Relational Contracts, Efficiency Wages, and Employment Dynamics*

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Abstract
This paper studies the optimal dynamic provision of incentives in employment relationships with rents for the worker. In a model of relational contracts with limited liability, we show that the optimal relational contract generates definitive and joint implications on how job security, the pay level, and the sensitivity of pay to performance change over time as the employment relationship progresses. Our results also shed light on how turnover and pay dynamics change as job and firm characteristics vary.

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1 Introduction

A prominent line of research has emphasized the importance and consequences of moral hazard within firms; see Gibbons and Waldman (1999), Hart and Holmstrom (1987), and Prendergast (1999) for reviews. In employment relationships, the moral hazard problem has three salient features. First, employment relationships are typically ongoing, and many last a “lifetime.” The repeated interaction between the firm and the worker implies that the moral hazard problem has a dynamic aspect. Second, legal and institutional constraints restrict the degree to which a worker can be punished. In other words, workers are protected by a limited liability constraint. Finally, the costs of and difficulties in specifying and verifying performance often render formal contracts infeasible. Instead, the firms rely on "relational contracting" whereby workers are rewarded or punished at the firm’s discretion.

This paper develops a moral hazard model that combines these three features. The main purpose of the paper is to explore the consequence of optimal relational contracts with limited liability on employment dynamics. Our model provides definitive and joint implications on how job security, the pay level, and the sensitivity of pay to performance change over time as the employment relationship progresses. These patterns reveal the economic mechanism behind the firms’ ability to optimally extract rents from workers in a dynamic setting while continuing to offer work incentives.

The second purpose of this paper is to study how the employment dynamics are affected by job and firm characteristics. We focus on the joint changes in turnover and pay as the underlying economic environment varies. The two main sources of variation in our study are the firm’s ability to commit to future promises and the complexity of the job. Our analysis suggests that as the job becomes more complex, the average pay increases but the job security decreases. Our analysis also suggests that in firms that are better able to commit to future promises, workers have both higher average pay and higher job security.

Specifically, we study a model of relational contracts with imperfect monitoring, which is an infinitely repeated principal-agent model where outputs are publicly observable but not contractible. The effort is the worker’s private information and it stochastically affects the output. Unlike early models of dynamic moral hazard such as Lazear (1979), Shapiro-Stiglitz (1984),

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1 Hall (1982) finds that the average tenure of U.S. workers is eight years and that a quarter of the workforce what he calls "lifetime jobs," which are jobs lasting longer than 20 years. Beaudry and DiNardo (1992) also show that the labor market is better described by firms offering long-term contracts than by the spot markets.

2 We use "limited liability" to describe any restriction that sets a lower bound on the worker’s pay. One example is minimum wage. Another example is the court’s unwillingness to enforce "liquidated damage" in contracts, i.e., provisions that call for larger penalties than then damage done; see Dickens et al (1990) for a discussion. The limited liability constraint has been recognized as an important factor affecting incentives: formal models include Sappington (1983), Innes (1990), and Jewitt, Kadan, and Swinkels (2008).

3 For a definitive treatment of relational contracts with observable actions, see MacLeod and Malcomson (1988).
Akerlof and Katz (1989), the output may be low even if the worker puts in effort. In addition, we model the worker’s limited liability constraint by requiring that the agent’s pay each period not fall below an exogenously given wage floor. This assumption sets us apart from the standard relational contracting model with imperfect public monitoring such as that of Levin (2003). To better relate the model to empirical patterns of employment dynamics without overcomplicating the analysis, we assume that both the output and the effort are binary.

Since no formal contracts are allowed, we follow Levin (2003) to define each relational contract as a Perfect Public Equilibrium (PPE) of the model and the optimal relational contract as the PPE that maximizes the principal’s payoff. We completely characterize the set of PPE payoffs using the method developed by Abreu, Pearce, Stacchetti (1990) and derive the implications of the optimal relational contract on employment dynamics. Moreover, we solve for the model when long-term contracts are feasible, and we compare the difference between the optimal long-term contract and the optimal relational contract on employment dynamics.

Our results show that limited liability drastically changes the nature and structure of the optimal relational contract. Without limited liability, Levin (2003) implies that the optimal relational contract in our setting is efficient and stationary, and that the firm can extract all of the surplus in the relationship.\textsuperscript{1} When limited liability is present, rents must be given to the worker to induce effort. In designing the optimal dynamic incentive structure, the firm faces a tradeoff between surplus maximization and rent extraction. The optimal relational contract is no longer efficient nor is it stationary. But the structure of the optimal relational contract is surprisingly simple.

In particular, the employment relationship begins with a “probation phase,” during which the worker puts in effort and receives a constant wage equaling the wage floor. If the output history has been sufficiently favorable, the worker transitions into the “regular phase,” during which he is incentivized with pay for performance. When the surplus in the relationship is sufficiently high, the optimal relational contract in the regular phase can be implemented by a sequence of stationary contracts just as in Levin (2003): the worker receives a fixed amount of bonus for each high output. If the output history has been sufficiently unfavorable, inefficiency occurs. There are two types of inefficiency, depending on the exogenous parameters of the model.\textsuperscript{5} If the wage floor is low, the inefficiency takes the form of temporary suspension of production: there are periods in which the worker is paid the wage floor but does not put in effort. If the wage floor is high, the inefficiency takes the form of permanent employee termination.

In the high wage floor case, our model gives joint predictions on turnover and pay dynamics. In terms of turnover, the model predicts that the turnover rate is initially increasing with respect

\textsuperscript{4}These properties also hold for the optimal long-term contract.

\textsuperscript{5}For each set of parameter values, inefficiency occurs in a unique way.
to time on the job and is eventually decreasing to zero. In particular, the model can generate an inverse-U-shaped turnover rate as found in data; see for example Farber (1994). In terms of the pay level, the model predicts that the pay level is deferred in that there is a discrete jump in the average pay level after the agent transitions from the probation phase into the regular phase. Moreover, this increase is associated with a discrete jump in the sensitivity of pay to performance.

This paper is mainly related to three strands of literature. First, the employment dynamics in our model reflects features of optimal rent extraction under a dynamic setting. Since the worker has ex ante rents in the relationship, our model belongs to the class of efficiency-wages models. Our model adds to classic efficiency wages models such as Shapiro and Stiglitz (1984) by featuring a) a stochastic production (on the equilibrium) and b) an explicit modelling of the limited liability constraint. The combination of these two assumptions helps generate equilibrium turnover and allows us to make predictions on pay and turnover jointly. In addition, if the probability of high output given effort is used as a measure of job complexity, our paper sheds light on how pay and turnover change jointly with respect to the routineness of the job. When the jobs are routine (so that the output is always high given effort), we show these jobs have better job security but are also associated with lower average pay. In other words, there is a negative correlation between pay and job security when the job’s complexity changes.

Second, our paper is related to a recent, vibrant literature on dynamic contracts with limited liability. In particular, in the case where the principal can commit to a long-term contract, DeMarzo and Fishman (2007) show that termination can arise in the optimal long-term contract when the agent has private information about the cash flow and has a limited liability constraint. In addition, many papers have taken the optimal contracting view and studied its implications in diverse economic situations: see, for example, Biais, Mariotti, Plantin, and Rochet (2007) and DeMarzo and Sannikov (2006) for application in finance; Clementi and Hopenhayn (2006) in industrial organization and firm dynamics; Myerson (2008) in political economy; and Lewis (2009) and Lewis and Ottaviani (2008) in search theory. Ours can be considered as applying the optimal contracting view to labor economics, where we focus on how pay and turnover rate change with time on the job.

In the main body of the paper, however, the employment dynamics results from the optimal relational contract instead of the optimal long-term contract as we assume that the principal cannot commit to long-term contracts. This relational contracting approach reflects our view that the costs of drafting and enforcing long-term contracts are often too high to be feasible, and one focus of the paper is to study when and how employment dynamics are affected by firms’ abilities to keep promises. In Section 5.2, we characterize the optimal long-term contract and compare it with the optimal relational contract. We show that when the job is routine
or the relationship has high surplus, the employment dynamics is not affected by the firm’s commitment power. However, when the relationship has low surplus, our results suggest that stronger commitment power leads firms to provide to their workers with both higher pay and higher job security. Our results cast new light on empirical findings that link employment dynamics with firm characteristics.

Finally, our model adds to the growing literature that studies the dynamics of relational contracts. Classic models of relational contracts, such as Bull (1987), MacLeod and Malcomson (1988), and Levin (2003, first part), have focused on the conditions under which cooperation can be sustained, and the optimal relational contracts in these models are stationary. Several more recent papers have examined the evolution of relationship when additional frictions are present. Halac (2008) studies a model in which the principal’s type is private information. In Yang (2009), the agent’s type is private information. Fuchs (2007) and Levin (2003, second part) both analyze the case in which the output is privately observed by the principal. In Chassang (forthcoming), the efficient production function is unknown, and the agent’s moral hazard problem is linked to an experimentation problem.

In our model, the source of friction is the limited liability of the agent, so (monetary) transfer between the principal and the agent is constrained. A closely related paper on relational contract with two-sided limited liability is Thomas and Worrall (2007). The main difference is that they have a partnership game, so outputs depend on the joint efforts of the two parties. In addition, effort are perfectly observed so the relationship does not terminate. They focus on the dynamics of effort provision and show that effort may be overprovided at the earlier stage of the relationship. Another related paper is Padro i Miguel and Yared (2009), who in a political-economy setting, study a similar repeated principal-agent model without commitment or transfer. Without transfer between the players, they focus on how the surplus is destroyed in terms of the likelihood, duration, and intensity of the punishment.

The rest of the paper is organized as follows. We set up the model in Section 2. The PPE payoff set and the optimal relational contract are characterized in Section 3. Section 4 studies the empirical implications of the optimal relational contract when the principal can commit. Section 5 explores the non-commitment case. Section 6 concludes.

2 Setup

There is one principal and one agent. Both are risk neutral, infinitely lived, and have a common discount factor of $\delta$. Time is discrete and indexed by $t \in \{1, 2, ..., \infty\}$.

At the beginning of each period $t$, the principal decides whether to offer a contract to the agent: $d^P_t \in \{0, 1\}$. If no contract is offered ($d^P_t = 0$), the two parties receive their outside
options in this period. The agent’s per period outside option is $u$; the principal’s per period outside option is $v$. If a contract is offered, it specifies a legally enforceable wage $w_t \geq w$, where $w \in R$ is an exogenously given wage floor. When $w = 0$, the stage game becomes a standard model of moral hazard with limited liability.

In many relational contracting models, the contract also includes a discretionary bonus at the end of the period. In our setup, the discretionary bonus can be thought of as being postponed until the beginning of the next period and it becomes part of the wage offered. These two setups give rise to the same set of equilibrium payoffs, and the results obtained here can be directly translated into a version with a discretionary bonus. The current setup is chosen to simplify notation and to facilitate the comparison to efficiency wages models.

