Online Appendix to accompany Baliga, Hanany and Klibanoff, "Polarization and Ambiguity", *American Economic Review*

This Appendix contains all proofs not included in the main text and some further results on the direction of updating.

A Proofs not in the Main Text

Proof. [Proof of Theorem 1] Bayesian updating is only well-defined following positive probability signals. Therefore, assume $\sum_i \check{\eta}(\theta_i)\pi_{\theta_i}(x) > 0$ and $\sum_i \hat{\eta}(\theta_i)\pi_{\theta_i}(x) > 0$. We use proof by contradiction. Suppose two individuals use Bayesian updating and that $\check{\eta}$ stochastically dominates $\check{\nu}$ and $\hat{\nu}$ stochastically dominates $\hat{\eta}$ with at least one dominance strict (i.e., that polarization occurs). Observe that $\check{\eta}$ stochastically dominates $\check{\nu}$ implies $\check{\eta}(\theta_1) \leq \check{\nu}(\theta_1) = \frac{\check{\eta}(\theta_1)\pi_{\theta_1}(x)}{\sum_i \check{\eta}(\theta_i)\pi_{\theta_i}(x)}$ and $\check{\eta}\left(\theta_{|\Theta|}\right) \geq \check{\nu}\left(\theta_{|\Theta|}\right) = \frac{\check{\eta}\left(\theta_{|\Theta|}\right)\pi_{\theta_{|\Theta|}}(x)}{\sum_i \check{\eta}(\theta_i)\pi_{\theta_i}(x)}$. Simplifying, this implies

$$\pi_{\theta_1}(x) \geq \sum_i \check{\eta}(\theta_i)\pi_{\theta_i}(x) \geq \pi_{\theta_{|\Theta|}}(x). \tag{6}$$

Similarly, observe that $\hat{\nu}$ stochastically dominates $\hat{\eta}$ implies $\hat{\eta}(\theta_1) \geq \hat{\nu}(\theta_1) = \frac{\hat{\eta}(\theta_1)\pi_{\theta_1}(x)}{\sum_i \hat{\eta}(\theta_i)\pi_{\theta_i}(x)}$ and $\hat{\eta}\left(\theta_{|\Theta|}\right) \leq \hat{\nu}\left(\theta_{|\Theta|}\right) = \frac{\hat{\eta}\left(\theta_{|\Theta|}\right)\pi_{\theta_{|\Theta|}}(x)}{\sum_i \hat{\eta}(\theta_i)\pi_{\theta_i}(x)}$. Simplifying, this implies

$$\pi_{\theta_1}(x) \leq \sum_i \hat{\eta}(\theta_i)\pi_{\theta_i}(x) \leq \pi_{\theta_{|\Theta|}}(x). \tag{7}$$

The only way for (6) and (7) to be satisfied simultaneously is when

$$\pi_{\theta_1}(x) = \sum_i \check{\eta}(\theta_i)\pi_{\theta_i}(x) = \sum_i \hat{\eta}(\theta_i)\pi_{\theta_i}(x) = \pi_{\theta_{|\Theta|}}(x). \tag{8}$$

Notice that under (8) $\hat{\eta}(\theta_1) = \hat{\nu}(\theta_1)$, $\hat{\eta}\left(\theta_{|\Theta|}\right) = \hat{\nu}\left(\theta_{|\Theta|}\right)$, $\check{\eta}(\theta_1) = \check{\nu}(\theta_1)$ and $\check{\eta}\left(\theta_{|\Theta|}\right) = \check{\nu}\left(\theta_{|\Theta|}\right)$. Given $\sum_i \check{\eta}(\theta_i)\pi_{\theta_i}(x) = \sum_i \hat{\eta}(\theta_i)\pi_{\theta_i}(x)$, consider the induction hypothesis that, for some $1 \leq n < |\Theta|$,

$$\hat{\eta}(\theta_i) = \hat{\nu}(\theta_i) \text{ and } \check{\eta}(\theta_i) = \check{\nu}(\theta_i) \text{ for } i = 1, ..., n.$$

