Abstract

This paper extends the study of optimal relational contracts and ownership structures to a market setting where agents can search for new partners. In the context of a general equilibrium model with random matching, we investigate the effects of market structure on the level of surplus that can be sustained in relational contracts and the optimal integration decision. We then extend the model in two directions. First, we endogenize entry into the market and find that the mechanism analyzed here can give rise to a novel source of multiple equilibria. Second, we expand the set of possible ownership structures to include ownership of inputs as well as ownership of output, with input ownership serving to influence the costs of separation.

1 Introduction

The prevalence and importance of relational contracts (self-enforcing contracts not enforced by the rule of the court but by the concerns of the parties for their future interests) have been emphasized both inside and outside the economics literature. Inside economics, relational contracts have been discussed informally by Klein, Williamson, and others, and Macauley (1963) and MacNeil (1978) are prominent examples of discussions of relational contracts outside economics. Formal economic models of relational contracts, including Bull (1987), Macleod and Malcomson (1989), and Levin (2003) have focused on the conditions under which a relational contract is sustainable. To date,
however, little study has focused the role of the parties’ options in the outside market in shaping the structure of the optimal relational contract.

Market conditions often imply an asymmetry in parties’ replaceability. An agent on the short side of the market can not only extract a large share of the surplus; he will also have an easier time starting a new relationship with a substitute partner should the current relationship turn sour, and this in turn increases his temptation to renege within a relational contract. Consider, for example, an entrepreneur who has specialized knowledge of the technology to produce a new product but requires an venture capitalist to provide funds and business expertise. Suppose the entrepreneur is the only person who knows how to produce the product but that there are many venture capitalists able to provide the same service. In this setting, the venture capitalist is much more replaceable than the entrepreneur: the entrepreneur can easily find an alternate venture capitalist, but the venture capitalist will not be able to find another entrepreneur to produce the same product.

This paper extends the property rights (Grossman & Hart, 1986; Hart & Moore, 1990) and the relational (Baker, Gibbons & Murphy, 2002; 2006) theories of the firm to a market setting where parties have the option to terminate "sour" relationships rather than play a static Nash equilibrium forever. Section 2 models a relational contract between a single upstream and a single downstream firm in the spirit of Baker, Gibbons & Murphy (2002)–hereafter BGM (2002)–allowing for the possibility of replacement in a reduced form way. Section 3 is the core of the paper and places the partnership in a large market characterized by a random matching technology. Replacement costs stem from the time required for the parties to find new matches, and we relate aggregate replacement costs to market structure, i.e., the ratio of participants on one side of the market to participants on the other. Under a reasonable matching technology, balanced markets have higher total replacement costs for the two members of a partnership, but these higher costs can help to sustain relational contracts by discouraging deviations. We also show in Section 3 that market structure influences the optimal boundaries of the firm.

Sections 4 and 5 explore two extensions to the basic model. In section 4, we endogenize entry on one side of the market and show that multiple equilibria can easily occur. While other search models can generate multiple equilibria in entry (Manning, 2003), we believe that the mechanism discussed here is novel. When there are very few upstream firms in the market, upstream firms are able to capture a large share of each relationship’s surplus. However, the magnitude of this surplus can be quite small, since the market is unbalanced and replacement costs are low. This effect can mean that upstream firms benefit from entry by their competitors, holding the number of downstream firms fixed. In effect, an economy can be stuck in a low-entry trap where one side of the market suffers from the fact that it can too easily exploit the other.

Section 5 expands the set of possible ownership structures by introducing an input that can be held by either party independently of output ownership. The owner of the input has the right
to carry it forward into future relationships; it can therefore act as a "hostage" and facilitate long-term cooperation (Williamson, 1983). Interestingly, we show in Section 5 that Williamson’s insight—that hostages can help to sustain relationships—is not always true in our setting. When *ex post* bargaining over assets following the relationship’s dissolution is efficient, input ownership does not affect the sum of reneging temptations or total surplus in the relationship, so the sustainability of the relational contract is unaffected. The irreplaceable party is willing to terminate the relationship, purchase the asset and resell it to a new partner, all with no loss of surplus. When *ex post* bargaining is inefficient, we show that Williamson’s insight is restored, and higher surplus can be obtained by assigning the input to the more replaceable party. Contrary to the transactions cost theory of the firm (Coase, 1937; Williamson, 1975 and others), we emphasize the positive effects of haggling costs in sustaining efficient production. Section 5 also investigates the interaction between input and output ownership in a special parametric case.

Finally, Section 6 concludes with some brief comments on directions for future research.

2 Relational Contracts with Replacement

Our baseline model in this section closely follows BGM (2002) but allows parties to separate and form new relationships after a deviation rather than reverting to static production. For now we abstract from the process of forming these new relationships; section 3 examines this process in detail in a general equilibrium setting.

2.1 Players and Production

There are two types of firms, upstream (US) firms and downstream (DS) firms. There is a continuous measure $M$ of US firms and measure $N$ of DS firms. All firms are infinitely lived and share the common interest rate $r$. Unmatched US and DS firms are randomly matched pairwise according to a matching technology to be described below; matched players are subject to both endogeneous and exogenous separation.

Each period that an US firm stays matched with a DS firm, it takes a vector of actions $a = (a_1, a_2, ..., a_n)$, the cost of which is $c(a)$. The actions affect both the DS firm’s use value of the product, which is either $Q_L$ or $Q_H$, and the alternative use value in an outside market, which is either $P_L$ or $P_H$. In particular, probability distributions of the use value to the DS firm and the alternative use value to the outside market are

$$Q = \begin{cases} Q_H \text{ with probability } q(a) \\ Q_L \text{ with probability } 1 - q(a) \end{cases}$$

$$P = \begin{cases} P_H \text{ with probability } p(a) \\ P_L \text{ with probability } 1 - p(a) \end{cases}.$$
with \( Q \) and \( P \) determined independently. Assume that \( Q_H > Q_L > P_H > P_L \) and that \( c(0) = 0, q(0) = 0 \) and \( p(0) = 0 \).

Finally, we assume the instantaneous payoff of any unmatched US or DS firm is zero.

### 2.2 Information Structure and Timeline between Matched Firms

The US firm’s actions \( a \) are unobservable. The use values \( Q \) and \( P \) are observable by both parties but are not verifiable and cannot be contracted upon; however, the ownership of output is contractible. Finally, we assume that matched US and DS firms do not observe each other’s histories in previous matches.

The timeline is as follows. In the period when an US firm and a DS firm are matched, the DS firm offers a contract \( C = (o, s, b)^1 \), \( b = (b_{LL}, b_{LH}, b_{HL}, b_{HH}) \), which specifies the contractible ownership of the output \( o \in \{US, DS\} \) throughout the relationship, a contractible per-period fixed wage \( s \in \mathbb{R} \), if \( o = DS \), a per-period net bonus \( b_{ij} \) from DS to US which is contingent (but not contractible) on the value of the output to the DS firm \( Q_i \) and the value of the output to the outside market \( P_j \). After the contract is accepted, in each period, sequentially, \( s \) is paid, US chooses \( a \), and \( P \) and \( Q \) are realized. If \( o = DS \), then DS decides unilaterally whether to pay the bonus when \( b_{ij} > 0 \) and US decides unilaterally whether to pay its "penalty" when \( b_{ij} < 0 \). If \( o = US \), then after the realization of \( Q_i \) and \( P_j \) both firms simultaneously agree or disagree to sign a contract in which DS makes a net transfer of \( b_{ij} \) to US and US transfers the output to DS. If either party rejects the contract, the parties engage in symmetric Nash bargaining over the output with US’s outside option being \( P_j \), resulting in a sale price of \( \frac{1}{2}(Q_i + P_j) \). At the end of the period, the players each choose whether to continue the relationship or dissolve it, and nature decides to dissolve the relationship exogenously with probability \( \varepsilon \). If either player or nature decides to dissolve the relationship, then both players return to the matching market in the following period; otherwise US and DS repeat the game following the same contract.

### 2.3 Strategies and Equilibrium Concept

To define the strategies and equilibrium concept, we first define the public histories for the players. For the DS and US that have been engaged in \( T \) periods of relationship, the public history is \( H^T = C, d^1_{US}, \{P_t, Q_t, b_t, (d^2_{US,t}, d^2_{DS,t})\}_{t=1}^T \), where \( d^1_{US} \in \{0, 1\} \) denotes US firm’s decision to accept or reject DS firm’s contract offer, and \( (d^2_{US,t}, d^2_{DS,t}) \in \{0, 1\} \times \{0, 1\} \) denote US and DS firms’ simultaneous decisions on whether to break up from the relationship in period \( t \). Let \( H = \bigcup_T H^T \).

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1Based on Levin (2003), we restrict attention to stationary relational contracts.
Next, the strategies for US and DS are defined as follows. The strategy of DS specifies a contract offer $C \in \{US, DS\} \times \mathbb{R}^5$ and a mapping which for every period $T$ maps $C \cup d^T_{US} \cup H^{T-1} \cup \{P_j, Q_i\}$ to a period $T$ transfer payment $b^US_T \in \mathbb{R}$ and $C \cup d^T_{US} \cup H^{T-1} \cup \{P_j, Q_i\} \cup b_T$ to a breakup decision $d^2_{DS,T} \in \{0, 1\}$. The strategy of US is a mapping which for every period $T$ maps $C \cup d^T_{US} \cup H^{T-1}$ to an action $a \in \mathbb{R}^n$, $C \cup d^1_{US} \cup H^{T-1} \cup \{P_j, Q_i\}$ to a transfer payment $b^PS_T \in \mathbb{R}$, and $C \cup d^1_{US} \cup H^{T-1} \cup \{P_j, Q_i\} \cup b_T$ to $d^2_{US,T} \in \{0, 1\}$. (We denote the set of period $T$ voluntary transfers $(b^US_T, b^DS_T)$ by $b_T$.)

Finally, a relational contract between US and DS consists of a complete plan for US and a complete plan for DS. A relational contract between US and DS is self-enforcing if it describes a perfect public equilibrium in the relationship—in other words, we restrict the strategies of US and DS to be independent of private histories. The equilibrium of the game between the matched US and DS firms consists of a set of self-enforcing relational contracts between US and DS.

In our equilibrium selection, we assume that every time newly matched firms form a new relationship, they maximize the total surplus from the relationship and equally divide the surplus according to symmetric Nash bargaining, with outside options being equal to the utility each party could obtain by returning to the match market. Such equilibrium selection, which implicitly imposes stationarity, is reasonable given our assumption that matched firms do not observe each other’s histories in previous matches.\(^2\)

### 2.4 Analysis

A contract $(o, s, b)$ may be supported with or without relational contract. When there is no relational contract, we call the regime spot governance; otherwise, we call it relational governance. Like BGM (2002), then, we have four combinations of output ownership and governance regimes: (i) spot outsourcing, (ii) spot employment, (iii) relational outsourcing, and (iv) relational employment. Our analysis focuses on the sustainability of relational contracts, treating spot outsourcing and spot employment as two of the firms’ outside options in the relational contract. The final outside option is to break up from the relationship and enter the matching market in the following period. As in BGM (2002), we assume that the most efficient of these three outside options is chosen after a deviation.