If the principal offers a contract, the agent chooses $d_t^A \in \{0, 1\}$. If he rejects the contract ($d_t^A = 0$), the two parties receive their outside options. Otherwise, the relationship starts and the principal pays out wage $w_t$. The agent then chooses effort $e_t \in \{0, 1\}$, and output $Y_t \in \{0, y\}$ is realized. If the agent works ($e_t = 1$), he incurs a cost of effort $c$, and $Y_t$ is $y$ with probability $p \in (0, 1)$ and 0 with probability $1 - p$. If the agent shirks ($e_t = 0$), no effort cost is incurred, and $Y_t$ is $y$ with probability $q < p$. To make the analysis interesting, we assume that the value of the relationship exceeds the sum of outside options if and only the agent works:

$$py - c > u + v > qy.$$ 

In terms of the information structure, the effort of the agent is his private information. Outputs are publicly observable. In addition, we assume that a public randomization device is available. In particular, both the principal and the agent observe at the end of the period the realization of a random variable, $x_t \in [0, 1]$. We summarize the timing in the graph below.

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6 Malcomson and MacLeod (1998) provides a formal proof for the symmetric information case. Their proof can be adapted to the current case.

7 The public randomization device is a commonly-made assumption in models of repeated games to convexify the equilibrium payoffs; see for example Mailath and Samuelson (2006), Section 3.4 for a discussion of its roles.
At the beginning of any period $t$, the expected payoffs to the principal and the agent are given by

$$v_t = (1 - \delta) \sum_{\tau=t}^{N} \delta^{N-t} [(1 - d_P^t d_A^t) u_t + d_P^t d_A^t (y(q + (p - q)e_t) - w_t)];$$

$$u_t = (1 - \delta) \sum_{\tau=t}^{N} \delta^{N-t} [(1 - d_P^t d_A^t) u_t + d_P^t d_A^t (w_t - c e_t)],$$

where we multiply through by $(1 - \delta)$ to express the payoffs as per period averages.

Our environment is an infinitely repeated game with imperfect public monitoring, and we follow the literature to use public perfect equilibrium (PPE) as the solution concept.\(^8\) A PPE is strategy profile such that a) players use strategies that depend on the public history, and b) the strategy profiles following any public history form a Nash Equilibrium.

Formally, we denote $h_t = \{d_P^t, w_t, d_A^t, y_t, x_t\}$ as the public events that occur in period $t$. Let $h^t = \{h_n\}_{n=1}^{t-1}$ be a public history path at the beginning of period $t$, and $h^1 = \emptyset$. Let $H^t = \{h^t\}$ be the set of public history paths till time $t$, and define $H = \cup_t H^t$ as the set of public histories.

In period $t$, the principal’s actions are specified by

$$D_P^t : H^t \to \{0, 1\};$$

$$W_t : H^t \cup \{d_P^t\} \to [w, \infty),$$

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\(^8\)This model is a game of imperfect monitoring with "product structure", in that the output depends on the agent’s effort alone. It follows that our restriction to PPEs is without loss of generality; see Fudenberg and Levine (1994).
where $D_t^P$ is a function that determines whether to offer a contract and $W_t$ is the function that determines the wage level. The principal’s public strategy, $s^P$, is given by $\{D_t^P, W_t\}_{t=1}^{\infty}$. The agent’s actions are specified by

$$D_t^A : H_t \cup \{w_t\} \to \{0, 1\},$$
$$E_t : H_t \cup \{w_t\} \to \{0, 1\},$$

where $D_t^A$ is a function that determines whether to accept the contract and $E_t$ is a function that determines the effort level. The agent’s public strategy, $s^A$, is given by $\{D_t^A, E_t\}_{t=1}^{\infty}$.

Let $v(s^P, s^A|h_t)$ and $u(s^P, s^A|h_t)$ be the principal’s and the agent’s expected payoffs following public history $h_t$. A strategy profile $(s^P, s^A)$ is a PPE if and only if following any public history $h_t$,

$$s^P \in \arg \max_{\tilde{s}^P} v(\tilde{s}^P, s^A|h_t);$$
$$s^A \in \arg \max_{\tilde{s}^A} v(s^P, \tilde{s}^A|h_t).$$

The optimal relational contract is the PPE that maximizes the principal’s payoff at the beginning of the first period.

3 Analysis

We solve for the optimal relational contract in this section. First, in Subsection 3.1 we review the benchmark model of relational contracts without limited liability. This review helps make clear the additional trade-off the principal faces when the limited liability constraint is introduced. In Subsection 3.2, we characterize the PPE payoff set using the method developed by Abreu, Pearce, and Stacchetti (1990) (APS hereafter). In subsection 3.3, we derive the properties of the optimal relational contract from the PPE payoff set. With a few exceptions, formal derivations are contained in the Appendix.

3.1 Benchmark: Relational Contracts without Limited Liability

Without limited liability, our model becomes a special case of Levin (2003) with an inconsequential timing difference that we discuss towards the end of this subsection. Levin (2003) shows that the optimal relational contract can be implemented by a sequence of short-term contracts that are stationary over time. When outputs are binary, each contract specifies a base wage, $w_s$, and a discretionary bonus, $b_s$, if the output is high. To induce effort, we need the (normalized) benefit of effort to exceed its cost:

$$(1 - \delta)(p - q)b_s \geq (1 - \delta)c, \quad (1)$$
where recall \( p - q \) is the additional success probability through effort, so the left hand side is the agent’s benefit of effort. In a relational contract, for the principal not to renege on the bonus, a necessary condition is that the bonus cannot exceed the expected future surplus in the relationship:

\[
(1 - \delta)b_s \leq \delta(py - c - u - v).
\]

Combining these two inequalities, a necessary condition for the existence of an efficient relational contract becomes:

\[
(py - c) - (u + v) \geq k \equiv \frac{(1 - \delta)c}{\delta(p - q)},
\]

where \( k \) can be thought of as the minimum reward for high output that sustains effort. Levin (2003) shows that the necessary condition is also sufficient.

When Eq(2) holds, the optimal relational contract can be found using a two-step procedure. First, the bonus \( b_s \) must be big enough to induce effort. Thus, we set

\[
b_s = \frac{c}{p - q}
\]

so the worker is just willing to put in effort. Second, the base wage \( w_s \) must be set to extract the entire surplus of the relationship and, therefore, leaves the agent with no rent. Thus, we set

\[
w_s = u - \frac{qc}{p - q}
\]

so that the agent’s payoff is exactly equal to his outside option: \( w_s + pb_s - c = u \). Note that in this two-step procedure, there is no conflict between surplus maximization and rent extraction for the principal.

In our setup, the principal does not give a bonus at the end of a period. Instead, the bonus is essentially paid out as part of the wage in the following period. This allows for a direct translation of the results in Levin (2003) to ours. In particular, the optimal "stationary" relational contract becomes

\[
\begin{align*}
w_1 &= w_s \\
n_t &= w_s + 1\{Y_{t-1} = y\} \left( \frac{b_s}{\delta} \right), \text{ for } t > 1,
\end{align*}
\]

where paying out \( b_s \) at the end of a period is equivalent to paying out \( \frac{b_s}{\delta} \) at the beginning of the following period.

To facilitate a comparison to the case with limited liability, Figure 2 below illustrates the set of the PPE payoffs without limited liability. Denote \((u, v)\) as a payoff pair where the agent
receives \( u \) and the principal receives \( v \). Denote \( f(u) \) as the principal’s highest possible PPE payoff when the agent receives \( u \). We refer to \( f \) as the PPE payoff frontier. Figure 2 shows that, when there is no limited liability, \( f \) is a negative 45 degree line segment originating from the payoff pair given by the optimal relational contract \((u, py - c - u)\), and ending in \((v, py - c - v)\).

To see why the PPE payoff frontier has a slope of \(-1\), note that the principal can always reset to base wage in period 1 to reallocate the payoffs between the principal and agent (and keep the rest of the strategies fixed), as long as the resulting payoffs to both parties are greater than or equal to their outside options. More formally, if \((u, v)\) is a PPE payoff, then all the payoffs in the set

\[
\{(u', v')|u' + v' = u + v, \ u' \geq u, \ v' \geq v\}
\]

are also PPE payoffs.

In summary, when the limited liability constraint is absent, every point on the PPE payoff frontier is efficient (with a joint value of \(py - c\)). The optimal relational contract is an efficient PPE that leaves the agent with no rent. The optimal relational contract can be implemented by a sequence of stationary contracts with a base wage \(w_s = u - \frac{qc}{p-q}\).

### 3.2 Frontier of the PPE Payoff Set with the Limited Liability Constraint

For the limited liability constraint to matter, we assume

\[
w > u - \frac{qc}{p-q}
\]  
(3)
so the stationary relational contract in Subsection 3.1 is infeasible, and therefore, it is impossible to extract all of the rents from the agent. The optimal relational contract captures how the principal can best extract rent from the agent while maintaining the agent’s incentive to work. Unlike the case without limited liability, the optimal relational contract is inefficient and non-stationary. In particular, the PPE payoff frontier is no longer a line segment with a slope of $-1$.

Instead, the PPE payoff frontier is divided into three regions. Each region corresponds to a distinct phase of the employment relationship. Theorem 1 describes these regions precisely. Below, we discuss informally the equilibrium play and the transitions of the continuation values associated with each of the regions. Figure 3 illustrates the PPE payoff frontier for the case where both the surplus and the wage floor are sufficiently high (to be made precise below).

**The Right Region**

To the right of a high threshold $(u_e)$, the PPE payoff frontier is again a line segment with a slope of $-1$, just as in the case without the limited liability constraint. Each payoff in this region is efficient when the surplus in the relationship is sufficiently high, i.e.,

$$py - \frac{[1 - \delta(1 - p)]c}{\delta(p - q)} \geq v + w. \quad (4)$$

In this case, each payoff can be implemented by a sequence of stationary contracts with a base wage and a discretionary performance bonus. The stationarity implies, in particular, that this region is self-generating: once the PPE payoff pair falls in this region, the continuation values stay in it forever.

Unlike the case without limited liability, this region does not extend all the way to the left to $u$. Instead, it stops at $u_e$, where the payoff pair $(u_e, f(u_e))$ is implemented by a sequence of
stationary contracts with a base wage of $w$ and a discretionary bonus of $\frac{c}{p-q}$. Note that at $u_e$, the base wage already hits the wage floor $w$ and cannot be lowered further. Under the stationary contracts, an agent with value $u_e$ receives $w$ and puts in effort in period 1. If the output is high, the agent’s continuation payoff goes up to $u_e + k$ in the next period; if the output is low, the agent’s continuation payoff remains at $u_e$.

When the surplus in the relationship is low, the slope of the frontier in this region remains at $-1$. However, the payoffs on the frontier are no longer efficient nor can they be implemented by a sequence of stationary contracts. The difference arises because the bonus used to induce effort in the high surplus case ($b = \frac{c}{p-q}$) will lead the principal to renege when the surplus is low. To induce effort in this region, the principal pays a smaller bonus for a high output and increases the probability of future punishment (such as termination) for a low output. The smaller bonus prevents the principal from reneging. The possibility of future punishment implies that this region is no longer self-generating: a low output moves the agent’s continuation value out of this region to the left.

The Left Region

To the left of threshold $u_0$, the PPE payoff frontier is characterized by the straight line

$$f(u) = f(u) + \frac{u - w}{u_0 - u} (f(u_0) - f(u)).$$

In this region, the agent’s payoff is so low that it is impossible to induce effort from him. In other words, the maximum future rent to the agent that is consistent with such a low payoff is too low to provide sufficient incentive for effort. Consequently, the payoff frontier is obtained from a randomization between $(u, f(u))$ and $(u_0, f(u_0))$. If the randomization outcome gives the agent $u_0$, which happens with probability $(u - u) / (u_0 - u)$, then the agent will continue to exert effort. Otherwise, the agent receives $u$, no effort is put in, and inefficiency occurs in the relationship.

