Managing Conflicts in Relational Contracts*

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Abstract

A manager and a worker are in an infinitely repeated relationship in which the manager privately observes her opportunity costs of paying the worker. We show that the optimal relational contract generates periodic conflicts during which effort and expected profits decline gradually but recover instantaneously. To manage a conflict, the manager uses a combination of informal promises and formal commitments that evolves with the duration of the conflict. Finally, we show that liquidity constraints limit the manager's ability to manage conflicts but may also induce the worker to respond to a conflict by providing more effort rather than less.

Keywords: relational contracts, imperfect monitoring

JEL classifications: C73, D23, D82, J33

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1 Introduction

Relational contracts often suffer from conflicts during which workers punish managers for broken promises. A common cause for such conflicts is disagreement over the availability and efficient use of funds. In a typical conflict of this sort, workers demand that a bonus be paid while managers insist that the necessary funds are either non-existent or would be better spent on something else, such as an exceptional investment opportunity.

One source of disagreement over the availability and efficient use of funds is asymmetric information. Managers are typically better informed about the challenges and opportunities that their firms face and therefore often have private information about the opportunity costs of paying their workers. The aim of this paper is to explore optimal relational contracts in such a setting. For this purpose, we examine the repeated relationship between a manager and a worker in which the manager's opportunity costs of paying the worker are stochastic and privately observed by her. In the optimal relational contract, the manager promises a bonus if opportunity costs are low but none if they are high. Conflicts therefore arise whenever the manager does not pay the bonus. To manage these conflicts, the manager relies on a combination of informal promises and formal commitments that evolves with the duration of the conflict. As a result, effort and expected profits decline during a conflict only gradually and then recover instantaneously. The same pattern is repeated over time. The relationship between the manager and the worker therefore never terminates, nor does it reach a steady state. Instead, it cycles indefinitely.

The Lincoln Electric Company provides an example of the type of situation that motivates this paper. In the early 1990s, Lincoln Electric was a leading manufacturer of welding machines, well-known for its promise to share a significant fraction of profits with its factory workers. In 1992, Lincoln's U.S. business had generated a significant profit and as a result its U.S. workers expected to be paid their bonus. Mounting losses in its recently acquired foreign operations, however, more than wiped out U.S. profits. This presented CEO Donald Hastings with a dilemma: "*Our 3,000 U.S. workers would expect to receive, as a group, more than $50 million. If we were in default, we might not be able to pay them. But if we didn't pay the bonus, the whole company might unravel*" (Hastings 1999, p.164). To prevent the company from unraveling, Hastings decided to borrow $52.1 million and pay the bonus.

Why would Hastings have to take the seemingly inefficient step of borrowing money to pay the bonus? After all, the bonus was explicitly a "cash-sharing bonus" and U.S. workers had a long history of accepting fluctuations in the bonus in response to fluctuations in U.S. profits. The reason, it seems, was that U.S. workers were unable to observe foreign losses and therefore did not know whether U.S. profits really were needed to cover those losses. This explains why shortly after he paid the bonus, Hastings "[...] *instituted a financial education program so that employees would understand that no money was being hidden from them* [...]" (Hastings 1999, p.172).

The Lincoln Electric case illustrates the issues that arise if a manager is privately informed about the opportunity costs of paying her worker. In such a setting, if the manager does not pay a bonus, the worker cannot observe her motives. Is the manager not paying the bonus because it is more efficient to spend resources on something else, as she claims? Or is she just making up an excuse to extract some of the worker's rents? To keep the manager honest, the worker must then punish her whenever she does not pay a bonus. As a result, the manager faces a trade-off between the current benefits of adapting bonus payments to their opportunity costs and the future costs the worker inflicts on her if she does not pay a bonus. In short, the manager faces a trade-off between the benefits of adaptation and the costs of conflict.

To explore this trade-off, we examine a firm that consists of a risk-neutral owner-manager and a risk-neutral but liquidity-constrained worker. Output and effort are observable but not contractible. At the beginning of every period the manager offers the worker a contractible wage and a non-contractible bonus. After accepting the offer, the worker decides on his effort level. Effort is continuous and imposes a cost on the worker. Finally, output is realized and the manager decides how much to pay the worker. So far this is a relational contracting model with public information that is well understood (MacLeod and Malcomson 1989). The only change we make to this standard model is to assume that the manager's opportunity costs of paying the worker are stochastic and privately observed by her. In particular, just before the manager decides how much to pay the worker, she observes whether the firm has been hit by a shock—in which case opportunity costs are high—or not—in which case they are low.

In this setting, it is optimal for the manager to motivate the worker by promising to pay him a bonus. The bonus, however, is contingent on opportunity costs being low. To keep the manager honest, the worker punishes her when she does not pay a bonus and rewards her when she does. In particular, if the manager does not pay the bonus this period, expected profits will be lower next period, unless they are already at their lower bound, in which case they stay there. In contrast, expected profits immediately jump to their upper bound if the manager does pay the bonus. Expected profits therefore cycle indefinitely and the relationship never terminates. These cycles differ in length depending on the number of consecutive shock periods the firm experiences. They all, however, follow the same pattern in which downturns are gradual and recoveries instantaneous. In our setting, it is therefore not optimal for the parties to alternate between high profit "cooperation" phases and low profit "punishment" phases, as in Green and Porter (1984). Nor is it optimal for them to adopt termination contracts in which the failure to pay the bonus triggers the termination of their relationship with some probability, as in Levin (2003).

One reason for why downturns are gradual and recoveries instantaneous is that joint surplus is increasing in expected profits. Essentially, the larger is the manager's stake in the relationship, the less tempted she is to renege, and thus the more effort the worker is willing to provide. Rewarding the manager for paying a bonus is therefore not only good for her incentives to act truthfully but is also efficiency enhancing. As a result, it is optimal for expected profits to jump to their upper bound when the manager pays the bonus. In contrast, punishing the manager for not paying the bonus destroys joint surplus. Since the production function is concave, the most efficient way to punish the manager is then to reduce profits gradually.

To see how the optimal punishment is implemented, consider a period in which expected profits are not at their lower bound and suppose the manager does not pay the bonus. In the next period, the manager will then have to offer the worker a larger bonus. Moreover, the manager will either have to accept a reduction in effort or offer a higher wage. In particular, the optimal punishment calls for a reduction in effort if expected profits are high and an increase in the wage if they are low. The reason for this pattern is again the concavity of the production function, which makes it more efficient to punish the manager through a reduction in effort if expected profits are high and through an increase in wages if they are low.

The optimal punishment gives rise to conflicts that go through up to three distinct phases. Initially, the manager responds to a conflict by offering the worker a larger and larger bonus but she does not commit to a wage. In response, the worker provides less and less effort. Once expected profits are sufficiently low, the conflict enters its second phase during which the manager complements his promise to pay larger and larger bonuses with the formal commitment to also pay higher and higher wages. In response, the worker no longer reduces effort. Finally, expected profits reach their lower bound and the conflict enters its final phase. During this phase the bonus and the wage stay constant at their maximized levels and the worker continues to provide the same level of effort. The final phase of the conflict continues until a period in which the firm is not hit by a shock, at which point the conflict ends and expected profits return to their upper bound.

A key assumption in our model is that the firm is not liquidity-constrained. The manager can therefore always pay the worker any positive amount, even if the opportunity costs of doing so may be high. In our main extension we relax this assumption. We show that liquidity constraints limit the manager's ability to manage conflicts, which slows down recoveries and may lead to termination. They can also, however, induce the worker to respond to a conflict by providing more effort rather than less. Essentially, the worker understands that more effort relaxes the firm's liquidity constraint, which, in turn, allows the manager to pay him a larger bonus.

To illustrate the role of liquidity constraints in managing relational contracts, we return to the Lincoln Electric case. In early 1993, a few months after he had borrowed the necessary funds to pay his workers, CEO Hastings realized that European losses would once again wipe out U.S. profits. The covenants in the debt that he took on the previous year, however, prevented him from again borrowing the necessary funds to pay the bonus:

"[...] *we turned to our U.S. employees for help. I presented a 21-point plan to the board that called for our U.S. factories to boost production dramatically* [...]. *'We blew it,' I said* [to the U.S. employees]*. 'Now we need you to bail the company out. If we violate the covenants, banks won't lend us money. And if they don't lend us money, there will be no bonus in December'*" (Hastings 1999, pp.171-172).

According to Hastings, his "*statement appealed not only to* [the U.S. workers'] *loyalty but also to what James F. Lincoln called their 'intelligent selfishness'*" (Hastings 1999, p.172). And, apparently, it worked:

"*Thanks to the Herculean effort in the factories and in the field, we were able to increase revenues and profits enough in the United States to avoid violating our loan covenants*" (Hastings 1999, p.178).

As a result, Hastings was able to renew the covenants, which, in turn, allowed him to once again borrow the necessary funds and pay the bonus. In line with the reasoning that we sketched above, therefore, Lincoln Electric's U.S. workers increased their efforts so as to relax the firm's liquidity constraints, which, in turn, ensured that they were paid their bonus.

2 Related Literature

There is a large literature that examines relational contracts both between and within firms; see MacLeod (2007) and Malcomson (forthcoming) for recent reviews. Our paper contributes to the branch of this literature that studies the actions managers can take to sustain relational contracts better, such as the timing of payments (MacLeod and Malcomson 1998), the design of explicit contracts (Baker, Gibbons, and Murphy 1994, Che and Yoo 2001), the allocation of ownership rights (Baker, Gibbons, and Murphy 2002, Rayo 2007), the differential treatment of workers (Levin 2002), the grouping of tasks (Mukherjee and Vasconcelos 2011), and others. In contrast to these papers, our focus is on how to manage conflicts once they arise, rather than on how to prevent them in the first place.

Our paper contributes to the branch of the literature on relational contracts in which the manager is privately informed about aspects of the relationship. In this context, three closely related papers are Levin (2003), Fuchs (2007), and Englmaier and Segal (2011). In the second part of Levin (2003), the manager privately observes the worker's performance. In contrast to our setting, the worker's effort is privately observed, which makes joint punishments necessary. Levin shows that the optimal contract is stationary and can be implemented through a termination contract. In the same setting, Fuchs (2007) allows for private monitoring and shows that while the optimal contract can still be implemented through a termination contract, it is no longer stationary. In both models efficient transfers are always available. In contrast, in our setting efficient transfers are not always available and as a result termination contracts are not optimal.

Englmaier and Segal (2011) correspond to a version of our model in which effort is binary and the manager's opportunity costs in a shock period are infinite. They focus on a particular relational contract in which the parties alternate between phases with high and low effort. In contrast, we characterize the optimal relational contract and show that under such a contract conflicts evolve more gradually.

Another closely related paper, albeit in a different context, is Yared (2010). He characterizes the optimal sequential equilibria in a game between an aggressive country that demands transfers from a peaceful one, where the peaceful country is privately informed about the costs of paying the transfers. In this setting, the aggressive country can punish the peaceful one through the binary choice of going to war or not. In contrast, in our setting the worker can punish the manager by providing less effort or demanding higher wages, both of which are continuous choices. As a result, optimal temporary conflicts are more gradual in our setting than optimal temporary wars in Yared's paper. Moreover, while in his setting the parties eventually engage in permanent war, termination is never optimal in ours.

Our paper also contributes to the recent and growing literature on dynamics within relational contracts. Chassang (2010) studies a model of exploration with private information and shows that the relationship is path-dependent and can settle in different long-run equilibria. Fong and Li (2010) study a moral hazard problem in which the worker has limited liability and explore patterns of the worker's job security, pay level, and the sensitivity of pay to performance. Padro i Miquel and Yared (2011) examine a political economy model and study the likelihood, duration, and intensity of war. Thomas and Worrall (2010) examine a partnership game with perfect information and two-sided limited liability. They show that the relationship becomes more efficient over time as the division of future rents becomes more equal. Dynamics also arise in models of relational contracts in which agents have private and fixed types; see, for example, Halac (forthcoming), Watson (1999, 2002), and Yang (2011). In these papers, dynamics arise when the principal updates her beliefs about the agent's type.

In terms of its analytical structure, our model is related to the literature on dynamic games with hidden information; see, for example, Abdulkadiroglu and Bagwell (2010), Athey and Bagwell (2001, 2008), Athey, Bagwell and Sanchirico (2004), Hauser and Hopenhayn (2008), Mobius (2001), and for surveys, Mailath and Samuelson (2006) and Samuelson (2006). A distinct feature of our model is the occasional availability of efficient transfers. As a result, the dynamics in our model feature gradual downturns or instantaneous recoveries.

Our model is also related to the literature on dynamic contracting between banks and privately informed entrepreneurs (DeMarzo and Fishman 2007, DeMarzo and Sannikov 2006, Biais et al. 2007, and Clementi and Hopenhayn 2006). In contrast to this literature, we focus on a setting in which long-term contracts are not feasible. The unavailability of long-term contracts is crucial for our results. In particular, we show in Section 7 that if the long-term contracts were feasible, the parties could approximate first-best.

Finally, since the efficiency of transfers depends on the state of the world, our model is related to the large literature on risk sharing. Kocherlakota (1996) and Ligon, Thomas, and Worrall (2002) explore efficient risk sharing between risk-averse agents when information is public and commitment is limited. Hertel (2004) examines the case with two-sided asymmetric information without commitment. Thomas and Worrall (1990) study a one-sided asymmetric information problem with commitment. This literature typically assumes that the player's endowments are exogenously given and path-independent. In our model, instead, output depends on the worker's effort and thus on how it was divided in the past.

3 The Model

A firm consists of a risk-neutral owner-manager and a risk-neutral but liquidity-constrained worker. The manager and the worker are in an infinitely repeated relationship. Time is discrete and denoted by $t = \{1, 2, ..., \infty\}$.

At the beginning of any period t the manager makes the worker an offer. The offer consists of a contractible commitment to pay wage $w_t \geq 0$ and a non-contractible promise to pay bonuses $b_{s,t}$ and $b_{n,t}$, where s and n stand for "shock" and "no-shock." The worker either accepts the offer or rejects it. We denote the worker's decision by d_t, where $d_t = 0$ if he rejects the offer and $d_t = 1$ if he accepts it. If the worker rejects the offer, the manager and the worker realize their per-period outside options $\underline{\pi} > 0$ and $\underline{u} > 0$ and time moves on to period $t + 1$.

If, instead, the worker accepts the manager's offer, he next decides on his effort level $e_t \geq 0$. Effort is costly to the worker and we denote his effort costs by $c(e_t)$. The cost function is strictly increasing and strictly convex with $c(0) = c'(0) = 0$ and $\lim_{e\to\infty} c'(e) = \infty$. After the worker provides effort e_t, the manager realizes output $y(e_t)$. The output function is strictly increasing and strictly concave with $y(0) = 0$. Effort e_t, effort costs $c(e_t)$, and output $y(e_t)$ are observable but not contractible.

After the manager realizes output $y(e_t)$, she privately observes the state of the world $\Theta_t \in \{s, n\}$, where, as mentioned above, s and n stand for "shock" and "no-shock." The states are drawn independently across time from a binary distribution. The probability with which a shock state occurs is given by $\theta \in (0, 1)$. The state of the world determines the opportunity cost of paying the worker: if the firm is not hit by a shock, paying the worker an amount of $w + b$ costs the manager $w + b$; if, instead, the firm is hit by a shock, paying the worker $w + b$ costs the manager $(1 + \alpha)(w + b)$, where $\alpha \in (0, \infty)$. We do not model explicitly why opportunity costs may be high. As discussed above, however, managers do sometimes face high opportunity costs of paying their workers. This may be the case, for instance, because they need to borrow money to make their payments, as in the Lincoln Electric case.

After the manager observes the state of the world, she pays the worker the wage w_t and a bonus $b_t \geq 0$. Since the promised bonus is not contractible, the payment b_t can be different from the promises $b_{n,t}$ and $b_{s,t}$.

Finally, at the end of period t, the manager and the worker observe the realization x_t of a public randomization device. This allows the players to publicly randomize at the beginning of any period $t + 1 \geq 2$ based on the realization of the public randomization device observed at the end of the previous period. To allow the players to publicly randomize in period 1, we assume that they can also observe a realization of the randomization device at the beginning of that period. We denote this realization by x_0. The existence of a public randomization device is a common assumption in the literature and is made to convexify the set of equilibrium payoffs. The timing is summarized in Figure 1.

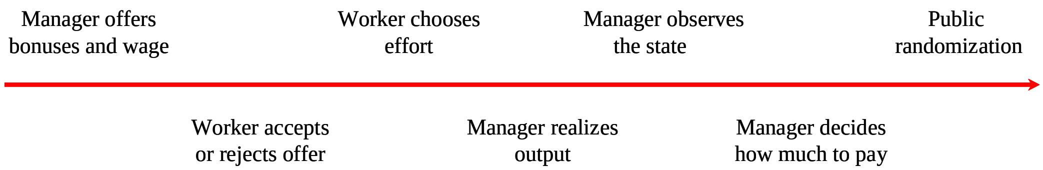


Figure 1: Timeline

The manager and the worker share the same discount factor $\delta \in (0,1)$. At the beginning of any period t, their respective expected payoffs are therefore given by

$$\pi_t = (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathrm{E}\left[d_\tau\left[y\left(e_\tau\right)-\left(1+1_{\{\Theta_\tau=s\}}\alpha\right)\left(w_\tau+b_\tau\right)\right]+(1-d_\tau)\underline{\pi}\right]$$

and

$$u_t = (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathrm{E}\left[d_\tau\left(w_\tau+b_\tau-c\left(e_\tau\right)\right)+(1-d_\tau)\underline{u}\right].$$

Note that we multiply the right-hand side of each expression by $(1-\delta)$ to express profits and payoffs as per-period averages.

We follow the literature on imperfect public monitoring and define a "relational contract" as a pure-strategy Perfect Public Equilibrium (henceforth PPE) in which the manager and the worker play public strategies and, following every history, the strategies are a Nash Equilibrium. Public strategies are strategies in which the players condition their actions only on publicly available information. In particular, the manager's strategy does not depend on her past private information. Our restriction to pure strategy is without loss of generality because our game has only one-sided private information and is therefore a game with the product structure (see, for instance, p.310 in Mailath and Samuelson (2006)). In this case, there is no need to consider private strategies since every sequential equilibrium outcome is also a PPE outcome (see, for instance, p.330 in Mailath and Samuelson (2006)).

Formally, let $h_{t+1} = \{w_\tau, b_{n,\tau}, b_{s,\tau}, d_\tau, e_\tau, b_\tau, x_\tau\}_{\tau=1}^{t}$ denote the public history at the beginning of any period $t+1$ and let H_{t+1}denote the set of period $t+1$ public histories. Note that $H_1 = \Phi$. A public strategy for the manager is a sequence of functions $\{W_t, B_{s,t}, B_{n,t}, B_t\}_{t=1}^{\infty}$, where $W_t : H_t \to [0,\infty)$, $B_{s,t} : H_t \to [0,\infty)$, $B_{n,t} : H_t \to [0,\infty)$, and $B_t : H_t \cup \{w_t, b_{s,t}, b_{n,t}, d_t, e_t, \theta_t\} \to [0,\infty)$. Similarly, a public strategy for the worker is a sequence of functions $\{D_t, E_t\}_{t=1}^{\infty}$, where $D_t : H_t \cup \{w_t, b_{s,t}, b_{n,t}\} \to \{0,1\}$ and $E_t : H_t \cup \{w_t, b_{s,t}, b_{n,t}, d_t\} \to [0,\infty)$.

We define an "optimal relational contract" as a PPE with payoffs that are not Pareto-dominated by any other PPE. Note that when the discount factor is sufficiently small, the only relational contract is a trivial one in which the parties always take their outside options $\underline{\pi}$ and $\underline{u}$. To make the analysis interesting, we assume that the parties are sufficiently patient so that a non-trivial relational contract exists. A sufficient condition for a non-trivial relational contract to exist is

$$y\left(e^{FB}\right) - \underline{\pi} - (1+\alpha\theta)\left(c\left(e^{FB}\right) + \underline{u}\right) \geq \frac{1-\delta}{\delta}(1+\alpha\theta)\, c\left(e^{FB}\right),$$

where e^{FB} is the first-best effort that maximizes $y(e) - c(e)$.

4 Preliminaries

In this section we first list the constraints that have to be satisfied for a payoff pair to be in the PPE payoff set. We then consider a relaxed problem that ignores one of these constraints. For this relaxed problem we characterize properties of the PPE payoff set in Section 4.2 and properties of the optimal relational contract in Section 4.3. These properties of the relaxed problem will then allow us to characterize the optimal relational contract in Section 5.

4.1 The Constraints

We denote the set of PPE payoffs by E. Each payoff pair $(\pi, u) \in E$ is associated with a profile of actions (e, w, b_s, b_n) and continuation payoffs (π_s, π_n, u_s, u_n), where π_s and π_n are the manager's continuation payoffs associated with shock and no-shock states and u_s and u_n are defined analogously. We say that a PPE payoff pair (π, u) can be "supported" by pure actions if there exists a profile of actions (e, w, b_s, b_n) and continuation payoffs (π_s, π_n, u_s, u_n) that satisfy the following three sets of constraints.

Feasibility: For the actions to be feasible, the wage, bonuses, and effort have to be non-negative. Specifically, the non-negativity constraints are given by

$$w \geq 0, \tag{NN$_\mathrm{W}$}$$

$$b_n \geq 0, \tag{NN$_\mathrm{N}$}$$

$$b_s \geq 0, \tag{NN$_\mathrm{S}$}$$

and

$$e \geq 0. \tag{NN$_\mathrm{e}$}$$

Moreover, for the continuation payoffs to be feasible, they also need to be PPE payoffs. The self-enforcing constraints are therefore given by

$$(\pi_n, u_n) \in E \tag{SE$_\mathrm{N}$}$$

in a no-shock period and

$$(\pi_s, u_s) \in E \tag{SE$_{\text{S}}$}$$

in a shock period.