Before deriving the set of sustainable relational contracts, we specify the details of spot outsourcing and spot employment and derive the total surplus to the matched firms under such production modes.

**Spot outsourcing** We adopt BGM’s convention that when the US and DS firm engage in spot outsourcing, the US and DS firms negotiate on the price of the output according to Nash bargaining

\(^2\)This assumption has the same spirit as the assumption in MacLeod & Malcomson (1998).
with equal bargaining power. Therefore, the price of transaction is \((Q_i + P_j)/2\). Anticipating that, the upstream firm take action \(a^{SO}\) to solve

\[
\max_a \frac{Q_L + q(a) \Delta Q}{2} + \frac{P_L + p(a) \Delta P}{2} - c(a) = u_{US}^{SO}.
\]

The downstream’s total payoff is

\[
u_{DS}^{SO} = \frac{E(Q_i - P_j|a = a^{SO})}{2}.
\]

The total surplus per period under spot outsourcing is therefore

\[
s^{SO} = u_{US}^{SO} + u_{DS}^{SO} = Q_L + q(a^{SO}) \Delta Q - c(a^{SO}).
\]

**Spot employment** When matched firms engage in spot employment, the upstream firm will set \(a = 0\) anticipating the downstream firm’s lack of incentive to pay. This means that total surplus per period is

\[
s^{SE} = Q_L.
\]

### 2.5 Relational Contracts

In this model, separation is costly because it takes time for the parties to find new matches. We derive equilibrium replacement costs in section 3. For now, we denote the discounted expected payoff of party \(k\) as \(U_k^c\) if \(k\) is in a contract of type \(c\) (where \(c \in \{SO, SE, R\}\)) and \(X_k\) if \(k\) is unmatched; we define the cost to party \(k\) of separation from a relational contract as \(R_k \equiv U_k^R - X_k^R\) and the total replacement cost as \(R \equiv R_{DS} + R_{US}\). The discounted value of the surplus generated by contract \(c\) is \(S^c\). In general, we use lower case letters to denote per period payoffs and upper case letters to denote discounted values. Thus \(u_k^c\) is \(k\)’s expected payoff per period in a contract of type \(c\) and \(s^c \equiv u_{DS}^c + u_{US}^c\) is the surplus per period generated by this contract.

Proposition 1 formulates the problem solved by the optimal relational contract:

**Proposition 1** The optimal relational contract solves the following problem:

\[
\max_{o,s,b,a} s^R = Q_L + q(a) \Delta Q - c(a)
\]
subject to

$$
\begin{align*}
\min \left\{ \max \{b_{ij}\} - \min \{b_{ij}\}, \max \left\{ b_{ij} - \frac{1}{2} (Q_i - P_j) \right\} - \min \left\{ b_{ij} - \frac{1}{2} (Q_i - P_j) \right\} \right\} \\
\leq \min \left\{ R, \frac{s^R - \max \{ s^{SE}, s^{SO} \}}{r} \left( 1 - \frac{\varepsilon R}{s^R} \right) \right\}
\end{align*}
$$

(1)

and

$$
a = \arg \max_a s + b_{LL} (1 - q(a)) (1 - p(a)) + b_{HL} q(a) (1 - p(a)) + b_{LH} (1 - q(a)) p(a) + b_{HH} pq - c(a) .
$$

Proof Consider a relational contract \((o, s, b)\). Suppose that \(o = DS\) (i.e., we are in a relational employment regime) and that separation is more efficient than static production after a deviation. In that case, the downstream firm will honor every possible bonus payment if and only if

$$
\max \{b_{ij}\} \leq R_{DS} 
$$

(3)

while the upstream firm will honor every bonus payment if and only if

$$
- \min \{b_{ij}\} \leq R_{US} 
$$

(4)

If the parties will revert to static production after a deviation instead, then the downstream firm will honor every bonus if and only if

$$
- \max \{b_{ij}\} + \frac{U_{DS}^R}{1 + r} \geq \max \left\{ \frac{U_{DS}^{SE}}{1 + r}, \frac{U_{DS}^{SO}}{1 + r} \right\}
$$

(5)

and the upstream firm will do so if and only if

$$
\min \{b_{ij}\} + \frac{U_{US}^R}{1 + r} \geq \max \left\{ \frac{U_{US}^{SE}}{1 + r}, \frac{U_{US}^{SO}}{1 + r} \right\}
$$

(6)
Combining 3, 5, 4 and 6, a relational employment contract is sustainable if and only if
\[
\max \{b_{ij}\} - \min \{b_{ij}\} \leq \min\{R, \frac{S^R - \max\{S^{SE}, S^{SO}\}}{1 + r}\}
\]

Since the loss in surplus when a contract of type \(c\) terminates exogenously is \((s^c/s^R)^3\), we can rewrite this expression as
\[
\max \{b_{ij}\} - \min \{b_{ij}\} \leq \min\{R, \frac{S^R - \max\{s^{SE}, s^{SO}\}}{r} \left(1 - \frac{\varepsilon R}{s^R}\right)\}
\]
(7)

Suppose now that \(o = US\) (i.e., the parties are engaged in a relational outsourcing contract). Because US owns the output, if either firm reneges on a bonus payment then the two firms will Nash bargain of the current period’s output and agree on a price of \((P_j + Q_i)/2\), giving DS and US the instantaneous payoffs of \((Q_i - P_j)/2\) and \((P_j + Q_i)/2\). By the same logic as above, DS will honor the relational contract by paying the bonus following all realizations of \(P\) and \(Q\) if and only if, for all \((i, j)\),
\[
Q_i - b_{ij} + \frac{U^R_{DS}}{1 + r} \geq \frac{Q_i - P_j}{2} + \max \left\{\frac{U^{SE}_{DS}}{1 + r'}, \frac{U^{SO}_{DS}}{1 + r'}, \frac{U^R_{DS}}{1 + r'} - R_{DS}\right\},
\]
i.e.,
\[
\frac{U^R_{DS}}{1 + r} \geq \max\{b_{ij} - \frac{Q_i + P_j}{2}\} + \max \left\{\frac{U^{SE}_{US}}{1 + r'}, \frac{U^{SO}_{US}}{1 + r'}, \frac{U^R_{US}}{1 + r'} - R_{US}\right\}.
\]
(8)
US will honor the relational contract by giving up the output for the bonus following all realizations of \(P\) and \(Q\) if and only if, for all \((i, j)\),
\[
b_{ij} + \frac{U^R_{US}}{1 + r} \geq \frac{P_j + Q_i}{2} + \max \left\{\frac{U^{SE}_{US}}{1 + r'}, \frac{U^{SO}_{US}}{1 + r'}, \frac{U^R_{US}}{1 + r'} - R_{US}\right\},
\]
i.e.,
\[
\min\{b_{ij} - \frac{P_j + Q_i}{2}\} + \frac{U^R_{US}}{1 + r} \geq \max \left\{\frac{U^{SE}_{US}}{1 + r'}, \frac{U^{SO}_{US}}{1 + r'}, \frac{U^R_{US}}{1 + r'} - R_{US}\right\}.
\]
(9)

\(^3\)This expression arises from the assumption that the loss of surplus after exogenous separation stems solely from time spent in search. Since search costs are not dependent on past contracts when the nature of these contracts is private information, separation costs from contract \(c\) must be proportional to the surplus per period it generates, \(s^c\).
Combining (8) and (??), a relational outsourcing contract is sustainable if and only if

\[
\max \left\{ b_{ij} - \frac{1}{2} (Q_i - P_j) \right\} - \min \left\{ b_{ij} - \frac{1}{2} (Q_i - P_j) \right\} \leq \min \left\{ R, \frac{S^R - \max \{ S^SE, S^SO \}}{1 + r} \right\} \tag{10}
\]

\[
= \min \left\{ R, \frac{s^R - \max \{ s^SE, s^SO \}}{r} \right\} \left( 1 - \frac{\varepsilon R}{s^R} \right) \tag{11}
\]

Next, conditional on any admissible bonus structure \( b \), setting \( o = \hat{o}(b) \) minimizes the total reneging temptation, where

\[
\hat{o}(b) = \begin{cases} D S & \text{if } \max \{ b_{ij} \} - \min \{ b_{ij} \} \leq \max \left\{ b_{ij} - \frac{1}{2} (Q_i - P_j) \right\} - \min \left\{ b_{ij} - \frac{1}{2} (Q_i - P_j) \right\}, \\ U S & \text{otherwise.} \end{cases}
\]

Therefore, the firms’ incentives to pay bonuses in all realizations of \( P \) and \( Q \) will be satisfied if and only if (1) is satisfied.

Finally, under the assumption that any firm designated to make a bonus payment after \( P \) and \( Q \) are realized will adhere to the relational contract, the upstream firm will choose \( a \) to solve

\[
\max_a s + b_{LL} (1 - q(a)) (1 - p(a)) + b_{HL} q(a) (1 - p(a)) + b_{LH} (1 - q(a)) p(a) + b_{HH} pq - c(a). 
\]

Q.E.D.

Proposition 1 tells us that the optimal relational contract chooses a governance structure to minimize the total reneging temptation. The total reneging temptation is bounded above by the most efficient alternative for the parties–spot outsourcing or spot employment as in BGM (2002) or separation and replacement with new partners.

3 General Equilibrium with Random Matching

In this section, we provide a micro-foundation for the replacement cost by introducing an equilibrium matching framework with exogenous market participation on both sides. We show that when replacement dominates static production after a deviation, the surplus from relational contracts increases as the two sides of the market become more balanced (in the sense that \( M/N \) approaches 1).

3.1 A Two-Sided Matching Market

Suppose there are \( M \) upstream firms and \( N \) downstream firms in the economy. Suppose in equilibrium there are \( K \) matched pairs. In each period, with probability \( \varepsilon \) each of the existing pair breaks
apart. In each period, each of the unpaired firms match with each other according to the matching function

\[ m(M - K, N - K) = \alpha \sqrt{(M - K)(N - K)}, \]

where \( \alpha \) specifies the the efficiency of matching.\(^4\)

We study the steady state of this economy, where the number of matched pairs is constant. Denote as \( \lambda_{DS} \) the probability that an unmatched DS firm finds an US firm in the steady state, and denote as \( \lambda_{US} \) the probability that an unmatched US firm finds an DS firm. In the steady state, the number of dissolved pair of firms must equal to the number of newly formed firms:

\[ K \varepsilon = (N - K)\lambda_{DS} = (M - K)\lambda_{US} = \alpha \sqrt{(M - K)(N - K)}. \]

This implies that, in steady state, the number of matched pairs is given by

\[ K = \frac{M + N - \sqrt{(M + N)^2 - 4MN(1 - \rho^2)}}{2(1 - \rho^2)}, \]

where \( \rho = \varepsilon/\alpha. \)\(^5\)

To study how the ratio \( M/N \) affects relational contracts, the following lemma will be useful. Lemma 1 states that the sum of matching probabilities is U-Shaped with respect to \( M/N, \) with a minimum when the ratio is equal to 1.