When the agent receives $u$ and inefficiency occurs, there are two types of equilibrium play. The first is permanent termination of the relationship. In this case, the principal receives her outside option $v$ in all future periods. Termination occurs when the wage floor is sufficiently high. In Theorem 1, we show that there exists a wage floor $w^* \in (u - \frac{qc}{p-q}, u)$ such that the agent is punished through termination if and only if $w > w^*$. This result follows because a higher wage floor gives the agent bigger rent in the relationship, so termination becomes a more effective tool for inducing effort. In addition, bigger rent for the agent means that the principal’s payoff in the relationship is smaller, so termination is less costly for her. Both forces make termination the more effective method of punishment when the wage floor is higher.

When $w < w^*$, the principal can more effectively punish the agent by asking the agent
to exert low effort for one period and paying the agent \( w \). After that period, the agent’s continuation value will be equal to \( \left[ u - (1 - \delta) \frac{w}{\delta} \right] / \delta \) regardless of the outcome.\(^9\) With this type of punishment, the principal’s payoff this period is \( qy - w \) and his continuation payoff is \( f \left( \frac{u - (1 - \delta) w}{\delta} \right) \). We call this punishment temporary suspension of production (TSP) because the agent is expected to put in effort after the punishment. When TSP occurs, the agent’s payoff in the period \( (w) \) is lower than his static maxmin payoff in the period \( (u) \). This can be part of an equilibrium because the agent expects to receive payoffs exceeding \( u \) in the future.

To see why the principal may prefer TSP over termination when the wage floor is low, note that the principal’s payoff during TSP is \( qy - w \), which can exceed his outside option via termination \( (v) \) when the wage floor, \( w \), is sufficiently low. In this case, the principal definitely prefers punishing with TSP. But even if \( qy - w < v \), the principal may still prefer TSP since his continuation payoff \( f \left( \frac{u - (1 - \delta) w}{\delta} \right) \) through TSP always exceeds \( v \). The following figure depicts how a convex combination of \( (w, qy - w) \) and \( (\frac{u - (1 - \delta) w}{\delta}, f(\frac{u - (1 - \delta) w}{\delta})) \) can give the principal a payoff higher than \( v \) (while giving the agent a payoff of \( u \)) even when \( qy - w < v \).

![Figure 4: Determination of \( f(u) \) when \( f(u) > v \)](image)

Finally, to determine the boundary of the left region \( (u_0) \), we introduce a linear function \( L(u) \), which is defined by the following equation:

\[
\begin{align*}
    u &= (1 - \delta)(w - c) + \delta[(1 - p)L(u) + p(L(u) + k)].
\end{align*}
\]  

\(^9\)Note that this is the continuation value that keeps the agent’s payoff at \( w \):

\[
(1 - \delta) w + \delta \frac{w - (1 - \delta) w}{\delta} = u.
\]
In other words, \( L(u) \) corresponds to the agent’s continuation payoff next period if he is paid \( w \) this period, he puts in effort, but the outcome is low. For a relational contract to induce effort from the agent, the agent’s payoff, \( u \), must satisfy that \( L(u) \geq u \). In Theorem 1, we show that the boundary of this left region, \( u_0 \), satisfies exactly \( L(u_0) = u \). Alternatively, we have

\[
u_0 = (1 - \delta)(w - c) + \delta(u + pk).
\]

The Middle Region

In the middle region, each point on the PPE payoff frontier is implemented as follows. The principal offers a wage equal to the wage floor \( w \) and the agent puts in effort. For an agent with payoff \( u \), if the output is low, the agent’s continuation value moves to the left to \( L(u) \), as defined in \( (5) \). If the output is high, the agent’s continuation value moves to the right to \( L(u) + k \). Since the difference in the continuation values is \( k \), the agent is just willing to put in effort. In addition, for values in the middle region, we have

\[
L(u) < u < L(u) + k.
\]

Therefore, the middle region is not self-generating: if the outputs have been consecutively low, the agent’s continuation payoff will eventually drift to the left region. If the outputs are consecutively high, the agent’s continuation payoff will eventually drift to the right region.

In the middle region, the principal’s payoff \( f(u) \) satisfies

\[
f(u) = (1 - \delta)(py - w) + \delta[pf(L(u) + k) + (1 - p)f(L(u))].
\]

This functional equation implies that the principal’s continuation values remain on the PPE payoff frontier. Note that this feature also holds in the other two regions as well. Therefore, as long as an equilibrium payoff is on the frontier, its continuation values following any history remain on the frontier. This implies that the optimal relational contract is sequentially optimal, just as in the case without limited liability.

The most interesting feature of the middle region is that the agent’s wage is always equal to \( w \). This feature reflects a general principle on how to dynamically provide incentives in jobs with rent: the principal should delay bonus payments as part of the reward as much as possible. To see why paying a wage exceeding \( w \) is suboptimal, note that when this happens, the principal can lower the wage payment and also lower the probability of punishing the agent when his continuation falls into the left region to keep the agent indifferent. Such change strictly benefits the principal because it keeps the agent’s payoff unchanged and at the same time increases the total surplus. The total surplus is increased because the change reduces the chance that
inefficiency occurs in the future. In general, when a high output is realized, by substituting a bonus payment with a reduction in the probability of future punishment as part of the reward, the principal can increase her payoff by increasing the surplus of the relationship.

To summarize, the frontier of the PPE payoff satisfies

$$f(u) = \begin{cases} 
  f(u) + \frac{u-u_0}{u_0-u}(f(u_0) - f(u)) & \text{if } u \in [u_0, u] \\
  (1-\delta)(p_y - w) + \delta\beta f(L(u) + k) + (1-p)f(L(u)) & \text{if } u \in [u_0, u_e] \\
  f(u_e) + u_e - u & \text{if } u \in [u_e, u_{\text{max}}],
\end{cases} \tag{8}$$

where $u_0 = (1-\delta)(w - c) + \delta(u + pk)$,

$$u_e = \begin{cases} 
  u_r \equiv (w - c) + \frac{\delta pk}{1-\delta} & \text{if } py - \frac{1-\delta(1-p)c}{\delta(p-q)} \geq v + w \\
  (1-\delta)(w - c) + \delta[u_{\text{max}} - (1-p)k] & \text{if } py - \frac{1-\delta(1-p)c}{\delta(p-q)} < v + w
\end{cases}$$

for some $u_{\text{max}} \leq py - c - v$ and

$$f(u) = \begin{cases} 
  (1-\delta)(qy - w) + \delta f\left(\frac{u-(1-\delta)w}{\delta}\right) & \text{if } w < w^* \\
  v & \text{if } w \geq w^*,
\end{cases}$$

for some $w^* < u$.

In addition, for $f$ to be nontrivial, we need $u_0 \leq u_e$ and $f(u_{\text{max}}) \geq v$. These two conditions translate into

$$py - c - u - v \geq \frac{1-p\delta}{\delta} \frac{c}{p-q} \geq w - c - u$$

and lead to the following theorem.

**Theorem 1** When $py - c - u - v \geq \frac{1-p\delta}{\delta} \frac{c}{p-q} \geq w - c - u$, there exists a unique $u_{\text{max}} \leq py - c - v$ (which holds in equality when $u_r + k \leq py - c - v$) and a unique function $f$ that satisfy (8), and $f$ is the frontier of the PPE payoff set.

Equation (8) expresses $f$ as a solution to a functional equation. In general, the existence and uniqueness of solutions to functional equations can be difficult to verify. In our case, we can view the right hand side of equation (8) as an operator on $f$. It can be shown that this operator is monotone and nonexpansive, and, thus, has a unique fixed point.

In some cases, we can calculate $f$ explicitly. For example, suppose the relationship has high surplus and

$$p \leq q + \frac{\delta(u - w)}{(1-\delta-\delta q)c}. \tag{9}$$

In this case, a single high output moves the agent’s continuation payoff to the right region. Under this condition, the middle region is partitioned by a sequence of thresholds $\{u_n\}_{n=1}^{\infty}$, and the
payoff frontier between any two adjacent points is a straight line; see Figure 5. The thresholds are obtained as follows: if the agent’s expected payoff is \( u_n \), his continuation payoff following a low output moves to the left to \( u_{n-1} \), i.e.,

\[
L(u_n) = u_{n-1}.
\]

This implies that an agent with payoff \( u_n \) is guaranteed to stay in the relationship for at least \( n \) more periods, but he will be terminated if the next \( n+1 \) outputs are all low. As \( n \) goes to infinity, \( u_n \) converges to \( u_e \), which is the lowest payoff at which the agent is given tenure.

![Figure 5: Pareto Frontier w/ Countable Number of Kinks](attachment:image.png)

We will return to this example in the next section when discussing turnover dynamics. We show there that this example leads to an up-or-out rule: the agent is terminated if he is not tenured by a certain date. In addition to its empirical implications, this example also illustrates a theoretical point of the non-differentiability of the payoff frontier in repeated principal-agent models. In particular, at each threshold \( u_n \), the payoff frontier \( f \) has a kink, and is thus not differentiable. The non-differentiability of the payoff frontier has been pointed out by Thomas and Worrall (1994). Here we have an example with a countable number of non-differentiable points. Interestingly, these nondifferentiable points converge to \( u_e \), which is differentiable and has a slope \(-1\), as we will show in the next subsection. Corollary 1 gives the formula explicitly.

**Corollary 1:** If (4) and (9) hold, then

\[
f(u) = \begin{cases} 
  \nu + s_0(u - \nu) & \text{if } u \in [\nu, u_0] \\
  \nu + \frac{(u_0 - \nu)}{1 - \delta (1 - p)} \left(s_0 - \frac{\nu (1 - s_{n+1})}{1 - \delta} - \delta^{n+1} s_n (1 - p) + s_{n+1}(u - u_n)\right) & \text{if } u \in [u_n, u_{n+1}] \\
  f(u_e) + u_e - u & \text{if } u \in [u_e, u_e + f(u_e) - \nu],
\end{cases}
\]
\[ u_0 = (1 - \delta)(w - c) + \delta(u + pk) \]
\[ u_n = u_0 + \frac{\delta(1 - \delta^n)}{1 - \delta}(u_0 - u) \]
\[ u_e = (w - c) + \frac{\delta pk}{(1 - \delta)} \]
\[ s_0 = \frac{(1 - \delta)(py - w) + \delta((1 - p)v + p(py - c - (u + k))) - v}{(1 - \delta)(w - c) + \delta(u + pk) - u} \]
\[ s_n = s_0 - (1 + s_0)(p + (1 - (1 - p)^{n+1})) \]
\[ f(u_e) = py - c - ((w - c) + \frac{\delta pk}{(1 - \delta)}) \]

3.3 Optimal Relational Contract

Theorem 1 provides information on the actions played at each point on the PPE payoff frontier and the continuation values associated with each outcome. To characterize the optimal relational contract, it remains to determine which point on the PPE payoff frontier gives the principal the highest payoff.

In Figure 5, the principal’s payoff is maximized in the middle region. The key result in this subsection is that this property is more general. The optimal relational contract always starts in the middle region and is, thus, inefficient.