No Deviation: To ensure that neither party deviates, we need to consider both off- and on-schedule deviations. Off-schedule deviations are deviations that can be publicly observed. There is no loss of generality in assuming that if an off-schedule deviation occurs, the parties terminate their relationship, as this is the worst possible equilibrium that gives each party its minmax payoff.

The manager deviates off-schedule when he pays a bonus that is different from either b_s or b_n. If the manager does deviate off-schedule, she cannot do better than to pay a zero bonus. The non-reneging constraints are therefore given by

$$\delta\pi_n - \delta\underline{\pi} \geq (1-\delta)\, b_n \tag{NR$_{\text{N}}$}$$

in a no-shock period and

$$\delta\pi_s - \delta\underline{\pi} \geq (1-\delta)\,(1+\alpha)\, b_s \tag{NR$_{\text{S}}$}$$

in a shock period.

The worker deviates off-schedule when he rejects the manager's offer or provides an effort level that is different from e. The individual rationality constraint

$$u \geq \underline{u} \tag{IR$_{\text{W}}$}$$

ensures that the worker does not reject the manager's offer. If the worker accepts the offer but deviates by providing an effort level that is different from e, he cannot do better than to provide no effort. The worker's incentive constraint is therefore given by

$$u \geq (1-\delta)\, w + \delta\underline{u}. \tag{IC$_{\text{W}}$}$$

In contrast to off-schedule deviations, on-schedule deviations are only privately observed. Since the worker does not have any private information, he cannot engage in any on-schedule deviations. The manager, however, can do so by either paying b_n in a shock period or b_s in a no-shock period. The manager's truth-telling constraints are therefore given by

$$\delta\,(\pi_n - \pi_s) \geq (1-\delta)\,(b_n - b_s) \tag{TT$_{\text{N}}$}$$

in a no-shock period and

$$\delta\,(\pi_n - \pi_s) \leq (1+\alpha)\,(1-\delta)\,(b_n - b_s) \tag{TT$_{\text{S}}$}$$

in a shock period.

Promise Keeping: Finally, the consistency of the PPE payoff decomposition requires that the parties' payoffs are equal to the weighted sum of current and future payoffs. The promise-keeping constraints are therefore given by

$$\pi = \theta\left\{(1-\delta)\left[y\left(e\right)-(1+\alpha)\left(w+b_s\right)\right]+\delta\pi_s\right\}+(1-\theta)\left\{(1-\delta)\left[y\left(e\right)-w-b_n\right]+\delta\pi_n\right\} \quad (\mathrm{PK_M})$$

for the manager and

$$u = \theta\left[(1-\delta)\left(w+b_s\right)+\delta u_s\right]+(1-\theta)\left[(1-\delta)\left(w+b_n\right)+\delta u_n\right]-(1-\delta)\,c\left(e\right) \qquad (\mathrm{PK_W})$$

for the worker.

4.2 Properties of the PPE Payoff Set

Before characterizing the optimal relational contract that satisfies all the constraints in the previous section, we first turn to a relaxed problem that ignores the worker's incentive constraint $\mathrm{IC_W}$. To streamline the exposition, however, we keep the same notation as above.

Specifically, we now use the technique developed by Abreu, Pearce, and Stacchetti (1990) to characterize the PPE payoff set E of the relaxed problem. For this purpose, we define the payoff frontier as

$$u(\pi) \equiv \sup\{u' : (\pi, u') \in E\}.$$

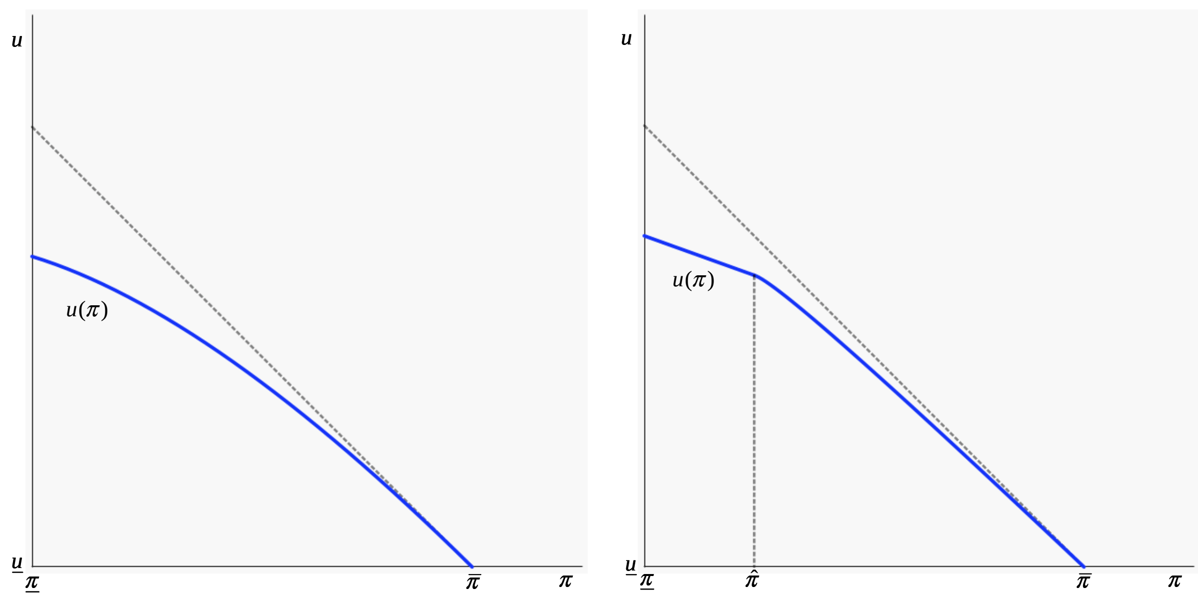
Our first lemma establishes several properties of the PPE payoff set E.

LEMMA 1. *The PPE payoff set E has the following properties: (i.) it is compact, (ii.) payoff pair (π, u) belongs to E if and only if $\pi \in [\underline{\pi}, \overline{\pi}]$ and $u \in [\underline{u}, u(\pi)]$, and (iii.) the extremal point $u(\overline{\pi})$ satisfies $u(\overline{\pi}) = \underline{u}$.*

A key implication of this lemma is that the PPE payoff set E is fully characterized by its frontier $u(\pi)$. To see why, notice that for any $u(\pi)$, the points below it can be sustained by randomizing among $(\underline{\pi}, \underline{u})$, $(\overline{\pi}, \underline{u})$, and $(\pi, u(\pi))$. Since the PPE frontier fully characterizes the PPE payoff set, we now turn to the next lemma, which establishes several of its properties.

LEMMA 2. *The PPE frontier $u(\pi)$ has the following properties: for all $\pi \in [\underline{\pi}, \overline{\pi}]$, (i.) payoffs $(\pi, u(\pi))$ can be supported by pure actions in the stage game other than taking the outside options, (ii.) u is concave, and (iii.) u is differentiable with $-1 < u'(\pi) \leq -1/(1+\alpha\theta)$.*

The first part of the lemma shows that payoffs on the PPE frontier can be supported by pure actions, that is, they do not require any public randomization. This fact will allow us to represent the PPE frontier recursively below. This property follows from the concavity of the output function $y(e)$. Essentially, any PPE that is supported by randomization between two effort levels can be improved upon by having the worker provide an appropriately chosen intermediate level of effort. The concavity of the PPE frontier follows immediately from the availability of a public randomization device and its differentiability follows from the continuity of effort. Finally, the bounds on the derivative imply that the PPE frontier is downward sloping and that joint surplus $\pi + u(\pi)$ is increasing in expected profits π and thus maximized at $\overline{\pi}$. Figures 2a and b illustrate these properties.



Figures 2a and 2b: The PPE Frontier

In both figures the PPE frontier is downward sloping and joint surplus is maximized at $\overline{\pi}$. The difference between the two figures is that in Figure 2a the derivative of the PPE frontier never reaches its upper bound $u'(\pi) = -1/(1+\alpha\theta)$. In Figure 2b, instead, the derivative does reach this upper bound at a critical level of expected profits $\widehat{\pi}$ and as a result the PPE frontier is linear for all $\pi \leq \widehat{\pi}$. This critical level of expected profits is defined by

$$\widehat{\pi} = (1-\delta)y(\widehat{e}) + \delta\underline{\pi}, \tag{1}$$

where $\widehat{e}$ is the unique effort level that solves

$$\frac{c'(\widehat{e})}{y'(\widehat{e})} = \frac{1}{1+\alpha\theta}. \tag{2}$$

Notice that $\widehat{\pi}$ can be smaller than $\underline{\pi}$, as in Figure 2a, or it can be larger, as in Figure 2b. We will see in the next section that $\widehat{\pi}$ is the threshold level of expected profits below which the manager pays a positive wage and above which the wage is zero.

4.3 Properties of Optimal Relational Contracts

We now turn to properties of the optimal relational contract. We continue to focus on the relaxed problem that ignores the $\mathrm{IC_W}$ constraint. The next lemma shows that any optimal relational contract is sequentially optimal.

LEMMA 3. *Any optimal relational contract is sequentially optimal, that is, for any PPE payoff pair* $(\pi, u(\pi))$*, the associated continuation payoffs satisfy* $u_s = u(\pi_s)$ *and* $u_n = u(\pi_n)$.

Essentially, since the worker's actions are publicly observable, it is not necessary to punish him by moving below the PPE frontier. This feature of our model is similar, for instance, to Spear and Srivastava (1987) and the first part of Levin (2003). In contrast, joint punishments are necessary in models with two-sided private information, such as Green and Porter (1984), Athey and Bagwell (2001), and the second part of Levin (2003).

Recall from Lemma 2 that payoffs on the frontier are sustained by pure actions. Lemma 3 then implies that the optimal relational contract does not involve any public randomization. In what follows we therefore refer to an optimal relational contract as one in which no public randomization device is used.

Next we can simplify and combine some of the constraints in Section 4.1. In particular, the next lemma will allow us to eliminate the shock and no-shock bonuses from the above constraints.

LEMMA 4. *For any optimal relational contract with payoffs* $(\pi, u(\pi))$*, there exists a set of actions and continuation payoffs supporting it such that (i.) the bonuses satisfy* $b_n(\pi) \geq b_s(\pi) = 0$ *and (ii.) the manager's truth-telling constraint in a no-shock period* $\mathrm{TT_N}$ *is binding.*

Since it is less expensive for the manager to pay the worker in a no-shock period than in a shock period, it is natural that $b_n(\pi) \geq b_s(\pi)$. As a result, one can always replicate a PPE in which $b_s(\pi) > 0$ with an economically equivalent one in which the shock bonus is zero. To do so, one simply has to add $b_s(\pi)$ to the wage and subtract it from the no-shock bonus. Finally, the fact that $\mathrm{TT_N}$ is binding follows from the concavity of the PPE frontier $u(\pi)$. Essentially, if $\mathrm{TT_N}$ were not binding, one could always reduce π_n and increase π_s in such a way that π remains the same and all the relevant constraints continue to be satisfied. Since the PPE frontier is concave, however, such a change would make the worker better off.

The lemma allows us to eliminate $b_s(\pi)$ from the constraints in Section 4.1 by setting it equal to zero. Furthermore, we can now eliminate a number of these constraints. For instance, it is immediate that if TT_N is binding, TT_S is satisfied. The next lemma shows that we can represent the PPE frontier by focusing on only four of the above constraints.

LEMMA 5. *The PPE frontier $u(\pi)$ is recursively defined by the following problem. For all $\pi \in [\underline{\pi}, \overline{\pi}]$,*

$$\pi + u(\pi) = \max_{e,w,\pi_s,\pi_n} (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) - (1-\delta)\theta\alpha w$$

such that

$$w \geq 0, \tag{NN_W}$$

$$\underline{\pi} \leq \pi_n \leq \overline{\pi}, \tag{SE_N}$$

$$\underline{\pi} \leq \pi_s \leq \overline{\pi}, \tag{SE_S}$$

and

$$\pi = (1-\delta)y(e) + \delta\pi_s - (1-\delta)(1+\theta\alpha)w. \tag{PK_M}$$

Notice that there might be multiple functions $u(\pi)$ that solve this problem. This type of multiplicity is common in games of relational contracts (see, for instance, Baker, Gibbons, and Murphy (1994)). When multiple solutions exist, it is immediate that the PPE frontier is given by the largest one.

In the next section we characterize the solution to this problem and show that it satisfies the IC_W constraint that we ignored so far. The solution to the problem in the lemma therefore characterizes the optimal relational contracts for the full problem.

5 The Optimal Relational Contract

In this section we characterize the optimal relational contract. We saw in the previous section that there is no loss of generality in setting the shock bonus equal to zero. We therefore now define an optimal relational contract to be a PPE that is not Pareto-dominated by any other PPE and in which $b_s = 0$. Our first proposition characterizes the optimal relational contract and shows that it is unique. In the proof of the proposition we solve the relaxed problem in Lemma 5 and then show that it satisfies the IC_W constraint that we ignored so far.

PROPOSITION 1. *For any level of expected profits π, there exists a unique optimal relational contract that gives the worker $u(\pi)$. Under the optimal relational contract:*

(*i.*) *The no-shock bonus* $b_n^*(\pi)$ *is given by*

$$b_n^*(\pi) = \frac{\delta}{1-\delta}\left(\overline{\pi} - \pi_s^*(\pi)\right) > 0 \textit{ for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

(*ii.*) *If the firm is hit by a shock, the continuation profit* $\pi_s^*(\pi)$ *satisfies*

$$\pi_s^*(\underline{\pi}) = \underline{\pi} \textit{ and } \pi_s^*(\pi) < \pi \textit{ for all } \pi \in (\underline{\pi}, \overline{\pi}].$$

(*iii.*) *If the firm is not hit by a shock, the continuation profit is given by*

$$\pi_n^*(\pi) = \overline{\pi} \quad \textit{for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

(*iv.*) *Effort* $e^*(\pi)$ *is given by the unique effort level* e *that solves*

$$\frac{c'(e)}{y'(e)} = -u'(\pi) \textit{ for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

(*v.*) *The wage is given by*

$$w^*(\pi) = \max\left[\frac{\widehat{\pi} - \pi}{(1-\delta)(1+\alpha\theta)},\ 0\right] \textit{ for all } \pi \in [\underline{\pi}, \overline{\pi}].$$

The optimal relational contract reflects a balancing of incentives. In particular, the manager must give the worker an incentive to provide effort, while the worker must give the manager an incentive to make any promised payments. Part (i.) shows that the manager motivates the worker to provide effort by promising him a bonus. The promise, however, is contingent on opportunity costs being low. The worker must therefore motivate the manager to act truthfully.

To motivate the manager to act truthfully, the worker punishes her if she does not pay the bonus, and rewards her if she does. In particular, part (ii.) shows that if the manager does not pay the bonus, expected profits will be strictly lower next period, unless they are already at their lower bound $\underline{\pi}$, in which case they stay there. In contrast, part (iii.) shows that if the manager does pay the bonus, expected profits immediately jump to their upper bound $\overline{\pi}$ in the next period. Expected profits therefore cycle indefinitely and the relationship never terminates. These cycles differ in length depending on the number of consecutive shock periods the firm experiences. They all, however, follow the same pattern in which downturns are gradual and recoveries instantaneous. These dynamics are illustrated in Figure 3 which plots expected profits for an arbitrary sequence of shock periods—indicated by red squares—and no-shock periods—indicated by blue dots.

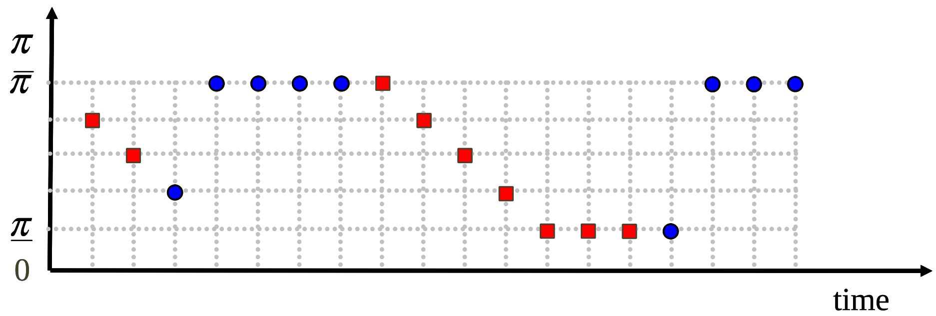


Figure 3: Evolution of Expected Profits

To understand why downturns are gradual and recoveries are instantaneous, recall that joint surplus $\pi + u(\pi)$ is increasing in expected profits. Rewarding the manager for paying the bonus is therefore both good for her incentives to act truthfully and efficiency enhancing. As a result, it is optimal for expected profits to jump to their upper bound if the manager pays the bonus. In contrast, punishing the manager for not paying the bonus involves a destruction of joint surplus. In principle, the worker could punish the manager for not paying the bonus by insisting that expected profits plummet to $\underline{\pi}$ in the next period. One way to do so would be to terminate the relationship. As discussed after Lemma 3, however, termination is not optimal in our setting since the public observability of the worker's actions makes joint punishments unnecessary. Another way to force expected profits to $\underline{\pi}$ would be to reduce effort significantly for a number of periods without, however, terminating the relationship. The dynamics would then be reminiscent of those in Green and Porter (1984) in which the parties alternate between cooperation and non-cooperation phases. Because of the concavity of the production function, however, such a punishment is not optimal in our setting. Instead, the optimal punishment involves changes to the bonus, effort, and wages that lead to a gradual reduction in expected profits.

Specifically, part (i.) shows that the optimal punishment involves an increase in the bonus that the manager has to pay the worker in the next period. Since the bonus only has to be paid if opportunity costs turn out to be low, this part of the punishment is efficient. But since the opportunity costs are only observed by the manager, the optimal punishment must also involve changes in the wage, effort, or both.

The last two parts of the proposition show that if expected profits are high, the optimal punishment involves a reduction in effort but no change in the wage; and if expected profits are low, the optimal punishment involves an increase in the wage but no change in effort. To see this formally, recall from Section 4.2 that $\widehat{\pi}$ is the critical level of expected profits below which the derivative of the PPE frontier is at its upper bound. Part (iv.) then implies that effort is the same for all $\pi \leq \widehat{\pi}$ and increasing in expected profits for all $\pi > \widehat{\pi}$. At the same time, part (v.) implies that wages are decreasing in expected profits for all $\pi \leq \widehat{\pi}$ and zero for all $\pi > \widehat{\pi}$. Essentially, because of the concavity of the production function, there is a threshold level of profits above which it is more efficient to punish the manager through a reduction in effort and below which it is more efficient to do so through an increase in the wage.

To see this more clearly, consider two ways in which expected profits can be reduced by some $\varepsilon > 0$. One way to do so is to reduce effort by $\varepsilon / y'(e)$ and the other is to increase wages by $\varepsilon / (1 + \alpha\theta)$. The reduction in effort increases the worker's expected payoff by $\varepsilon c'(e) / y'(e)$, while the increase in wages increases it by $\varepsilon / (1 + \alpha\theta)$. Recall now from Section 4.2 that the effort level $\widehat{e}$ that is associated with expected profits $\widehat{\pi}$ is defined by $c'(\widehat{e}) / y'(\widehat{e}) = 1 / (1 + \alpha\theta)$. For expected profits above $\widehat{\pi}$ it is therefore more efficient to punish the manager through a reduction in effort, and for expected profits below $\widehat{\pi}$ it is more efficient to do so through an increase in the wage.

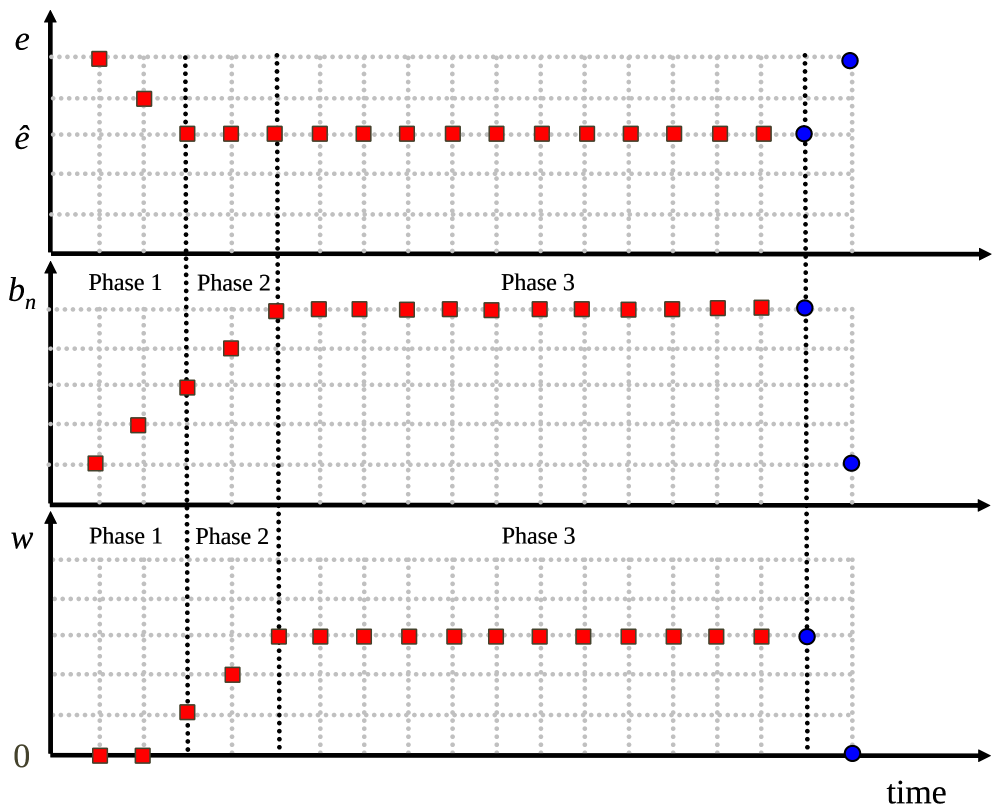


Figure 4: Evolution of Effort, Bonuses, and Wages

To illustrate the evolution of bonuses, effort, and wages over time, suppose that expected profits are at their upper bound $\overline{\pi}$ and that the firm is then hit by shocks in a large number of consecutive periods. The resulting conflict goes through three distinct phases, which are illustrated in Figure 4. As in Figure 3, red squares indicate shock periods and blue dots indicate no-shock periods.