**Lemma 1:** Let \( \Lambda = \lambda_{US} + \lambda_{DS}. \) Then we have

\[ \frac{d\Lambda}{d(M/N)} < 0 \quad \text{if} \quad M < N \]
\[ \frac{d\Lambda}{d(M/N)} > 0 \quad \text{if} \quad M > N \]

**Proof.** First, we can show that

\[ \Lambda = \lambda_{US} + \lambda_{DS} = \alpha \left( \sqrt{\frac{N - K}{M - K}} + \sqrt{\frac{M - K}{N - K}} \right). \]

Now supposing that \( N > M, \) it is easy to see that

\[ \frac{d\Lambda}{d(N-K)} > 0. \]

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\(^4\)We choose this matching function for simplicity and symmetry. The main results are robust to more general matching functions.

\(^5\)It can be shown that \( K = \frac{M + N + \sqrt{(M + N)^2 - 4MN(1 - \rho^2)}}{2(1 - \rho^2)} > \min\{M, N\} \) and cannot be a solution. This expression holds for both \( \rho > 1 \) and \( \rho < 1. \)
To prove the Lemma, it then suffices to show that
\[ d\left( \frac{N - K}{M - K} \right) / d\left( \frac{N}{M} \right) > 0. \]

Some algebra shows that
\[ \frac{N - K}{M - K} = 1 + \frac{(\frac{N}{M} - 1)2(1 - \rho^2)}{2(1 - \rho^2) - 1 - \frac{N}{M} + \sqrt{\left( \frac{N}{M} - 1 \right)^2 + 4\rho^2 \frac{N}{M}}} \]
and it can be checked that \( \frac{N - K}{M - K} \) is increasing in \( \frac{N}{M} \) when \( \frac{N}{M} > 1 \).

The proof when \( N < M \) is analogous. Q.E.D. ■

It is worth noting that in the special case of \( \alpha = \varepsilon \), the expressions for the matching probabilities can be written in a particularly simple form:

\[ \lambda_{US} = \varepsilon \sqrt{\frac{N}{M}}; \quad \lambda_{DS} = \varepsilon \sqrt{\frac{M}{N}}; \]

These expressions will be used in Section 3.2.

### 3.2 Replacement Cost

In this subsection, we derive an expression for the equilibrium replacement cost \( R \), and we show that the replacement cost is larger when the two sides of the market is more balanced, i.e., the ratio of the number of upstream firms to downstream firms is closer to 1.

As before, let \( X_k \) be the total discounted expected payoff of party \( k \in \{US, DS\} \) if unmatched and \( U_k \) be the expected discounted payoff of party \( k \in \{US, DS\} \) if matched. We can write the total replacement cost as

\[ R = \frac{1}{1 + r} \sum_{k=US,DS} (U_k - X_k). \]

We assume that if party \( k \) is unmatched at the beginning of a period, then with probability \( 1 - \lambda_k \) the party fails to find a match and receives its outside option \( \bar{u}_k = 0 \). With probability \( \lambda_k \), the party finds a match and can start production. When a match is formed, we assume that the two parties divide the surplus through Nash Bargaining with equal bargaining power. At the end of the period, each of the matched pairs dissolves with probability \( \varepsilon \).

Lemma 2 computes the link between surplus, replacement cost, and total discounted expected payoff of matched and unmatched firms.
Lemma 2: For $k \in \{US, DS\}$, the replacement cost and the expected total payoffs for unmatched and matched firms satisfy

\[
R = \frac{(2 - \lambda_{US} - \lambda_{DS})(1 + r)}{2(r + \varepsilon) + (1 - \varepsilon)(\lambda_{US} + \lambda_{DS})} s^R; \\
U_k = \frac{(1 + r)}{r} \frac{r + \lambda_k}{2(r + \varepsilon) + (1 - \varepsilon)(\lambda_{US} + \lambda_{DS})} s^R; \\
X_k = \frac{(1 + r)}{r} \frac{(1 + r)\lambda_k}{2(r + \varepsilon) + (1 - \varepsilon)(\lambda_{US} + \lambda_{DS})} s^R;
\]

where $s^R$ is the surplus per period within the optimal relational contract.

Proof. Since an unmatched party finds a match with probability $\lambda_k$, we have

\[
X_k = \lambda_k U_k + (1 - \lambda_k)(\pi_k + \frac{1}{1 + r} X_k).
\]

This implies that

\[
X_k = \frac{1 + r}{r + \lambda_k} [\lambda_k U_k + (1 - \lambda_k)\pi_k] \\
U_k - X_k = \frac{1 - \lambda_k}{r + \lambda_k} [rU_k - (1 + r)\pi_k].
\]

Therefore, the replacement cost $R$ satisfies

\[
(1 + r)R = \sum_{k=US,DS} \frac{1 - \lambda_k}{r + \lambda_k} [rU_k - (1 + r)\pi_k]
\]

Now letting $\pi_k = 0$, we have

\[
R = \frac{1 - \lambda_{US}}{r + \lambda_{US}} \frac{r}{1 + r} U_{US} + \frac{1 - \lambda_{DS}}{r + \lambda_{DS}} \frac{r}{1 + r} U_{DS}
\]

(15)

Note that party $k'$s outside option (when $\pi_k = 0$) during the Nash bargaining that follows immediately upon a new match is

\[
\frac{1}{1 + r} X_k = \frac{\lambda_k}{r + \lambda_k} U_k
\]

This implies that the net Nash bargaining surplus is equal to

\[
\sum_{k=US,DS} U_k - \frac{\lambda_k}{r + \lambda_k} U_k = \sum_{k=US,DS} \frac{r}{r + \lambda_k} U_k.
\]
Nash Bargaining with equal bargaining power then implies that

\[
U_{DS} = \frac{1}{1+r} X_{DS} + \frac{1}{2} \sum_{k=US,DS} \frac{r}{r+\lambda_k} U_k \\
U_{US} = \frac{1}{1+r} X_{US} + \frac{1}{2} \sum_{k=US,DS} \frac{r}{r+\lambda_k} U_k.
\]

Substituting for \( \frac{1}{1+r} X_k = \frac{\lambda_k}{r+\lambda_k} U_k \), we have

\[
\frac{r}{r+\lambda_{DS}} U_{DS} = \sum_{k=US,DS} \frac{r}{r+\lambda_k} U_k \\
\frac{r}{r+\lambda_{US}} U_{US} = \sum_{k=US,DS} \frac{r}{r+\lambda_k} U_k,
\]

and this implies that

\[
\frac{U_{DS}}{U_{US}} = \frac{r+\lambda_{DS}}{r+\lambda_{US}}. \tag{16}
\]

Since each matched pair dissolves in equilibrium with probability \( \varepsilon \), we can write

\[
U_{US} + U_{DS} = s^R + \frac{1}{1+r} [(1-\varepsilon)(U_{US} + U_{DS}) + \varepsilon (X_{US} + X_{DS})] \\
= s^R + \frac{1}{1+r} [(U_{US} + U_{DS}) - \varepsilon (U_{US} + U_{DS} - (X_{US} + X_{DS}))] \\
= s^R + \frac{1}{1+r} [(U_{US} + U_{DS}) - \varepsilon R].
\]

where \( R \) is the loss from replacement.

The above equation, together with eq (15) and eq (16), allow us to solve for \( U_{US}, X_{US}, \) and \( R \) as a function of \( s^R \); we can then obtain the expected total discounted payoffs of DS firms by using

\[
U_{DS} = \frac{r+\lambda_{DS}}{r+\lambda_{US}} U_{US}; \\
X_{DS} = \frac{(1+r)\lambda_{DS}}{r+\lambda_{US}} U_{US}.
\]

Equation 14 follows from straightforward algebra on the above. \( Q.E.D. \) ■

Lemma 2 establish the link between the replacement cost and the surplus in a relational contract. The replacement cost is proportional to the surplus, where the factor of proportionality

\[
\frac{(2 - \lambda_{US} - \lambda_{DS})(1+r)}{2(r+\varepsilon) + (1-\varepsilon)(\lambda_{US} + \lambda_{DS})}
\]
is decreasing in the sum of the matching probabilities. Noting that by Lemma 1, the sum of the matching probabilities is smaller when the ratio of US firms to DS firms is closer to 1, this implies that the replacement cost is larger when the market is more balanced (holding the $s^R$ fixed).

Since Proposition 1 relates $s^R$ to $R$, we deduce the relationship between market balance and the surplus supportable in relational contracts.

**Proposition 2:** Suppose that replacement is more efficient than static production after a deviation from the relational contract, i.e.,

$$R < \frac{s^R - \max \left\{ s^{SE} s^{SO} \right\}}{r} \left( 1 - \frac{\varepsilon R}{s^R} \right)$$

Then

$$\frac{ds^R}{d(M/N)} \geq 0 \quad \text{if } M < N$$

$$\frac{ds^R}{d(M/N)} \leq 0 \quad \text{if } M > N^6$$

When static production is more efficient than replacement, on the other hand,

$$\frac{ds^R}{d(M/N)} \leq 0 \quad \text{if } M < N$$

$$\frac{ds^R}{d(M/N)} \geq 0 \quad \text{if } M > N$$

**Proof.** By Lemma 2, we can write the replacement cost as

$$R = \frac{(2 - \Lambda)(1 + r)}{2(r + \varepsilon) + (1 - \varepsilon)\Lambda} s^R,$$

where $\Lambda = \lambda_{US} + \lambda_{DS}$ is the sum of matching probabilities. It is clear that $\frac{(2 - \Lambda)(1 + r)}{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}$ decreases with $\Lambda$. So by Lemma 1, $\frac{(2 - \Lambda)(1 + r)}{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}$ increases when $\frac{M}{N}$ becomes closer to 1.

Now consider a relational employment contract with bonus payments $\{b_{ij}\}$. By Proposition 1, a necessary and sufficient condition for this contract to be self-enforcing is that

$$\max \{b_{ij}\} - \min \{b_{ij}\} \leq \min \left\{ R, \frac{s^R - \max \left\{ s^{SE} s^{SO} \right\}}{r} \left( 1 - \frac{\varepsilon R}{s^R} \right) \right\}$$
As $\frac{M}{N}$ approaches 1, the left hand side of the above inequality does not change. If replacement dominates static production, then the right hand side increases, so that the same relational contract continues to be sustainable. It follows that total surplus from the relational contract $s^R$ must weakly increase. On the other hand, if static production dominates replacement then the right hand side of this inequality is decreasing as $\frac{M}{N}$ approaches 1, and $s^R$ must weakly decrease.

A similar argument can be applied to relational outsourcing contracts.