**Proposition 1:** Suppose (4) holds. Then \( f'(u_e) = -1 \) so there exists \( u < u_e \) such that

\[ f(u) > f(u_e). \]

**Proof.** Because \( f \) is concave, then for almost all \( u \in [u_0, u_e] \), Theorem 1 implies that

\[ f'(u) = pf'(L(u) + k) + (1 - p)f'(L(u)). \]  \( \text{(10)} \)

Take \( u_0' < u_e \) such that \( f(u_0') \) is differentiable and \( L(u_0') + k > u_e \). Take \( u_1' \) such that \( L(u_1') = u_0' \). According to (10) and the fact that \( u_0' < u_e < u_0' + k \), within the interval of \([u_0', u_e]\), we can find \( u_1' \) such that

\[ f'(u_1') \leq -p + (1 - p)f'(u_0'). \]

This procedure can continue forever, i.e., we can find \( u_{n+1}' \in (u_n', u_e) \) such that

\[ f'(u_{n+1}') \leq -p + (1 - p)f'(u_n'). \]
Moreover, for any \( u'_n \) such that \( f'(u'_n) = -1 + \varepsilon \), there exists \( u'_{n+1} \in (u'_n, u_e) \) such that
\[
-1 + \varepsilon > f'(u'_{n+1}) > -1.
\]

Therefore, \( f'(u_e-) = \lim_{n \to \infty} f'(u'_n) = -1 \).

Once we establish that \( f'(u_e) = -1 \), then it is immediate that there exists \( u < u_e \) such that \( f(u) > f(u_e) \). \( \blacksquare \)

From Theorem 1, we know that the right derivative of \( f(u_e) \) is \(-1\). Proposition 1 essentially says that when (4) holds, the left derivative of \( f(u_e) \) is also \(-1\). Thus in the optimal relational contract, the agent’s payoff falls strictly below \( u_e \). Since \( u_e \) is the smallest payoff of the agent that the joint surplus is maximized, Proposition 1 then implies that the optimal relational contract is inefficient. Moreover, Proposition 1 implies that the marginal cost of rent extraction starts at zero\(^{10}\) when the surplus in the relationship is high, so the principal always sacrifices some efficiency to extract rent from the agent.

The tradeoff between efficiency and rent extraction also exists in static moral-hazard model with limited liability. But rent extraction in a dynamic setting leads to qualitative differences. Suppose the wage floor is sufficiently high for the inefficiency to take the form of termination. In stationary relational contracts, in each period the agent earns a base wage, receives a performance bonus for high output, and faces a fixed probability of termination for low output. The principal may extract more rent from the worker by increasing this fixed probability of termination. However, termination in such a stationary way is costly for the principal: the entire relationship may end after period 1. In fact, termination with a fixed probability can impose such a high cost to the principal that for some parameters the optimal stationary relational contracts use no termination. In other words, the optimal stationary relational contract may actually be efficient.

In contrast, Proposition 1 implies that the marginal cost of using termination starts at 0 in the optimal relational contract (without the stationarity restriction). Therefore, termination always occurs with positive probability and the optimal relational contract is inefficient. Moreover, it is nonstationary: the termination probability differs across periods and is path-dependent. For example, Proposition 1 implies that the optimal relational contract starts in the middle region and the termination probability can be zero following period 1.

Taking the standard carrot-and-stick metaphor, rent extraction in a stationary way requires \textit{in each period} giving out a carrot for high output and using a stick (with some probability) for low output. Both are costly for the principal. Optimal dynamic rent extraction implies that neither the carrot nor the stick should be used immediately. Instead, the relationship begins in the middle region, with the principal simply holding the stick. The principal raises

\(^{10}f'(u_e) = -1\) implies that the marginal value being destroyed from moving to the left of \( u_e \) is 0.
the stick higher (i.e., increases the probability of future termination) following each high output and lowers it otherwise. Following a sufficiently bad output history, the stick is raised high enough (the agent’s continuation value falls in the left region) and the principal will use the stick with positive probability. Following a sufficiently good output history, the stick is held low enough (the agent’s continuation value falls in the right region) and carrots are given out for high outputs. When the surplus in the relationship is high, the principal will eventually put down the stick following a sufficiently favorable output history. After that, the agent is incentivized exclusively with carrots.

### 4 Employment Dynamics

This section explores the empirical implications of the optimal relational contract. To study turnover dynamics, we assume in this section

\[ w > w^*, \]  

where \( w^* < u \) is the cutoff above which punishment takes the form of termination.\(^{11}\) Below, we state the predictions on employment dynamics. Subsection 4.2 relates our predictions to empirical findings and alternative theories on employment dynamics.

#### 4.1 Predictions on Employment Dynamics

The optimal relational contract gives a specific prediction on the pattern of employment dynamics. In particular, the worker starts the employment relationship in a “probation phase” in which he receives \( w \) regardless of his performance, and depending on his performance, he either transitions into the regular phase or is terminated. In the regular phase, the worker’s pay depends on his performance. If the relationship has a high surplus so that (4) holds, then once the worker transitions into the regular phase, the continuation optimal relational contract can be implemented by a sequence of stationary contracts and the worker is no longer terminated. Proposition 2 characterizes the employment structure formally.

**Proposition 2:** Suppose \( w > w^* \). In the optimal relational contract, the set of histories can be partitioned into \( H = H_1 \cup H_2 \cup H_3 \), such that

(i): Employment starts with the probation phase \((H_1)\), during which the worker’s pay is equal

\(^{11}\)Recall that when \( w < w^* \), the worker is punished by temporary suspension of production (TSP). The popular press sometimes reports stories about sports stars being benched following poor performance or disciplinary issues. If the stars have a smaller rents than the regular players (possibly because more public information is available about the stars), then our theory suggests that, following poor performances, the stars are more likely to be benched, while the other players are more likely to be terminated.
to $w$:

$$h^1 \in H_1;$$

$$w(h^t) = w; \text{ for } h^t \in H_1.$$

(ii): With positive probability, the relationship is terminated, after which both the principal and the agent receive their outside options $(u, v)$.

$$\Pr(h^t \in H_2) > 0 \text{ for some } t.$$  

$$u_t(h^t|h^t) \in H_2 = w;$$

$$v_t(h^t|h^t) \in H_2 = v.$$

(iii): (regular phase): With positive probability, the worker transitions into the regular phase $(H_3)$,

$$\Pr(h^t \in H_3) > 0 \text{ for some } t.$$  

When $h^t \in H_3$, if (4) holds, the optimal relational contract can be implemented in the following way:

$$w(h^{t+1}) = \begin{cases} 
 w & \text{if } y_t = 0; \\
 w + \frac{c}{(p-q)} & \text{if } y_t = y.
\end{cases}$$

In other words, the optimal relational contract in $H_3$ may be stationary.

When $h^t \in H_3$, if (4) fails, then the agent’s wage is given by

$$w_t = w + \max\{u_t - u_e, 0\},$$

where $u_e$ is determined by Theorem 1, and $u_t$ is the agent’s continuation value.

(iv): In the long run, the worker stays in the probation phase with probability 0. The relationship terminates with probability 1 if and only if the relationship has a low surplus ((4) fails)

$$\lim_{t \to \infty} \Pr(h^t \in H_1) = 0;$$

$$\lim_{t \to \infty} \Pr(h^t \in H_2) = 1 \text{ if (4) fails};$$

$$\lim_{t \to \infty} \Pr(h^t \in H_2) < 1 \text{ if (4) holds};$$

Proof. Define $H_3$ as the set of histories such that the agent’s continuation payoff $u \geq u_e$. Define $H_2$ as the set of histories such that the agent’s continuation payoff $u = u$. By Theorem 1, it is clear that $H_2 \cap H_3 = \emptyset$. Define $H_1 = H \setminus (H_2 \cup H_3)$. 

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From Theorem 1 and Proposition 1, we know that the game starts in $H_1$ directly gives (i). Since $(u, v)$ is the unique PPE payoff in which the agent’s payoff is $u$ and it is supported by taking the outside option forever, once the agent’s payoff reaches $u$, he stays there forever. The wage formula in (iii) follows directly from Theorem 1.

To prove the remaining of the proposition, it suffices to show that $L(u^*) < u^*$, and $L(u^*) + k > u^*$, where $u^*$ is the agent’s payoff associated with the optimal relational contract.\(^{12}\) (Once this is proven, we have $L(u) < u$ for all $u < u^*$, and $L(u) + k > u$ for all $u > u^*$, and the rest follows from the standard argument in the probability theory.) Given $L(u_e) = u_e$, and since $u^* < u_e$, we immediately have $L(u_e) < u$ for all $u < u^*$. and $L(u_e) + k > u$ for all $u > u^*$; and the rest follows from the standard argument in the probability theory.)

Given $L(u_e) = u_e$, and since $u^* < u_e$, we immediately have $L(u^*) < u^*$. To see that $L(u^*) + k > u^*$, note that $f'_-(u^*) \geq 0$ and $f'_+(u^*) \leq 0$. For small $\varepsilon > 0$, we have

$$f(u^* + \varepsilon) - f(u^*) = \delta p(f(L(u^* + \varepsilon) + k) - f(L(u^*) + k)) + (1 - p)(f(L(u^* + \varepsilon)) - f(L(u^*)).$$

Sending $\varepsilon$ to 0, we have

$$f'_+(u^*) = pf'_+(L(u^*)) + (1 - p)f'_+(L(u^*)) + k$$

If $L(u^*) + k \leq u^*$, then $L(u^*) < u^*$, so $f'_+(L(u^*)) > 0$. Now consider two cases. In case 1, $L(u^*) + k < u^*$, and the left hand side is strictly positive, violating the condition $f'_+(u^*) \leq 0$. In case 2, $L(u^*) + k = u^*$, and we have $f'_+(u^*) = f'_+(L(u^*)) > 0$, again violating the condition $f'_+(u^*) \leq 0$.

The three phases of the employment dynamics correspond to the three regions in the PPE payoff set. Proposition 1 implies that the optimal relational contract starts in the middle region, which corresponds to the probation phase. While in the probation phase, the agent’s continuation values vary according to the outputs. If the output is high, the agent’s continuation value moves the right. Otherwise, it moves to the left. When the continuation value moves across the right threshold $(u_e)$ the agent receives permanent employment when the surplus is high and the continuation optimal relational contract can be implemented by a sequence of stationary contracts. When the continuation value crosses the left threshold $(u_0)$, termination occurs with positive probability. This employment pattern give predictions on turnover, pay level, and the sensitivity of pay to performance, which we discuss below.

**Turnover Dynamics**

In terms of turnover dynamics, Proposition 2 has two predictions. First, the turnover rates are initially increasing with time on the job. This low turnover rate at the beginning of the employment relationship is a feature of the optimal relational contract. As mentioned in Section

\(^{12}\) If there are multiple agent’s payoffs that maximize the principal’s payoff, take $u^*$ as the smallest one.
3, to optimally extract rents from the worker, the principal does not terminate the agent immediately for a low output but instead raises the agent's termination probability in the future. This explains why the turnover rate is initially low.

Second, the turnover rates eventually drop down to zero. This follows because workers who have not received permanent employment always face some risk of termination (possibly in the future). As time passes, these workers will either be terminated or receive permanent employment. It follows that for workers who remain on the job, the fraction of them who have received permanent employment increases over time, and the turnover rate eventually moves towards zero.