In the initial phase of the conflict, the bonus is increasing and effort is decreasing. Wages, however, stay constant at zero. Once expected profits fall below $\widehat{\pi}$, the conflict enters its second phase. During this phase the bonus is still increasing. Wages, however, are now also increasing, while effort is constant at $\widehat{e} > 0$. Eventually, expected profits hit their lower bound, at which point the conflict enters its final phase. During this phase, the bonus and the wage stay constant at their maximized levels and effort stays constant at $\widehat{e}$. This final phase of the conflict continues until the parties reach a period in which the firm is not hit by a shock. In that period the manager finally pays the promised bonus and expected profits return to their upper bound $\overline{\pi}$.

The prediction that the firm sometimes does not pay any wages is at odds with the fact that most workers have a contracted wage which has to be paid except in the case of bankruptcy. One way to account for wages would be to allow for the worker to be risk averse, in which case the manager would find it optimal to pay him some amount every period. Alternatively one could assume that the firm has to pay the worker a minimum wage every period. Such a minimum wage would not affect the key features of the optimal relational contract and the dynamics that we described above. Another prediction that is worth discussing is that the manager makes the largest payments to the worker at the end of a severe conflict. This prediction depends crucially on the assumption that the firm is not liquidity-constrained. We will see in the next section that if the firm is liquidity-constrained, the manager may be forced to spread the payment over multiple periods, making the changes in the predicted payments less stark.

Finally, changes in the underlying parameters, such as the size and probability of the shock α and θ, the outside options $\underline{u}$ and $\underline{\pi}$, and the discount factor δ, do not affect the key features of the optimal relational contract and the dynamics that we described above. As one would expect, however, such changes do affect the set of payoffs that can be sustained as PPEs. In particular, it can be shown that the PPE frontier $u(\pi)$ and the maximum sustainable profits $\overline{\pi}$ are decreasing in α, θ, $\underline{u}$, and $\underline{\pi}$ and increasing in δ. Other things equal, the manager and the worker are therefore better off the smaller and less frequent the shocks are, the lower the outside options are, and the more patient the parties are.

6 The Effects of Liquidity Constraints

The Lincoln Electric case we discussed in the Introduction suggests that liquidity constraints can have significant effects on managers' ability to manage conflicts. In this section we explore this issue by allowing for the firm to be liquidity-constrained.

Specifically, we now assume that if the firm realizes output $y(e)$ and is not hit by a shock, the manager can pay the worker at most $(1+m)\,y(e)$, where the parameter $m \geq 0$ captures the extent to which the firm is liquidity-constrained. Liquidity constraints make the PPE frontier non-differentiable and thus substantially complicate the characterization of the optimal relational contract. To make the analysis more tractable, we now assume that the size of the shock α is equal to infinity. An immediate implication of this assumption is that wages and the shock bonus are always equal to zero. The liquidity constraint is therefore given by

$$b_n \leq (1+m)\,y(e)\,. \tag{LC}$$

Finally, we assume that $(1+m)\,\underline{\pi}/\delta$ is strictly smaller than the maximal expected profits if the firm is not liquidity-constrained. If this condition did not hold, the liquidity constraint LC would never bind. We can now establish the following lemma.

LEMMA 6. *There exist critical levels of expected profits $\pi_0 \leq \pi_1 \leq \pi_2 < \overline{\pi}$ such that:*

(i.) if $\pi \in [\underline{\pi}, \pi_0]$, then $u(\pi)$ is supported by randomization between $(\underline{\pi}, \underline{u})$ and $(\pi_0, u(\pi_0))$,

(ii.) if $\pi \in [\pi_1, \overline{\pi}]$, then the self-enforcing constraint $\pi_n \leq \overline{\pi}$ is binding, and

(iii.) if $\pi \in [\pi_0, \pi_2)$, then the liquidity constraint $b_n \leq (1+m)\,y(e)$ is binding.

Part (i.) shows that, in contrast to the model without liquidity constraints, termination can now be part of the optimal relational contract. Specifically, for any $\pi \in [\underline{\pi}, \pi_0]$, the manager and the worker publicly randomize between terminating their relationship and playing the strategies that deliver expected payoffs π_0 and $u(\pi_0)$. One way to implement termination is for the firm to always offer a zero wage and no bonus and for the worker to always turn down this offer; moreover, in case either party deviates from these actions, the worker always provides zero effort and the firm never pays a bonus. The relationship therefore terminates eventually as long as $\pi_0 > \underline{\pi}$. A pair of sufficient conditions for $\pi_0 > \underline{\pi}$ is given by $m < \theta/(1-\theta)$ and

$$\underline{u} > \frac{(1-\theta)(1+m)}{\theta-(1-\theta)\,m}\left(1-\frac{c'(\underline{e})}{y'(\underline{e})}\right)\underline{\pi} - c(\underline{e})\,,$$

where $\underline{e}$ is the effort level for which $y(\underline{e}) = \underline{\pi}$. In the appendix we show that these conditions ensure that the joint surplus at $\underline{\pi}$ is larger under termination than if the manager and the worker were to continue their relationship. As a result, termination must occur. Notice that these conditions are more likely to be satisfied the smaller m, suggesting that termination is more likely to occur, the more liquidity-constrained the firm is.

Parts (ii.) and (iii.) show that to the right of π_0 there are three regions. If expected profits are sufficiently high, the liquidity constraint is not binding. The manager is then able to pay a sufficiently large bonus for expected profits to return to $\overline{\pi}$ after a no-shock period. If, in contrast, expected profits are sufficiently low, the self-enforcing constraint is not binding, which implies that expected profits do not return to $\overline{\pi}$ after a no-shock period. This occurs because effort and output are too low for the manager to be able to pay a sufficiently large bonus. Finally, for intermediate values of expected profits, both the liquidity constraint and the self-enforcing constraint are binding. We will see below that in this region, a reduction in expected profits is associated with an increase in effort. A pair of sufficient conditions for the intermediate region to exist is given by

$$(1+m)^2 (1-\theta)\,\theta < m < \theta/(1-\theta)\,.$$

In the appendix we derive these conditions by contradiction. Specifically, suppose that the intermediate region does not exist. We can then show that the payoff frontier has a kink at the boundary between the left and the right region. We can also show, however, that under the above conditions the payoff frontier must be differentiable. Since this is a contradiction, the middle region must exist.

We can now state our next proposition, which characterizes the optimal relational contract when the firm is liquidity-constrained.

PROPOSITION 2. *For any level of expected profits π, there exists a unique optimal relational contract that gives the worker $u(\pi)$. Under the optimal relational contract:*

(*i.*) *The no-shock bonus $b_n^*(\pi)$ is given by*

$$b_n^*(\pi) = \begin{cases} (1+m)\,y\,(e^*(\pi)) > 0 & \textit{for all } \pi \in [\pi_0, \pi_2) \textit{ and} \\ \frac{\delta}{1-\delta}\left(\overline{\pi} - \pi_s^*(\pi)\right) > 0 & \textit{for all } \pi \in [\pi_2, \overline{\pi}]. \end{cases}$$

(*ii.*) *If the firm is hit by a shock, the continuation profit $\pi_s^*(\pi)$ satisfies*

$$\pi_s^*(\pi_0) = \underline{\pi} \textit{ and } \pi_s^*(\pi) < \pi \textit{ for all } \pi \in (\pi_0, \overline{\pi}]\,.$$

(*iii.*) *If the firm is not hit by a shock, the continuation profit $\pi_n^*(\pi)$ is given by*

$$\pi_n^*(\pi) = \begin{cases} \left[\pi + (1-\delta)\,my\,(e^*(\pi))\right]/\delta & \textit{for all } \pi \in [\pi_0, \pi_1) \textit{ and} \\ \overline{\pi} & \textit{for all } \pi \in [\pi_1, \overline{\pi}]. \end{cases}$$

(*iv.*) *Effort is given by the unique effort level* $e^*(\pi)$ *that solves*

$$\frac{c'(e)}{y'(e)} = \begin{cases} -u'_+(\pi) + (1+m)(1-\theta)\left(1 + u'_+(\pi_n^*(\pi))\right) & \textit{for all } \pi \in [\pi_0, \pi_1), \\ -u'(\pi) & \textit{for all } \pi \in [\pi_2, \overline{\pi}], \end{cases}$$

and

$$y(e) = (\delta\overline{\pi} - \pi) / ((1-\delta)m) \ \textit{for all } \pi \in [\pi_1, \pi_2),$$

where $u'_+(\cdot)$ *denotes the right derivative.*

The proposition shows that most of the features of the optimal relational contract are not affected by liquidity constraints. In particular, part (i.) shows that the manager still motivates the worker by promising him a strictly positive bonus. Notice, however, that when the liquidity constraint is binding, that is, $\pi \in [\pi_0, \pi_2)$, there is a limit to the size of the bonus. In this region, the manager would like to pay a larger bonus but is constrained to paying $b_n^*(\pi) = (1+m)\,y(e^*(\pi))$.

Part (ii.) shows that, as in the case without liquidity constraints, the worker punishes the manager for not paying bonuses by gradually reducing expected profits. While downturns are still gradual, however, part (iii.) shows that recoveries may no longer be instantaneous. Specifically, when $\pi \in [\pi_0, \pi_1)$, $\pi_n^*(\pi)$ is strictly less than $\overline{\pi}$, even though the manager is paying the largest possible bonus. When $\pi \in [\pi_1, \pi_2)$ this bonus is sufficient to compensate the worker for letting expected profits return to $\overline{\pi}$. When $\pi \in [\pi_0, \pi_1)$, however, the largest possible bonus is too small to compensate the worker. The manager then needs to spread the bonus payments over multiple periods. Liquidity constraints therefore slow down recoveries from sufficiently severe downturns.

Together with the Lemma 6, part (ii.) also implies that the relationship terminates if $\pi_0 > \underline{\pi}$ and the firm is hit by shocks in sufficiently many consecutive periods. The reason is that after a sufficiently severe downturn the manager's reward for paying a bonus is limited since it then takes multiple periods for expected profits to return to their upper bound. To ensure that the manager stays truthful, the worker therefore has to increase the punishment for the manager not paying the bonus. Since expected profits are already small, the only way to do so is to increase the threat of termination.

Finally, part (iv.) shows how liquidity constraints affect effort provision. When the liquidity constraint is not binding, that is, $\pi \in [\pi_2, \overline{\pi}]$, the expression that determines effort is the same as in the model without liquidity constraints. In particular, the ratio of marginal effort costs to marginal output is again equal to the negative of the slope of the payoff frontier. In contrast, when the liquidity constraint is binding, that is, $\pi \in [\pi_0, \pi_2)$, the ratio of marginal effort costs to marginal output is always strictly larger than the negative of the slope of the payoff frontier. The reason is that an increase in effort now has the additional benefit of relaxing the liquidity constraint. If $\pi \in [\pi_0, \pi_1)$, this additional benefit is captured by the second term on the right hand side of the expression in part (iv.). And if $\pi \in [\pi_1, \pi_2)$, effort is raised just enough for expected profits to return to $\overline{\pi}$ following a no-shock period. Notice that in this region effort is decreasing in expected profits. The worker, therefore, responds to a reduction in expected profits by providing more effort rather than less. Essentially, the worker understands that providing more effort relaxes the firm's liquidity constraint which, in turn, allows the manager to pay him a larger bonus if the firm is not hit by another shock. As discussed in the Introduction, this reasoning is broadly consistent with the experience at Lincoln Electric.

In summary, liquidity constraints limit the manager's ability to manage conflicts, which slows down recoveries and can lead to termination. Liquidity constraints, however, can also induce the worker to respond to a conflict by providing more effort rather than less.

7 Discussion

In this section we revisit key features and assumptions of our model and examine them in more detail. We focus on our main model without liquidity constraints.

7.1 The Failure to Achieve First-Best

We show in Appendix C (Proposition C1) that the Folk Theorem holds in our setting. Specifically, we show that as the discount factor δ goes to one, the limit set of the PPE payoff contains the interior of the set of feasible payoffs. Joint surplus therefore converges towards first-best as the parties become increasingly patient. It is important to note, however, that as long as the discount factor δ is strictly less than one, joint surplus is strictly less than first-best. For any $\delta < 1$ the optimal relational contract is therefore inefficient. This is in line with related repeated games such as Hertel (2004) in which first-best also cannot be achieved. In contrast, first-best can be achieved, for instance, in Athey and Bagwell (2001).

The reason for the parties' inability to achieve first-best is that the worker can never be sure that opportunity costs are low. The fact that the worker can never be sure that opportunity costs are high, in contrast, does not matter for the parties' inability to achieve first-best. To see this, suppose that whenever the firm is not hit by a shock, there is some probability $p \in [0, 1)$ with which it becomes publicly known that the firm's opportunity costs are low. And whenever the firm is hit by a shock, there is some probability $q \in [0, 1)$ with which it becomes publicly known that the firm's opportunity costs are high. If $p = q = 0$, this model is the same as our main model.

And if either p or q were equal to one, the state would be publicly observed and there would be no need for the manager to be punished on the equilibrium path. We discuss this public information benchmark in the next section. In Appendix C (Proposition C2) we show that in the setting in which $p \in [0,1)$ and $q \in [0,1)$, first-best can be achieved for sufficiently high discount factors if and only if $p > 0$. Essentially, when $p > 0$, the manager does not pay the worker when the firm is hit by a shock but promises him a larger bonus in the next period in which it is publicly observed that the firm's opportunity costs are low. Since the occurrence of such an event is publicly observable, the manager's promise is credible and first-best is feasible.

Firms that ask their workers to accept cuts to their compensation often open their books to prove that those cuts really are necessary (see, for instance, the Introduction and Englmaier and Segal (2011)). The above argument suggests that firms should not only open their books during hard times, in the hope of avoiding worker punishments. Instead, it may be even more important for firms to keep their books open during good times, so as to make punishments less costly.

7.2 Benchmarks: Public Information and Long-Term Contracts

The first benchmark we examine is one in which shocks are publicly observed. In Appendix C (Proposition C3) we examine this case and characterize the PPE payoff set. For every payoff pair on the PPE frontier we then characterize the optimal relational contract supporting it. As in our main model, expected profits, effort, and joint surplus jump to their upper bounds following a no-shock period. Because of the public observability of the opportunity costs, however, shocks no longer lead to a reduction in expected profits. Instead, following a shock period, expected profits, effort, and joint surplus remain unchanged. They therefore reach their highest achievable levels with probability one and then stay there forever. Finally, if the manager and the worker are patient enough, those highest achievable levels are equal to first-best. The evolution of the relationship between the manager and the worker therefore depends crucially on whether shocks are publicly observed.

The second benchmark we examine is one in which the manager can commit to a long-term contract. Suppose that in any period t the manager first observes the state $\Theta_t \in \{n, s\}$ and then makes an announcement $m_t \in \{n, s\}$ about the state. Suppose also that before the first period, the manager can commit to a contract that, for any period t, maps her announcements $(m_1, m_2, ..., m_t)$ into the bonus b_t that she has to pay the worker at the end of period t.

A long-term contract does not allow the parties to achieve first-best. It does, however, allow them to approximate first-best. To see this, let $\tau(t)$ denote the number of consecutive periods immediately preceding t in which the manager did not pay the worker. Now consider a contract with three features. First, the contract asks the worker to provide first-best effort in all periods. If the worker ever does not provide first-best effort, the manager will never again pay him. Second, the contract specifies that if, in period t, the manager announces that the firm has not been hit by a shock, she pays the worker a bonus

$$b_t(\tau(t)) = \left(1 + \frac{1}{\delta} + \frac{1}{\delta^2} + ... + \frac{1}{\delta^{\tau(t)}}\right)\left(\underline{u} + c\left(e^{FB}\right)\right).$$

And third, the contract specifies the maximum number of consecutive periods $T \geq 1$ that the manager can go without paying the worker the bonus. In particular, if, in period t, the manager announces that the firm has been hit by a shock and if $\tau(t) < T$, the manager does not have to pay the worker. If, however, $\tau(t) = T$, the manager has to pay the worker a bonus $b_t(T)$.

In Appendix C (Proposition C4) we show that under such a contract, the worker always provides first-best effort and the manager always announces the state truthfully. Essentially, under this contract the manager has to pay the worker $\left(\underline{u} + c\left(e^{FB}\right)\right)$ per period, independent of her announcements. By lying about the state, the manager can therefore affect the timing of payments but not their net present value.

This contract does not achieve first-best since it induces inefficient payment whenever the firm is hit by shocks in T consecutive periods. By agreeing to a large T, however, the parties can come arbitrarily close to achieving first-best. The evolution of the relationship between the manager and the worker therefore depends crucially on whether the manager is able to commit to a long-term contract. And as we saw above, it also depends on whether shocks are privately observed.

8 Conclusions

In a well-known article in *The New Yorker*, Stewart (1993) describes the upheavals at the investment bank Credit Suisse First Boston (CSFB) after consecutive years of disappointing bonus payments. Problems started in 1991 when traders demanded that management pay them a higher bonus. Management, however, stood firm, insisting that a higher bonus was not justified because of the need to "*build capital.*" To appease the traders, management then simply "*promised that 1992 would be different—that salaries and bonuses would again be competitive.*" Traders were forthcoming in expressing their disappointment but their retaliations were limited. The traders' behavior changed the following year, however, when bonus payments were once again below expectations. This time "*many traders seemed to drag their heels, further depressing the firm's earnings*" and "*defections* [...] *increased as soon as First Boston actually began paying bonuses.*" This response forced management to adapt its compensation policy by formally committing to "*guaranteed pay raises*," in some cases as much as 100%.

At the heart of the conflict at CSFB was uncertainty, and possibly private information, about the opportunity costs of bonus payments. In particular, while there was no disagreement about the traders' performance, there was disagreement about the extent to which bonus payments should be contingent on the need to "build capital." The aim of this paper was to explore the conflicts that arise in such a setting. In our model, it is optimal for the manager to make payments contingent on their opportunity costs, even though this makes conflicts inevitable. As in the CSFB example, the manager responds to such conflicts by adapting compensation to their duration, moving from the informal—promising that 1992 will be different—to the formal—committing to guaranteed pay raises. Because the manager responds to a conflict by changing the compensation she offers the worker, conflicts evolve gradually. This is again illustrated in the CSFB example in which traders did not switch from cooperation to punishment abruptly. Instead, the relationship deteriorated gradually in response to repeated disagreements about bonus pay. Finally, in our main model, expected profits cycle indefinitely. The relationship between the manager and the worker therefore never terminates, nor does it reach a steady state. This is in contrast to the CSFB example where many traders did leave. Termination, however, can also arise in our setting once we allow for the firm to be liquidity-constrained.

To discuss the empirical implications and testability of our model, it is useful to note that the basic structure of the stage game is closely related to a "trust game." In a standard trust game, the "proposer" first decides on the size of a monetary gift that he makes to the "responder." The gift is then increased by some amount after which the responder decides how much to give back to the proposer. One can therefore view our model as an infinitely repeated trust game in which the responder faces shocks to the costs of giving back. There is an extensive literature in experimental economics that examines trust games. This suggests that one could test our model in a laboratory setting. Two predictions, in particular, are clear cut. First, the evolution of trust—as measured by the size of the gift—depends crucially on whether shocks are publicly observed. If shocks are publicly observed, trust increases over time until it tops out at some level. If shocks are instead privately observed, trust evolves through booms and busts. Second, the long-term prospects of a relationship depend on whether the responder is liquidity-constrained. If the responder is not liquidity-constrained, the relationship continues forever. But if she is liquidity-constrained, it is certain to terminate eventually.

We have cast our model in the context of an employment relationship. The main ingredients of the model—repeated interaction, limited commitment, and inefficient transfers—are also relevant in many other economic settings. One example is the lending relationship between an entrepreneur and an investor who are not able to commit to long-term contracts. The entrepreneur can have private information about her marginal value of money and the investor can adjust his future financing terms based on the payment history of the entrepreneur. Another example is that of long-term and informal supplier relationships in which buyers face shocks to their ability to pay their suppliers. In 1995, for instance, Continental Airlines was close to bankruptcy and its "*most pressing need was to shore up its cash position. The airline* [...] *was only able to make its January 1995 payroll when* [its CEO] *Bethune successfully begged Boeing to return cash deposits on aircraft whose delivery he had deferred*" (Frank 2009). A final example involves the informal insurance relationships among farmers in developing countries. There is some evidence that the farmers' income is private information (see, for example, Kinnan (2011)). While most of the literature has focused on moral hazard and insurance issues separately, our model suggests that these issues are related since insurance decisions affect future production choices.

9 Appendix A: Main Model

Proof of Lemma 1: Part (i.): Note that $\underline{\pi}$ and $\underline{u}$ are the manager's and the worker's minmax payoffs and that they can be sustained as PPE payoffs by having each party take their outside option in each period. It is then immediate that in any PPE the bonus payments, wages, and effort are bounded. As a result, we can restrict the manager's and the worker's actions to compact sets. Standard arguments then imply that the PPE payoff set E is compact so that

$$u(\pi) = \max\{u, (\pi, u) \in E\}.$$

Part (ii.): Since there is a public randomization device, any payoff on the line segment between $(\underline{\pi}, \underline{u})$ and $(\overline{\pi}, \underline{u})$ can be supported as a PPE payoff. It then follows that randomization between $(\pi, \underline{u})$ and $(\pi, u(\pi))$ allows us to obtain any payoff (π, u') for all $u' \in [\underline{u}, u(\pi)]$.