Q.E.D. ■

Proposition 2 shows that market balance can affect the sustainability of relational contracts through total replacement costs. When static production dominates replacement, a deviation does not trigger replacement costs; however, high replacement costs effectively increase the discount rate by magnifying the effects of exogenous separation and therefore make relational contracts harder to sustain. In this case, relational surplus is highest when the market is unbalanced. When replacement dominates static production, on the other hand, high replacement costs act directly as a deterrent against deviations. In this case, highly unbalanced markets can prevent effective relational contracts from operating. This logic is most easily grasped when considering an extreme case of market unbalance. Suppose that there is one upstream firm and infinitely many downstream firms. In this case, the upstream firm is totally irreplaceable and can easily find a new downstream firm when the existing relational contract breaks up. Neither party has much to fear from breakup: the upstream firm can immediately find a new partner, and the downstream firm is receiving negligible rents (because it has no bargaining power). When breakup is not costly, only static production will be sustainable.

For the remainder of this paper, we focus on the scenario where replacement dominates static production. This case holds when static production is very inefficient or replacement costs are not too high, and it corresponds to the presumption that sour relationships will typically dissolve.

3.3 Market Structure and Optimal Ownership Structures

In this subsection, we examine the relationship between market structure ($M/N$) and the relative surplus generated by relational employment and outsourcing contracts. We impose the following parametric forms:

\[
q(a) = \sum_{i=1}^{n} q_i a_i
\]
\[
p(a) = \sum_{i=1}^{n} p_i a_i
\]
\[
c(a) = \sum_{i=1}^{n} \frac{1}{2} a_i^2
\]
Following BGM (2002), we also restrict the structure of bonus payments to be

\[ b_{ij} = b_i + \beta_j \]

and define \( \Delta b = b_H - b_L \) and \( \Delta \beta = \beta_H - \beta_L \). In this case, the upstream party’s first best actions are

\[ a^F_{i} = q_i \Delta Q \]

while the actions actually chosen will be

\[ a^R_{i} = q_i \Delta b + p_i \Delta \beta \]

Under these parametric forms, we can establish the following proposition:

**Proposition 3** Suppose that replacement dominates static production after a deviation. Then higher replacement costs make employment attractive relative to outsourcing. Formally, if \( R_1 < R_2 \), then it will never be optimal to use employment when \( R = R_1 \) and outsourcing when \( R = R_2 \).

**Proof.** When replacement is the efficient choice after a deviation, then (by the envelope theorem) the derivative of per period surplus with respect to \( R \) is

\[
\begin{align*}
    s^{RE'}(R) &= \eta^{RE} \\
    s^{RO'}(R) &= \eta^{RO}
\end{align*}
\]

where \( \eta^m \) is the Lagrange multiplier from the optimization of \( s^m \). Since the optimization of \( s^R \) with respect to \( \Delta b \) implies

\[ \eta = \sum_{i=1}^{n} (\Delta Q q_i^2 - q_i a_i) \]

\( s^{RE}(R) - s^{RO}(R) \) is increasing in \( R \) whenever

\[ \sum_{i=1}^{n} q_i (a^{RE}_i - a^{RO}_i) < 0 \]

Because \( a^{RE}_i - a^{RO}_i < 0 \) for all \( i \),\(^7\) the above inequality is clearly true.

\[ \square \]

\(^7\)See Result 2 in Baker, Gibbons and Murphy (2002).
Corollary 1  When the market is more balanced \((M/N \text{ is closer to } 1)\), we are more likely to see employment contracts.

Proof.  This follows directly from the fact that replacement costs are higher in more balanced markets. ■

The intuition for Proposition 3 stems from the fact that overall effort levels are higher when the upstream firm owns output–outsourcing contracts have high incentives relative to employment contracts, which induce both productive efforts to raise \(q\) and unproductive efforts to raise \(p\). Higher replacement costs allow relational contracts to have stronger incentives irrespective of output ownership, but the relaxation of this constraint is more valuable when the base levels of effort are low due to convex effort costs.

4 Endogenous Entry and Multiple Equilibria

In the previous section, the numbers of US and DS firms were fixed. In this section, we endogenize \(M\) and \(N\) and show that multiple equilibria can arise.

In a two-sided market, it is common to obtain multiple equilibria through interactive positive feedback between the two sides of the market: workers are willing to enter only when there are sufficiently many jobs being offered, and firms are willing to enter only when enough workers are looking for jobs. (See for example Manning (2003) for a discussion of models with this logic.)

Multiple equilibria arise in our model for reasons independent of such positive feedback. To distinguish this model from other multiple equilibrium models with positive feedback, we assume that the number of DS firms is fixed (corresponding to an entry cost of 0 for the first \(N\) firms and infinite entry costs for all others). Multiple equilibria arise because the surplus from relational contracts depends the balance between the two sides of the market. When very few US firms are in the market, the market is unbalanced, surplus from relational contracts is low, and other US firms may not want to enter. When more US firms enter, the market becomes more balanced, the surplus from relational contracts increases, and this makes it more attractive for further US firms to enter. In other words, there are increasing return to entry when there are few US firms relative to DS firms. Since individual US firms do not take this positive externality into account, the economy can be stuck in an equilibrium with few US firms or no US firms at all!

On the other hand, once sufficiently many US firms have entered, there can be over entry in this economy. The reason is that when there are more US firms than DS firms, new entries of US firms make the market more unbalanced, and this decreases the surplus from relational contracts. Since individual US firms do not take into account this negative entry externality, there can be excessive entries of US firms as well.
4.1 Setup

To formalize these ideas, we consider the following example. Suppose the entry cost of DS firms is 0, and there are a total of \( N \) DS firms in the economy. Suppose the entry cost of US firms is 0 for the first \( \varepsilon^2 N \) firms, and it is \( c > 0 \) for the rest of the US firms. The maximum number of US firm is bounded by \( \frac{N}{\varepsilon^2} \).

Now assume that the production function is given by

\[
q(a) = qa; \\
p(a) = 0; \\
c(a) = \frac{1}{2}a^2.
\]

For simplicity, we assume that the US and DS parties will engage in relational outsourcing contracts\(^9\), and in case of a deviation, the relationship will be dissolved and the parties will look for new matches in the market (i.e., replacement dominates static production).

To further simplify the expressions, we assume that \( \alpha = \varepsilon \). If in equilibrium \( M \) US firms enter the market, by Lemma 1 we have

\[
\lambda_{US} = \varepsilon \sqrt{\frac{N}{M}}; \quad \lambda_{DS} = \varepsilon \sqrt{\frac{M}{N}}.
\]

Recalling that \( \Lambda = \lambda_{US} + \lambda_{DS} \) is the sum of matching probabilities, we can write

\[
\lambda_{US} = \Lambda + \frac{\sqrt{\Lambda^2 - 4\varepsilon^2}}{2} \quad \text{for } M < N \\
= \Lambda - \frac{\sqrt{\Lambda^2 - 4\varepsilon^2}}{2} \quad \text{for } M \geq N
\]

Note that the assumptions on the entry cost of US firms guarantees that \( \lambda_{US} \) and \( \lambda_{DS} \) are between 0 and 1. It follows that the range of \( \Lambda \) is between \( 2\varepsilon \) and \( 1 + \varepsilon^2 \).

4.2 Analysis

Before we study the entry decision of US firms, we first derive expressions for the surplus of unmatched firms as a function of the sum of matching probabilities.

By Levin (03), we can restrict our analysis of relational contracts to stationary ones. Since \( p(a) = 0 \), we may assume that the relational contract specifies a bonus \( b \) if \( Q = Q_H \). In this case,\(^8\)

---

\(^8\)These assumptions are made for technical reasons. Essentially, they guarantee that in the steady state we have \( \lambda_{US} \) and \( \lambda_{DS} \) bounded by 1.

\(^9\)When \( p(a) = 0 \), BGM (2002) show that the relational outsourcing dominates relational employment.
the DS firm’s maximization problem per period is reduced to

\[
\max_a b^a q^a - \frac{1}{2} a^2.
\]

Therefore, we have

\[
a(b) = q^b \\
s^R(b) = q^2 b - \frac{1}{2} q^2 b^2,
\]

where \(a(b)\) is the agent’s action and \(s^R(b)\) is the total surplus in the relationship per period.

Note that the first best is given by

\[
b^{FB} = 1 \\
a^{FB} = q \\
s^R(FB) = \frac{1}{2} q^2.
\]

For this to be sustainable, Proposition 1 implies that we need

\[
\frac{1}{2} b^{FB} = \frac{1}{2} \leq R.
\]

From Lemma 2, we know that \(R = \frac{(2 - \Lambda)(1 + r)}{2(r + \varepsilon) + (1 - \varepsilon)\Lambda} s^R\). It follows that the first best is attainable when

\[
q^2 \geq \frac{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}{(2 - \Lambda)(1 + r)}.
\]

In this case, we know from Lemma 2 that the expected payoff of entering the market for a US firm is

\[
X_{US} = \begin{cases} 
\frac{q^2(1 + r)^2}{4r} \frac{(\Lambda + \sqrt{\Lambda^2 - 4\varepsilon^2})}{2(r + \varepsilon) + (1 - \varepsilon)\Lambda} & \text{if } M < N \\
\frac{q^2(1 + r)^2}{4r} \frac{(\Lambda - \sqrt{\Lambda^2 - 4\varepsilon^2})}{2(r + \varepsilon) + (1 - \varepsilon)\Lambda} & \text{if } M > N.
\end{cases}
\]

When the first best cannot be achieved, the optimal relational contact must satisfy

\[
\frac{b}{2} = R,
\]

because otherwise the parties can increase \(b\) and achieve a higher level of effort and thus a higher
payoff. Using the DS firm’s effort response, we have

\[ s^R = 2q^2(R - R^2). \]

Note that by Lemma 2, we also have

\[ s_R = \frac{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}{(2 - \Lambda)(1 + r)}R. \]

These two equations imply that

\[ R = \frac{2q^2 - \frac{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}{(2 - \Lambda)(1 + r)}}{2q^2}. \]

Now using Lemma 2, we see that when first best level of effort cannot be sustained, we have

\[ X_{US} = \frac{(1 + r)\lambda_{US}}{(2 - \Lambda)r}R \]

\[ = \frac{1}{2rq^2} \left( \frac{\Lambda + \sqrt{\Lambda^2 - 4\varepsilon^2}}{2(2 - \Lambda)}(2(1 + r)q^2 - \frac{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}{2 - \Lambda}) \right) \text{ if } M < N \]

\[ = \frac{1}{2rq^2} \left( \frac{\Lambda - \sqrt{\Lambda^2 - 4\varepsilon^2}}{2(2 - \Lambda)}(2(1 + r)q^2 - \frac{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}{2 - \Lambda}) \right) \text{ if } M > N \]

It is also clear from the expression above that when \( q^2 < \frac{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}{2(2 - \Lambda)(1 + r)} \), no relational contracts can be sustained. Together with the condition that first best can we achieved when \( q^2 \geq \frac{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}{(2 - \Lambda)(1 + r)} \), we have the following result.

**Proposition 4:** The sustainable surplus in the relational outsourcing contract satisfies the following:

(i): If \( q^2 \geq \frac{2(r + \varepsilon) + (1 - \varepsilon)(1 + \varepsilon^2)}{(1 - \varepsilon^3)(1 + r)} \), first best actions can be sustained in the relational contract regardless of the number of firms in the economy.