Combining these two predictions, the model suggests that the turnover rate may be inverse-U-shaped with job tenure. We should note, however, the turnover rate is not necessarily inverse-U-shaped. Due to the discreteness of the time periods, the turnover rate may not be very smooth and can have multiple peaks. Nevertheless, the following provides an example where the turnover rate is a degenerate inverse-U: the turnover rates are zero both before and after a fixed date, and it is positive on that date.

**Corollary 2:** If \((4)\) and \((9)\) hold, there exists \(T^*\) such that the turnover rate is 0 for \(t < T^*\) and is again 0 for \(t > T^* + 1\). Generically, turnover happens only in \(T^*\).

**Proof.** When \((4)\) and \((9)\) hold, Corollary 1 gives an explicit formula of \(f\). There are two cases to consider. In case 1, there's a unique \(u_n\) that maximizes \(f(u)\). In this case, if any of the output in the first \(n + 2\) periods is positive, the agent receives permanent employment. Otherwise, the agent's continuation payoff moves to \(u_{n+1-t}\) in period \(t\), and is terminated at time \(t = n + 2\).

In case 2, there exists \(n\) such that \(f(u)\) is maximized in \([u_{n-1}, u_n]\). In this case, if no positive outcome has been generated, the agent's continuation will be in \([u_{n-t}, u_{n+1-t}]\) in time \(t\). And the agent will be terminated either in time \(t = n + 1\) or \(t = n + 2\).

The turnover process in Corollary 2 follows an "up-or-out" rule: the worker is terminated at a fixed date if he does not receive tenure by then. Note that the condition in Corollary 2 is more likely to be satisfied when the probability of success \((p)\) is small. In this regard, Corollary 2 appears to fit some segment of the academic labor market. A few (small-probability) home-run publications bring an assistant professor to tenure, and failure to do so before a fixed date leads to termination.

**Pay Dynamics**

In terms of pay level, Proposition 2 predicts that pay is deferred that the average pay is immediately higher once the worker transitions out of the probation case. The logic is straightforward: during the probation phase, the firm rewards for high output with future job security.
instead of bonus, and this results in lower average pay. For each individual worker, the model predicts deferred pay between phases of employment and not within. At the aggregate level, since the durations of the probation phases are stochastic, the model predicts that the average pay rises with time on the job. Moreover, it predicts that the increase in the average pay eventually goes to zero as most workers will have transitioned out of the probation phase.

In terms of the sensitivity of pay to performance, the model predicts that the (short-term) pay is insensitive to performance during the probation phase and responds to performance only in the regular phase. This happens because the rewards for good performance in a worker’s earlier career take the form of increased future job security as opposed to bonus payments. Just as in pay level, the increase in pay sensitivity occurs between, and not within, phases. But at the aggregate level, the average pay sensitivity increases with time on the job as more workers receive permanent employment over time.

4.2 Empirical Findings and Related Theories

In terms of turnover dynamics, the highlight of our model is that it can generate an inverse-U-shaped turnover rate, which fits well with empirical findings. Farber (1994) uses NLSY data and finds that the monthly hazard rate of job ending is not monotone decreasing. Rather, it increases to a maximum at three months and declines thereafter. The inverse-U-shaped pattern is typically explained by the matching model of Jovanovic (1979). In the Jovanovic model, a worker learns about the match quality with his employer over time, and the worker leaves for a new employer if the expected match quality falls below a (time-dependent) threshold. Turnover in Jovanovic’s model is efficient and, thus, the model does not distinguish voluntary and involuntary turnovers.

In contrast, the separation in our model is ex post inefficient, and since the worker has rents in the job, turnover in our model is better characterized as involuntary turnovers.\(^\text{13}\) In the CEO labor market, the distinction between quits and dismissals have been emphasized in the empirical literature. Brookman and Thistle (2009) find that the inverse-U-shaped pattern holds for both forced and unforced CEO turnovers. However, Farber (1994) does not distinguish between voluntary and involuntary turnovers, and it will be useful to study whether the inverse-U-shape pattern also folds for involuntary turnovers in the labor market in general.

Terminations also occur in two notable models of relational contracts. In Fuchs (2007), the source of termination is two-sided moral hazard. In his model, the output is the principal’s private information. When the principal reports a low output, it could either be that the output is low (which calls for punishing the agent) or that the principal has lied (which calls

\(^{13}\)The worker strictly prefers working for the firm than his outside option when \(w > u\). When \(w < u\), it is less clear whether the nature of turnover is voluntary or involuntary in this model.
for punishing the principal). The inability to identify which player has led to the bad outcome implies that surplus destruction is necessary and termination will occur (when the probability of high output $p < 1$). It is also possible that the termination probability starts out at zero in Fuchs (2007) especially when the surplus in the relationship is high. However, Fuchs (2007) is a model of repeated game with private monitoring, and there are great technical difficulties in solving such games. The turnover pattern associated with the optimal relational contract remains unknown. In Yang (2009), the source of termination is worker heterogeneity. In his model, high-ability workers always produce high outputs in equilibrium. A low output, therefore, is perfectly indicative of low ability. Yang (2009) shows that there exists a type of pooling equilibrium in which the worker’s ability is gradually revealed. A worker is immediately terminated following a low output. In Yang (2009), the turnover rate is monotone decreasing and depends on the proportion of low-ability workers.

In terms of pay level, our model predicts, broadly, that the average pay is increasing with job tenure, which is a well-documented empirical finding; see for example Rubinstein and Weiss (2007) for a recent survey. The universe of theories that generates this prediction is vast and includes (at least) firm-specific human capital, selection, screening, and learning and insurance. Most relatedly, it is well-known since the seminal works by Lazear (1979, 1981) that agency theory can generate upward-sloping wage profiles. In Lazear’s models, the upward-sloping wage profiles help prevent the worker from shirking close to the retirement age, and workers in his models do not have rents in the employment relationship. In contrast, the upward-sloping wage profile arises in our model as a feature of optimal rent extraction. One distinctive prediction of our model is a discrete increase in the expected pay following the probation phase.\footnote{Models of screening can also generate a discrete jump in wages following the probation phase. Unlike our model, however, such discrete jumps are not always necessary.}

In terms of pay sensitivity, the model predicts that pay becomes more sensitive to performance over time. The empirical evidence on this issue is broadly, though not exclusively, supportive. Hashimoto (1979) finds that the bonus to wage ratio is increasing with experience in Japanese firms. Gibbons and Murphy (1992) show that the pays of older CEOs are more sensitive to stock-market performance. Gompers and Lerner (1999) document that the sensitivity of pay to performance is smaller for newer venture capitalists. Misra, Coughlan and Narasimhan (2005) find that the salary to total compensation ratio is decreasing with salesperson seniority. However, Khan and Sherer (1990) find that bonuses are more sensitive to performance for less senior managers.\footnote{Unlike the rest of the authors cited above, Kahn and Sherer (1990) base their findings on a single firm. Workers in this firm have considerable job security: the annual discharge rate is 0.5%, one-tenth of the industry average.}

The increasing sensitivity of pay to performance is often explained by career-concern models; see for example Gibbons and Murphy (1992) and Gompers and Lerner (1999). In these models,
pay for performance and the outside market are substitutes in incentive provision. Over time, the worker’s ability becomes better known to the market and the incentive from the market weakens. To maintain proper incentives for the worker, pay becomes more sensitive to performance. In our model, in contrast, pay for performance and future job security are substitutes in incentive provision. Over time, pay becomes more sensitive to performance to substitute for the weakened job-security incentive. We think these two mechanisms complement each other and both can be relevant in practice.

Recently, Garrett and Pavan (2009, 2010) show that pay can also become more sensitive to performance over time in environments in which outputs depend both on the agent’s effort and ability and the agent’s ability is his private information and changes over time. Using a dynamic mechanism approach, they show that the optimal long-term contracts can be implemented by a sequence of (path-dependent) linear contracts and the average slopes of the linear contracts increase over time. Similar to our paper, the increased sensitivity of pay to performance also reflects dynamic rent extraction. Unlike our paper, there is no limited liability constraint and the agent’s rent comes from better information about his ability. In particular, efforts are distorted downwards to induce truth-telling from the agent. More interestingly, the distortions become smaller over time because abilities in the further future are less correlated with the current ability, and, thus, the effort distortions in the future are less effective in inducing truth-telling today. Garrett and Pavan (2010) allows the principal to replace the current agent and restart the relationship with a new agent drawn from a pool of infinitely many potential agents. In this case, the principal fires agents of lower ability. In addition, the optimal retention policy becomes more lenient over time, and there is excessive retention in the long run.

In all of the models of cited above, less is known about how turnover and pay dynamics are affected by exogenous variables. One unique feature of our model is its ability to predict how turnover and pay dynamics should change with respect to job characteristics and the firm’s ability to commit, which we discuss next.

5 Job and Firm Characteristics

In this section, we study how employment dynamics change as the characteristics of the job and the firm vary. In Subsection 5.1, we study the employment dynamics for a routine production process, i.e., the outputs are always high if the worker puts in effort. In Subsection 5.2, we study how employment dynamics change with respect to the firm’s ability to commit to its promises. Our analysis shows that to the extent that larger firms can better commit to future promises, they can also provide higher wages and better job security for their workers.
5.1 P=1 Case

As mentioned in the setup section, this model departs from classic efficiency models by adding both limited liability \((w \geq w)\) and stochastic production \((p < 1)\). The combination of these two assumptions helps make joint predictions on turnover and pay dynamics. In this subsection, we study the case of \(p = 1\), which better describes production processes that are more routine or standard. The purpose of this subsection is to clarify the role of limited liability and to better relate our model to classic efficiency-wage models. In addition, to the extent that \(p\) is related to job characteristics, we discuss how turnover and pay dynamics vary with \(p\).

The PPE payoff frontier for \(p = 1\) can be characterized in the same way as in the main body of analysis \((p < 1)\), so we skip the proofs and only describe the results here. When \(p = 1\), the PPE payoff frontier can be partitioned into two regions, divided by a threshold \(u_s\). To the right of \(u_s\), the frontier is efficient and has a slope of \(-1\). To the left of \(u_s\), the PPE frontier is again a straight line (with a positive slope) linking \((u, f(u))\), to \((u_s, py - c - u_s)\). The threshold, \(u_s\), is given by

\[
u_s = \max\{w - c, u_0\},\]

where \(u_0 = (1 - \delta)(w - c) + \delta(u + k)\) corresponds to the boundary of the left region of the PPE in the main body of the analysis (by setting \(p = 1\)). The optimal relational contract gives the agent a payoff of \(u_s\) and the principal a payoff of \(py - c - u_s\).

A special feature of the \(p = 1\) case is that the optimal relational contract is efficient. In particular, the agent always puts in effort, and since effort always leads to a high output when \(p = 1\), the relationship never ends. This result is in line with classic efficiency-wage models such as Shapiro and Stiglitz (1984) and Akerlof and Katz (1989) in which a worker will not be caught shirking when he puts in effort. More generally, turnovers do not occur in equilibrium (except for exogenous reasons) when cheating can be immediately and perfectly detected. Fuchs (2007) shows that when \(p = 1\), the relational contract is efficient even if the output is the principal’s private information. In this case, a report of low effort from the principal indicates that either the principal or the agent has cheated. Fuchs (2007) also shows that when \(p = 1\), the optimal relational contract can be implemented by "one-period review contracts," in which the principal reports the output every period.