Part (iii.): Suppose to the contrary that $u(\overline{\pi}) > \underline{u}$. Note that since $(\overline{\pi}, u(\overline{\pi}))$ is an extremal point of E it must be sustained by pure actions in period 1. Consider a PPE with payoffs $(\overline{\pi}, u(\overline{\pi}))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n). Now consider an alternative strategy profile with the same first-period continuation payoffs (π_s, π_n, u_s, u_n) but in which first-period actions are given by $(\widehat{e}, b_s, b_n, w)$, where $\widehat{e} = e + \varepsilon$. It follows from the promise keeping constraints $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this alternative strategy profile the payoffs are given by

$$\widehat{\pi} = \theta\left((1-\delta)\left(y\left(\widehat{e}\right) - (1+\alpha)(w+b_s)\right) + \delta\pi_s\right) + (1-\theta)\left((1-\delta)\left(y\left(\widehat{e}\right) - w - b_n\right) + \delta\pi_n\right)$$

and

$$\widehat{u} = \theta((1-\delta)\left(w+b_s\right) + \delta u_s) + (1-\theta)\left((1-\delta)\left(w+b_n\right) + \delta u_n\right) - (1-\delta)\, c\left(\widehat{e}\right).$$

Notice that since y is increasing, we have that $\widehat{\pi} > \overline{\pi}$. Moreover, for small enough ε, $\widehat{u} \geq \underline{u}$. It can be checked that this alternative strategy profile satisfies all the constraints in Section 4.1. (with the exception of the $\mathrm{IC_W}$ constraint that we ignore throughout) and therefore constitutes a PPE. Since $\widehat{\pi} > \overline{\pi}$ this contradicts the definition of $\overline{\pi}$. ■

Proof of Lemma 2: Part (i.): We proceed in two steps. First, we show that, for any $\pi_1 < \pi_2$, if both $u(\pi_1)$ and $u(\pi_2)$ can be sustained by pure actions in the stage game other than taking the outside option, then $u(\pi)$ can also be sustained by pure actions for any $\pi \in (\pi_1, \pi_2)$. Second, we then show that $u(\underline{\pi})$ is supported by a pure action in the stage game without taking the outside option. Since we know from the proof of Lemma 1 that $u(\overline{\pi})$ can sustained by pure actions, the result follows.

We prove the first step by contradiction. Consider any $\pi_1 < \pi_2$ such that $u(\pi_1)$ and $u(\pi_2)$ can be sustained by pure actions in the stage game. Take any $\pi \in (\pi_1, \pi_2)$ and suppose to the contrary that $(\pi, u(\pi))$ is not sustained by pure actions. Then there exists a $\rho \in (0,1)$, a $\widetilde{\pi}_1 \in [\pi_1, \pi)$, and a $\widetilde{\pi}_2 \in (\pi, \pi_2]$ such that (i.) $(\widetilde{\pi}_1, u(\widetilde{\pi}_1))$ and $(\widetilde{\pi}_2, u(\widetilde{\pi}_2))$ are sustained by pure actions, (ii.) $\pi = \rho\widetilde{\pi}_1 + (1-\rho)\widetilde{\pi}_2$, and (iii.) $u(\pi) = \rho u(\widetilde{\pi}_1) + (1-\rho) u(\widetilde{\pi}_2)$. Now consider a PPE with payoffs $(\widetilde{\pi}_j, u(\widetilde{\pi}_j))$, for $j = 1, 2$, and associated first-period actions $(e_j, b_{s_j}, b_{n_j}, w_j)$ and first-period continuation payoffs $(\pi_{s_j}, \pi_{n_j}, u_{s_j}, u_{n_j})$. Define $\widehat{e}$ as the effort level satisfying

$$y(\widehat{e}) = \rho y(e_1) + (1-\rho)y(e_2).$$

Since y is strictly concave, we have that

$$y(\rho e_1 + (1-\rho)e_2) > \rho y(e_1) + (1-\rho)y(e_2) = y(\widehat{e}).$$

And since y is strictly increasing this implies that

$$\widehat{e} < \rho e_1 + (1-\rho)e_2.$$

Now consider an alternative strategy profile with first-period actions $(\widehat{e}, \widehat{w}, \widehat{b}_s, \widehat{b}_n)$ and first-period continuation payoffs $(\widehat{\pi}_s, \widehat{\pi}_n, \widehat{u}_s, \widehat{u}_n)$, where $\widehat{w} = \rho w_1 + (1-\rho)w_2$ and where $\widehat{b}_s, \widehat{b}_n, \widehat{\pi}_s, \widehat{\pi}_n, \widehat{u}_s$,and $\widehat{u}_n$ are defined analogously.

Also, let $\widehat{w} = \rho w_1 + (1-\rho)w_2$ and define $\widehat{b}_s, \widehat{b}_n, \widehat{\pi}_s, \widehat{\pi}_n, \widehat{u}_s$, and $\widehat{u}_n$ analogously. It follows from the promise keeping constraints $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this alternative strategy profile the payoffs are given by $\widehat{\pi} = \rho\widetilde{\pi}_1 + (1-\rho)\widetilde{\pi}_2$ and

$$\widehat{u} = \rho u(\widetilde{\pi}_1) + (1-\rho)u(\widetilde{\pi}_2) + (1-\delta)\left(\rho c(e_1) + (1-\rho)\, c(e_2) - c(e)\right).$$

Since $c(e)$ is strictly increasing and strictly convex, it follows that $\widehat{u} > \rho u(\widetilde{\pi}_1) + (1-\rho)u(\widetilde{\pi}_2)$. It can be checked that this alternative profile satisfies all the constraints in Section 4.1. and therefore constitutes a PPE. Since $\widehat{u} > \rho u(\widetilde{\pi}_1) + (1-\rho)u(\widetilde{\pi}_2)$, this proves the first step.

To prove the second step, suppose to the contrary that $u(\underline{\pi})$ cannot be sustained by pure actions other than taking the outside options. Since $(\underline{\pi}, u(\underline{\pi}))$ is an extremal point, it cannot be sustained by randomizations. The parties must therefore take their outside options in period 1, which implies that $u(\underline{\pi}) = \underline{u}$.

Now choose a payoff pair $(\pi, u(\pi))$ that is sustained by pure actions other than the outside option. Notice that such a payoff pair must exist since $(\overline{\pi}, u(\overline{\pi}))$ is an extremal point that is sustained by pure actions. Suppose $(\pi, u(\pi))$ is obtained by a PPE with first-period actions

(e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n). Next, consider an alternative strategy profile with the same first-period continuation payoffs (π_s, π_n, u_s, u_n) but different first-period actions $(e, \widehat{w}, b_s, b_n)$, where $\widehat{w} = w + \varepsilon$ for some $\varepsilon > 0$. This strategy profile generates payoffs $(\widehat{\pi}, \widehat{u})$, where $\widehat{\pi} = \pi - (1-\delta)(1+\alpha\theta)\varepsilon$ and $\widehat{u} = u(\pi) + (1-\delta)\varepsilon$. It can be checked that as long as $\widehat{\pi} \geq \underline{\pi}$ or, equivalently, $\varepsilon \leq (\pi - \underline{\pi}) / ((1-\delta)(1+\alpha\theta))$, this alternative strategy profile satisfies all of the constraints in Section 4.1 and therefore constitute a PPE. Let $\varepsilon = (\pi - \underline{\pi}) / ((1-\delta)(1+\alpha\theta))$. The above then implies that $(\underline{\pi}, u(\pi) + (\pi - \underline{\pi})/(1+\alpha\theta))$ is a PPE payoff, which contradicts $u(\underline{\pi}) = \underline{u}$.

Finally, notice that by sending ε to zero, the above implies that $u'_{-}(\pi) \leq -1/(1+\alpha\theta)$ for all $\pi > \underline{\pi}$. Below we will use this fact to prove part (iii.).

Part (ii.): Concavity follows directly from the availability of the public randomization device.

Part (iii.): Consider a PPE with payoffs $(\pi, u(\pi))$, where $\pi \in (\underline{\pi}, \overline{\pi})$, and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n). Now consider an alternative profile with the same first-period continuation payoffs (π_s, π_n, u_s, u_n) but in which first-period actions are given by $(\widehat{e}, w, b_s, b_n)$, where $\widehat{e} = e + \varepsilon$ for some $\varepsilon > 0$. It follows from $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this strategy profile the payoffs are given by $\widehat{\pi} = \pi + (1-\delta)(y(e+\varepsilon) - y(e))$ and $\widehat{u} = u(\pi) - (1-\delta)(c(e+\varepsilon) - c(e))$. For small enough ε, the alternative strategy profile satisfies all the constraints in Section 4.1. (with the exception of the $\mathrm{IC_W}$ constraint that we ignore throughout) and therefore constitutes a PPE.

Since $u(\pi)$ is the frontier, it must be that

$$u(\pi + (1-\delta)(y(e+\varepsilon) - y(e))) \geq \widehat{u} = u(\pi) - (1-\delta)(c(e+\varepsilon) - c(e)).$$

Sending ε to zero, we then obtain that

$$-\frac{c'(e)}{y'(e)} \leq u'_{+}(\pi).$$

Since u is concave and, as shown in the proof of part (ii.), $u'_{-}(\pi) < -1/(1+\alpha\theta)$, we have $u'_{+}(\pi) \leq u'_{-}(\pi) < -1/(1+\alpha\theta)$. This implies that

$$\frac{c'(e)}{y'(e)} \geq \frac{1}{1+\alpha\theta}.$$

Since $c'(e)/y'(e) = 0$ for $e = 0$, we therefore have $e > 0$. Next, consider another alternative strategy profile with first-period actions $(\widehat{e}, w, b_s, b_n)$, where $\widehat{e} = e - \varepsilon$ for small $\varepsilon > 0$, and first-period continuation payoffs (π_s, π_n, u_s, u_n). Proceeding in exactly the same way, it can be shown that $u'_{-}(\pi) \leq -c'(e)/y'(e)$. This implies that $u'_{+}(\pi) = u'_{-}(\pi)$ and thus u is differentiable. Since we have $u'_{-}(\pi) \leq -1/(1+\alpha\theta)$ this concludes the proof except for the claim that $-1 < u'(\pi)$. For simplicity of exposition we will prove this part after Lemma 5. The proofs of Lemmas 3-5 do not make use of this claim. ■

Proof of Lemma 3: Recall from Lemma 1 that the PPE payoff set is compact. Consider a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n). Suppose to the the contrary of the claim that $u_s < u(\pi_s)$. Now consider an alternative strategy profile with the same first-period actions but in which first-period continuation payoffs are given by $(\pi_s, \pi_n, \widehat{u}_s, u_n)$, where $\widehat{u}_s = u_s + \varepsilon$ and where $\varepsilon > 0$ is small enough such that $u_s + \varepsilon \leq u(\pi_s)$. It follows from the promise keeping constraints $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this alternative strategy profile the payoffs are given by $\widehat{\pi} = \pi$ and $\widehat{u} = u(\pi) + \delta(1-\theta)\varepsilon > u(\pi)$. It can be checked that this alternative strategy profile satisfies all the constraints in Section 4.1 (with the exception of the $\mathrm{IC_W}$ constraint that we ignore throughout) and therefore constitutes a PPE. Since $\widehat{u} > u(\pi)$ this contradicts the definition of $u(\pi)$. Thus it must be that $u_s = u(\pi_s)$. The proof for $u_n = u(\pi_n)$ is analogous. ■

Proof of Lemma 4: Part (i.): By combining the truth-telling constraints $\mathrm{TT_N}$ and $\mathrm{TT_S}$ we have

$$(1+\alpha)(1-\delta)(b_n - b_s) \geq \delta(\pi_n - \pi_s) \geq (1-\delta)(b_n - b_s).$$

Since $(1+\alpha) > 0$, this implies that $b_n - b_s \geq 0$.

Next, consider a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs (π_s, π_n, u_s, u_n) and suppose that $b_s > 0$. Now consider an alternative strategy profile with the same continuation payoffs (π_s, π_n, u_s, u_n) but in which first-period actions are given by $(e, \widehat{w}, \widehat{b}_s, \widehat{b}_n)$, where $\widehat{w} = w + b_s$, $\widehat{b}_s = 0$, and $\widehat{b}_n = b_n - b_s$. It follows from the promise keeping constraints $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this alternative strategy profile the payoffs are given by $(\pi, u(\pi))$. It can be checked that this alternative strategy profile satisfies all the constraints in Section 4.1. (with the exception of the $\mathrm{IC_W}$ constraint that we ignore throughout) and therefore constitutes a PPE. This proves part (i.).

Part (ii.): Consider a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w, b_s, b_n) and first-period continuation payoffs $(\pi_s, \pi_n, u(\pi_s), u(\pi_n))$. Suppose that for this PPE the $\mathrm{TT_N}$ is slack, that is, $\delta(\pi_n - \pi_s) > (1-\delta)(b_n - b_s)$. Together with the non-reneging constraint $\mathrm{NR_S}$ this implies that $\mathrm{NR_N}$ is slack. Now consider an alternative strategy profile with the same first-period actions (e, w, b_s, b_n) but in which first-period continuation payoffs are given by $(\widehat{\pi}_s, \widehat{\pi}_n, u(\widehat{\pi}_s), u(\widehat{\pi}_n))$, where $\widehat{\pi}_s = \pi_s + (1-\theta)\varepsilon$ and $\widehat{\pi}_n = \pi_n - \theta\varepsilon$ for $\varepsilon > 0$. It follows from the promise keeping constraints $\mathrm{PK_M}$ and $\mathrm{PK_W}$ that under this strategy profile the payoffs are given by $\widehat{\pi} = \pi$ and

$$\widehat{u} = (1-\delta)(w + \theta b_s + (1-\theta) b_n - c(e)) + \delta(\theta u(\widehat{\pi}_s) + (1-\theta) u(\widehat{\pi}_n)).$$

From the concavity of u it then follows that

$$\widehat{u} \geq (1-\delta)(w + \theta b_s + (1-\theta) b_n - c(e)) + \delta(\theta u(\pi_s) + (1-\theta) u(\pi_n)) = u(\pi).$$

It can be checked that for sufficiently small ε this alternative strategy profile satisfies all the constraints in Section 4.1. (with the exception of the $\mathrm{IC_W}$ constraint that we ignore throughout) and therefore constitutes a PPE. Since $\widehat{u} \geq u(\pi)$ this implies that for any PPE with payoffs $(\pi, u(\pi))$ for which $\mathrm{TT_N}$ is not binding there exists another PPE for which $\mathrm{TT_N}$ is binding and which gives the parties weakly larger payoffs. ■

Proof of Lemma 5: By Lemma 2, every point on the PPE frontier can be supported by pure actions other than the outside options. Lemma 3 and the definition of u then imply that $\pi + u(\pi)$ is given by

$$\pi + u(\pi) = \max_{e,w,\pi_s,\pi_n} (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) - (1-\delta)\theta\alpha w$$

subject to the constraints in Section 4.1. (with the exception of the $\mathrm{IC_W}$ constraint that we ignore throughout). Notice that we have used Lemma 4 to substitute out b_s and b_n in the objective function.

To reduce the number of constraints, consider first the feasibility constraints. By Lemma 4 the non-negative constraints $\mathrm{NN_S}$ is no longer relevant. At the end of this proof we show that the non-negativity constraint $\mathrm{NN_N}$ is also no longer relevant. Moreover, in the proof of Lemma 2 we showed that $c'(e)/y'(e) \geq 1/(1+\alpha\theta)$. We therefore have $e(\pi) > 0$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$ which implies that $\mathrm{NN_e}$ is slack.

Next, Lemma 3 implies that $\mathrm{SE_N}$ and $\mathrm{SE_S}$ can be reduced to $\underline{\pi} \leq \pi_s \leq \overline{\pi}$ and $\underline{\pi} \leq \pi_n \leq \overline{\pi}$. The only relevant feasibility constraints are therefore given by $\mathrm{NN_W}$, $\mathrm{SE_N}$, and $\mathrm{SE_S}$.

Next, we examine the no deviation constraints with the exception of the $\mathrm{IC_W}$ constraint which we ignore throughout. Part (i.) of Lemma 4 and $\mathrm{SE_S}$ imply that $\mathrm{NR_N}$ and $\mathrm{NR_S}$ are satisfied. Part (ii.) of Lemma 4 further implies that $\mathrm{TT_N}$ and $\mathrm{TT_S}$ are satisfied.

Next, we turn to the promise-keeping constraints. We obtain the version of the $\mathrm{PK_M}$ constraint in the lemma by using Lemma 4 to substitute b_s and b_n out of the original $\mathrm{PK_M}$ constraint. The $\mathrm{PK_W}$ constraint is satisfied because $\pi + u(\pi)$ is the solution to the functional equation given by the constrained maximization problem.

Finally, consider the non-negativity constraint $\mathrm{NN_N}$. Notice first that $\mathrm{NN_N}$ is equivalent to $\pi_s \leq \pi_n$. Now consider a PPE with payoffs $(\pi, u(\pi))$ and associated first-period actions (e, w) and first-period continuation payoffs (π_s, π_n). Suppose to the contrary of the claim that $\pi_n < \pi_s$. There then exists an alternative strategy profile that satisfies all the constraints in Lemma 5 and generates a strictly larger joint surplus, leading to a contradiction. Specifically, consider an alternative strategy profile with the same first-period actions (e, w) but in which the first-period continuation payoffs are given by $(\pi_s, \widehat{\pi}_n)$, where $\widehat{\pi}_n = \pi_n + \varepsilon$ for some $\varepsilon > 0$. Since, by Lemma 2, $1 + u'(\pi) > 0$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$, the alternative strategy profile generates a strictly larger joint surplus $\pi + u(\pi)$. Moreover, it can be checked that this alternative strategy profile satisfies all the constraints in Lemma 5. As claimed above, the non-negativity constraint $\mathrm{NN_N}$ is therefore no longer relevant. ■

Proof of the last part of Lemma 2 ($-1 < u'(\pi)$): The Lagrangian associated with the maximization problem in Lemma 5 is given by

$$\begin{aligned} \pi + u(\pi) &= L = (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) - (1-\delta)\theta\alpha w \\ &\quad + \lambda_1(\pi - (1-\delta) y(e) - \delta\pi_s + (1-\delta)(1+\theta\alpha) w) \\ &\quad + \lambda_2(\delta\pi_s - \delta\underline{\pi}) + \lambda_3(1-\delta)w + \lambda_4(\delta\overline{\pi} - \delta\pi_n), \end{aligned}$$

where we use the fact that $\pi_s \leq \pi_n$ (as shown in the proof of Lemma 5) to eliminate the constraints $\pi_s \leq \overline{\pi}$ and $\pi_n \leq \underline{\pi}$. Note that this is a well-defined concave program. The first order conditions with respect to π_n, π_s, w, and e are given by

$$(1-\theta)(1 + u'(\pi_n)) - \lambda_4 = 0, \tag{$\mathrm{FOC_N}$}$$

$$\theta(1 + u'(\pi_s)) - \lambda_1 + \lambda_2 = 0, \tag{$\mathrm{FOC_S}$}$$

$$-\theta\alpha + \lambda_1(1 + \theta\alpha) + \lambda_3 = 0, \tag{$\mathrm{FOC_W}$}$$

and

$$y'(e) - c'(e) - \lambda_1 y'(e) = 0. \tag{$\mathrm{FOC_e}$}$$

Furthermore, the envelope condition is given by

$$1 + u'(\pi) = \lambda_1. \tag{envelope}$$

To prove that $u'(\pi) > -1$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$, note that $\mathrm{FOC_S}$ and $\mathrm{FOC_N}$ imply that

$$\begin{aligned} \lambda_1 &= \theta(1 + u'(\pi_s)) + \lambda_2 \\ &\geq \theta(1 + u'(\pi_n)) \\ &\geq 0. \end{aligned}$$

It therefore follows from the envelope condition that $1 + u'(\pi) \geq 0$ for $\pi \in [\underline{\pi}, \overline{\pi}]$.

To finish the proof, we need to rule out that $u'(\pi) = -1$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$. Suppose the contrary and define $\Gamma \equiv \{\pi \in [\underline{\pi}, \overline{\pi}] \,|u'(\pi) = -1\}$. We now establish three facts. First, if $\pi \in \Gamma$ then $\pi_s(\pi) \in \Gamma$. This follows from $\mathrm{FOC_S}$, $\lambda_1 = 0$, and $\lambda_2 \geq 0$. Second, if $\pi \in \Gamma$ then $e(\pi) = e^{FB}$, where e^{FB} is the first-best effort level that solves $y'(e) = c'(e)$. This follows from $\mathrm{FOC_e}$ and $\lambda_1 = 0$. Finally, if $\pi \in \Gamma$ then $w(\pi) = 0$. To see this, note that since $\lambda_1 = 0$, $\mathrm{FOC_W}$ implies that $\lambda_3 > 0$. It then follows from the complementarity slackness condition with respect to $\mathrm{NN_W}$ that $w(\pi) = 0$.

The three facts above imply that the manager can maximize her pay by always claiming to have been hit by a shock. This is a contradiction since the $(-c(e^{FB}))$ would then be smaller than his outside option. ■

Proof of Proposition 1: We first continue to ignore the $\mathrm{IC_W}$ constraint. For this relaxed problem we first prove parts (i.) to (v.) and then show that the optimal relational contract is unique. Finally, we show that the optimal relational contract of the relaxed problem satisfies the $\mathrm{IC_W}$ constraint.