(ii): If \( \frac{2(r + \varepsilon) + (1 - \varepsilon)(1 + \varepsilon^2)}{(1 - \varepsilon^3)(1 + r)} < q^2 < \frac{2(r + \varepsilon) + (1 - \varepsilon)(1 + \varepsilon^2)}{(1 - \varepsilon^2)(1 + r)} \), then there exists a \( \Lambda^* \) such that first best actions can be sustained in the relational contract for \( \Lambda < \Lambda^* \). When \( \Lambda > \Lambda^* \), relational contracts are possible but only second best level of effort can be sustained.

(iii): If \( \frac{r + \varepsilon}{(1 - \varepsilon)(1 + r)} < q^2 < \frac{(r + \varepsilon) + (1 - \varepsilon)(1 + \varepsilon^2)}{(1 - \varepsilon^2)(1 + r)} \), then there exists a \( \Lambda_1^* < \Lambda_2^* \) such that first best actions can be sustained in the relational contract for \( \Lambda < \Lambda_1^* \). When \( \Lambda_1^* < \Lambda < \Lambda_2^* \), relational contracts are possible but only second best level of effort can be sustained. When \( \Lambda > \Lambda_2^* \), no relational contract is possible.

(iv): If \( \frac{(r + \varepsilon)}{(1 - \varepsilon)(1 + r)} < q^2 < \frac{(r + \varepsilon)(1 - \varepsilon)}{(1 - \varepsilon)(1 + r)} \), then there exists a \( \Lambda_3^* \) such that when \( \Lambda < \Lambda_3^* \), relational contracts are possible but only second best level of effort can be sustained. When \( \Lambda > \Lambda_3^* \), no relational contract is possible.

\(^{10}\) Assuming that \( 2r < 1 - \varepsilon - \varepsilon^2 - \varepsilon^3 \). Otherwise, this set is empty.
(v): If \( q^2 < \frac{(r+\varepsilon)+(1-\varepsilon)\varepsilon}{2(1-\varepsilon)(1+r)} \), no relational contract is possible.

**Proof.** Straightforward calculations. 

Proposition 4 enables us to calculate the equilibrium number of US firms that enter. Some cases are straightforward. First, when \( q^2 > \frac{(2(r+\varepsilon)+(1-\varepsilon)(1+\varepsilon^2))}{(1-\varepsilon^2)(1+r)} \), Proposition 3 implies that first best actions will always be achieved. It is easy to see that the expected payoff of entering US firms, \( X_{US} \), is strictly decreasing in \( M \). Therefore, there will be a unique level of \( M \) at which \( X_{US}(M) = c \).

Second, when \( q^2 < \frac{(r+\varepsilon)+(1-\varepsilon)\varepsilon}{(1-\varepsilon)(1+r)} \), which implies that \( q^2 < \frac{2(r+\varepsilon)+(1-\varepsilon)\Lambda}{2(2-\Lambda)(1+r)} \) for all feasible levels of \( \Lambda \), no relational contract can be sustained and there will be no entry of US firms other than the \( \varepsilon N \) ones that have entry costs of 0.

Cases (ii), (iii), and (iv) are more interesting. Since the analysis for the three cases are similar in spirit, we provide below only the analysis of (iii), which gives the richest set of possibilities for the outcomes from relational contract.

**Proposition 5:** Suppose \( \frac{(r+\varepsilon)+(1-\varepsilon)\varepsilon}{(1-\varepsilon)(1+r)} < q^2 < \frac{(2(r+\varepsilon)+(1-\varepsilon)(1+\varepsilon^2))}{(1-\varepsilon^2)(1+r)} \). Then there exists a \( c^* \) such that if the entry cost \( c > c^* \), no US firms enter other than the ones with zero cost of entry. There exists a \( c_* \) such that if \( c < c_* \), there exists a unique equilibrium with positive entry. If \( c_* < c < c^* \), there are exactly three equilibria, of which only two are stable.

**Proof.** See Appendix.

To understand Proposition 5, note that the value of \( X_{US}(M) \) depends on the ratio \( M/N \). The first best action can be achieved when \( \frac{\Lambda^*-\sqrt{\Lambda^*-4\varepsilon^2}}{2\varepsilon} N \leq M < \frac{\Lambda^*+\sqrt{\Lambda^*+4\varepsilon^2}}{2\varepsilon} N \); this is the case where the market is very balanced. In this region, an increase in the number of upstream firms does not change the surplus in the relationship but decreases the value of \( X_{US}(M) \) because it makes it harder for an upstream firm to be matched with a downstream firm (also increasing the share of the surplus captured by downstream firms).

When \( M > \frac{\Lambda^*+\sqrt{\Lambda^*+4\varepsilon^2}}{2\varepsilon} N \), first best actions can no longer be achieved: the market is imbalanced due to too many upstream firms. In this region, an increase in the number of upstream firms decreases \( X_{US}(M) \) in two ways. First, it is harder for an upstream firm to be matched with a downstream firm. Second, it makes the relationship more difficult to sustain and decreases total surplus.

Finally, when \( M < \frac{\Lambda^*+\sqrt{\Lambda^*+4\varepsilon^2}}{2\varepsilon} N \), an increase in the number of upstream firms has two opposing effects. On the one hand, again it is harder for an upstream firm to be matched with a downstream firm and this effect decreases \( X_{US}(M) \). On the other hand, more upstream firms
makes the market more balanced and this can increase the total surplus in relational contracts. This second effect increases the value of $X_{US}(M)$. It is possible for the second force to dominate, so $X_{US}$ is increasing over some range.

Figure 1 illustrates the relationship between $X_{US}$ and $\Lambda$ (which is increasing in $M$) when $r = 0.1$, $\varepsilon = 0.1$, and $q^2 = 0.6$. At low levels of entry, the returns to US firms are increasing in the number of US firms in the market, and we observe multiple equilibria for a broad range of entry costs $c$.

Proposition 5 suggests a novel rationale for big-push policies. When there are too few US firms entering the market, coordination will be important in pushing the economy over the entry hurdle, and the government may consider subsidizing entry in order to escape the low-entry equilibrium. However, free entry can be inefficient in this model. When the number of US firms is already high, further entry may unbalance the market, diminishing the surplus in relational contracts. This negative externality is not taken into account by entering US firms. Thus while the model stresses the potential for a government role in ensuring market balance, policy-makers may have a difficult time knowing in which direction to push.

5 Input Ownership

In this section, we study one mechanism–input ownership–that parties can use to compensate for a market structure that inhibits relational contracts. While the optimality of different ownership structures has been studied extensively in the literature, the focus has typically been on other types of ownership. The central right embodied in ownership of a firm and its assets is most often taken to be the right to make decisions about the production process (Grossman and Hart (1986), Hart and Moore (1990), Baker, Gibbons & Murphy (2006)). That is, ownership means that an agent gets to decide how much to produce, what inputs to use and who will undertake individual stages of production (e.g., marketing). This aspect of ownership is typically described as residual control rights. Alternatively, the central element of ownership can be viewed as a legal claim on output. In this conception, ownership of a firm confers the ability to unilaterally sell the firm’s output and appropriate the revenue. In Baker, Gibbons & Murphy (2002), for example, ownership of an upstream firm by a downstream firm limits the former party’s wasteful attempts to increase the assets sale value to alternative buyers but tempts the downstream firm to take the output without paying promised bonuses. We term this aspect output ownership.

A third and less studied aspect of ownership is control over inputs that enhance productivity. Control over such inputs confers the critical ability to dissolve the current relationship and continue the game with a new partner; lack of control means that future relationships will be less productive. We term this third dimension input ownership, and it is the focus of this section.\footnote{In Halonen (2002), "ownership" covers both output and inputs, but these rights cannot be separated.} These three
elements of ownership are often tied together but need not be. Shareholders, for example, have strong claims on output but much weaker control rights and limited ownership of inputs (though they do own the firms’ brand name). An employee who develops strong, personalized relationships with clients may have a high level of input ownership—he can plausibly leave the firm and take its clients with her—but few control rights and no claim on output.

Abstracting from control rights, we can expand the distinction between employment and outsourcing into a two-by-two matrix with four possible (relational) ownership structures:

<table>
<thead>
<tr>
<th></th>
<th>Upstream owns output</th>
<th>Downstream owns output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upstream owns input</td>
<td>Outsourcing</td>
<td>Cottage Industry</td>
</tr>
<tr>
<td>Downstream owns input</td>
<td>Licensing</td>
<td>Employment</td>
</tr>
</tbody>
</table>

In what we classify as a typical employment relationship, the employer (downstream firm) owns the rights to both output and most inputs; the typical outsourcing relationship is characterized by complete separation of the two firms, with the upstream party owning both output and inputs. However, two intermediate ownership forms are also possible. We denote the situation in which the downstream firm owns output but the "employee" owns his own inputs as cottage industry, though it could equally refer to an employee with important human capital that is alienable ex ante but not ex post (e.g., personalized relationships with clients). We denote the reverse situation, with the upstream firm owning its own output but borrowing inputs from the downstream firm as licensing (e.g., the downstream firm might own a patent that it licenses to the upstream firm). In this section, we begin the process of extending the analysis of optimal ownership structures to the four cases listed above.\(^\text{12}\)

### 5.1 Model Setup

We return to the model in section 3 with exogenous participation on both sides of the market. Consider a unique upstream party, called the Entrepreneur, who owns an input.\(^\text{13}\) To fix ideas, suppose that this input is a patent that decreases the cost of production by some constant amount \(z\). With the input, the value of production to a downstream party is

\[
Q_H + z \quad \text{with probability } q(a) \\
Q_L + z \quad \text{with probability } 1 - q(a)
\]

\(^{12}\)For ease of exposition, we continue to refer to contracts where US owns the output as relational outsourcing when input ownership is not at issue. Similarly, we continue to refer to contracts where DS owns the output as relational employment.

\(^{13}\)The analysis is the same if the initial owner of the input is a downstream firm.
and the value to the outside market is

\[ P_H + z \quad \text{with probability } p(a) \]
\[ P_L + z \quad \text{with probability } 1 - p(a) \]

We abstract from the process of innovation and assume that the patent is unique, with no further patent development being possible. The Entrepreneur meets downstream firms with the same frequency as other upstream firms. When the Entrepreneur meets a downstream firm, they negotiate a relational contract that specifies a base payment \( s \), bonus payments \( \{b_{ij}\} \), output ownership \( o_{Out} \in \{US, DS\} \) and ownership of the input \( o_{In} \in \{US, DS\} \). Neither party is liquidity constrained.