While turnover does not occur for \(p = 1\), the wage dynamics under the optimal relational contract depends on the value of \(u_s\). When the wage floor is high \((w - c - u > k)\), \(u_s = w - c\). In this case, the optimal relational contract is uniquely implemented by having the agent receiving \(w\) each period and keeping him if and only if the outputs have always been high. Note that

---

16. More formally, \(f(u) = py - c - u\) for \(u \in [u_s, py - c - u_s]\).

17. Just as in the main body of the analysis, \(f(u) = py - c - u\) if \((u, u')\) lies above the line segment formed by \((w, py)\) and \((u_s, py - c - u_s)\), i.e., if \(\frac{u - w}{u_s - w} < \frac{py - c - u}{py - c - u_s}\).

---
\( w - c - u \) is the agent’s rent in the relationship if he receives \( w \) each period and puts in effort and
\[
k = \frac{(1 - \delta) c}{\delta (p - q)}
\] is the minimum reward necessary to induce effort. When \( w - c - u > k \), the future rents from staying in the relationship and receiving \( w \) is sufficient to induce effort from the agent.

When the wage floor is low (\( w - c - u < k \)), \( u_s = u_0 \). In this case, the optimal relational contract can be implemented as follows.\(^{18}\) The agent starts the employment relationship with a probation base that lasts for a fixed \( N \) periods.\(^{19}\) During the probation phase, the agent’s wage is \( w \). In period \( N + 1 \), the agent’s wage is randomized between \( w + \frac{c}{\delta (1 - q)} \) and \( w \).\(^{20}\) From period \( N + 2 \) on, the agent’s wage stays at \( w + \frac{c}{\delta (1 - q)} \). When \( w - c - u < k \), paying the agent \( w \) (and requiring effort) does not provide enough rents for effort. To sustain a relational contract with effort, the principal needs to pay higher wages. But paying \( w + \frac{c}{\delta (1 - q)} \) in all periods leaves the agent with too much rents, and the principal extracts rents from the agent by having a probation phase. The duration of the probation phase reflects the amount of rents in the relationship, and higher rents leads to longer durations. In the previous case with \( w - c - u > k \), the amount of rents in the relationship is so high that the probation is infinitely long (\( N = \infty \)).

The \( p = 1 \) case corresponds to jobs that are relatively routine. Our analysis suggests the probation period can arise in these jobs as a tool for rent extraction. For these jobs, the typical explanations for probation such as learning about workers, screening out bad workers, and encouraging workers to invest in firm-specific human capital are less likely to apply. Interestingly, our explanation appears to fit well with the use of a probation period in Henry Ford’s five-dollar-day Program, in which the workers are paid more than twice the prevailing wage. To qualify for this program, a worker is required to have worked for Ford for at least a six-month probation period during which he receives the prevailing market wage. Raff and Summers (1989) argue that the use of probation in this example is consistent with the efficiency-wage hypothesis, which sometimes involves deferred compensation.\(^{21}\) Explanations for probation based on learning, screening, or improving worker qualities are less relevant in this context because the majority of the jobs at Ford during that time were simple and routine. Raff and Summers (1989) provide the following citation from Meyer (1981), p. 41:

"Division of labor has been carried to such a point that an overwhelming majority of jobs consist of a very few simple operations. In most cases a complete mastery of movements does not take more than from five to ten minutes. All the training that

\(^{18}\) In this case, the optimal relational contracts can actually be implemented in many different ways. The pattern we describe is the unique limit of the cases with \( p \) approaching 1. When \( p < 1 \), there is a unique equilibrium employment pattern as described in Proposition 2.

\(^{19}\) \( N \) is the smallest integer such that
\[
\sum_{t=1}^{\infty} \delta^{t-1} (w + 1_{(t>N+1)} \frac{c}{\delta (1 - q)}) \leq u_0.
\]

\(^{20}\) The randomization ensures that the worker’s expected payoff is \( u_0 \).

\(^{21}\) On the other hand, we are unaware of efficiency wage models that explicitly generate the probation period. In fact, the reason that wages are deferred in our model is different from that in earlier efficiency wage models.
a man receives in connection with his job consists of one or two demonstrations by his foreman or a worker who has been doing the job. After these demonstrations he is considered a fully qualified "production man." All that he has to do now is to autonomize these few operations so that speed may rapidly be increased."

If $p$ is taken as a measure of the job’s complexity, our model suggests that more complex jobs lead to higher average pay but lower job security. In particular, consider two jobs. The first job is routine: $p = 1$. The second job is less so: $p < 1$. Our analysis predicts that, for job 1, the worker is never terminated and his long-run wage is either $w$ or $w + \frac{c}{\delta(1-q)}$, depending on the level of rents in the relationship. For job 2, Proposition 2 implies that the worker may be terminated, and the worker’s average wage following probation is $w + \frac{pc}{\delta(p-q)}$. Note that for $q > 0$, $w + \frac{pc}{\delta(p-q)}$ strictly exceeds $w + \frac{c}{\delta(1-q)}$. In other words, the average long-run wage for the less-routine job is higher. However, the less-routine job also has lower job security. These results suggest that job security and average pay can be negatively correlated as the job becomes more routine.

In general, jobs vary in multiple dimensions, and differences in all these dimensions affect employment dynamics. Even within our model, in addition to job complexity, employment dynamics are affected by the cost of effort required ($c$), the sensitivity of output to effort (as measured by $q$ controlling for $p$), the transacting parties’ outside options ($u$ and $v$) and discount factors ($\delta$), and the exogenously determined minimum amount of rents necessary for the job (as measured by $w$). Since each of these factors affects turnover and pay jointly, in interpreting empirical findings as evidence for or against efficiency wage models, it is important to identify the source of variation. In the next subsection, for example, we show that pay and job security can be positively correlated if the source of variation arises from the firm’s commitment power.

### 5.2 Commitment Power

Until now, we have analyzed the model under relational contracting. In this subsection, we characterize the optimal long-term contract and compare how employment dynamics differ under the commitment (optimal long-term contract) case and non-commitment (optimal relational contract) case. Theorem 2 gives a characterization of the optimal long-term contract and the resulting employment dynamics.

22 In his review of relational contracts and reputation, MacLeod (2007) denotes $p = 1$ as the normal goods case, and he refers to cases with small $p$ as the innovative goods case. MacLeod (2007) develops a model that nests these two cases and he shows how types of employment contract varies with $p$. Here, in contrast, we focus on the joint variation between turnover and pay and $p$ changes.

23 Given the dynamic nature of the model, comparative statics (except for the long-run average pay) with respect to these parameters are difficult to obtain. One result we can show is that the relationship is less likely to be terminated if the principal’s outside option decreases.
**Theorem 2:** Suppose \( w \geq u \) and \( py - w - v \geq \frac{pc}{p-q} \). When the firm can commit to long-term contracts, the Pareto frontier is given by the unique function that solves the following equation

\[
f(u) = \begin{cases} 
    v + \frac{u-u_0}{u_0-u}(f(u_0) - v) & \text{if } u \in [u, u_0] \\
    (1-\delta)(py - w) + \delta[pf(L(u) + k) + (1-p)f(L(u))] & \text{if } u \in [u_0, u_e] \\
    f(u_e) + u_e - u & \text{if } u \in [u_e, u_e + f(u_e) - v],
\end{cases}
\]

where \( u_e = (w-c) + \frac{\delta pk}{(1-\delta)} \), \( f(u_e) = py - c - ((w-c) + \frac{\delta pk}{(1-\delta)}) \), and \( u_0 = (1-\delta)(w-c) + \delta(u + pk) \).

In particular, the set of histories can be partitioned into \( \bar{H} = \bar{H}_1 \cup \bar{H}_2 \cup \bar{H}_3 \), such that

(i): Employment starts with the probation phase \( \bar{H}_1 \), during which the worker’s pay is equal to \( w \):

\[
    h^1 \in \bar{H}_1; \\
    w(h^1) = w; \text{ for } h^t \in \bar{H}_1.
\]

(ii): With positive probability, the relationship is terminated, after which both the principal and the agent receive their outside options \((u, v)\).

\[
\begin{align*}
    \Pr(h^t \in \bar{H}_2) &> 0 \text{ for some } t. \\
    u_t(h^t | h^t) &\in \bar{H}_2 = u; \\
    v_t(h^t | h^t) &\in \bar{H}_2 = v.
\end{align*}
\]

(iii): With positive probability, the worker receives tenure, following which the optimal relational contract can be implemented by a sequence of stationary contracts:

\[
\begin{align*}
    \Pr(h^t \in \bar{H}_3) &> 0 \text{ for some } t. \\
    w(h^{t+1}) &\begin{cases} 
        = w & \text{if } y_t = 0; \\
        = w + \frac{c}{\delta(p-q)} & \text{if } y_t = y.
    \end{cases}
\end{align*}
\]

In other words, the optimal relational contract in \( \bar{H}_3 \) may be stationary.

(iv): In the long run, the worker stays in the probation phase with probability 0. The tenure
and the termination phase are absorbing:

\[
\begin{align*}
\lim_{t \to \infty} Pr(h^t \in \mathcal{H}_1) &= 0; \\
\lim_{t \to \infty} Pr(h^t \in \mathcal{H}_2) &= 0; \\
\lim_{t \to \infty} Pr(h^t \in \mathcal{H}_3) &= 0.
\end{align*}
\]

**Proof.** When the agent’s expected payoff is \(u \geq u_e\), commitment implies that the optimal long-term contract can give the principal a payoff of \(py - c - u\), i.e., no surplus needs to be destroyed to the right of \(u_e\). The rest of the proof then proceeds in the same way as in Theorem 1 and Proposition 2. ■

Theorem 2 implies that employment dynamics under the optimal long-term contract share many commonalities with that under the optimal relational contract. In particular, employment starts with a probation phase, during which the worker’s wage is equal to the wage floor and is thus insensitive to performance. The commonality arises because the underlying logic for the optimal rent extraction remains the same. In particular, when choosing between bonus and future job security to reward the agent, the firm will again frontload job security as much as possible. In other words, the worker is incentivized with future job security at the beginning of the relationship in both cases. In fact, when the relationship has high surplus (\((4)\) holds), commitment is not an issue for the principal and it does not affect the employment dynamics.

The difference in employment dynamics appears when the relationship has a low surplus (\((4)\) fails). Under long-term contracts, the firm is able to commit to frontloading job security and backloading pay. Under relational contracts, however, the degree of frontloading is constrained by the firm’s future stake in the relationship. With a low surplus, using discretionary bonus alone to induce effort will lead the firm to renege on the bonus. Therefore, regardless of the history path, the firm needs to (at least partially) use the threat of job insecurity to induce effort in relational contracts. This implies the agent will not receive tenure and a sufficiently long sequence of failures will lead to termination. In this case, we say that the frontloading of job security is incomplete. Since a long streak of failures will almost surely occur for sufficiently long periods of time, the agent will eventually be terminated.

For relationships with low surplus, the lack of commitment leads to lower average pay and lower average sensitivity of pay to performance in the long run. When firms can commit, almost all of the remaining workers eventually have tenure. For these workers, high outputs are rewarded exclusively with bonus payment. When firms cannot commit, in contrast, workers are never given tenure and part of the reward for each high output is given by a higher future job security. This results in that bonus occurs less frequently, and even when it does occur, its
size is smaller than that in the commitment case. Therefore, both the average pay and average sensitivity of pay to performance is lower in the long run when the firm cannot commit and the relationship has a low surplus.