Parts (i.) to (v.): The expression for b_n in part (i.) follows from the $\mathrm{TT_N}$ constraint. The fact that $b_n > 0$ follows from $\pi_s^* < \overline{\pi}$ which is shown in part (ii.).

For part (ii.) consider $\mathrm{FOC_S}$. Suppose first that $\lambda_2 > 0$. In this case, the complementarity condition associated with $\mathrm{SE_S}$ implies that $\pi_s = \underline{\pi}$. Suppose next that $\lambda_2 = 0$. In this case

$$1 + u'(\pi_s) = \frac{1}{\theta}\lambda_1 = \frac{1}{\theta}(1 + u'(\pi)),$$

where the first equality follows from $\mathrm{FOC_S}$ and the second equality follows from the envelope condition. Since Lemma 2 implies that $1 + u'(\pi) > 0$, we have that $u'(\pi_s) > u'(\pi)$. Part (ii.) then follows from the concavity of u.

Part (iii.) follows from $1 + u'(\pi) > 0$ and $\mathrm{FOC_N}$.

Part (iv.) follows from combining $\mathrm{FOC_e}$ and the envelope condition. Note that since $c'(e)/y'(e)$ is strictly increasing, $e^*(\pi)$ is unique.

For part (v.), recall from Lemma 2 that $u'(\pi) \leq -1/(1+\alpha\theta)$. Suppose first that $u'(\pi) < -1/(1+\alpha\theta)$ for all $\pi \in [\underline{\pi}, \overline{\pi}]$. In this case, $\mathrm{FOC_W}$ and the envelope condition imply that $w \equiv 0$. Suppose next that there exists a line segment with $u'(\pi) = -1/(1+\alpha\theta)$. Recall that $\widehat{e}$ is the unique effort level satisfying $c'(\widehat{e})/y'(\widehat{e}) = 1/(1+\alpha\theta)$. Part (iv.) implies that on the line segment

$e=\widehat{e}$. Moreover, we must have $\pi_s=\underline{\pi}$. To see this, note that $\mathrm{FOC_S}$ implies that

$$\begin{aligned}\lambda_2 &= \lambda_1-\theta(1+u'(\pi_s)) \\ &= \frac{\alpha\theta}{1+\alpha\theta}-\theta(1+u'(\pi_s)) \\ &\geq (1-\theta)\frac{\alpha\theta}{1+\alpha\theta},\end{aligned}$$

where the inequality follows from $u'(\pi_s)\leq -1/(1+\alpha\theta)$. Since $\lambda_2>0$, the complementary slackness with respect to $\mathrm{SE_S}$ then implies that $\pi_s=\underline{\pi}$. Since $\pi_s(\pi)=\underline{\pi}$ and $e(\pi)=\widehat{e}$ for π on the line segment, the $\mathrm{PK_M}$ constraint implies that $w^*(\pi)=((1-\delta)y(\widehat{e})+\delta\underline{\pi}-\pi)/((1-\delta)(1+\alpha\theta))$. This proves part (v.).

Uniqueness: Note that in all of the derivations above, $e^*(\pi),w^*(\pi)$, and $\pi_n^*(\pi)$ are unique. Moreover, $\mathrm{PK_M}$ implies that $\pi_s^*(\pi)$ is unique. This proves that the optimal relational contract is unique as long as $\mathrm{TT_N}$ is binding. In Lemma 4 we showed that for any optimal relational contract there exists an equivalent one in which $\mathrm{TT_N}$ is binding. Next we show that $\mathrm{TT_N}$ has to be binding for all optimal relational contracts.

For this purpose, suppose to the contrary that there exists an optimal relational contract for which $\mathrm{TT_N}$ is not binding. This implies that there exists a PPE with payoffs $(\pi,u(\pi))$ and associated first-period actions (e,w,b_s,b_n) and first-period continuation payoffs $(\pi_s,\pi_n,u(\pi_s),u(\pi_n))$ such that

$$\delta(\pi_n-\pi_s)>(1-\delta)(b_n-b_s).$$

Now consider an alternative strategy profile with the same first-period actions (e,w,b_s,b_n) but in which first-period continuation payoffs are given by $(\widehat{\pi}_s,\widehat{\pi}_n,u(\widehat{\pi}_s),u(\widehat{\pi}_n))$, where $\widehat{\pi}_s=\pi_s+(1-\theta)\varepsilon$ and $\widehat{\pi}_n=\pi_n-\theta\varepsilon$. It can be checked that when $\varepsilon=[\delta(\pi_n-\pi_s)-(1-\delta)(b_n-b_s)]/\delta$, this alternative strategy profile satisfies all the constraints in Section 4.1. and is therefore constitutes a PPE with payoffs $(\pi,u(\pi))$. Moreover, the $\mathrm{TT_N}$ constraint is binding since

$$\delta(\widehat{\pi}_n-\widehat{\pi}_s)=\delta(\pi_n-\pi_s)-\delta\varepsilon=(1-\delta)(b_n-b_s).$$

Notice that under this alternative strategy profile $\widehat{\pi}_n<\overline{\pi}$. This is a contradiction since we saw above that when $\mathrm{TT_N}$ is binding, the optimal relational contract must have $\pi_n^*(\pi)=\overline{\pi}$.

Checking that $\mathrm{IC_W}$ is satisfied: Recall that the $\mathrm{IC_W}$ constraint is given by $\delta\underline{u}+(1-\delta)w^*(\pi)\leq u(\pi)$. The constraint is clearly satisfied when $w=0$, that is, for all $\pi\geq\widehat{\pi}$.

For $\pi<\widehat{\pi}$, part (v.) implies that

$$\frac{\mathrm{d}(\delta\underline{u}+(1-\delta)w^*(\pi))}{\mathrm{d}\pi}=-\frac{1}{1+\alpha\theta}=u'(\pi),$$

which in turn implies that

$$\begin{aligned} & u(\pi) - (\delta \underline{u} + (1-\delta) w^*(\pi)) \\ = \; & u(\widehat{\pi}) - (\delta \underline{u} + (1-\delta) w^*(\widehat{\pi})) \\ > \; & 0. \end{aligned}$$

This shows that $\mathrm{IC_W}$ is satisfied for all $\pi \in [\underline{\pi}, \overline{\pi}]$. ■

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10 Appendix B (For Online Publication): The Effects of Liquidity Constraints

In this appendix we prove the results in Section 6 that analyzes the model with liquidity constraints. Specifically, the firm is subject to the liquidity constraint that

$$\max\{w + b_s, w + b_n\} \leq (1 + m)y$$

for some $m > 0$. This constraint significantly complicates the analysis. In particular, the PPE frontier is no longer differentiable. To make the analysis more tractable, we assume that the manager cannot pay the worker in a shock state ($\alpha = \infty$). Consequently, this implies that $w \equiv 0$ and $b_s \equiv 0$.

Since the analysis with the liquidity constraints follows similar steps as the main model, we omit the proofs for results that are obtained from identical arguments. Below, we first describe the basic properties of the PPE frontier in the background subsection and then prove the main results in the dynamics subsection. The last subsection provides several sufficient conditions that allow for further characterization of the dynamics.

10.1 Background

Proceeding in the same way as the main model, we can show that when the PPE frontier is sustainable by pure actions (other than the outside options), the joint surplus $\pi + u(\pi)$ is defined recursively by the following problem:

$$\pi + u(\pi) = \max_{e,\pi_s,\pi_n} (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) \tag{3}$$

such that

$$\delta\pi_n \leq \pi + (1-\delta)my(e), \tag{LIQ$_\mathrm{F}$}$$

$$\underline{\pi} \leq \pi_n \leq \overline{\pi}, \tag{LSE$_\mathrm{N}$}$$

$$\underline{\pi} \leq \pi_s \leq \overline{\pi}, \tag{LSE$_\mathrm{S}$}$$

and

$$\pi = (1-\delta)y(e) + \delta\pi_s. \tag{LPK$_\mathrm{M}$}$$

The problem above corresponds with that in Lemma 5 with several modifications. In particular, $\mathrm{LIQ_F}$ is the extra liquidity constraint for the firm. In addition, $w \equiv 0$ implies that $\mathrm{NN_W}$ is no longer needed. Moreover, w does not appear in the LPK_{M}, and relatedly, the truth-telling condition under the no-shock state is given by

$$\delta\left(\pi_n-\pi_s\right)=\left(1-\delta\right)b_n. \tag{LTT_{N}}$$

In addition to these modifications, there are several other differences between the main model and the model with liquidity constraint. First, unlike the main model, it is no longer true that each point on the PPE frontier can be sustained by pure actions.

LEMMA B1. *There exists a critical level of expected profits $\pi_0 \in [\underline{\pi},\overline{\pi})$ such that for all $\pi \geq \pi_0$ the PPE frontier $u(\pi)$ is supported by pure actions and for all $\pi \in (\underline{\pi},\pi_0)$ it is supported by randomization. Specifically, for any $\pi < \pi_0$ the manager and the worker randomize between terminating their relationship and playing the strategies that deliver expected payoffs π_0 and $u(\pi_0)$.*

Proof: Define π_0 as the smallest payoff of the manager at which $u(\pi_0)$ can be sustained by pure actions other than the outside options. To see that π_0 is well defined, let $\Pi = \{\pi \in [\underline{\pi},\overline{\pi}] | (\pi,u(\pi))$ can be supported by pure actions other than the outside options$\}$. We need to show that the set Π is non-empty and closed. Note that $\overline{\pi} \in \Pi$ because $(\overline{\pi},u(\overline{\pi}))$, being an extremal point of the PPE payoff set, can be sustained by pure actions, and since $\overline{\pi} \neq \underline{\pi}$, the pure actions that support $(\overline{\pi},u(\overline{\pi}))$ is not the outside options. To see that Π is closed, consider a convergent sequence $\{\pi_j\}_{j=1}^{\infty} \subset \Pi$ such that $\lim \pi_j = \pi$. Let $(e_j, b_{n_j}, \pi_{s_j}, \pi_{n_j}, u_{s_j}, u_{n_j})$ be the first-period actions and first-period continuation payoffs associated with $(\pi_j, u\left(\pi_j\right))$. Since the actions and continuation payoffs maximize the PPE payoff, the Maximum Theorem implies that the actions and continuation payoffs are upper hemi-continuous in expected profits. We therefore have $\lim e_j = e(\pi)$, $\lim \pi_{s_j} = \pi_s(\pi)$, $\lim \pi_{n_j} = \pi_n(\pi)$, $\lim b_{n_j} = b_n(\pi)$, $\lim u_{n_j} = u_n(\pi)$, and $\lim u_{s_j} = u_s(\pi)$. It can be checked that the profile $(e(\pi), b_n(\pi), \pi_s(\pi), \pi_n(\pi), u_s(\pi), u_n(\pi))$ satisfies all the constraints and supports $(\pi,u(\pi))$, which implies that $\pi \in \Pi$. This shows that π_0 is well defined.

Repeating the proof in Lemma 2, it is immediate that Π is convex and thus $\Pi = [\pi_0,\overline{\pi}]$. The part that for all $\pi \geq \pi_0$ the PPE frontier $u(\pi)$ is supported by pure actions follows from the definition of π_0 and the convexity of Π. It remains to show that for any $\pi \in (\underline{\pi},\pi_0)$ the manager and the worker randomize between terminating their relationship and playing the strategies that deliver expected payoffs π_0 and $u(\pi_0)$. Note that if $\pi_0 = \underline{\pi}$, the randomization region does not exist and there is nothing to prove. Therefore, let us assume that $\pi_0 > \underline{\pi}$. Notice that $(\underline{\pi},u(\underline{\pi}))$ is an extremal point of the PPE payoff set, and since $\pi_0 > \underline{\pi}$, $(\underline{\pi},u(\underline{\pi}))$ must be supported by the outside options, and therefore, $(\underline{\pi},u(\underline{\pi})) = (\underline{\pi},\underline{u})$.

Now for any $\pi \in (\underline{\pi},\pi_0)$, $(\pi,u(\pi))$ is sustained by randomization by the definition of π_0. Since the PPE payoff set is two-dimensional and convex and $(\pi, u(\pi))$ is at its boundary, $(\pi, u(\pi))$ can be expressed as the linear combination of two extremal points. Denote these two points by $(\pi', u(\pi'))$ and $(\pi'', u(\pi''))$, where $\pi' < \pi < \pi''$. Since these are both extremal points they are sustained by pure actions. Moreover, since $\pi' < \pi < \pi_0$ and $u(\pi')$ is supported by pure actions, we must have $\pi' = \underline{\pi}$ by the definition of π_0. Now by the concavity of u, it is clear that we can have $\pi'' = \pi_0$. We provide the argument that π'' must equal to π_0 following Lemma B2, whose proof does not depend on the uniqueness of π''. ■

Second, the PPE frontier u is no longer differentiable for all π for some parameter value m. However, since u is again concave (given the public randomization device), both the left and the right derivatives exist. This implies that the results written as equalities of derivatives can be replaced with a pair of corresponding inequalities involving left and right derivatives.

Third, while it remains true that $u'_{-}(\pi) > -1$ for all π, we no longer always have $\pi_n = \overline{\pi}$. When LIQ_F binds, $\pi_n < \overline{\pi}$. The lemma below gives the exact expression for π_n and proves that $u'_{-}(\pi) > -1$ for all π.

LEMMA B2. $u'_{-}(\pi) > -1$, *and the continuation payoff following a no-shock period is given by the following:*

$$\pi_n^* (\pi) = \min\{\overline{\pi}, \frac{1}{\delta} (\pi + (1-\delta)my)\}.$$

Proof: The Lagrangian of the joint surplus associated with the recursive problem is given by

$$\begin{aligned} L \; = \; & (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\pi_n + u(\pi_n)) \\ & + \lambda_1[\pi - (1-\delta) y(e) - \delta\pi_s] \\ & + \lambda_2(\delta\pi_s - \delta\underline{\pi}) + \lambda_3[\pi + (1-\delta) my(e) - \delta\pi_n] + \lambda_4(\delta\overline{\pi} - \delta\pi_n). \end{aligned}$$

The first-order conditions with respect to π_s, π_n and e are given by

$$\theta(1 + u'_{-}(\pi_s)) \geq \lambda_1 - \lambda_2 \geq \theta(1 + u'_{+}(\pi_s)) \tag{4}$$

$$(1-\theta)(1 + u'_{-}(\pi_n)) \geq \lambda_3 + \lambda_4 \geq (1-\theta)(1 + u'_{+}(\pi_n)) \tag{5}$$

$$(1 - \lambda_1 + m\lambda_3) y'(e) - c'(e) = 0. \tag{6}$$

The envelope condition is given by

$$1 + u'_{-}(\pi) \geq \lambda_1 + \lambda_3 \geq 1 + u'_{+}(\pi). \tag{7}$$

Notice that if $u'_-(\pi) > -1$ for all $\pi \in [\pi_0, \overline{\pi})$, the expression for $\pi_n^*(\pi)$ immediately follows. To see this, suppose to the contrary that for some π,

$$\pi_n^*(\pi) < \min\{\overline{\pi}, \frac{1}{\delta}[\pi + (1-\delta)my(e(\pi))]\}.$$

On the one hand, this implies that $\lambda_4(\pi) = \lambda_3(\pi) = 0$ since the associated constraints are not binding. As a result, $u'_+(\pi_n) \leq -1$ by equation (5). On the other hand, consider $\hat{\pi}_n = \pi_n + \varepsilon$ for $\varepsilon > 0$. Since $\pi_n < \overline{\pi}$, we can make ε small enough so that $\hat{\pi}_n < \overline{\pi}$. By the concavity of u, we have

$$u'_+(\pi_n) \geq u'_-(\hat{\pi}_n) > -1,$$

where the second inequality comes from the fact that $u'_-(\pi) > -1$ for all $\pi \in [\pi_0, \overline{\pi})$. This is a contradiction because we have shown that $u'_+(\pi_n) \leq -1$.

Now we prove that $u'_-(\pi) > -1$ for all $\pi \in [\pi_0, \overline{\pi})$ in two steps. In step 1, we show that $u'_-(\pi) \geq -1$ for all $\pi \in [\pi_0, \overline{\pi})$. Suppose to the contrary that $u'_-(\pi) < -1$ for some $\pi \in [\pi_0, \overline{\pi})$. Then there is a PPE with payoff $(\pi, u(\pi))$ such that its first-period actions are denoted by (e, b_n) and its first-period continuation payoffs are denoted by (π_s, π_n, u_s, u_n). Notice that $\lambda_3(\pi)$ has to be zero because otherwise we have

$$\pi_n = \frac{1}{\delta}[\pi + (1-\delta)\, my\,(e)] > \pi.$$

Since $u'_-(\pi_n) \geq -1$ by equation (5), this implies that $u'_-(\pi) \geq u'_-(\pi_n) \geq -1$, contradicting the claim that $u'_-(\pi) < -1$.

Next, since $\lambda_2(\pi) \geq 0$, (4) and (7) imply that

$$1 + u'_-(\pi) \geq \lambda_1 \geq \theta(1 + u'_+(\pi_s)).$$

Since $u'_-(\pi) < -1$, the above inequality means that $u'_+(\pi_s) < -1$. Notice that if $\pi_s < \pi_n$, the concavity of u implies that $u'_+(\pi_s) \geq u'_-(\pi_n) \geq -1$, which contradicts $u'_+(\pi_s) < -1$.

As a result, we must have $\pi_s = \pi_n$. $\mathrm{LTT_N}$ then implies that $b_n = 0$, and the worker's promise-keeping condition then gives that

$$u(\pi) = \delta u(\pi_s) - (1-\delta)c(e).$$

This implies $u(\pi_s) > u(\pi)$, and consequently, $\pi_s < \pi$ given u is concave and $u'_-(\pi) < -1$.

Now consider an alternative strategy profile with first-period actions (e, b_n) and first-period continuation payoffs $(\hat{\pi}_s, \hat{\pi}_n, \hat{u}_s, \hat{u}_n)$, where $\hat{\pi}_k = \pi_k + \varepsilon$ and $\hat{u}_k = u(\hat{\pi}_k)$, for $k = s, n$ and $\varepsilon > 0$.

It can be checked that for small enough ε, the alternative strategy profile is a PPE with payoffs $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned}\hat{\pi} &= \pi + \delta\varepsilon \\ \hat{u} &= u(\pi) + \delta[u(\hat{\pi}_s) - u(\pi_s)].\end{aligned}$$

By the definition of u, we have $\hat{u} \leq u(\hat{\pi})$. Sending ε to zero, we get

$$u'_+(\pi) \geq u'_+(\pi_s).$$

This means $u'_+(\pi) = u'_+(\pi_s)$ since we have shown that $\pi_s < \pi$. Hence, the PPE frontier contains a line segment in $[\pi_s, \pi]$ with a slope strictly below -1. Define the left end point of the line segment as π_l. For all $\pi > \pi_l$, it must be that $\pi_s^*(\pi) = \pi_n^*(\pi) = \pi_l$ because otherwise it contradicts $u'_-(\pi_n) \geq -1$. Moreover, equation (6) and the envelope condition (7) implies that $e(\pi)$ is constant since $\lambda_3(\pi) = 0$. But this contradicts $\mathrm{LPK_M}$

$$\pi = (1-\delta)\, y(e) + \delta\pi_s,$$

since the left hand side is strictly increasing in π and the right hand side is constant. This finishes proving that $u'_-(\pi) \geq -1$ for all $\pi \in [\pi_0, \overline{\pi})$.

Now in step 2, we show that it is impossible that $u'_-(\pi) = -1$ for some $\pi \in [\pi_0, \overline{\pi})$. Suppose the contrary. Then the PPE frontier contains a line segment with slope -1 on the right side. Again define π_l as the left end point of this segment. In order to derive a contradiction, we first show the following result:

$$\text{for any } \pi > \pi_l, \text{ it must be that } e^*(\pi) \equiv e^{FB} \text{ and } \pi_s(\pi) > \pi_l. \tag{8}$$

Note that if $\pi > \pi_l$, it is easy to see that $\lambda_3 = \lambda_4 = 0$ (because otherwise $u'_-(\pi) \geq u'_-(\pi_n) > -1$). In addition, since u is differentiable at π, and $u'(\pi) = -1$, it must be that $\lambda_1 = 0$. Then from (6) we have $e^*(\pi) = e^{FB}$, which solves $y'\left(e^{FB}\right) = c'\left(e^{FB}\right)$. To see that $\pi_s(\pi) > \pi_l$, suppose $\pi_s(\pi) < \pi_l$, then $u'_+(\pi_s) \geq u'_-(\pi_s + \varepsilon) > -1$, for $0 < \varepsilon < \pi_l - \pi_s$. This contradicts condition (4),

$$0 \geq \theta(1 + u'_+(\pi_s)) - \lambda_1 + \lambda_2,$$

due to the facts that $\lambda_1 = 0$ and $\lambda_2 \geq 0$. Suppose $\pi_s(\pi) = \pi_l$, since $e(\pi) = e^{FB}$ for all $\pi > \pi_l$, we have $\pi_s(\pi - \varepsilon) < \pi_s(\pi) = \pi_l$ for some small positive ε. This becomes the case when $\pi_s(\pi) < \pi_l$ for some $\pi > \pi_l$. Similar to the previous analysis, we can derive a contradiction, which completes the proof for result (8).

Finally, result (8) implies that the manager can maximize her pay by always announcing that the state is a shock state, because the worker always makes an effort of e^{FB} and the manager never makes a payment. But this is a contradiction because in this case, the worker's payoff is $-c(e^{FB})$, which is smaller than his outside option. ■

Now we can prove the uniqueness of π'' in Lemma B1.