At the end of any period where separation has occurred, the owner of the input has the right to carry it into the matching market and use it in his next match. However, the separating parties can engage in bargaining over the asset, where the bargaining process leads to a payment of \( \pi_k \) when the purchasing party is of type \( k \) and a receipt of \( \pi_k - C_T \) by the selling party. We interpret \( C_T \) as a reduced-form bargaining cost; it might arise from direct transaction costs (e.g., government fees for title transfer), asymmetric information (e.g., Matouschek, 2004), or general haggling costs (Williamson, 1975; Klein, Crawford & Alchian, 1978). In most of this section, we assume that \( M < N \), so that the upstream firms are the short side of the market. This means that the input is worth more to an unmatched upstream firm than to an unmatched downstream firm. When \( o_{In} = US \), there will be no ex post bargaining over ownership of the input, while when \( o_{In} = DS \) bargaining occurs for sufficiently small \( C_T \).

Finally, we continue to assume that separation is more efficient than static production following a deviation.

### 5.2 Input Ownership and Replacement Costs

In this setup, input ownership affects relational contracts through its effect on replacement costs. When \( o_{In} = US \), there is no ex post bargaining, and so replacement costs have the same form as in section 3:

\[
R = R_0 = \frac{(2 - \Lambda)(1 + r)}{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}(s^R + z)
\]

where \( s^R \) is the per period surplus of a relational contract for pairs without an input. When
In DS, the parties suffer an additional replacement cost after separation—either DS keeps the input and receives less value from it than US or they jointly pay the transfer cost $C_T$. In the former case, the joint loss is

$$\Theta(r, \lambda_{US}, \lambda_{DS}) = \frac{(1+r)\lambda_{US}}{r + \lambda_{US}} (U_{US}^* - U_{US}) - \frac{(1+r)\lambda_{DS}}{r + \lambda_{DS}} (U_{DS}^* - U_{DS}) > 0$$

where $U_{k}^*$ is the utility of a type $k$ who owns the input at the moment when he finds a match. For the remainder of this section, we assume that $C_T$ is always less than $\Theta(r, \lambda_{US}, \lambda_{DS})$. This assumption simplifies the calculations but does not substantively affect the results. When $a_{In} = DS$, then, total replacement costs are

$$R = R_0 + C_T$$

This leads to our next result:

**Proposition 6** When ex post bargaining over asset ownership is efficient ($C_T = 0$), input ownership is irrelevant.

The proof follows immediately from the fact that $R$ is the same for $a_{In} = US$ and $a_{In} = DS$, since input ownership affects relational contracts solely through $R$. Intuitively, we might expect that stronger relational contracts can be sustained by giving the input to the weaker (downstream) party. In that way, the downstream party is dissuaded from reneging through his difficulty in finding a replacement and the upstream party is dissuaded by the fact that he will have to purchase the input if he is to carry it into his next relationship. With efficient ex post bargaining, input ownership allows the parties to transfer reneging temptation across one another. However, the relational contracts or section 2 already allow for full transferability of reneging temptation, so only factors that affect the sum of reneging temptations matter. When $C_T = 0$, input ownership has no effect on this sum.

Under inefficient ex post bargaining, $R$ is higher when the input is owned by the downstream firm. This has both a positive and a negative effect: it can allow the parties to sustain more efficient relational contracts by discouraging deviations, but it imposes higher costs when separation occurs along the equilibrium path. The optimal input ownership structure maximizes the (stock) surplus from the relational contract

$$S^R = (s^R + z) + \frac{1}{1+r}[(1-\epsilon)S^R + \epsilon(S^R - R)] \quad (19)$$

$$\Rightarrow S^R = \frac{1+r}{r}(s^R(R) + z) - \frac{\epsilon}{r}R$$

25
where \( s^{R}(R) \geq 0 \).

An immediate result is that the upstream firm will own the input if first best production is attainable under that structure. We restrict further analyses to a special linear-quadratic case of the model.

### 5.3 A Linear-Quadratic Special Case

In this subsection, we assume that

\[
q(a) = a_1, \\
p(a) = a_2, \\
c(a) = \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2
\]

Following BGM (2002), we further assume that relational incentive contracts take the linear form

\[
b_{ij} = b_i + \beta_j
\]

and define \( \Delta b = b_H - b_L \) and \( \Delta \beta = \beta_H - \beta_L \). In this case, optimal actions for the upstream firm are

\[
a_1 = \Delta b, \\
a_2 = \Delta \beta
\]

and the first best can be attained if

\[
a_{1}^{FB} = \Delta Q, \\
a_{2}^{FB} = 0.
\]
5.3.1 Downstream Ownership of Output

When replacement costs are a binding constraint, the optimal relational employment contract solves

$$\max_{\Delta b, \Delta \beta} \Delta Q a_1 - \frac{1}{2} a_1^2 - \frac{1}{2} a_2^2$$

subject to

$$a_1 = \Delta b$$
$$a_2 = \Delta \beta$$
$$|\Delta b| + |\Delta \beta| \leq R$$

Clearly the optimal contract has $\Delta \beta^{RE} = 0$ and $\Delta b^{RE} = R$, so surplus per period is

$$s^{RE}(R) = Q_L + \Delta QR - \frac{1}{2} R^2 \quad \text{if } R < \Delta Q$$
$$= Q_L + \frac{1}{2} \Delta Q^2 \quad \text{if } R \geq \Delta Q$$

where $Q_L + \frac{1}{2} \Delta Q^2$ is first best surplus.

5.3.2 Upstream Ownership of Output

In this case, the optimal contract solves

$$\max_{\Delta b, \Delta \beta} \Delta Q a_1 - \frac{1}{2} a_1^2 - \frac{1}{2} a_2^2$$

subject to

$$a_1 = \Delta b$$
$$a_2 = \Delta \beta$$
$$\left|\Delta b - \frac{1}{2} \Delta Q\right| + \left|\Delta \beta - \frac{1}{2} \Delta P\right| \leq R$$
There is no gain to setting $\Delta b < \frac{1}{2} \Delta Q$ or $\Delta \beta > \frac{1}{2} \Delta P$, so the last constraint can be rewritten as

$$\Delta b - \Delta \beta \leq R + \frac{1}{2} \Delta Q - \frac{1}{2} \Delta P$$

When $R \geq \frac{1}{2}(\Delta Q + \Delta P)$, first best is attainable. If not, the solution to this program depends on the sign of $\Delta Q - \Delta P$. If $\Delta Q > \Delta P$, then

$$\begin{align*}
\Delta b^{RO} &= \min\left\{ \frac{3}{4} \Delta Q - \frac{1}{4} \Delta P + \frac{1}{2} R, \frac{1}{2} \Delta Q + R \right\} \\
\Delta \beta^{RO} &= \min\left\{ \frac{1}{4} \Delta Q + \frac{1}{4} \Delta P - \frac{1}{2} R, \frac{1}{2} \Delta P \right\}
\end{align*}$$

while if $\Delta Q < \Delta P$,

$$\begin{align*}
\Delta b^{RO} &= \max\left\{ \frac{3}{4} \Delta Q - \frac{1}{4} \Delta P + \frac{1}{2} R, \frac{1}{2} \Delta Q \right\} \\
\Delta \beta^{RO} &= \max\left\{ \frac{1}{4} \Delta Q + \frac{1}{4} \Delta P - \frac{1}{2} R, \frac{1}{2} \Delta P - R \right\}
\end{align*}$$

(In each case, we get the first elements inside the brackets whenever $R > \frac{1}{2} |\Delta Q - \Delta P|$ and the second elements otherwise.)

Some tedious calculations then show that

$$s^{RO}(R) = s^{FB}$$

if $R \geq \frac{1}{2}(\Delta Q + \Delta P)$,

$$s^{RO}(R) = Q_L + \frac{7}{16} \Delta Q^2 - \frac{1}{8} \Delta Q \Delta P - \frac{1}{16} \Delta P^2 + \frac{1}{4}(\Delta Q + \Delta P)R - \frac{1}{4} R^2$$

if $\frac{1}{2}(\Delta Q + \Delta P) > R > \frac{1}{2} |\Delta Q - \Delta P|$, and

$$s^{RO}(R) = Q_L + \frac{3}{8} \Delta Q^2 - \frac{1}{8} \Delta P^2 + \frac{1}{2} \max\{\Delta Q, \Delta P\} R - \frac{1}{2} R^2$$
if \( R \leq \frac{1}{2} |\Delta Q - \Delta P| \).

### 5.3.3 Optimal Output Ownership

**Proposition 7** Suppose \( R \leq \min\{\Delta Q, \frac{1}{2}(\Delta Q + \Delta P)\} \), so that first best is unattainable. When

\[
\frac{\Delta P}{\Delta Q} \leq 1, \quad s^{RO} > s^{RE}. \quad \text{When} \quad 1 < \frac{\Delta P}{\Delta Q} \leq 3 - \sqrt{2}, \quad s^{RO} > s^{RE} \quad \text{if} \quad R/\Delta Q < f\left(\frac{\Delta P}{\Delta Q}\right), \quad \text{where}
\]

\[
f(x) = 1 - \frac{1 + \sqrt{2}}{2}(x - 1)
\]

and \( s^{RO} \leq s^{RE} \) otherwise. When \( 3 - \sqrt{2} < \frac{\Delta P}{\Delta Q} < \sqrt{3} \), \( s^{RO} > s^{RE} \) if \( R/\Delta Q < g\left(\frac{\Delta P}{\Delta Q}\right) \), where

\[
g(x) = \frac{x^2 - 3}{4(x - 2)}
\]

and \( s^{RO} \leq s^{RE} \) otherwise. When \( \frac{\Delta P}{\Delta Q} \geq \sqrt{3} \), \( s^{RO} \leq s^{RE} \).

**Proof.** See appendix. ■

From Proposition 3, we know that higher replacement costs increase surplus faster when output is owned by the downstream firm; the above result reflects this fact. Figure 2 displays the surplus generated under each output ownership structure as a function of \( R \) (for parameters \( Q_L = \Delta Q = 1 \) and \( \Delta P = 1.5 \)). Total surplus from relational contracts is the upper envelope of surplus from relational outsourcing (the green line) and surplus from relational employment (the red line).\(^{14}\)

### 5.3.4 Incorporating Input Ownership

Because optimal output ownership depends on \( R \) and input ownership affects \( R \), the two ownership decisions are not separable—when \( M/N < 1 \), downstream input ownership raises \( R \) and makes downstream output ownership more attractive. This fact suggests that there is a benefit to studying input and output ownership jointly. Figure 3 illustrates the interaction between input and output ownership, displaying the optimal ownership structure as a function of \( \Delta P \) and \( R_0 \) (with

\(^{14}\)The non-concavity of \( s^R(R) \) at the point where optimal output ownership switches is a generic feature of the model.
Two features are worth noting. First, there is a general tendency for upstream output ownership to occur when \( \Delta P \) is small and upstream input ownership to occur when replacement costs are high; the latter stems from the fact that \( s^R(R) \) is concave over the bulk of its range (so that the benefit of increasing replacement costs by assigning the input to the downstream firm are typically decreasing in \( R_0 \)). Secondly, over the intermediate rage of \( \Delta P \) where neither output ownership regime is always dominant, we often see switching from employment to outsourcing as replacement costs decrease; this reflects the effect of replacement costs on output ownership described in Proposition 3. Finally, when replacement costs are in the neighborhood of 0.25 and \( \Delta P \) is roughly 1.55, the region where Employment is optimal "juts into" the Outsourcing region. This irregular shape reflects the interaction between input and output ownership described above: low replacement costs induce the parties to assign the input to DS, which in turn induces them to assign output ownership to DS as well.