In addition to lowering long-run average pay and pay sensitivity, the lack of commitment also affects how the employment dynamics evolves over time. In particular, with commitment, a worker’s expected pay and pay sensitivity are (weakly) increasing with time on the job: they jump up once the worker transitions into the tenure case and they do not drop again. Without commitment, however, the expected pay and pay sensitivity are not always increasing with time on the job. After the worker transitions into the bonus phase, a sufficiently poor output sequence will bring him out of it, resulting in lower expected pay and pay sensitivity.

Our discussion on the role of commitment in affecting pay and job security sheds new light on the empirical findings that relate firm characteristics with employment dynamics. For example, Krueger (1991) examines the differences in employment dynamics between company-owned and franchisee-owned fast food restaurants. For lower-level managers, the starting pay is about the same at the company-owned and franchisee-owned restaurants, but the pay rises more rapidly at company-owned restaurants. The steeper earning profile contributes to an average 9% higher pay at company-owned restaurants. In addition, the tenure of these managers is half a year longer at company-owned restaurants. For full-time crew workers, however, the pay difference is less than 2% and there is no statistical difference in job tenure.

Krueger (1991) argues that the pay difference arises because company-owned restaurants can monitor their employees less well and have to rely on paying efficiency wages. While monitoring costs are clearly relevant, this explanation alone does not explain the difference in turnover rates, nor does it explain why pay and turnover dynamics differ for low-level managers but not for crew workers.

Our discussion suggests that difference in commitment power can provide an unified explanation for the empirical findings. Since the companies are larger in size than the franchisees, they are better able to commit to future promises. Therefore, our model suggests that, first, company-owned stores can better backload wage payments, leading to a steeper earning profile. Second, higher commitment power enables the company-owned stores to better use future job security as part of the reward, leading to better job security. Finally, our theory implies that these differences exist only for jobs that are complex \( p < 1 \). When the jobs are routine \( p = 1 \), which arguably is the case for the crew workers, our theory implies that there is no difference in employment dynamics.

\[ \text{Recently, Ji and Weil (2009)) suggest that franchisees care less about the brand reputation.} \]
\[ \text{When } p = 1, \text{ our analysis in the previous section shows that there is no equilibrium turnover and the relationship is efficient (when it can be sustained). Commitment does not affect the employment dynamics in this case.} \]
In our explanation above, we have implicitly assumed that the wage floors at the company-owned and franchisee-owned restaurants are the same. This assumption appears empirically plausible since the starting pay, which is a proxy for the wage floor, is about the same in both types of restaurants. For some restaurant workers, this wage floor is just the minimum wage, but not all workers have minimum wage as their starting pay. For these workers, it is desirable to have a theory for what determines their wage floor, which is currently left as exogenous. Without it, it remains an empirical puzzle on why company-owned restaurants don’t reduce their starting pay for their workers, who have more rents than those in the franchisee-owned restaurants. Nevertheless, to the extent that the starting pay can be viewed as the wage floor, our model shows how pay and job security evolve over time.

6 Conclusion

This paper studies a model of relational contracts with limited liability. The optimal relational contract generates several implications on employment dynamics. In particular, the turnover rate is initially increasing and is eventually decreasing. In addition, there is a discrete jump both in average pay level and in pay sensitivity to performance after the worker transitions out of the probation phase. Our analysis also sheds light on how turnover and pay dynamics jointly vary with the complexity of the job and the firm’s ability to commit to future promises.

In the current model, the outside options of the firm and the worker are treated as exogenous. A useful extension is to embed the model into a general equilibrium framework. Interestingly, this model may generate multiple self-fulfilling equilibrium turnover patterns. In particular, consider an economy in which vacant firms and unemployed workers match randomly. When it becomes easier to form new employment relationships, the surplus in the existing relationship is lowered due to the higher outside options. Our analysis implies that the relationships are more likely to dissolve when the surplus is lower. This increases the number of vacant firms and unemployed workers, making it even easier to form new employment relationships. Such multiplicity may help shed light on the large cross-country differences in employment patterns.

\[26\] Higher pay at larger firms (controlling for personal fixed effects) is often thought of as evidence against this type of bond-posting efficiency wage models: see for example Katz (1986) and Oi and Idson (1999). The reasoning is that workers in larger firms can better post bonds, and the larger bonds should reduce, not increase, the ex ante rents. But if there are reasons for the same starting pay across firms, our model suggests that the average pay at larger firms should be higher than smaller ones. One possibility for the same starting pay is that there’s an industry norm that attaches the starting pay to jobs. Lowering the starting pay from the industry norm will be viewed as unfair, and firms instead compete on nonpecuniary dimensions.
References


7 Appendix

First, we show that the set of PPE payoff is completely characterized by the Pareto frontier.

**Lemma 0**: Let $u_{\text{max}}$ be the maximum PPE payoff of the agent. Then the PPE payoff set $E$ is given by

$$E = \{(u, v) : u \in [u, u_{\text{max}}], v \in [v, f(u)]\}$$

**Proof.** First, note that the payoff pair $(u, v)$ (meaning that the agent’s normalized expected payoff is $u$ and the principal’s normalized payoff is $v$) is in the PPE payoff set. This payoff is supported by an equilibrium in which on the equilibrium path, the principal and the agent do not start a relationship, and off the equilibrium path, the agent never puts in effort and both the principal and the agent do the start the relationship in the future.

Second, it follows by convexity of the PPE payoff set that any payoff on the line segment between $(u, v)$ and $(u, f(u))$ can be supported as a PPE payoff, and this is the left boundary of the PPE payoff set. Third, because there is no limit in the amount of transfer the principal can make to the agent at the beginning of period 1, it is easily seen that the lower boundary of the PPE set is given by the horizontal line at $v$. Finally, convexity implies that any equilibrium payoff between $(u, v)$ and $(u, f(u))$ can be obtained. ■

Now we follow seven steps to prove Theorem 1 which characterizes the Pareto Frontier. First, we show that there exists a threshold $u_e$ such that $f$ has a slope of $-1$ to the right of $u_e$ (Lemma 1) and has a slope larger than $-1$ to the left of $u_e$ (Corollary 1a). Second, we show that to the left of $u_e$, only the minimum wage $w$ will be paid out regardless of the output (Lemma 2). Third, we show that there exists a threshold $u_0$ (the value of which is determined in Lemma 4) such that to the left of $u_0$, $f$ is a straight line between $(u, f(u))$ and $(u_0, f(u_0))$ (Lemma 3). Following a sequence of bad outcomes, both the principal and agent will be punished. Our fourth step is to show that there exists a threshold $w^*$ such that if $w \geq w^*$, then punishment will be in the form of termination in which case the principal receives $f(u) = v$ and if $w < w^*$, then punishment will be done through temporary suspension of production and in which case the principal receive $f(u) > v$ (Lemma 5). Fifth, we determine the value of $u_e$ (Lemmas 6a-6b). Sixth, we show that the value of $f$ between $u_0$ and $u_e$ is the fixed point of a contraction mapping indexed by the value of $f(u_0)$ (Lemma 7). Steps 1-6 presumes the existence of $f$. In Step 7, we establish the existence and uniqueness of $f(u)$.

Denote $u_e$ as the smallest PPE payoff of the agent that maximizes the sum of the equilibrium payoffs of the principal and agent. More precisely, define $M := \{u : f(u) + u \geq f(u') + u'\}$ and $u_e := \{u \in M : u \leq u' \forall u' \in M\}$. First, $(u_e, f(u_e))$ exists because the payoff set is compact and $f$ is continuous in the compact set. Next, we show that

**Lemma 1** $f(u_{\text{max}}) = v$ and $f(u) = f(u_e) + u_e - u$ for $u \in [u_e, u_{\text{max}}]$. 

38
Proof. First, we establish that \( f(u_{\text{max}}) = v \). Suppose \( f(u_{\text{max}}) > v \). Then the principal can increase the first period transfer to the agent by \( f(u_{\text{max}}) - v \) and raise the agent’s payoff to \( u_{\text{max}} + f(u_{\text{max}}) - v \). This contradicts the definition of \( u_{\text{max}} \).

Notice that \( u + f(u) \geq u_e + f(u_e) \) for \( u > u_e \) because principal can freely pay the agent extra in the first period; and \( u + f(u) \leq u_e + f(u_e) \) holds by definition of \( u_e \). This completes the proof. ■

The following corollary follows Lemma 1, definition of \( u_e \), and concavity of \( f \).

**Corollary 1a** \( f'(u) = -1 \) for \( u > u_e \) and \( f'(u) > -1 \) for \( u < u_e \).

**Lemma 2** Suppose \((u, v) = (u, f(u))\). Then following any history \( h \) that happens with positive probability, \( w_h = w \) if \( u_h \in [u, u_e] \).

**Proof.** Suppose \( u \in (u, u_e] \) and the first period wage satisfies \( w_1 > w \). Then, there exists \( \varepsilon > 0 \) such that it is still an equilibrium if the principal lowers the first period wage to \( w_1 - \varepsilon \) while holding everything else unchanged. This implies

\[
f(u - \varepsilon) \geq f(u) + \varepsilon.
\]

If \( u = u_e \), then this immediately contradicts the definition of \( u_e \). Moreover, this also contradicts that \( f'(u) > -1 \) for \( u < u_e \). The proof is completed by noting that \((u, v) = (u, f(u))\) implies \((u_h, v_h) = (u_h, f(u_h))\) following any history that happens with a positive probability. ■

Now define \( u_0 \) as the smallest \( u \) in which \((u, f(u))\) is obtained by requiring the agent to put in effort in period 1. The next lemma shows that \( f \) is a straight line between \((u, f(u))\) and \((u_0, f(u_0))\).

**Lemma 3** For \( u \in [u, u_0] \),

\[
f(u) = f(u) + \frac{u - u}{u_0 - u}(f(u_0) - f(u)).
\]

**Proof.** Note that \( f \) is a concave function. The statement amounts to showing that \( f \) does not have an extremal point in \((u, u_0)\).

Suppose instead there exists \( u \in (u, u_0) \) such that \((u, f(u))\) is an extremal point. In particular, this implies that \((u, f(u))\) is obtained through pure strategy in period 1. Since \( u < u_0 \), the first period play is either \((u, v)\) or \((w, qy - w)\). Let the agent’s (deterministic) continuation payoff in the second period be \( r(u) \). Consider three possible cases: \( r(u) = u_0 \), \( r(u) > u_0 \), and \( r(u) < u_0 \).

If \( r(u) = u \), then it means that the agent’s payoff is \( u \) and it is achieved by him receiving \( w = u \) every period and not exerting effort. Therefore, the principal’s payoff is \( qy - w = qy - u \leq v + u - u < v \). This cannot be an equilibrium payoff.
Suppose \( r(u) > u \). Now consider an alternative equilibrium strategy that replicates the play for \((u, f(u))\) in period 1, and uses a continuation of \((r(u) - \varepsilon, f(r(u) - \varepsilon))\), for some small enough \( \varepsilon > 0 \). This new strategy gives the agent a payoff of \( u - \delta \varepsilon \) and the principal a payoff of \( f(u) + \delta (f(r(u) - \varepsilon) - f(r(u))) \).