Proof: Suppose the contrary. Let $u = L(x)$ be the line that passes through $(\underline{\pi}, \underline{u})$ and $(\pi, u(\pi))$, for $x \in [\underline{\pi}, \overline{\pi}]$. Then we have that $u(\pi) = L(\pi)$ and $u(\pi'') = L(\pi'')$. We will derive a contradiction in three steps.

In step 1, we show that $u(x) = L(x)$ for all $x \in [\underline{\pi}, \pi'']$. To see this, note that the payoffs on the line $u = L(x)$ are PPE payoffs, and therefore, $u(x) \geq L(x)$ by the definition of u. Now if $u(x') > L(x')$ for some x', then let $(\pi, \hat{u})$ be the weighted average of $(\underline{\pi}, \underline{u})$ and $(x', u(x'))$, and we obtain that $\hat{u} > L(\pi) = u(\pi)$, which is a contradiction. Thus, the PPE frontier contains a line segment in the left. Let $\pi_r \geq \pi'' > \pi_0$ be the right end point of this segment. Since $u(\overline{\pi}) = \underline{u}$, it cannot be that $\pi_r = \overline{\pi}$.

In step 2, we show that $e(x) \equiv e$ for all $x \in [\pi_0, \pi_r]$. Suppose to the contrary that $e(x_1) \neq e(x_2)$, then following the proof of part (ii.) of Lemma 2, we can find a PPE with payoffs $(\rho x_1 + (1-\rho)x_2, \hat{u})$ for $\rho \in (0,1)$, such that $\hat{u} > u(\rho x_1 + (1-\rho)x_2)$, which contradicts the definition of u.

In step 3, let $(\pi_r, u(\pi_r))$ be supported by a PPE with first-period actions (e_r, b_{nr}) and first-period continuation payoffs $(\pi_{sr}, \pi_{nr}, u_{sr}, u_{nr})$. We show that $\pi_{nr} \leq \pi_r$. Suppose the contrary. The definition of π_r then implies that $u'_{-}(\pi_{nr}) < u'_{-}(\pi_r)$. Since $e(x) \equiv e$ for all $x \in [\pi_0, \pi_r]$ by step 2, the worker's promise-keeping condition $(\pi = (1-\delta)\, y\,(e) + \delta\pi_s)$ then implies that π_s strictly increases with π. Since $\pi_s(\pi_0) \geq \underline{\pi}$, it must be that $\pi_{sr} > \underline{\pi}$. Now consider an alternative strategy profile with first-period actions (e_r, b_{nr}) and first-period continuation payoffs $(\hat{\pi}_{sr}, \hat{\pi}_{nr}, \hat{u}_{sr}, \hat{u}_{nr})$ where $\hat{\pi}_{sr} = \pi_{sr} - \varepsilon$, and $\hat{\pi}_{nr} = \pi_{nr} - \varepsilon$, for $\varepsilon > 0$. When ε is small enough, it can be checked that the alternative strategy profile is a PPE with associated payoff $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned} \hat{\pi} &= \pi_r - \delta\varepsilon \\ \hat{u} &= u(\pi_r) + \theta\delta[u(\hat{\pi}_{sr}) - u(\pi_{sr})] + (1-\theta)\delta[u(\hat{\pi}_{nr}) - u(\pi_{nr})]. \end{aligned}$$

The definition of u gives $\hat{u} \leq u(\hat{\pi})$. Sending ε to zero, we get

$$u'_{-}(\pi_r) \leq \theta u'_{-}(\pi_{sr}) + (1-\theta)u'_{-}(\pi_{nr}).$$

By the definition of π_r and the concavity of u, the above inequality holds only when $u'_{-}(\pi_{sr}) = u'_{-}(\pi_{nr}) = u'_{-}(\pi_r)$, a contradiction.

Finally, the result of step 3 contradicts Lemma B1, which states that

$$\pi_n^*(\pi) = \min\{\overline{\pi}, \frac{1}{\delta}[\pi + (1-\delta)my(e^*(\pi))]\} > \pi, \text{ for all } \pi < \overline{\pi}. \quad \blacksquare$$

Next, the proof that $e > 0$ in the model with liquidity constraints is different from that in the main model. We state this result as a separate lemma below.

LEMMA B3. *For each* $\pi \geq \pi_0$, $e^*(\pi) > 0$.

Proof: We first prove that $e^*(\pi) > 0$ for $\pi > \pi_0$. Suppose to the contrary that there exists a $\pi > \pi_0$ with $e^*(\pi) = 0$. The liquidity constraint then implies that $b_n = 0$, and $\mathrm{LTT_N}$ implies that $\pi_s^*(\pi) = \pi_n^*(\pi)$. As a result, $\mathrm{LPK_M}$ implies that $\pi_s^*(\pi) = \pi_n^*(\pi) = \pi/\delta > \pi$. We now derive a contradiction in three steps.

In step 1, we show that $u'_-(\pi) = u'_-(\pi/\delta)$, so u is a line segment in $[\pi, \pi/\delta]$. To see this, suppose to the contrary that $u'_-(\pi) > u'_-(\pi/\delta)$. Now consider an alternative strategy profile with first-period actions (e, b_n) and first-period continuation payoffs $(\hat{\pi}_s, \hat{\pi}_n, \hat{u}_s, \hat{u}_n)$, where $\hat{\pi}_k = \pi_k - \varepsilon$ and $\hat{u}_k = u(\hat{\pi}_k)$, for $k = s, n$. It can be checked that for small enough ε, the alternative strategy profile is PPE with $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned} \hat{\pi} &= \pi - \delta\varepsilon \\ \hat{u} &= u(\pi) + \delta[u(\pi/\delta - \varepsilon) - u(\pi/\delta)]. \end{aligned}$$

By the definition of u, we have

$$u(\pi - \delta\varepsilon) \geq u(\pi) + \delta[u(\pi/\delta - \varepsilon) - u(\pi/\delta)].$$

Sending ε to 0, we get that

$$u'_-(\pi) \leq u'_-(\pi/\delta),$$

which contradicts the assumption that $u'_-(\pi) > u'_-(\pi/\delta)$.

In step 2, we show that u is a line segment in $[\pi_0, \pi/\delta]$. To see this, for each $\pi' < \pi$, consider a strategy profile with first-period actions $(\hat{e}, \hat{b}_n)$ and first-period continuation payoffs $(\hat{\pi}_s, \hat{\pi}_n, \hat{u}_s, \hat{u}_n)$ such that $\hat{e}(\pi') = \hat{b}_n(\pi') = 0$, $\hat{\pi}_k(\pi') = \pi'/\delta$, and $\hat{u}_k = u(\hat{\pi}_k)$, for $k = s, n$. It can be checked that these strategy profiles are PPEs and that their payoffs lie on the left extension of the line segment between $(\pi, u(\pi))$ and $(\pi/\delta, u(\pi/\delta))$. The concavity of u then implies that these payoffs are on the PPE frontier. It follows that u is a line segment in $[\pi_0, \pi/\delta]$. This implies that the PPE frontier contains a line segment in the left. Let π_r be the right end point of this segment.

In step 3, we derive a contradiction on the left derivative of u for payoffs near $\delta\pi_r$. To do this, first note that the same construction as in step 2 implies that $e^*(\delta\pi_r) = 0$, $\pi_s^*(\delta\pi_r) = \pi_n^*(\delta\pi_r) = \pi_r$. It then follows that for any $\pi' \in (\delta\pi_r, \delta\pi_r + \varepsilon)$ for small enough $\varepsilon > 0$, we must have $\pi_n^*(\pi') > \pi_r$ by $\mathrm{LIQ_F}$ and $\pi_s^*(\pi') > \pi_0$ by the continuity of π_s^*. This implies that $u'_-(\pi_n^*(\pi')) < u'_-(\pi')$ by the definition of π_r. In addition, $\pi_s^*(\pi') > \pi_0$, so $u'_-(\pi_s^*(\pi'))$ exists. Moreover, $u'_-(\pi_s^*(\pi')) \leq u'_-(\pi')$ since u is a line segment in $[\pi_0, \pi_r]$. The inequalities on π_s^* and π_n^* then imply that

$$(1-\theta)u'_-(\pi_n^*(\pi')) + \theta u'_-(\pi_s^*(\pi')) < u'_-(\pi').$$

Now, starting at $(\pi', u(\pi'))$, consider an alternative strategy profile with the same first-period actions, but whose first-period continuation payoffs are given by $(\pi_s^*(\pi') - \varepsilon, u(\pi_s^*(\pi') - \varepsilon))$ and $(\pi_n^*(\pi') - \varepsilon, u(\pi_n^*(\pi') - \varepsilon))$. For small enough ε, it can be checked that this alternative strategy profile is a PPE. Sending ε to 0, we get

$$(1-\theta)u'_-(\pi_n^*(\pi')) + \theta u'_-(\pi_s^*(\pi')) \geq u'_-(\pi'),$$

which contradicts the earlier inequality. This proves that we cannot have $e(\pi) = 0$ for $\pi > \pi_0$.

Finally, suppose to the contrary that $e^*(\pi_0) = 0$. As a result $\pi_s^*(\pi_0) = \pi_n^*(\pi_0) > \pi_0$. The same argument as above then implies that u is a line segment in $[\pi_0, \pi_n^*(\pi)]$, and we can derive the same contradiction as above. ■

The next lemma shows that the PPE frontier for $\pi \geq \pi_0$ can be divided into (at most) three regions. In the right region, the liquidity constraints are slack. In the left region, the liquidity constraints are binding and $\pi_n < \overline{\pi}$. In the middle region, the liquidity constraints are binding and $\pi_n = \overline{\pi}$.

LEMMA B4. *There exists π_1 and π_2 with $\pi_0 \leq \pi_1 \leq \pi_2 < \overline{\pi}$ such that the following holds: (i.) if $\pi > \pi_2$, $\pi_n^*(\pi) = \overline{\pi}$ and $\pi + (1-\delta)my(e^*) > \delta\overline{\pi}$, (ii.) if $\pi \in [\pi_1, \pi_2]$, $\pi_n^*(\pi) = \overline{\pi}$ and $\pi + (1-\delta)my(e^*) = \delta\overline{\pi}$, and (iii.) if $\pi < \pi_1$, $\pi_n^*(\pi) < \overline{\pi}$ and $\pi + (1-\delta)my(e^*) < \delta\overline{\pi}$.*

Proof: To prove part (i.), it suffices to show that for any $\pi' \geq \pi$, $\pi + (1-\delta)my(e^*(\pi)) > \delta\overline{\pi}$ implies $\pi' + (1-\delta)my(e^*(\pi')) > \delta\overline{\pi}$. Now take a manager's payoff π with $\pi + (1-\delta)my(e^*(\pi)) > \delta\overline{\pi}$, the same argument as in Lemma 2 shows that u is differentiable at π with

$$u'(\pi) = -\frac{c'(e^*(\pi))}{y'(e^*(\pi))}.$$

To see this, consider a PPE with payoff $(\pi, u(\pi))$ and first-period actions (e, b_n) and first-period continuation payoffs
Consider an alternative strategy profile with the same first-period continuation payoffs, but with a

different first-period actions $(\hat{e}, b_n)$ where $\hat{e} = e + \varepsilon$ for some $\varepsilon > 0$. For small enough ε, it can be checked that this alternative strategy profile is a PPE with payoffs $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned} \hat{\pi} &= \pi + (1-\delta)\left(y(e+\varepsilon) - y(e)\right) \\ \hat{u} &= u(\pi) - (1-\delta)\left(c(e+\varepsilon) - c(e)\right). \end{aligned}$$

The definition of u implies that $\hat{u} \leq u(\hat{\pi})$. As a result,

$$u\left(\pi + (1-\delta)\left(y(e+\varepsilon) - y(e)\right)\right) \geq u(\pi) - (1-\delta)\left(c(e+\varepsilon) - c(e)\right).$$

Sending ε to zero, we obtain

$$u'_{+}(\pi) \geq -\frac{c'(e)}{y'(e)}.$$

Next, consider an alternative strategy profile with again the same first-period continuation payoffs, but with a different first-period actions $(\hat{e}, b_n)$ where $\hat{e} = e - \varepsilon$ for some $\varepsilon > 0$. By Lemma B3, $e > 0$. Therefore, if ε is sufficiently small, we have that (i.) $\hat{e} > 0$; (ii.) $\pi + (1-\delta)my(\hat{e}) > \delta\overline{\pi}$. One can then check that this alternative strategy is a PPE with payoff $(\hat{\pi}, \hat{u})$, where

$$\begin{aligned} \hat{\pi} &= \pi + (1-\delta)\left(y(e-\varepsilon) - y(e)\right) \\ \hat{u} &= u(\pi) - (1-\delta)\left(c(e-\varepsilon) - c(e)\right). \end{aligned}$$

Similar to the analysis above, we have $\hat{u} \leq u(\hat{\pi})$, which implies

$$u\left(\pi + (1-\delta)\left(y(e-\varepsilon) - y(e)\right)\right) \geq u(\pi) - (1-\delta)\left(c(e-\varepsilon) - c(e)\right).$$

Sending ε to zero, we obtain

$$u'_{-}(\pi) \leq -c'(e)/y'(e).$$

Finally, the concavity of u implies that $u'_{-}(\pi) \geq u'_{+}(\pi)$, which, combined with the above results, gives that

$$u'_{-}(\pi) = u'_{+}(\pi) = -c'(e)/y'(e).$$

Given that $u'(\pi) = -c'(e)/y'(e)$, the concavity of u then implies that

$$\frac{c'(e(\pi'))}{y'(e(\pi'))} > \frac{c'(e(\pi))}{y'(e(\pi))}.$$

As a result, $e(\pi') \geq e(\pi)$, and, thus, $\pi' + (1-\delta)my(e(\pi')) > \delta\overline{\pi}$. This proves part (i.).

Given part (i.), we prove parts (ii.) and (iii.) simultaneously by showing that if $\pi_n(\pi) = \overline{\pi}$, then for all $\pi' > \pi$, $\pi_n(\pi') = \overline{\pi}$. Suppose the contrary. This implies that there exists a pair of $\pi' > \pi$ such that

$$\pi'_n = \pi' + (1-\delta)my(e\left(\pi'\right)) < \pi + (1-\delta)my(e(\pi)) = \overline{\pi}.$$

Now consider an alternative strategy profile with first-period actions $(\widehat{e}, \widehat{b}_n)$ and first-period continuation payoffs $(\widehat{\pi}_s, \widehat{\pi}_n)$ such that $\widehat{e} = e(\pi') + \varepsilon$, $\widehat{\pi}_s = \pi_s(\pi')$, and $\widehat{\pi}_n$ and $\widehat{b}_n$ are chosen appropriately so that the liquidity constraint remains to bind. It can be checked that this alternative strategy profile constitutes a PPE, and its payoffs fall weakly below the PPE frontier. Sending ε to zero, we get

$$\frac{c'\left(e\left(\pi'\right)\right)}{y'\left(e\left(\pi'\right)\right)} \geq (m+1)(1-\theta)(1+u'_+(\pi_n(\pi')) - u'_+(\pi').$$

Similarly, at π, decrease $e(\pi)$ to $e(\pi)-\varepsilon$, keep $\pi_s(\pi)$ the same, and decrease $\pi_n(\pi)$ correspondingly, we get

$$\frac{c'\left(e\left(\pi\right)\right)}{y'\left(e\left(\pi\right)\right)} \leq (m+1)(1-\theta)(1+u'_-(\overline{\pi})) - u'_-(\pi).$$

Since u is concave, we have $u'_+\left(\pi_n(\pi')\right) > u'_-(\overline{\pi})$ and $u'_+(\pi') \leq u'_-(\pi)$. The two inequalities above then imply that

$$\frac{c'\left(e\left(\pi'\right)\right)}{y'\left(e\left(\pi'\right)\right)} \geq \frac{c'\left(e\left(\pi\right)\right)}{y'\left(e\left(\pi\right)\right)},$$

and, thus, $e(\pi') \geq e(\pi)$. But this contradicts

$$\pi' + (1-\delta)my(e\left(\pi'\right)) < \pi + (1-\delta)my(e\left(\pi\right)). \quad \blacksquare$$

Notice that Lemma 6 follows immediately from Lemma B1 and B4. Also notice that the right region always exist (so that $\pi_2 < \overline{\pi}$) because $\pi + (1-\delta)my(e\left(\pi\right)) > \delta\overline{\pi}$ for all $\pi > \delta\overline{\pi}$. In contrast, the middle region or the left region does not always exist. This can occur, for example, when m is large and when $\underline{\pi}$ is large. In this case, the firm's liquidity constraint is always slack, and we return to the main model. At the end of this section, we provide sufficient conditions for the existence of the left and the middle region.

10.2 Dynamics of the Optimal Relational Contract

In this subsection, we study the dynamics of the optimal relational contract. The following lemma characterizes the effort and continuation payoff functions associated with the optimal relational contract.

LEMMA B5. *There exists a unique set of effort and continuation payoffs* $(e^*(\pi), \pi^*_s(\pi), \pi^*_n(\pi))$ *that satisfies the following:*

(*i.*) *For* $\pi > \pi_2$, *the PPE frontier is differentiable with*

$$\frac{c'\left(e^*\right)}{y'\left(e^*\right)} = -u'(\pi)$$

and

$$\theta u'_{+}\left(\pi_s^*\right)-(1-\theta)\leq u'\left(\pi\right)\leq\theta u'_{-}\left(\pi_s^*\right)-(1-\theta).$$

In this region, both e^* *and* π_s^* *weakly increase with* π.

(ii.) For $\pi\in[\pi_1,\pi_2]$, *if* $m\neq 0$, *then*

$$y\left(e^*\right)=\frac{\delta\overline{\pi}-\pi}{(1-\delta)m}\ \text{and}\ \pi_s^*=\frac{(m+1)\pi-\delta\overline{\pi}}{\delta m}.$$

In this region, e^* *strictly decreases with* π *and* π_s^* *strictly increases with* π.

If $m=0$, *then* $\pi_1=\pi_2=\delta\overline{\pi}$. u *is not differentiable at* $\delta\overline{\pi}$, *and* e^* *and* π_s^* *satisfy*

$$-u'_{+}(\delta\overline{\pi})\leq\frac{c'\left(e^*\right)}{y'\left(e^*\right)}\leq -u'_{-}(\delta\overline{\pi})$$

and

$$\theta u'_{+}\left(\pi_s^*\right)-(1-\theta)\leq u'\left(\pi\right)\leq\theta u'_{-}\left(\pi_s^*\right)-(1-\theta).$$

(iii.) For $\pi\in[\pi_0,\pi_1)$, e^*, π_s^*, *and* π_n^* *satisfy*

$$\begin{aligned}(1+m)\left(1-\theta\right)\left(1+u'_{+}\left(\pi_n^*\right)\right)-\frac{c'\left(e^*\right)}{y'\left(e^*\right)} &\leq u'_{+}\left(\pi\right)\leq u'_{-}\left(\pi\right)\\ &\leq (1+m)\left(1-\theta\right)\left(1+u'_{-}\left(\pi_n^*\right)\right)-\frac{c'\left(e^*\right)}{y'\left(e^*\right)}.\end{aligned}\tag{L-e-n}$$

When $\pi_s^*>\underline{\pi}$,

$$\begin{aligned}(1+m)(\theta u'_{+}\left(\pi_s^*\right)-(1-\theta))+\frac{c'\left(e^*\right)}{y'\left(e^*\right)} &\leq mu'_{+}\left(\pi\right)\leq mu'_{-}\left(\pi\right)\\ &\leq (1+m)(\theta u'_{-}\left(\pi_s^*\right)-(1-\theta))+\frac{c'\left(e^*\right)}{y'\left(e^*\right)},\end{aligned}\tag{L-e-s}$$

and

$$\theta u'_{+}\left(\pi_s^*\right)+(1-\theta)\,u'_{+}\left(\pi_n^*\right)\leq u'_{+}\left(\pi\right)\leq u'_{-}\left(\pi\right)\leq\theta u'_{-}\left(\pi_s^*\right)+(1-\theta)\,u'_{-}\left(\pi_n^*\right).\tag{L-s-n}$$

In this region, π_s^* *weakly increases in* π.

Proof: Notice that for $\pi>\pi_2$, the differentiability of the payoff frontier and that $u'(\pi)=-c'\left(e^*\right)/y'\left(e^*\right)$ are both established in the proof of Lemma B4. In addition, the inequalities in this lemma are all equalities if u were differentiable. In this case, the equalities can be obtained directly from the Kuhn-Tucker conditions of Lagrangian associated with the constrained maximization problem (3). The formal proof of the inequalities is standard and is omitted here. Below, we show that u is not differentiable at $\delta\overline{\pi}$ and that π_s^* is weakly increasing in $[\pi_0,\pi_1]$.

To see that u is not differentiable at $\delta\overline{\pi}$ when $m=0$, note that $\pi=\delta\overline{\pi}$ is the only point in the middle region. On the one hand, part (i.) implies that

$$u'_{+}(\delta\overline{\pi})=-\frac{c'\left(e^{*}\right)}{y'\left(e^{*}\right)}.$$

On the other hand, L-e-n implies that

$$u'_{-}(\delta\overline{\pi})\geq-\frac{c'\left(e^{*}\right)}{y'\left(e^{*}\right)}+(1+m)\left(1-\theta\right)\left(1+u'_{+}\left(\pi_{n}^{*}\right)\right).$$

Notice that from Lemma B2, we have $u'_{+}\left(\pi_{n}^{*}\right)>-1$. Therefore, $u'_{-}(\delta\overline{\pi})>u'_{+}(\delta\overline{\pi})$, so u is not differentiable at $\delta\overline{\pi}$.