6 Conclusion

This paper places the study of optimal relational contracts and ownership structures into a market context, where the two sides of the market are asymmetric in their replaceability. We argue that market structure can play a key role in determining replacement costs, which in turn affect the set of sustainable relational contracts and the optimality of different ownership structures. We then explore two extensions to the basic model. When market entry is endogenous, we find that the mechanism studied here can lead to a novel source of multiple equilibria; this finding may have interesting applications to economic development. We also examine the potential for matched partners to counteract market forces by using ownership of inputs as a kind of hostage. We find that this strategy is feasible only when ex post bargaining is inefficient and that optimal input and output ownership decisions are not separable.

We believe that the basic approach adopted in this paper can be applied fruitfully to a number of economic issues. For example, the model of input ownership potentially provides a novel rationale for the protection of intellectual property. With no intellectual property protection, investors and partners of innovating entrepreneurs might find themselves easily replaced. This in turn encourages the entrepreneurs to take self-interested courses of action and makes it more difficult for them to find a partner in the first place. Enforceable intellectual property rights can be sold to investors, effectively making both parties hard to replace and sustaining more efficient relational contracts. However, it does not follow that patent law should be as clear and as strong as possible. As this

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15 Calculations behind Figure 3 are summarized in the Appendix.
16 A similar argument applies to the enforcement of non-compete clauses in employment contracts.
paper shows, assigning a patent to the investor facilitates relational contracting only if \textit{ex post} bargaining is inefficient. Insofar as ambiguous patent rights encourage parties to expend resources in their attempts to claim intellectual property rights (e.g., on legal fees), they can lower the sum of the parties’ payoffs following dissolution and support relational contracting. Thus while the high aggregate cost of resolving patent uncertainty is often malign, it may serve a useful function.

The model also points to an important role for credit constraints in hampering efficient production. Throughout the paper, we abstract from liquidity concerns and assume that the parties can make whatever upfront transfers are necessary to support the efficient allocation of assets. This may be a realistic assumption as long as we identify the replaceable party (who purchases the asset) as an investor and the irreplaceable party as an entrepreneur. More generally, however, input ownership by the replaceable party means that the long side of the market must make an up front payment to the short side of the market. Since the short side of the market will receive a smaller surplus, we might expect it to be particularly credit constrained. Market prices, then, will tend to lead naturally to wealth distributions that hamper the efficient allocation of productive assets and optimal relational contracts. Economies with poorly developed financial markets will be especially afflicted by this problem.

Finally, we would like to highlight one important dimension along which the model can be extended. Following Macleod and Malcomson (1998), we have assumed the current players’ histories are not observable to any replacements. In some situations this assumption may be realistic, but in other cases agents from one side of the market form institutions to share their experiences with past partners (e.g., credit ratings, social networks among venture capitalists). These institutions make the agents harder to replace and therefore potentially reduce the importance of assets as hostages. The role of different observability assumptions, particularly in combination with imperfect public monitoring of partners’ actions, is worth exploring in future work.
References


A Appendix

A.1 Proof of Proposition 5

Proposition 5: Suppose \( \frac{(r+\varepsilon)+(1-\varepsilon)\varepsilon}{(1-\varepsilon)(1+r)} < q^2 < \frac{(2(r+\varepsilon)+(1-\varepsilon)(1+\varepsilon^2))}{(1-\varepsilon^2)(1+r)} \). Then there exists a \( c^* \) such that if the entry cost \( c > c^* \), no US firms enter other than the ones with zero cost of entry. There exists a \( c_* \) such that if \( c < c_* \), there exists a unique equilibrium with positive entry. If \( c_* < c < c^* \), there are exactly three equilibria, of which only two are stable.

Proof. When \( \frac{(r+\varepsilon)+(1-\varepsilon)\varepsilon}{(1-\varepsilon)(1+r)} < q^2 < \frac{(2(r+\varepsilon)+(1-\varepsilon)(1+\varepsilon^2))}{(1-\varepsilon^2)(1+r)} \), whether the first best can be achieved depends on the value of \( \Lambda \). It can be shown that there exists \( \Lambda^* = \frac{2q^2(1+r)-2(r+\varepsilon)}{q^2(1+r)+(1-\varepsilon)} \) such that if \( \Lambda < \Lambda^* \), then the first best action can be sustained and otherwise it cannot. Therefore, summarizing the discussion above, we have

\[
X_{US}(M) = \begin{cases} 
\frac{1}{2rq^2} \left( \frac{\Lambda + \sqrt{\Lambda^2 - 4\varepsilon^2}}{2(1+r)} - (2(1+r)q^2 - \frac{2(r+\varepsilon) + (1-\varepsilon)\Lambda}{2-\Lambda}) \right) & \text{if } M < \frac{\Lambda^* - \sqrt{\Lambda^*^2 - 4\varepsilon^2}}{2\varepsilon} N; \\
\frac{q^2(1+r)^2}{4r} \left( \frac{\Lambda + \sqrt{\Lambda^2 - 4\varepsilon^2}}{2(r+\varepsilon) + (1-\varepsilon)\Lambda} \right) & \text{if } \frac{\Lambda^* - \sqrt{\Lambda^*^2 - 4\varepsilon^2}}{2\varepsilon} N \leq M \leq \frac{\Lambda^* + \sqrt{\Lambda^*^2 - 4\varepsilon^2}}{2\varepsilon} N; \\
\frac{1}{2rq^2} \left( \frac{\Lambda - \sqrt{\Lambda^2 - 4\varepsilon^2}}{2(1+r)} - (2(1+r)q^2 - \frac{2(r+\varepsilon) + (1-\varepsilon)\Lambda}{2-\Lambda}) \right) & \text{if } M \geq \frac{\Lambda^* + \sqrt{\Lambda^*^2 - 4\varepsilon^2}}{2\varepsilon} N. 
\end{cases}
\]
It is easily seen that if $M > \frac{\Lambda^{*} - \sqrt{\Lambda^{*} - 4\varepsilon^{2}}}{2\varepsilon} N$, we must have $dX_{US}(M)/dM < 0$. Therefore, we are interested in the case when $M < \frac{\Lambda^{*} - \sqrt{\Lambda^{*} - 4\varepsilon^{2}}}{2\varepsilon} N$. In this case, we have

$$X_{US}(\Lambda) = \frac{1}{2rq^{2}} \frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{2(2 - \Lambda)} \left(2(1 + r)q^{2} - \frac{2(r + \varepsilon) + (1 - \varepsilon)\Lambda}{2 - \Lambda}\right)$$

$$= \frac{1}{4rq^{2}} \frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} (2(1 + r)q^{2}(2 - \Lambda) - 2r(1 + \varepsilon) - (1 - \varepsilon)\Lambda)$$

$$= \frac{1}{4rq^{2}} \frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} (4(1 + r)q^{2} - 2r(1 + \varepsilon) - (2(1 + r)q^{2} + (1 - \varepsilon))\Lambda)$$

$$= \frac{1}{4rq^{2}} \frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} (A - BA),$$

where $A \equiv 4(1 + r)q^{2} - 2r(1 + \varepsilon)$, and $B \equiv 2(1 + r)q^{2} + (1 - \varepsilon)$. Note that $A = 2B - 2(1 + r)$.

Taking the derivative with respect to $\Lambda$, we have

$$\frac{d\frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} (A - BA)}{d\Lambda} = \frac{(1 + \frac{\Lambda}{\sqrt{\Lambda^{2} - 4\varepsilon^{2}}})(2 - \Lambda)^{2} + 2(\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}})(2 - \Lambda)}{(2 - \Lambda)^{4}} (A - BA)$$

$$- B \frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}}$$

$$= \frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} \left[\left(\frac{1}{\sqrt{\Lambda^{2} - 4\varepsilon^{2}}} + \frac{2}{2 - \Lambda}\right)(A - BA) - B\right].$$

There are two cases to consider now. First, if $A - BA < 0$, i.e. $\Lambda > \frac{A}{B}$, we must have

$$\frac{d\frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} (A - BA)}{d\Lambda} < 0.$$ 

Now suppose instead that $A < \frac{A}{B}$. In this case, we have

$$\frac{d\frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} (A - BA)}{d\Lambda}$$

$$= \frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} \left[\frac{A - BA}{\sqrt{\Lambda^{2} - 4\varepsilon^{2}}} + 2B \frac{\frac{A}{B} - \Lambda}{2 - \Lambda} - B\right]$$

$$< 0,$$

where the inequality follows because $\frac{\Lambda + \sqrt{\Lambda^{2} - 4\varepsilon^{2}}}{(2 - \Lambda)^{2}} > 0$, $\frac{A - BA}{\sqrt{\Lambda^{2} - 4\varepsilon^{2}}}$ decreases with $\Lambda$, and $\frac{\frac{A}{B} - \Lambda}{2 - \Lambda}$ also decreases with $\Lambda$ because $\frac{A}{B} = 2(1 - \frac{1 + r}{B}) < 2$.

This implies that there exists $M^{*}$ such that $X_{US}(M)$ is strictly increasing in $M$ for $M < M^{*}$ and $X_{US}(M)$ is strictly decreasing in $M$ for $M > M^{*}$. The rest of the proof follows easily.

Q.E.D.
A.2 Proof of Proposition 7

Proposition 7 Suppose $R \leq \min\{\Delta Q, \frac{1}{2}(\Delta Q + \Delta P)\}$, so that first best is unattainable. When

\[ \frac{\Delta P}{\Delta Q} \leq 1, \quad s^{RO} > s^{RE}. \]

When \( 1 < \frac{\Delta P}{\Delta Q} \leq 3 - \sqrt{2} \), \( s^{RO} > s^{RE} \) if \( R/\Delta Q < f(\frac{\Delta P}{\Delta Q}) \), where

\[
f(x) = 1 - \frac{1 + \sqrt{2}}{2}(x - 1)
\]

and \( s^{RO} < s^{RE} \) otherwise. When \( 3 - \sqrt{2} < \frac{\Delta P}{\Delta Q} < \sqrt{3} \), \( s^{RO} > s^{RE} \) if \( R/\Delta Q < g(\frac{\Delta P}{\Delta Q}) \), where

\[
g(x) = \frac{x^2 - 3}{4(x - 2)}
\]

and \( s^{RO} < s^{RE} \) otherwise. When \( \frac{\Delta P}{\Delta Q} \geq \sqrt{3} \), \( s^{RO} < s^{RE} \).

Proof. Let \( x \equiv \frac{\Delta P}{\Delta Q} \) for notational simplicity.