Under this hypothesis, $\check{\eta}$ stochastically dominates $\check{\nu}$ implies $\check{\eta}(\theta_{n+1}) \leq \check{\nu}(\theta_{n+1}) = \frac{\check{\eta}(\theta_{n+1})\pi_{\theta_{n+1}}(x)}{\sum_i \check{\eta}(\theta_i)\pi_{\theta_i}(x)}$ and $\hat{\nu}$ stochastically dominates $\hat{\eta}$ implies $\hat{\eta}(\theta_{n+1}) \geq \hat{\nu}(\theta_{n+1}) = \frac{\hat{\eta}(\theta_{n+1})\pi_{\theta_{n+1}}(x)}{\sum_i \hat{\eta}(\theta_i)\pi_{\theta_i}(x)} = \frac{\hat{\eta}(\theta_{n+1})\pi_{\theta_{n+1}}(x)}{\sum_i \check{\eta}(\theta_i)\pi_{\theta_i}(x)}$. Therefore,

$$\hat{\eta}(\theta_{n+1}) = \hat{\nu}(\theta_{n+1}) \text{ and } \check{\eta}(\theta_{n+1}) = \check{\nu}(\theta_{n+1}).$$

Since we showed above that the induction hypothesis holds for $n = 1$, we conclude that $\check{\eta}$ stochastically dominates $\check{\nu}$ and $\hat{\nu}$ stochastically dominates $\hat{\eta}$ implies $\check{\eta} = \check{\nu}$ and $\hat{\eta} = \hat{\nu}$. This contradicts our supposition of polarization. □

Proof. [Proof of Proposition 1] It is immediate from (1) that $\alpha^*(x) \in (0,1)$ since $\mu \in (0,1)$ and $\phi' > 0$. To prove (i), fix any $x \in \mathscr{X}$ and, from (2), observe that for any $y \in \mathscr{X}$, $\alpha^*(y)$ is a strictly increasing function of $\alpha^*(x)$ in any solution of the system of first-order conditions. This and the fact that ϕ is concave implies that the left-hand side of (1) is strictly increasing in $\alpha^*(x)$ and decreasing in $\frac{\pi_1(x)}{\pi_0(x)}$. The right-hand side of (1) is strictly increasing in μ and constant in $\alpha^*(x)$. Therefore, $\alpha^*(x)$ is well-defined and strictly increasing in μ and $\frac{\pi_1(x)}{\pi_0(x)}$. The first-order condition describing the best constant prediction, which we denote here by $\bar{\alpha}$, is

$$\frac{\bar{\alpha}}{1-\bar{\alpha}} \frac{\phi'[-(\bar{\alpha})^2]}{\phi'[-(1-\bar{\alpha})^2]} = \frac{\mu}{1-\mu}. \tag{9}$$

Again, concavity of ϕ implies that the left-hand side is strictly increasing in $\bar{\alpha}$ and thus the best constant prediction is strictly increasing in μ.

To prove (ii), let $\beta^*(x)$ denote the optimal prediction after observing x. By the first-order conditions for optimality, these predictions must satisfy

$$\frac{\beta^*(x)}{1-\beta^*(x)} \frac{\phi'[-(\beta^*(x))^2]}{\phi'[-(1-\beta^*(x))^2]} = \frac{\nu_x}{1-\nu_x}. \tag{10}$$

Therefore, by the same reasoning as in (i), $\beta^*(x)$ is strictly increasing in the posterior ν_x. Comparing (9) and (10) and using concavity of ϕ yields

$$\nu_x \gtreqless \mu$$

if and only if

$$\beta^*(x) \gtreqless \bar{\alpha}.$$

Finally, under dynamically consistent updating, from (3), the posteriors must satisfy

$$\frac{\alpha^*(x)}{1-\alpha^*(x)} \frac{\phi'[-(\alpha^*(x))^2]}{\phi'[-(1-\alpha^*(x))^2]} = \frac{\nu_x}{1-\nu_x}.$$

Therefore, $\beta^*(x) = \alpha^*(x)$.

□

Proof. [Proof of Proposition 2] Dynamically consistent updating implies that (3) is satisfied in addition to (1). Combining the two equalities yields,

$$\nu_x \gtreqless \mu$$

if and only if

$$\frac{\phi'[-(\alpha^*(x))^2]}{\phi'[-(1-\alpha^*(x))^2]} \gtreqless \frac{\phi'[E_{\pi_0}(-(\alpha^*(X))^2)]}{\phi'[E_{\pi_1}(-(1-\alpha^*(X))^2)]} \frac{\pi_0(x)}{\pi_1(x)}.$$

□

Proof. [Proof of Theorem 2] By Proposition 2,

$$\nu_{x^H} \geq \mu$$

if and only if

$$\frac{\phi'[-(\alpha^*\left(x^H\right))^2]}{\phi'[-(1-\alpha^*(x^H))^2]} \geq \frac{\phi'[E_{\pi_0}(-(\alpha^*(X))^2)]}{\phi'[E_{\pi_1}(-(1-\alpha^*(X))^2)]} \frac{\pi_0(x^H)}{\pi_1(x^H)}. \tag{11}$$