By the definition of \( f \), we must have

\[
f(u - \delta \varepsilon) \geq f(u) + \delta (f(r(u) - \varepsilon) - f(r(u))).
\]

On the other hand, by the concavity of \( f \), we have that

\[
f(u - \delta \varepsilon) \leq f(u) + \delta (f(r(u) - \varepsilon) - f(r(u))).
\]

Combining these two equations, we have

\[
f(u - \delta \varepsilon) = f(u) + \delta (f(r(u) - \varepsilon) - f(r(u)));
\]

but this implies the slope in \((u - \delta \varepsilon, u)\) and the slope in \((r(u) - \varepsilon, r(u))\) are the same. This is a contradiction.

If \( r(u) < u \), a similar approach will also show that

\[
f(u + \delta \varepsilon) = f(u) + \delta (f(r(u) + \varepsilon) - f(r(u)),
\]

so the slope in \((r(u), r(u) + \varepsilon)\) and the slope in \((u, u + \delta \varepsilon)\) are the same. Once again, this contradicts that \((u, f(u))\) is an extremal point. ■

The next lemma gives the exact value of \( u_0 \). In particular, it follows directly from

\[ L(u_0) = u. \]

\textbf{Lemma 4:}

\[ u_0 = (1 - \delta)(\underline{w} - c) + \delta(\underline{w} + pk) \]

\textbf{Proof.} It is clear \( L(u_0) \geq u \), where \( u \) is the agent’s maxmin payoff. Now if \( L(u_0) > u \), we argue that there exists a PPE payoff that gives the agent the payoff of \( u_0 - \varepsilon \) and the principal a payoff that lies on the line segment between \((\underline{w}, f(\underline{w}))\) and \((u_0, f(u_0))\), and this violates the definition of \( u_0 \). In particular, let \( s \) be the slope between \((\underline{w}, f(\underline{w}))\) and \((u_0, f(u_0))\). Then by the weak concavity of \( f \), we know that both \((L(u_0) - \varepsilon, f(L(u_0)) - s \varepsilon)\) and \((L(u_0) + k - \varepsilon, f(L(u_0) + k) - s \varepsilon)\) are PPE payoffs. And a strategy profile that pays the agent \( w_1 = \underline{w} \), requires the agent to put in effort in period 1, and promises the agent \((f(L(u_0)) - s \varepsilon)\) when the output is 0 and \((f(L(u_0)) + k - s \varepsilon)\) when the output is \( y \), will be an equilibrium. Since the agent receives a
payoff of \((u_0 - \varepsilon, f(u_0) - s\varepsilon)\) yet puts in effort in period 1, it contradicts the definition of \(u_0\). Therefore, \(L(u_0) = u\). ■

**Lemma 5** There exists \(w^* \leq u\) such that \(f(u) = \nu\) if and only if \(w \geq w^*\).

**Proof.** Here we make explicit that \(f\) depends on \(w\) by using the notation \(f(u, w)\). Suppose the agent’s continuation payoff is \(u\). Let \(r(u|w)\) be his next period continuation payoff if in this period he receives an instantaneous payoff of \(u\) and exerts no effort, and let \(r(u|u)\) be his next period continuation payoff if in this period he receives an instantaneous payoff of \(w\) and exerts no effort.

Step 1. We establish that \(f(u) = \nu\) if and only if

\[
(1 - \delta) (qy - w) + \delta f(r(u|w), w) < \nu.
\]

Since \((u, f(u))\) is an extremal point, the first period play is pure. There are two possible instantaneous payoff profiles in the first period: \((u, \nu)\) and \((w, qy - w)\) for some \(w \geq w^*\). By Lemma 2, \(w = w^*\).

If players’ instantaneous payoffs are \((u, \nu)\) in the first period, then \(r(u, u) = u\). This implies the principal’s continuation payoff is \(\nu\) when the agent’s continuation payoff is \(u\). If players’ instantaneous payoffs are \((w, qy - w)\), then the principal’s payoff is

\[
F(w) := (1 - \delta) (qy - w) + \delta f(r(u|w), w).
\]

Clearly \(f(u, w) = \max\{F(w), \nu\}\).

Step 2. Now we show that if \(f(u, w) > \nu\) when minimum wage is \(w\), then \(f(u, w - \varepsilon) > \nu\) for \(\varepsilon > 0\) must hold when minimum wage is \(w - \varepsilon\).

Suppose \((1 - \delta) (qy - w) + \delta f(r(w)) \geq \nu\). Notice that \(f\) is weakly decreasing in \(w\). Also notice that

\[
(1 - \delta) w + \delta r(u|w) = (1 - \delta) (w - \varepsilon) + \delta r(u|w - \varepsilon) = u
\]

\[
r\left(u|w - \varepsilon\right) = r\left(u|w\right) + \frac{1 - \delta}{\delta} \varepsilon.
\]

Then

\[
(1 - \delta) (qy - (w - \varepsilon)) + \delta f\left(r\left(u|w - \varepsilon\right), w - \varepsilon\right)
\geq (1 - \delta) (qy - (w - \varepsilon)) + \delta f\left(r\left(u|w\right) + \frac{1 - \delta}{\delta} \varepsilon, w\right)
\geq (1 - \delta) (qy) + \delta f\left(r\left(u|w\right), w\right) > \nu.
\]

The first inequality follows (13) and the fact that \(f\) is weakly decreasing in \(w\). The second inequality follows the fact that \(f_1(r(u|w), w) \geq -1\). This completes the proof. ■

In the following two lemmas, we pin down the value of \(u_\varepsilon\).
Lemma 6a: If \(py - \frac{[1-\delta(1-p)]c}{\delta(p-q)} \geq v + w\), then

\[ u_e = L(u_e) = (w - c) + \frac{\delta pk}{1-\delta}. \]

**Proof.** Note that \(u = (1 - \delta) (w - c) + \delta [pL(u) + (1 - p) (L(u) + k)]\) for \(u \geq u_0\), which can be rewritten as

\[ L(u) = \frac{u - (1 - \delta) (w - c) - \delta (1 - p) k}{\delta}. \]

Define \(u_r\) by \(L(u_r) = u_r\). Then

\[ u_r = \frac{u_r - (1 - \delta) (w - c) - \delta (1 - p) k}{\delta}. \]

The assumption of the lemma can be rewritten as \(u_r + k \leq py - c - v\). This implies it is an equilibrium in which the agent is paid a contractable wage \(w\) and a bonus \(k = (1 - \delta) c/ \delta(p-q)\) (in form of a higher wage in the following period) if and only if the outcome is good and exerts effort every period. In other words, \(f(u_r) = py - c - u_r\). By definition of \(u_e\), \(u_e \leq u_r\) and \(f(u_e) = py - c - u_e\). Since \(L'(u) = 1/\delta\), if \(u_e < u_r\), then \(L(u_e) < u_e\). By definition of \(u_e\), efficiency is not achieved at \(L(u_e)\).

On the other hand, for efficiency to hold at \(u_e\),

\[ (1 - \delta) (py - c) + \delta [(1 - p) [L(u_e) + f(L(u_e))]] + p [L(u_e + k) + f(L(u_e + k))] = py - c \]

This implies that

\[ [L(u_e) + f(L(u_e))] = [L(u_e + k) + f(L(u_e + k))] = py - c \]

This contradicts that efficiency is not achieved at \(L(u_e)\). \(\blacksquare\)

**Lemma 6b:** Suppose \(py - \frac{[1-\delta(1-p)]c}{\delta(p-q)} < v + w\), then

\[ u_e = (1 - \delta) (w - c) + \delta [u_{\text{max}} - (1 - p) k]. \]

**Proof.** Note that \(L(u_e) + k \leq u_{\text{max}}\) must hold. The assumption of the lemma implies that \(L(u_e) + k > u_{\text{max}}\). Since \(L(u_e) + k \leq u_{\text{max}}\), it follows that \(u_e < u_r\). This implies \(L(u_e) < u_e\).

Suppose \(L(u_e) + k < u_{\text{max}}\). Then consider the equilibrium profile \((u_e + \epsilon, f(u_e + \epsilon))\). We
know that the slope of \( f \) for \( u > u_e \) is \(-1\). This implies that

\[
f(u_e + \varepsilon) = f(u_e) - \varepsilon.
\]

On the other hand,

\[
f(u_e + \varepsilon) \geq (1 - \delta)(py - w) + \delta((1 - p)f(L(u_e + \varepsilon)) + pf(L(u_e + \varepsilon + k)).
\]

Now for small enough \( \varepsilon \), we have \( L(u_e + \varepsilon) < u_e \). Now since \( f'(u) > -1 \) for \( u < u_e \), the above implies that \( f'(u_e + \varepsilon) > -1 \) for small enough \( \varepsilon \). Therefore, we have

\[
f(u_e + \varepsilon) > f(u_e) - \varepsilon,
\]

and this contradicts the definition of \( u_e \). The lemma follows immediately \( L(u_e) + k = u_{\max} \).

**Lemma 7:** For \( u \in [u_0, u_e] \),

\[
f(u) = (1 - \delta)(py - w) + \delta[pf(L(u) + k) + (1 - p)f(L(u))].
\] (14)

**Proof.** This follows in two steps. The first step shows that for \( u \in [u_0, u_e] \), \( (u, f(u)) \) can be obtained by an equilibrium profile in which the first period play requires effort. Now suppose the contrary. Let \( u_l \) be the largest point such that \( (u_l, f(u_l)) \) lies on the line given by \( (u, f(u)) \) and \( (u_0, f(u_0)) \).

Suppose \( u \in [u_l, u_e] \). There are two cases to consider. In the first case, \( (u, f(u)) \) can be reached by a pure play in period 1. In this case, the first period play payoff is given by either \( (u, v) \) or \( (w, qy - w) \). In either case, \( (u, f(u)) \) is a convex combination between the instantenous payoff and some continuation payoff, which we denote by \( (u_r, f(u_r)) \). Due to concavity of \( f \), \( (u, f(u)) \) is dominated by a mixing between \( (u_0, f(u_0)) \) and \( (u_r, f(u_r)) \).

In the second case, the \( (u, f(u)) \) is reached through a mix. Since \( (u, f(u)) \) lives in a two dimension space and \( f \) is concave, we may assume

\[
(u, f(u)) = p_1((u_1, f(u_1)) + (1 - p_1)((u_2, f(u_2))
\]

for some \( p_1, u_1 \), and \( u_2 \), where \( (u_1, f(u_1)) \) and \( (u_2, f(u_2)) \) are reached through pure play in period 1. If \( (u_i, f(u_i)), i = 1, 2 \) has first period play that doesn’t require effort, the previous paragraph implies that \( u_i \notin [u_l, u_e] \). But then the concavity of \( f \) implies that if \( (u, f(u)) \) can be obtained by a linear combination with the agent’s payoff \( u_i \notin [u_l, u_e] \), then it can also be obtained by a linear combination of points with the agent’s payoff belonging to \([u_l, u_e]\).

To finish the first step, if \( u \in (u_0, u_l) \), then \( (u, f(u)) \) can be achieved by mixing between \( (u_0, f(u_0)) \) and \( (u_l, f(u_l)) \). And since both \( (u_0, f(u_0)) \) and \( (u_l, f(u_l)) \) can be achieved by requiring
effort in period 1, so can \((u, f(u))\).

In the second step, we note that the equation follows because to achieve the maximum payoff for the principal, a) the continuation payoff must lie on the Pareto frontier, and b) the distance of payoff between the good and bad outcomes for the agent needs not to exceed \(k\) by the concavity of \(f\). ■