CASE 1: \( \frac{\Delta P}{\Delta Q} \leq 1 \). First suppose that \( \frac{1}{2}(\Delta P + \Delta Q) > R > \frac{1}{4}(\Delta Q - \Delta P) \). Then

\[
s^{RO}(R) - s^{RE}(R) = \frac{1}{4}R^2 + (\frac{1}{4} \Delta P - \frac{3}{4} \Delta Q)R + (\frac{7}{16} \Delta Q^2 - \frac{1}{8} \Delta Q \Delta P - \frac{1}{16} \Delta P^2) \tag{20}
\]

which has roots

\[
R = \frac{3}{2} \Delta Q - \frac{1}{2} \Delta P \pm \frac{1}{\sqrt{2}} |\Delta Q - \Delta P| \tag{21}
\]

With \( \frac{\Delta P}{\Delta Q} \leq 1 \) and \( \frac{1}{2}(\Delta P + \Delta Q) > R > \frac{1}{4}(\Delta Q - \Delta P) \), it can be verified that \( s^{RO}(R) - s^{RE}(R) \) is always positive. If \( R \leq \frac{1}{2}(\Delta Q - \Delta P) \), we have

\[
s^{RO}(R) - s^{RE}(R) = \frac{3}{8} \Delta Q^2 - \frac{1}{8} \Delta P^2 - \frac{1}{2} \Delta Q R
\]

which is always positive for \( R \leq \frac{1}{2}(\Delta Q - \Delta P) \). Thus \( s^{RO}(R) > s^{RE}(R) \) in this case.

CASE 2: \( 1 < \frac{\Delta P}{\Delta Q} \leq 3 - \sqrt{2} \). First suppose that \( \Delta Q > R > \frac{1}{4}(\Delta P - \Delta Q) \). Then \( s^{RO}(R) - s^{RE}(R) \) is again determined by equation 20 and is positive when \( R \) is smaller than the lower root. The lower root can be rewritten as \( f(\frac{\Delta P}{\Delta Q}) \Delta Q \).
Suppose that $R \leq \frac{1}{2}(\Delta P - \Delta Q)$. Then

$$s^{RO}(R) - s^{RE}(R) = \frac{3}{8} \Delta Q^2 - \frac{1}{8} \Delta P^2 + \left(\frac{1}{2} \Delta P - \Delta Q\right) R$$

(22)

which is always positive over the relevant range of $R$.

CASE 3: $3 - \sqrt{2} < \frac{\Delta P}{\Delta Q} < \sqrt{3}$. First suppose that $\Delta Q > R > \frac{1}{2}(\Delta P - \Delta Q)$. Then $s^{RO}(R) - s^{RE}(R)$ is again determined by equation 20. Since $R$ is always larger than the lower root over this range, $s^{RO}(R) < s^{RE}(R)$.

Next suppose that $R \leq \frac{1}{2}(\Delta P - \Delta Q)$ Then $s^{RO}(R) - s^{RE}(R)$ is determined by equation 22. Based on this equation, $s^{RO}(R) > s^{RE}(R)$ iff

$$R < \frac{\Delta P^2 - 3\Delta Q^2}{4(\Delta P - 2\Delta Q)}$$

$$= \frac{x^2 - 3}{4(x - 2)} \Delta Q$$

$$= g(x) \Delta Q$$

In the range $3 - \sqrt{2} < x < \sqrt{3}$, $R < g(x) \Delta Q < \frac{1}{2} |\Delta Q - \Delta P|$ and so there is indeed a critical value $R_{\text{switch}} \equiv g(x) \Delta Q$ such that $s^{RO}(R) - s^{RE}(R) > 0$ if and only if $R < R_{\text{switch}}$.

CASE 4: $\frac{\Delta P}{\Delta Q} \geq \sqrt{3}$. First suppose that $\Delta Q > R > \frac{1}{2}(\Delta P - \Delta Q)$; then the same logic as in Case 3 applies and we have $s^{RO}(R) < s^{RE}(R)$ over the entire relevant range.

If $R \leq \frac{1}{2}(\Delta P - \Delta Q)$ and $\frac{\Delta P}{\Delta Q} < 2$, then $s^{RO}(R) > s^{RE}(R)$ iff $R < g(x) \Delta Q$, but this is impossible because it can be verified that $\frac{1}{2}(\Delta P - \Delta Q) \geq g(x) \Delta Q$.

When $\frac{\Delta P}{\Delta Q} = 2$, inspection of 22 reveals immediately that $s^{RO}(R) < s^{RE}(R)$.

Finally, suppose that $R \leq \frac{1}{2}(\Delta P - \Delta Q)$ and $\frac{\Delta P}{\Delta Q} > 2$. Then equation 22 implies that $s^{RO}(R) > s^{RE}(R)$ if and only if

$$R > \frac{\Delta P^2 - 3\Delta Q^2}{4(\Delta P - 2\Delta Q)}$$

$$= g(x) \Delta Q$$

However, we can show that $g(x) \Delta Q > \Delta Q$ for $x > 2$, so that the above condition can never be
satisfied where the first best is unattainable:
\[
g(x) - 1 = \frac{x^2 - 3 - 4(x - 2)}{4(x - 2)} = \frac{x^2 - 4x + 5}{4(x - 2)} > 0
\]

Q.E.D. ■

A.3 Calculations for Figure 3

Assume that \( Q_L = \Delta Q = 1 \) and \( \varepsilon = r = \frac{1}{4} \). Based on Proposition 7, we have the following:

CASE 1: \( \Delta P \leq 1 \), so that the input should go to the downstream firm when
\[
s^{RO}(R_0 + C_T) - s^{RO}(R_0) > \frac{\varepsilon}{1 + r}C_T
\]

Subcase A: When \( R_0 + C_T \leq \frac{1}{2}(\Delta Q - \Delta P) \), this becomes
\[
R_0 < \frac{3}{10} - \frac{1}{2}C_T
\]

Subcase B: When \( R_0 > \frac{1}{2}(\Delta Q - \Delta P) \), this becomes
\[
R_0 < \frac{3}{10} + \frac{1}{2}\Delta P - \frac{1}{2}C_T
\]

Subcase C: When \( R_0 \leq \frac{1}{2}(\Delta Q - \Delta P) < R_0 + C_T \), this becomes
\[
R_0^2 + [\Delta P - 1 - 2C_T]R_0 + \frac{1}{4}(1 - \Delta P)^2 + (1 + \Delta P)C_T - \frac{4}{5}C_T - C_T^2 > 0
\]

37
Subcase A: When $R_0 + C_T \leq \frac{1}{2}(\Delta P - \Delta Q)$, upstream owns the output regardless of input ownership and so the input should go to the downstream firm when

$$R_0 < \frac{3}{10} - \frac{1}{2} C_T$$

as in Case 1A.

Subcase B: $f(\frac{\Delta P}{\Delta Q}) > R_0 + C_T > R_0 > \frac{1}{2}(\Delta P - \Delta Q)$. Upstream again always owns the output, and the rule for input ownership is the same as in Case 1B:

$$R_0 < \frac{3}{10} + \frac{1}{2} \Delta P - \frac{1}{2} C_T$$

Subcase C: $R_0 \leq \frac{1}{2}(\Delta P - \Delta Q) < R_0 + C_T < f(\frac{\Delta P}{\Delta Q})$. The rule is the same as in Case 1C:

$$R_0^2 + [\Delta P - 1 - 2C_T]R_0 + \frac{1}{4}(1 - \Delta P)^2 + (1 + \Delta P)C_T - \frac{4}{5}C_T - C_T^2 > 0$$

Subcase D: $R_0 \geq f(\frac{\Delta P}{\Delta Q})$. In this case, output is always in the hands of the downstream firm, who gets the input as well when

$$s^{RE}(R_0 + C_T) - s^{RE}(R_0) > \frac{\epsilon}{1 + r} C_T$$

$$R_0 < \frac{4}{5} - \frac{1}{2} C_T$$

Subcase E: $\frac{1}{2}(\Delta P - \Delta Q) < R_0 < f(\frac{\Delta P}{\Delta Q}) \leq R_0 + C_T$. In this case, output goes to the down-
stream firm iff the input also goes to the downstream firm. The downstream firm gets both when

\[ s^{RE}(R_0 + C_T) - s^{RO}(R_0) > \frac{\epsilon}{1 + r} C_T \]

\[ -\frac{1}{4} R_0^2 + \left( \frac{3}{4} - \frac{1}{4} \Delta P - C_T \right) R_0 - \left[ \frac{7}{16} - \frac{1}{8} \Delta P - \frac{1}{16} \Delta P^2 + \frac{6}{5} C_T + \frac{1}{2} C_T^2 \right] > 0 \]

Subcase F: \( R_0 \leq \frac{1}{2}(\Delta P - \Delta Q) < f\left(\frac{\Delta P}{\Delta Q}\right) \leq R_0 + C_T \) (This case is unlikely to arise if \( C_T \) is "small." ) Again, input and output ownership go together, but now downstream gets both when

\[ R_0[1 - C_T - \frac{1}{2} \Delta P] > \frac{3}{8} - \frac{1}{8} \Delta P - \frac{4}{5} C_T + \frac{1}{2} C_T^2 \]

It can be shown that \( 1 - C_T - \frac{1}{2} \Delta P < 0 \) whenever \( C_T \geq f\left(\frac{\Delta P}{\Delta Q}\right) - \frac{1}{2}(\Delta P - \Delta Q) \) and \( 1 < \frac{\Delta P}{\Delta Q} \leq 3 - \sqrt{2} \). Thus we give both input and output to the downstream firm when

\[ R_0 < \frac{-\frac{3}{8} + \frac{1}{2} \Delta P + \frac{4}{5} C_T - \frac{1}{2} C_T^2}{\frac{1}{2} \Delta P + C_T - 1} \]

CASE 3: \( 3 - \sqrt{2} < \frac{\Delta P}{\Delta Q} < \sqrt{3} \).

Subcase A: When \( R_0 + C_T \leq g\left(\frac{\Delta P}{\Delta Q}\right) < \frac{1}{2}(\Delta P - \Delta Q) \), the upstream firm always gets the output, and so we follow the same rule as in Case 1A:

\[ R_0 < \frac{3}{10} - \frac{1}{2} C_T \]

Subcase B: When \( g\left(\frac{\Delta P}{\Delta Q}\right) < R_0 \), the downstream firm always gets the output, and so we follow the same rule as in Case 2D:

\[ R_0 < \frac{4}{5} - \frac{1}{2} C_T \]
Subcase C: When $R_0 \leq g(\frac{\Delta P}{\Delta Q}) < R_0 + C_T$, input and output ownership go together, and we follow the same rule as in Case 2F:

$$
R_0 < \frac{-3/5 + \frac{1}{5} \Delta P + \frac{4}{5} C_T - \frac{1}{2} C_T^2}{\frac{1}{2} \Delta P + C_T - 1}
$$

CASE 4: $\frac{\Delta P}{\Delta Q} \geq \sqrt{3}$. Output is always assigned to the downstream firm, and so the rule is the same as in Case 2D:

Proof.

$$
R_0 < \frac{4}{5} - \frac{1}{2} C_T
$$

■