Next, to see that π_{s}^{*} is weakly increasing for $\pi\in[\pi_{0},\pi_{1}]$, we assume that u is differentiable at π and $\pi_{s}^{*}\left(\pi\right)$ to ease exposition, and the argument can be adapted to the non-differentiable case. When u is differentiable at π and $\pi_{s}^{*}\left(\pi\right)$, L-e-s can be written as

$$-(1+m)\left(1-\theta\right)+(1+m)\theta u'\left(\pi_{s}^{*}\right)+\frac{c'\left(e^{*}\right)}{y'\left(e^{*}\right)}=mu'\left(\pi\right).$$

As π increases, the right hand side of the equation above weakly decreases. Now suppose to the contrary that π_{s}^{*} decreases. It follows that $u'\left(\pi_{s}^{*}\right)$ weakly increases. Moreover, when π increases and π_{s}^{*} decreases, e^{*} increases by the $\mathrm{LPK_M}$. Consequently, $c'\left(e^{*}\right)/y'\left(e^{*}\right)$ strictly increases. In summary, if π_{s}^{*} decreases, the left hand side strictly increases. This is a contradiction because the right hand side weakly decreases.

Finally, we show that $(e^{*}(\pi),\pi_{s}^{*}(\pi),\ \pi_{n}^{*}(\pi))$ is unique. Suppose first that $\pi>\pi_{2}$. Then $\pi_{n}^{*}(\pi)=\overline{\pi}$, $e^{*}(\pi)$ is uniquely determined by $u'\left(\pi\right)$, and $\pi_{s}^{*}(\pi)$ is uniquely determined by $\mathrm{LPK_M}$. Next, suppose that $\pi\in[\pi_{1},\pi_{2}]$. Then $\pi_{n}^{*}(\pi)=\overline{\pi}$, $e^{*}(\pi)$ is uniquely given by $\mathrm{LIQ_F}$, and $\pi_{s}^{*}(\pi)$ is uniquely given by $\mathrm{LPK_M}$. Finally, for $\pi\in[\pi_{0},\pi_{1})$ we prove uniqueness by contradiction. Suppose to the contrary that there are two different PPEs that both generate $\left(\pi,u\left(\pi\right)\right)$. Let the first-period action and the first-period continuation payoffs associated with these two PPEs as $(e_{1},b_{1},\pi_{s_{1}},\ \pi_{n_{1}},u_{s_{1}},u_{n_{1}})$ and $(e_{2},b_{2},\pi_{s_{2}},\pi_{n_{2}},u_{s_{2}},u_{n_{2}})$ respectively. It suffices to show that these two vectors are identical.

Define $\widetilde{e}$ as the effort level satisfying

$$y(\widetilde{e})=\frac{1}{2}y(e_{1})+\frac{1}{2}y(e_{2}).$$

Notice that since y is strictly increasing and strictly concave, $\widetilde{e}\leq\frac{1}{2}e_{1}+\frac{1}{2}e_{2}$. Next, define $\widetilde{b}_{n}=\frac{1}{2}b_{n_{1}}+\frac{1}{2}b_{n_{2}}$, $\widetilde{\pi}_{s}=\frac{1}{2}\pi_{s_{1}}+\frac{1}{2}\pi_{s_{2}}$, and $\widetilde{\pi}_{n},\widetilde{\pi}_{s}$ and $\widetilde{\pi}_{n}$ analogously. Now consider an alternative strategy profile with first-period actions $(\widetilde{e},\widetilde{b}_{n})$ and the first-period continuation payoffs $(\widetilde{\pi}_{s},\widetilde{\pi}_{n},\widetilde{u}_{s},\widetilde{u}_{n})$. It can be checked that the alternative strategy profile satisfies all the constraints in Section 10.1 and therefore constitute a PPE. Moreover, its payoff is given by $(\widetilde{\pi}, \widetilde{u})$, where $\widetilde{\pi} = \pi$, and

$$\widetilde{u} = u(\pi) + (1-\delta)\left(\frac{1}{2}c(e_1) + \frac{1}{2}c(e_2) - c(\widehat{e})\right).$$

Since $c(e)$ is strictly increasing and strictly convex, it follows that $\widetilde{u} > u(\pi)$ unless $e_1 = e_2$. The effort level $e^*(\pi)$ is therefore unique. The constraints LPK_M and LIQ_F then imply the uniqueness of $b_n^*(\pi), \pi_s^*(\pi)$ and $\pi_n^*(\pi)$. This implies that the two PPEs are identical and proves uniqueness.■

Since $\pi_s^*(\pi)$ is weakly increasing in π in all three regions, the continuity of $\pi_s^*(\pi)$ then implies that $\pi_s^*(\pi)$ is weakly increasing for all $\pi \in [\pi_0, \overline{\pi}]$. In contrast, $e^*(\pi)$ is decreasing in the middle region. In other words, the worker's effort level increases as the manager's payoff decreases.

The following two lemmas state the properties of the manager's continuation payoff functions for $\pi \geq \pi_0$ in the optimal relational contract.

LEMMA B6. *For all* $\pi \in [\pi_0, \overline{\pi})$, $\pi_n^*(\pi) > \pi$.

Proof: It is clear that $\pi_n^*(\pi) > \pi$ for $\pi \in [\pi_1, \overline{\pi})$ since $\pi_n^*(\pi) = \overline{\pi}$. For $\pi \in [\pi_0, \pi_1)$, suppose the contrary is true. The continuity of π_n^* then implies that there exists a largest $\pi < \overline{\pi}$ satisfying $\pi_n^*(\pi) = \pi$. Notice that $u'_+(\pi_s^*(\pi)) \geq u'_+(\pi_n^*(\pi)) = u'_+(\pi)$. Therefore, the first inequality in L-s-n implies that $u'_+(\pi_s^*) = u'_+(\pi_n^*)$. In other words, u is a line segment between $\pi_s^*(\pi)$ and $\pi_n^*(\pi)$. Define π_r as the right end point of this segment, and we must have $\pi_r < \overline{\pi}$. Because otherwise, the monotonicity of π_s^* implies that $u'_+(\pi_s^*(\pi')) = u'_+(\pi_n^*(\pi')) = u'_+(\pi')$ for all $\pi' > \pi$. This implies that $u'_-(\pi_s^*(\pi')) = -1$ for $\pi' > \pi_2$ by part (i.) of Lemma B5. But this contradicts $u'_-(\pi_s^*(\pi')) > -1$ by Lemma B2.

Now given $\pi_r < \overline{\pi}$, notice that the continuity of π_n^* then implies that there exists a $\pi'' \in (\pi, \pi_r)$ such that $\pi_n^*(\pi'') > \pi_r$. In addition, $\pi_s^*(\pi'') \in (\pi_s^*(\pi), \pi_r)$ since π_s^* increases with π. At π'', however, $u'_-(\pi_s^*(\pi'')) = u'_-(\pi'') > u'_-(\pi_s^*(\pi''))$, violating the last inequality in L-s-n. ■

LEMMA B7. *For all* $\pi \in (\pi_0, \overline{\pi}]$, $\pi_s^*(\pi) < \pi$.

Proof: It is clear that $\pi_s^*(\pi) < \pi$ for $\pi \geq \pi_1$. For $\pi < \pi_1$, suppose to the contrary there exists a manager's payoff π with $\pi_s^*(\pi) \geq \pi$. By L-s-n, u must be a line segment between π and $\pi_n^*(\pi)$. Let π_r be the right end point of this line segment. Using the same argument as in Lemma B6, we must have $\pi_r < \overline{\pi}$. Now by Lemma B6 and the monotonicity of π_s, there exists a $\pi' \in (\pi, \pi_r)$ such that $\pi_n^*(\pi') > \pi_r$ and $\pi_s^*(\pi') \in (\pi, \pi_+)$. This implies that $u'_-(\pi_s^*(\pi')) = u'_-(\pi') > u'_-(\pi_n^*(\pi'))$, again violating the last inequality in L-s-n. ■

The next lemma provides further information on the dynamics by characterizing π_0.

LEMMA B8. *$\pi_s^*(\pi_0) = \underline{\pi}$ if $\pi_0 > \underline{\pi}$.*

Proof: Suppose $\pi_0 > \underline{\pi}$ and to the contrary $\pi_s^*(\pi_0) > \underline{\pi}$. L-s-n implies that $u'(\pi_0) = u'(\pi_s^*) = u'(\pi_n^*)$. We same argument in Lemma B6 then implies that there exists a $\pi_r > \pi_0$ such that u is a line segment in $[\underline{\pi}, \pi_r]$. Notice that this is a contradiction by the proof of the uniqueness of π'' in the last part of Lemma B1. ■

Proof of Proposition 2: Notice that part (i.) follows from part (iii.) and LTT_N. Part (ii.) is a restatement of Lemma B7 and B8. Part (iii.) follows directly from Lemma B2 and B4. Finally, part (iv.) follows from Lemma B5. ■

10.3 Sufficient Conditions

In this subsection, we first provide a condition for the left region to exist, implying that the liquidity constraint is relevant. A sufficient condition for $\pi_0 > \underline{\pi}$ is given next. Finally, we provide a sufficient condition for the existence of the middle region.

Define $\overline{\pi}^u$ as the maximal equilibrium payoff of the manager in the main model.

LEMMA B9. *The PPE frontier contains more than the right region, i.e., ($\pi_2 > \underline{\pi}$) if and only if the following Condition L holds:*

$$\delta\overline{\pi}^u > (1+m)\underline{\pi}. \tag{L}$$

Proof: The PPE frontier contains more than the right region when the liquidity constraint is violated in the main model for some π. By Proposition 1, this is equivalent to that the liquidity constraint is violated at $\underline{\pi}$. In our main model, $\pi_n^*(\underline{\pi}) = \overline{\pi}^u$ and $\pi_s^*(\underline{\pi}) = \underline{\pi}$. In addition, NR_S states that

$$\underline{\pi} = \delta\pi_s^*(\underline{\pi}) + (1-\delta)y(e^*(\underline{\pi})).$$

This implies that $y(e^*(\underline{\pi})) = (1-\delta)\underline{\pi}$. Therefore, the liquidity constraint that $\delta\pi_n^* \leq \pi + (1-\delta)my(e^*)$ is equivalent to

$$\delta\overline{\pi}^u \leq (1+m)\underline{\pi}. \;\blacksquare$$

Next, we describe a sufficient condition for $\pi_0 > \underline{\pi}$.

LEMMA B10. *Suppose Condition L holds. $\pi_0 > \underline{\pi}$ if $m < \frac{\theta}{1-\theta}$ and*

$$\underline{u} > \frac{(1-\theta)(1+m)}{\theta - m(1-\theta)}\frac{y'(\underline{e}) - c'(\underline{e})}{y'(\underline{e})}\underline{\pi} - c(\underline{e}),$$

where $\underline{e}$ is the unique effort level satisfying $y(\underline{e}) = \underline{\pi}$.

Proof: To prove that $\pi_0 > \underline{\pi}$ if the conditions above hold, we proceed as if u were differentiable to simplify the exposition. The argument can be adapted for the non-differentiable case by replacing the equalities involving derivatives with inequalities involving left and right derivatives. Now suppose to the contrary $\pi_0 = \underline{\pi}$, define the Lagrangian as

$$\begin{aligned}\pi + u(\pi) &= L = (1-\delta)\left(y\left(e\right) - c\left(e\right)\right) + \theta\delta\left(\pi_s + u\left(\pi_s\right)\right) + (1-\theta)\,\delta\left(\pi_n + u\left(\pi_n\right)\right) \\ &\quad + \lambda_1(\pi - (1-\delta)\,y\left(e\right) - \delta\pi_s) \\ &\quad + \lambda_2(\pi + (1-\delta)my(e) - \delta\pi_n) \\ &\quad + \lambda_3(\delta\pi_s - \delta\underline{\pi}) + \lambda_4\delta(\overline{\pi} - \pi_n).\end{aligned}$$

The first-order conditions are given by

$$\theta(1 + u'\left(\pi_s\right)) - \lambda_1 + \lambda_3 = 0. \tag{FOC$_\mathrm{S}$}$$

$$(1-\theta)\left(1 + u'\left(\pi_n\right)\right) - \lambda_2 - \lambda_4 = 0. \tag{FOC$_\mathrm{N}$}$$

$$(1 - \lambda_1 + m\lambda_2)y'\left(e\right) - c'\left(e\right) = 0. \tag{FOC$_\mathrm{e}$}$$

The envelope condition is given by

$$1 + u'(\pi) = \lambda_1 + \lambda_2. \tag{envelope condition}$$

We now proceed in two steps. In step 1, we provide an upper bound to $u'(\underline{\pi})$. To do this, notice that Condition L implies that $\pi_n(\underline{\pi}) < \overline{\pi}$. $\mathrm{FOC_N}$ then implies that at $\pi = \underline{\pi}$,

$$(1-\theta)(1 + u'\left(\pi_n(\underline{\pi})\right)) = \lambda_2.$$

By the envelope condition,

$$1 + u'(\underline{\pi}) = \lambda_1 + \lambda_2.$$

Since u is concave, $u'(\underline{\pi}) \geq u'\left(\pi_n\right)$. This then implies that $\lambda_2 \leq (1-\theta)\lambda_1/\theta$, and it follows that

$$1 + u'(\underline{\pi}) \leq \lambda_1 + \frac{1-\theta}{\theta}\lambda_1 = \frac{\lambda_1}{\theta}.$$

Next, we provide an upper bound of λ_1. Note that at $\underline{\pi}$, Proposition 2 implies that $\pi_s(\underline{\pi}) = \underline{\pi}$, and consequently, by $\mathrm{LTT_N}$, $y(e(\underline{\pi})) = \underline{\pi}$. In other words, $e(\underline{\pi}) = \underline{e}$. $\mathrm{FOC_e}$ then implies that

$$\frac{y'\left(\underline{e}\right) - c'\left(\underline{e}\right)}{y'(\underline{e})} = \lambda_1 - m\lambda_2 \geq \frac{\theta - m(1-\theta)}{\theta}\lambda_1,$$

where the inequality follows because $\lambda_2 \leq (1-\theta)\lambda_1/\theta$ as shown above.

Combining the two inequalities above and noting $\theta - m(1-\theta) > 0$, we then obtain

$$1 + u'(\underline{\pi}) \leq \frac{\lambda_1}{\theta} \leq \frac{1}{\theta - m(1-\theta)} \frac{y'(\underline{e}) - c'(\underline{e})}{y'(\underline{e})},$$

which concludes step 1.

In step 2, we derive a contradiction on the joint payoff at $\underline{\pi}$ using the upper bound in step 1. Since the liquidity constraint binds at $\underline{\pi}$,

$$\delta\pi_n(\underline{\pi}) = \underline{\pi} + (1-\delta)my(\underline{e}) = (1 + (1-\delta)m)\underline{\pi}.$$

It follows that

$$\delta(\pi_n(\underline{\pi}) - \pi_s(\underline{\pi})) = (1-\delta)(1+m)\underline{\pi}.$$

The concavity of u then implies that

$$\begin{aligned} & u(\pi_n(\underline{\pi})) - u(\pi_s(\underline{\pi})) \\ \leq\ & u'(\pi_s(\underline{\pi}))(\pi_n(\underline{\pi}) - \pi_s(\underline{\pi})) \\ =\ & u'(\underline{\pi})\left(\frac{(1-\delta)(1+m)}{\delta}\right)\underline{\pi}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \underline{\pi} + u(\underline{\pi}) \\ =\ & (1-\delta)(y(\underline{e}) - c(\underline{e})) + \delta(\pi_s(\underline{\pi}) + u(\pi_s(\underline{\pi}))) \\ & + (1-\theta)\delta((\pi_n(\underline{\pi}) + u(\pi_n(\underline{\pi})) - ((\pi_s(\underline{\pi}) + u(\pi_s(\underline{\pi}))) \\ \leq\ & (1-\delta)(\overline{\pi} - \underline{c}) + \delta(\underline{\pi} + u(\underline{\pi})) + (1-\theta)(1-\delta)(1+m)(1+u'(\underline{\pi}))\underline{\pi}. \end{aligned}$$

Rearranging the above and substituting the inequality at the end of step 1, we get

$$\underline{\pi} + u(\underline{\pi}) \leq \overline{\pi} - \underline{c} + \frac{(1-\theta)(1+m)}{\theta - m(1-\theta)} \frac{y'(\underline{e}) - c'(\underline{e})}{y'(\underline{e})}\underline{\pi},$$

which contradicts the condition in the lemma. ■

Finally, we provide a sufficient condition for the middle region to exist.

LEMMA B11. *Suppose Condition L holds. The middle region exists, i.e.,* $(\pi_2 > \pi_1)$ *if* $m < \theta/(1-\theta)$ *and* $(1+m)^2(1-\theta)\theta/m < 1$.

Proof: Condition L implies that the PPE frontier contains more than the right region. Suppose to the contrary that the middle region does not exist. Let π_d be the payoff that divides the left and the right region. The same argument in Proposition 2 shows that u is not differentiable at π_d with

$$\begin{aligned} u'_{+}(\pi_d) &= -\frac{c'(e)}{y'(e)}, \text{ and} \\ u'_{-}(\pi_d) &= (1+m)(1-\theta)(1+u'(\overline{\pi})) - \frac{c'(e)}{y'(e)}. \end{aligned}$$

Let $\Delta u'(\pi_d) = u'_{+}(\pi_d) - u'_{-}(\pi_d) > 0$. L-e-s then implies that

$$\Delta u'(\pi_d) \leq \frac{(1+m)\theta}{m} \Delta u'(\pi_s(\pi_d)).$$

In other words, u is not differentiable at $\pi_s(\pi_d)$.

Note that by L-e-n, we have

$$\Delta u'(\pi_s(\pi_d)) \leq (1+m)(1-\theta)\Delta u'(\pi_n(\pi_s(\pi_d))).$$

This implies that u is not differentiable at $\pi_n(\pi_s(\pi_d))$.

Since u is differentiable for all $\pi \in (\pi_d, \overline{\pi}]$, the above implies that either $\pi_n(\pi_s(\pi_d)) = \pi_d$ or $\pi_n(\pi_s(\pi_d)) \in (\pi_s(\pi_d), \pi_d)$. In the later case, we can show, using the same argument as above, that either $\pi_n^2(\pi_s(\pi_d)) = \pi_d$ or $\pi_n^2(\pi_s(\pi_d)) \in (\pi_n(\pi_s(\pi_d)), \pi_d)$, where the superscript denotes that applying π_n twice. Since $\pi_n > \pi$, the sequence of π_n^k is monotone in k. It follows that there exists some K such that

$$\pi_d = \pi_n^K(\pi_s(\pi_d)).$$

Note that for all $k \leq K$, we have by above

$$\Delta u'(\pi_n^k(\pi_s(\pi_d))) \leq (1+m)(1-\theta)\Delta u'(\pi_n^{k+1}(\pi_s(\pi_d))).$$

Linking this chain of inequalities, we obtain

$$\Delta u'(\pi_d) \leq \frac{(1+m)\theta}{m}(1+m)^K(1-\theta)^K \Delta u'(\pi_d).$$

This is a contradiction because by assumption $(1+m)(1-\theta) < 1$, and $(1+m)^2(1-\theta)\theta/m < 1$, so for all $K \geq 1$

$$\frac{(1+m)\theta}{m}(1+m)^K(1-\theta)^K < 1. \quad \blacksquare$$

11 Appendix C (For Online Publication): Discussion

In this appendix, we prove the results in Section 7.

11.1 Folk Theorem and Conditions for First-Best

While the first-best is not achievable, the Folk Theorem holds: any interior payoff of the feasible payoff set belongs to the PPE payoff set as $\delta \to 1$.

PROPOSITION C1. *Define* $\pi^{FB} \equiv y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u}$. *For all* $\pi \in [\underline{\pi}, \pi^{FB})$ *and* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) \in (0,1)$ *such that for all* $\delta \geq \delta(\varepsilon)$,

$$u(\pi) + \pi > y\left(e^{FB}\right) - c\left(e^{FB}\right) - \varepsilon.$$

Proof: We prove the equivalent statement that for all π, the expected surplus destruction $D(\pi|\delta) \equiv y(e^{FB}) - c(e^{FB}) - \pi - u(\pi)$ goes to 0 as $\delta \to 1$. Since $u'(\pi) > -1$ by Lemma 2, $D(\pi|\delta)$ is decreasing in π and is thus maximized at $\pi = \underline{\pi}$. As a result, it suffices to show that $D(\underline{\pi}|\delta)$ goes to 0 as $\delta \to 1$.

Define d as the surplus destruction in the first-period play of the optimal relational contract at $\underline{\pi}$. It follows from Proposition 1 that

$$D(\underline{\pi}|\delta) = (1-\delta)\, d + \delta\left((1-\theta)\, D(\overline{\pi}|\delta) + \theta D(\underline{\pi}|\delta)\right),$$

or equivalently,

$$D(\underline{\pi}|\delta) = \frac{(1-\delta)\, d + \delta(1-\theta)\, D(\overline{\pi}|\delta)}{1-\delta\theta}.$$

This implies that

$$\lim_{\delta\to 1} D(\underline{\pi}|\delta) = \lim_{\delta\to 1} D(\overline{\pi}|\delta).$$

Since $D(\overline{\pi}|\delta) = y\left(e^{FB}\right) - c\left(e^{FB}\right) - \overline{\pi} - \underline{u} = \pi^{FB} - \overline{\pi}$, the proof is complete if we can show that for any $\varepsilon > 0$, $(\pi^{FB} - \varepsilon, \underline{u})$ can be sustained as a PPE payoff as $\delta \to 1$.

To do this, consider the following sequence of conjectured relational contracts in which the players take their outside options forever if any party deviates, and on the equilibrium path they choose $e = e^{FB}$, $w = 0$, and

$$b_t = \begin{cases} 0 & \textit{if } \Theta = s \textit{ and } t < t_n + T \\ \left(c\left(e^{FB}\right) + \underline{u}\right)\left(1 + \delta^{-1} + \delta^{-2} + \ldots + \delta^{-(t-1-t_n)}\right) & \textit{otherwise,} \end{cases}$$

where t_n is the largest previous period (t) in which the bonus (b_t) is positive and T is an exogenously given deadline.

