(\frac{p}{p + (1 - p)\rho} - \rho^*\right)(B + \delta V(0)) + c\right]
\]

\[
+ (1 - (p + (1 - p)\rho))\delta V(\rho^*)
\]

\[
= (1 - p)(B - \delta \rho^*(B + \delta V(0)))(\rho - \rho^*).
\]

The last equality follows from the linearity of \( V(\rho) \). Finally, (4) follows

\[
U(0) = w - c + p(B + \delta U(1))
\]

\[
= -\frac{p}{\rho^*}B + p(B + \delta(B + \delta U(0))(1 - \rho^*)).
\]

Next, \( w + \rho^*(B + \delta U(0)) = 0 \) implies

\[
c - \frac{p}{\rho^*}B + \rho^* \left[ B + \delta\frac{p(1 - \frac{1}{\rho^*} + \delta(1 - \rho^*))}{1 - p\delta^2(1 - \rho^*)}B\right] = 0
\]

or equivalently

\[
B = \frac{c/p}{\left(\frac{1}{\rho^*} - \frac{\rho^*}{p} + \delta(1 - \rho^*)\right)}.
\]

(5)

This choice of \( B \) ensures that at \( \rho^* \), the agent is indifferent between putting in effort or not, so we have \( V(\rho^*_+) = V(\rho^*) \). This leads to the expression of the value function derived based on the assumption that the agent puts in effort when \( \rho \leq \rho^* \) and does not do so when \( \rho > \rho^* \). This finishes the first step.

In the second step, we check that the effort decisions specified are optimal. To do that, we first need to make sure that for \( \rho \leq \rho^* \),

\[
w + \rho(B + \delta V(0))
\]

\[
\leq w - c + [(p + (1 - p)\rho)(B + \delta V(\frac{p}{p + (1 - p)\rho}))
\]

\[
= (1 - p)(B - \delta \rho^*(B + \delta V(0)))(\rho - \rho^*),
\]

Note that the above is satisfied if

\[
B + \delta V(0) \geq (1 - p)(B - \delta \rho^*(B + \delta V(0))).
\]
Let \( x = B + \delta V(0) \), and define \( T(x) = (1 - p)(B - \delta \rho^* x) \), then the above can be rewritten as

\[
T(x) \leq x.
\]

We also want to make sure that for \( \rho > \rho^* \), we have

\[
\begin{align*}
& w + \rho(B + \delta V(0)) \\
\geq & \ w - c + [(p + (1 - p)\rho)(B + \delta V(\frac{p}{p + (1 - p)\rho})) \\
= & \ w - c + [(p + (1 - p)\rho)(B + \delta(1 - p)(B - \delta \rho^*(B + \delta V(0)))(\frac{p}{p + (1 - p)\rho} - \rho^*))] \\
= & \ (1 - p)(B - \delta \rho^*(1 - p)(B - \delta \rho^*(B + \delta V(0))))(\rho - \rho^*).
\end{align*}
\]

If we again have \( x = B + \delta V(0) \) and \( T(x) = (1 - p)(B - \delta \rho^* x) \), then the slope of \( \rho \) in the expression above is given by \( T(T(x)) \), and we need

\[
T(T(x)) \leq x.
\]

Now note that \( T(x) \) is an affine function of \( x \) with slope \(-\delta \rho^*(1 - p) > -1\). Let \( x^* \) be such that \( T(x^*) = x^* \), then

\[
(1 - p)(B - \delta \rho^* x^*) = x^* \Rightarrow x^* = \frac{(1 - p)B}{1 + \delta \rho^*(1 - p)}.
\]

Now note that if \( x \geq x^* \), then

\[
T(x) \leq x.
\]

Moreover, since the slope of \( T(x) \) is equal to \(-(1 - p)\delta \rho^* > -1\), this implies that, for \( x > x^* \),

\[
\frac{T(x^*) - T(x)}{x - x^*} = \frac{x^* - T(x)}{x - x^*} < 1.
\]

By the linearity of \( T \), it follows that,

\[
\frac{T(T(x)) - T(T(x^*))}{T(x^*) - T(x)} = \frac{T(x^*) - T(x)}{x - x^*} < 1
\]

so that

\[
T(T(x)) - T(T(x^*)) \leq x - x^*.
\]
or

\[ T(T(x)) \leq x. \]

The discussion above implies that, as long as

\[ B + \delta V(0) = x \geq x^* = \frac{(1 - p)B}{1 + \delta \rho^*(1 - p)}, \]

the action profile is optimal. In other words, we need

\[ V(0) \geq -\frac{1}{\delta} \frac{(p + \delta \rho^*(1 - p))}{1 + \delta \rho^*(1 - p)} B. \]

Recalling from (4) that

\[ V(0) = \frac{p(1 - \frac{1}{\rho^*} + \delta(1 - \rho^*)}{1 - p\delta^2(1 - \rho^*)} B. \]

So

\[
V(0) + \frac{1}{\delta} \left( \frac{p + \delta \rho^*(1 - p)}{1 + \delta \rho^*(1 - p)} \right) B \\
= \frac{p(1 - \frac{1}{\rho^*} + \delta(1 - \rho^*)}{1 - p\delta^2(1 - \rho^*)} + \frac{1}{\delta} \left( \frac{p + \delta \rho^*(1 - p)}{1 + \delta \rho^*(1 - p)} \right) B \\
= \frac{B}{\delta(1 - p\delta^2(1 - \rho^*)) (1 + \delta \rho^*(1 - p))} \geq 0.
\]

This shows that the effort decisions are optimal.

Finally, to make sure that the agent always accepts the contract, we need to make sure (see the value function) that

\[ B + \delta V(0) > 0. \]

But from the above, we see that

\[ B + \delta V(0) \geq \frac{(1 - p)B}{1 + \delta \rho^*(1 - p)} > 0, \]

and this shows that the proposed value function is truly the value function if the agent’s action is truly optimal.
Following directly from (5),

\[
\frac{c}{Bp} = \frac{1 - \rho^*}{\rho^* - p} + \frac{\delta(1 - \rho^* - \delta(1 - \rho^*)\rho^*)}{1 - p\delta^2(1 - \rho^*)} \\
= \frac{\rho^* - p}{(1 - p)\rho^*} + \frac{\delta(1 - \rho^* - \delta(1 - \rho^*)\rho^*)}{1 - p\delta^2(1 - \rho^*)},
\]

where we have used \((\rho^*)^2(1 - p) = p(1 - \rho^*)\) in simplifying \(\frac{1}{\rho^*} - \frac{\rho^*}{p}\).

Now since

\[
(1 - p)\rho^{*2} + p\rho^* - p = 0, \\
\rho^* = -p + \frac{\sqrt{4p - 3p^2}}{2}.
\]

In other words, when \(p\) is small, \(\rho^*\) is roughly in the order of \(\sqrt{p}\).

It is clear that as \(p\) goes to 0, both \(\frac{\rho^* - p}{(1 - p)\rho^*}\) and \(\frac{(1 - \rho^* - \delta(1 - \rho^*)\rho^*)}{1 - p\delta^2(1 - \rho^*)}\) go to 1, so

\[
\lim_{p \to 0} \frac{c}{Bp} = 1 + \delta.
\]

This completes the proof. ■