This sequence of conjectured relational contracts is the same as the long-term contracts described in Proposition C4 with the extra requirement that the manager must not renege on the bonuses. When the manager can commit to the bonus, we show in Proposition C4 that the conjecture relational contracts above are incentive contractible for both players. We avoid repeating the argument here but mention that the proof of Proposition C4 does not depend on previous results.

Notice that for all T, each conjectured relational contract above gives the worker a payoff of $\underline{u}$. Define $\pi(T)$ as the associated manager's payoff. The proof of Proposition C4 implies that the normalized expected destruction of surplus is given by

$$\delta\theta\frac{1-\theta\delta}{(1-\delta)}\frac{\theta^T\left(1-\delta^T\right)}{\left(1-(\theta\delta)^{T+1}\right)}\alpha c\left(e^{FB}\right) < T\theta^T\alpha c\left(e^{FB}\right).$$

Notice that $T\theta^T\alpha c\left(e^{FB}\right)$ goes to 0 as $T\to\infty$. As a result, for any $\varepsilon>0$, there exists a $T(\varepsilon)$, independent of δ, such that $\pi(T(\varepsilon))>\pi^{FB}-\varepsilon$. Moreover, for the fixed $T(\varepsilon)$, the maximal bonus is bounded above by $2^T\left(c(e^{FB})+\underline{u}\right)$ for all $\delta\in(1/2,1)$. It follows that as $\delta\to 1$, it is incentive compatible for the manager to commit to the bonus, so the conjectured relational contract is a PPE. This shows that $\left(\pi^{FB}-\varepsilon,\underline{u}\right)$ can be sustained as a PPE payoff and finishes the proof. ■

Our next proposition shows that to obtain the first-best, it is necessary to have (partial) public observability of the *no-shock* states rather than the *shock* states. Specifically, suppose that when the firm is not hit by a shock, with probability $p\in[0,1)$ it becomes publicly known that the firm's opportunity costs are low. When the firm is hit by a shock, with probability $q\in[0,1)$ it becomes publicly known that the firm's opportunity costs are high. First-best can then be achieved (for sufficiently high discount factors) if and only if $p>0$.

PROPOSITION C2. *If $p=0$, there does not exist a PPE in which the joint payoff of the manager and worker is equal to $y\left(e^{FB}\right)-c\left(e^{FB}\right)$. Otherwise, when*

$$\delta\geq\frac{c\left(e^{FB}\right)+\underline{u}}{c\left(e^{FB}\right)+\underline{u}+(1-\theta)\,p\left(y\left(e^{FB}\right)-c\left(e^{FB}\right)-\underline{u}-\underline{\pi}\right)},$$

there exists a PPE such that the joint payoff of the manager and worker is equal to $y\left(e^{FB}\right)-c\left(e^{FB}\right)$.

Proof: Since $q\geq 0$, define b_{sk} as the bonus payment when it is publicly known that the firm is hit by a shock and b_{su} stands for the bonus payment when the shock is unknown to the worker. Similarly, define π_{sk} and π_{su} as the associated continuation payoffs.

First consider $p=0$. Suppose to the contrary that the first-best can be obtained. Let π_f be the smallest PPE payoff of the manager in which first-best is achieved. We derive a contradiction below by showing $\pi_f > y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u}$, which is the maximal feasible payoff of the manager that gives the worker a payoff of least $\underline{u}$.

Notice that if the first-best is obtained at π_f, we must have $w = b_{sk} = b_{su} = 0$ and $e = e^{FB}$. Moreover, the continuation payoffs must weakly exceed π_f since π_f is the smallest PPE payoff to obtain the first-best.

The promise-keeping constraint of the manager then implies that

$$\begin{aligned}
\pi_f &= (1-\delta)\left(y\left(e^{FB}\right) - (1-\theta)\, b_n\right) + \delta\left(\theta q \pi_{sk} + \theta(1-q)\,\pi_{su} + (1-\theta)\,\pi_n\right) \\
&\geq (1-\delta)\, y\left(e^{FB}\right) + \delta\left(\theta q \pi_{sk} + (1-\theta q)\,\pi_{su}\right) \\
&\geq (1-\delta)\, y\left(e^{FB}\right) + \delta \pi_f,
\end{aligned}$$

where the first inequality follows from the manager's truth-telling constraint in the no-shock state ($\delta\left(\pi_n - \pi_{su}\right) \geq (1-\delta)\left(b_n - b_{su}\right) = (1-\delta)\, b_n$) and the second inequality follows because the continuation payoffs weakly exceed π_f. But this implies $\pi_f \geq y\left(e^{FB}\right) > y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u}$, which is a contradiction as noted above.

Next, consider $p > 0$. We construct a PPE that reaches the first-best. In particular, along the equilibrium path, let $w = b_{sk} = b_{su} = 0$ and $e = e^{FB}$. Moreover, when it is publicly known that the firm's opportunity costs are low, the manager pays out a bonus $b_{nk} = \left(c\left(e^{FB}\right) + \underline{u}\right) / \left((1-\theta)\, p\right)$. Notice that the equilibrium play does not depend on the manager's private information. When any party deviates, the play reverts to the unique subgame perfect Nash equilibrium within each period and the parties take their outside options in all future periods.

To check that this an equilibrium, we need to show, first, that the worker will put in effort and is willing to participate:

$$c\left(e^{FB}\right) \leq (1-\theta)\, p b_{nk} \text{ and } -c\left(e^{FB}\right) + (1-\theta) p b_{nk} \geq \underline{u}.$$

Given $b_{nk} = \left(c\left(e^{FB}\right) + \underline{u}\right) / \left((1-\theta)\, p\right)$, both inequalities are clearly satisfied.

Moreover, we need to show that the manager will not renege on the bonus and is willing to participate. The participation constraint is clearly implied by the non-reneging constraint, which is given by

$$(1-\delta)\, b_{nk} \leq \delta\left(y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u} - \underline{\pi}\right).$$

Given $b_{nk} = \left(c\left(e^{FB}\right) + \underline{u}\right) / \left((1-\theta)\, p\right)$, the inequality above is equivalent to

$$\delta \geq \frac{c\left(e^{FB}\right) + \underline{u}}{c\left(e^{FB}\right) + \underline{u} + (1-\theta)\, p\left(y\left(e^{FB}\right) - c\left(e^{FB}\right) - \underline{u} - \underline{\pi}\right)},$$

which is the condition in the proposition. Therefore, the first-best can be obtained if the condition above is satisfied. ■

11.2 Benchmarks: Public Information and Long-term Contracts

In this subsection, we analyze the dynamics of the relationship when the state of the world is public information. We characterize the PPE frontier, and for each payoff pair on the frontier, we state the associated effort, base wage, bonuses, and the continuation payoffs. This essentially specifies the dynamics of the relationship. Since the analysis is similar to and simpler than that in the private information case, we only state and prove the main results.

LEMMA C1. *With public information, the PPE frontier satisfies the following. For each PPE payoff of the manager π,*

$$\pi + u(\pi) = \max_{e,w,\pi_s,b_n} (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\delta(\overline{\pi} + u(\overline{\pi})) - (1-\delta)\theta\alpha w \tag{Public Program}$$

subject to

$$\pi = (1-\delta)(y(e) - (1+\theta\alpha)w - (1-\theta)b_n) + \delta(\theta\pi_s + (1-\theta)\overline{\pi}), \tag{PK$_\mathrm{M}$}$$

$$w \geq 0, \ \ \textit{and} \tag{NN$_\mathrm{W}$}$$

$$(1-\delta)b_n \leq \delta(\overline{\pi} - \underline{\pi}). \tag{NR$_\mathrm{S}$}$$

Lemma C1 directly corresponds to Lemma 5 in the main model. As in the main model, $b_s \equiv 0$ and $\pi_n \equiv \overline{\pi}$, so they do not appear as choice variables. Notice that unlike the main model, there can be multiple choices for π_n. The multiplicity arises only when the first-best is achievable and does not affect the dynamics. We choose $\pi_n \equiv \overline{\pi}$ to better connect our results here to those in the main model. Another difference is that the maximization problem does not contain the truth-telling constraint in the no-shock state ($(1-\delta)(b_n - b_s) = \delta(\pi_n - \pi_s)$) since information is public. This implies that b_n is now a choice variable included in the program. As a result, we need to include the non-reneging constraint in the no-shock state. Since the proof of Lemma C1 is essentially the same as that of Lemma 5, we omit it here. The next proposition is the main result of this subsection.

PROPOSITION C3. *For any level of expected profit π, the PPE payoff frontier $u(\pi)$ and associated actions and continuation payoffs satisfy the followings:*

(*i.*) *For all $\pi \geq \underline{\pi}$,*

$$u'(\pi) \leq -\frac{1}{1+\theta\alpha}.$$

(ii.) *For all $\pi \geq \underline{\pi}$, the associated effort level is given by*

$$\frac{c'(e)}{y'(e)} = -u'(\pi).$$

(iii.) *(The middle region) When $u'(\pi) \in (-1, -1/(1+\theta\alpha))$, the associated wage, shock-state continuation payoff and the bonus are given by*

$$w = 0,\ \pi_s = \pi,\ \text{and } b_n = \frac{\delta}{1-\delta}(\overline{\pi} - \underline{\pi}).$$

In this region, $u'(\pi)$ is strictly decreasing.

(iv.) *(The right region) When $u'(\pi) = -1$, the PPE payoff frontier reaches the first-best at π, i.e., $u(\pi) = y(e^{FB}) - c(e^{FB}) - \pi$. The associated wage and effort are given by*

$$w = 0 \text{ and } e = e^{FB}.$$

There can be multiple choices of π_s and b_n. One such choice is

$$\pi_s = \pi \text{ and } b_n = \frac{(1-\delta)\,y(e) - (1+\delta\theta)\,\pi - \delta(1-\theta)\,\overline{\pi}}{(1-\delta)(1-\theta)},$$

where $\overline{\pi} = y(e^{FB}) - c(e^{FB}) - \underline{u}$.
The right region exists if and only if

$$\frac{1-\delta}{1-\theta}\left(c(e^{FB}) + \underline{u}\right) \leq \delta\left(y(e^{FB}) - c(e^{FB}) - \underline{u} - \underline{\pi}\right).$$

In this case, its left boundary is given by $\left((1-\delta)y(e^{FB}) + (1-\theta)\,\delta\underline{\pi}\right)/(1-\delta\theta)$.

(v.) *(The left region) When $u'(\pi) = -1/(1+\theta\alpha)$, the effort level and the bonus are given by*

$$e = \widehat{e},\ \text{and } b_n = \frac{\delta}{1-\delta}(\overline{\pi} - \underline{\pi}),$$

where recall $\widehat{e}$ is the unique effort level satisfying $c'(e)/y'(e) = 1/(1+\alpha\theta)$. There can be multiple choices of π_s and w. One such choice is

$$\pi_s = \pi \text{ and } w = \frac{1}{(1-\delta)(1+\alpha\theta)}\left((1-\delta)\,y(\widehat{e}) + \delta(1-\theta)\,\underline{\pi} - (1-\delta\theta)\,\pi\right).$$

The left region exists if $y(\widehat{e}) > \underline{\pi}$. In this case, the right boundary of this region is given by $((1-\delta)\,y(\widehat{e}) + \delta(1-\theta)\,\underline{\pi})/(1-\delta\theta)$.

Proof: Using Lemma C1, we define the Lagrangian associated with $\pi + u(\pi)$ as

$$\begin{aligned} L &= (1-\delta)(y(e) - c(e)) + \theta\delta(\pi_s + u(\pi_s)) + (1-\theta)\,\delta(\overline{\pi} + u(\overline{\pi})) - (1-\delta)\,\theta\alpha w \\ &\quad + \lambda_1(\pi - (1-\delta)(y(e) - (1+\theta\alpha)\,w - (1-\theta)\,b_n) - \delta(\theta\pi_s + (1-\theta)\,\overline{\pi})) \\ &\quad + \lambda_2(1-\delta)w + \lambda_3(\delta(\overline{\pi} - \underline{\pi}) - (1-\delta)b_n). \end{aligned}$$

The first-order conditions and the envelope condition are given by

$$1 + u'(\pi_s) = \lambda_1. \tag{FOC$_\mathrm{S}$}$$

$$-\theta\alpha + \lambda_1 (1 + \theta\alpha) + \lambda_2 = 0. \tag{FOC$_\mathrm{W}$}$$

$$y'(e) - c'(e) = \lambda_1 y'(e). \tag{FOC$_\mathrm{e}$}$$

$$\lambda_1 (1 - \theta) = \lambda_3. \tag{FOC$_\mathrm{N}$}$$

$$1 + u'(\pi) = \lambda_1. \tag{envelope}$$

For part (i.), notice that $\mathrm{FOC_W}$ implies $\lambda_1 \leq \theta\alpha/(1+\theta\alpha)$. The envelope condition then implies that $u'(\pi) \leq -1/(1+\alpha\theta)$.

For part (ii.), notice that $\mathrm{FOC_e}$ implies $\lambda_1 = (y'(e) - c'(e))/y'(e)$. Using this expression to substitute for λ_1 in the envelope condition, we obtain the formula for effort.

For part (iii.), we first derive the expression for b_n. Notice that if $u'(\pi) \in (-1, -1/(1+\theta\alpha))$, the envelope condition implies that $\lambda_1 > 0$. It then follows from $\mathrm{FOC_N}$ that $\lambda_3 > 0$. As a result, the complementarity slackness condition associated with λ_3 implies that $b_n = \delta(\overline{\pi} - \underline{\pi})/(1-\delta)$.

Next, we show that $w = 0$. To see this, notice that when $u'(\pi) < -1/(1+\theta\alpha)$, the envelope condition implies that $-\theta\alpha + \lambda_1(1+\theta\alpha) < 0$. Consequently, $\mathrm{FOC_W}$ implies that $\lambda_2 > 0$. As a result, the complementarity slackness condition associated with λ_2 implies that $w = 0$.

Finally, we show that $\pi_s = \pi$. Comparing the $\mathrm{FOC_S}$ with the envelope condition, we obtain that $u'(\pi_s) = u'(\pi)$. This implies that $\pi_s = \pi$ unless there is an interval of manager's payoffs in which $u'(\pi)$ is constant. To see that such an interval cannot exist, suppose the contrary. Notice that by using the expressions for b_n and w, we can rewrite $\mathrm{PK_M}$ as

$$\pi = (1-\delta)\, y(e) + \delta(\theta\pi_s + (1-\theta)\,\underline{\pi}).$$

Moreover, part (ii.) then implies that $y(e)$ is a constant in the interval. $\mathrm{PK_M}$ then implies that for each π in the interval, $d\pi_s/d\pi = 1/\delta\theta > 1$. But this is a contradiction unless the length of the interval is zero. This proves $\pi_s = \pi$.

For part (iv.), notice that when $u'(\pi) = -1$, the envelope condition implies that $\lambda_1 = 0$. It then follows from $\mathrm{FOC_W}$ that $\lambda_2 > 0$, and therefore, the associated complementarity condition implies that $w = 0$. Notice that $\lambda_1 = 0$ also implies $e = e^{FB}$ by $\mathrm{FOC_e}$, so efficient actions are taken in this region. Moreover, $\lambda_1 = 0$ implies that $u'(\pi_s) = -1$ by $\mathrm{FOC_S}$. Now by choosing $\pi_s = \pi$ and $\pi_n = \overline{\pi}$, the PPE payoff frontier in the region $[\pi, \overline{\pi}]$ becomes self-generating and reaches the first-best. Notice that the expression for b_n follows from $\mathrm{PK_M}$.

Finally, for this region to exist, a necessary and sufficient condition is that, at $\pi = \overline{\pi}$, the non-reneging constraint is satisfied. PK_{M} at $\overline{\pi}$ implies that

$$b_n(\overline{\pi}) = \frac{y\left(e^{FB}\right) - \overline{\pi}}{1-\theta} = \frac{c\left(e^{FB}\right) + \underline{u}}{1-\theta},$$

where the second inequality uses that the first-best is obtained at $\overline{\pi}$. Substituting this into the non-reneging constraint, we obtain the necessary and sufficient condition in part (iv.). Moreover, the left boundary of the region must satisfy that (a.) its shock-state continuation payoff must remains at the boundary and (b.) its non-reneging constraint must bind. Substituting these into the PK_{M}, we obtain the expression for the left boundary in part (iv.).

For part (v.), notice that $e = \widehat{e}$ by part (ii.). In addition, since $\lambda_1 > 0$, FOC_{N} implies that $\lambda_3 > 0$, and therefore, the associated complementarity slackness condition implies that $(1-\delta)\, b_n = \delta\left(\overline{\pi} - \underline{\pi}\right)$. The PK_{M} becomes $\pi = (1-\delta)\, y(\widehat{e}) - (1+\alpha\theta)\, w + \delta\left(\theta\pi_s + (1-\theta)\,\underline{\pi}\right)$ by substituting for e and b_n. Since $u'(\pi)$ is constant in this region, there is some flexibility is choosing π_s. To be consistent with our choice in the middle and the right region, we choose $\pi_s = \pi$. The PK_{M} above then gives the expression for w in the proposition.

Finally, for this region to exist, a necessary and sufficient condition is that at $\pi = \underline{\pi}$, we have $w > 0$. Note that $\pi_s(\underline{\pi}) = \underline{\pi}$, the PK_{M} above then implies that $w > 0$ is equivalent to $\underline{\pi} < y(\widehat{e})$. Moreover, the right boundary of this region must satisfy that (a.) its shock-state continuation payoff must remains on the boundary and (b.) $w = 0$. Substituting these into the PK_{M}, we obtain the expression for the right boundary in part (v.). ■

We finish by showing that if long-term contracts are feasible, the first-best can be arbitrarily approximated. Define $h^t \equiv \{y_1, ...y_t\}$ as the history of outputs, $m^t \equiv \{m_1, ..m_t\}$ as the history of reports, and denote $b_t\left(h^t, m^t\right)$ as the manager's payment to the worker in period t. Let $t_n(t)$ be the last period before t in which the manager reports a no-shock state, and $t_n(t) = 0$ if the manager has never reported a no-shock state.

PROPOSITION C4. *As T approaches ∞, the following sequence of contracts approaches first-best. The worker chooses $e_t = e^{FB}$. The manager reports the state of the world truthfully and pays out $b_t\left(h^t, m^t\right) = 0$ if $h^t \neq \{y^{FB}, .., y^{FB}\}$, or $m_t = s$ and $t < t_n(t) + T$ and otherwise*

$$b_t\left(h^t, m^t\right) = \left(c\left(e^{FB}\right) + \underline{u}\right)\left(1 + \delta^{-1} + ... + \delta^{-(t-1-t_n(t))}\right)$$

Proof: To simplify the exposition, we normalize $\underline{u}$ to zero. We first show that these contracts are incentive compatible for sufficiently large T and then show that the surplus destruction goes to zero under this sequence of contracts. To check the worker's incentive constraints for effort provisions are satisfied, notice that the worker's payoff is 0 if $e_t = e^{FB}$. Moreover, any other effort choice of the worker leads to a non-positive payoff. Therefore, it is optimal for the worker to accept the contract and choose $e_t = e^{FB}$ for all t.

To check the manager's incentive to report the state of the world truthfully, notice that the manager's payoff is equal to the value of the relationship since the worker's payoff is always zero. As a result, the manager's payoff is maximized when the surplus destruction, i.e., the expected payment in shock states, is minimized. This immediately implies that in a no-shock state the manager will report $m_t = n$ and pay the bonus.

To check that the manager will report $m_t = s$ in a shock state, first notice that the contract repeats itself whenever a no-shock state is reported. This implies that the optimal reporting strategy repeats itself following each restart. Since the manager will always report truthfully in a no-shock state (and, thus, triggers to contract to restart), her reporting strategy is then completely determined by her reports following τ consecutive shock periods for each $\tau \leq T$. Moreover, since the contract restarts itself following each no-shock state is reported, the manager's strategy is determined by the smallest number of consecutive shock-states following which a no-shock state is reported. Denote this number by N, and notice that if $N = T$, this means that the manager is truth-telling. Define D_i^N as the normalized expected surplus destruction (associated with N) if the manager has reported shock states in the past i^{th} periods. Notice that D_i^N satisfies the following equations:

$$\begin{aligned} D_i^N &= (1-\theta)\,\delta D_0^N + \theta\delta D_{i+1}^N, \quad \text{for } i < N, \\ D_N^N &= (1-\delta)\,\theta\left(1+\delta^{-1}+...+\delta^{-(N-1)}\right)\alpha c\left(e^{FB}\right)+\delta D_0^N. \end{aligned}$$

Solving these equations, we obtain

$$D_0^N = \delta\theta\frac{1-\theta\delta}{(1-\delta)}\frac{\theta^N\left(1-\delta^N\right)}{\left(1-(\theta\delta)^{N+1}\right)}\alpha c\left(e^{FB}\right).$$

Notice that $\theta^N\left(1-\delta^N\right)/\left(1-(\theta\delta)^{N+1}\right)$ is decreasing in N when $1-\theta > \delta^{N-1}$. Moreover, as N goes to infinity, $\theta^N\left(1-\delta^N\right)/\left(1-(\theta\delta)^{N+1}\right)$ goes to zero. Therefore, when T is sufficiently large, D_0^N is minimized at $N = T$. It follows that the manager's truth-telling constraints in the shock states are satisfied for sufficiently large T. This finishes showing that the contracts are incentive compatible.

Finally, it can be checked that D_0^T goes to zero as T goes to infinity, so this sequence of contracts approximates first-best. ■