For all $y \in \mathscr{X}$, since $\frac{\pi_1(x^H)}{\pi_0(x^H)} \geq \frac{\pi_1(y)}{\pi_0(y)}$, it follows from part (i) of Proposition 1 that

$$\alpha^*\left(x^H\right) \geq \alpha^*(y).$$

Therefore $(\alpha^*\left(x^H\right))^2 \geq E_{\pi_0}(\alpha^*(X))^2$ and $(1-\alpha^*\left(x^H\right))^2 \leq E_{\pi_1}(1-\alpha^*(X))^2$. As ϕ is concave, this implies

$$\frac{\phi'[-(\alpha^*\left(x^H\right))^2]}{\phi'[-(1-\alpha^*(x^H))^2]} \geq \frac{\phi'[E_{\pi_0}(-(\alpha^*(X))^2)]}{\phi'[E_{\pi_1}(-(1-\alpha^*(X))^2)]}. \tag{12}$$

Since $\frac{\pi_1(x^H)}{\pi_0(x^H)} \geq 1$, (11) follows. Thus $\nu_{x^H} \geq \mu$.

Furthermore, (3), (12) and (1) imply

$$\begin{aligned}
\frac{\nu_{x^H}}{1-\nu_{x^H}} &= \frac{\alpha^*\left(x^H\right)}{1-\alpha^*(x^H)} \frac{\phi'[-(\alpha^*\left(x^H\right))^2]}{\phi'[-(1-\alpha^*(x^H))^2]} \\
&\geq \frac{\alpha^*\left(x^H\right)}{1-\alpha^*(x^H)} \frac{\phi'[E_{\pi_0}(-(\alpha^*(X))^2)]}{\phi'[E_{\pi_1}(-(1-\alpha^*(X))^2)]} \\
&= \frac{\mu}{1-\mu} \frac{\pi_1(x^H)}{\pi_0(x^H)},
\end{aligned}$$

where the last expression is the posterior ratio generated by Bayesian updating of μ after observing x^H.

An analogous argument shows $\mu \geq \nu_{x^L}$ and

$$\frac{\nu_{x^L}}{1-\nu_{x^L}} \leq \frac{\mu}{1-\mu} \frac{\pi_1(x^L)}{\pi_0(x^L)}.$$

□

Proof. [Proof of Theorem 3] Recall that the optimal prediction $\alpha^*(x)$ is continuous and increasing in the prior probability of $\theta = 1$. Denote this probability by η. As the optimal prediction is 0 if $\eta = 0$ and 1 if $\eta = 1$, considering η close enough to 0 or η close enough to 1 is equivalent to considering $\alpha^*(x)$ close enough to 0 or 1 respectively. The proof strategy for determining updating for sufficiently extreme beliefs will be to consider updating for sufficiently extreme predictions.

Observe, by applying (1) and (3), that dynamically consistent updating of η after seeing $x \in \mathscr{X}$ will be shaded upward/equal to/shaded downward compared to Bayesian updating if and only if

$$\begin{aligned} &\phi'[-(\alpha^*(x))^2]\phi'[-\sum_{y\in\mathcal{X}}\pi_1(y)(1-\alpha^*(y))^2] \\ &\gtreqless \phi'[-(1-\alpha^*(x))^2]\phi'[-\sum_{y\in\mathcal{X}}\pi_0(y)(\alpha^*(y))^2]. \end{aligned} \tag{13}$$

From (2), $\alpha^*(y) = \beta_{\pi_1,\pi_0}(\alpha^*(x);y)$ where $\beta_{\pi_1,\pi_0} : [0,1]\times\mathscr{X} \to [0,1]$ is defined by $\beta_{\pi_1,\pi_0}(z;y) = \frac{z\frac{\pi_1(y)}{\pi_0(y)}}{z\frac{\pi_1(y)}{\pi_0(y)}+(1-z)\frac{\pi_1(x)}{\pi_0(x)}}$ for all $z \in [0,1]$ and $y \in \mathscr{X}$. Define the function $f : [0,1] \to \mathbb{R}$ such that

$$f(z) = \frac{\phi'[-\sum_{y\in\mathcal{X}}\pi_1(y)(1-\beta_{\pi_1,\pi_0}(z;y))^2]}{\phi'[-(1-z)^2]} - \frac{\phi'[-\sum_{y\in\mathcal{X}}\pi_0(y)(\beta_{\pi_1,\pi_0}(z;y))^2]}{\phi'(-z^2)}.$$

Under our assumptions, f is continuous and differentiable. By comparing f with (13), observe that when $z = \alpha^*(x) \in (0,1)$, the direction in which updating is shaded relative to Bayesian updating is determined by the sign of f. Therefore we want to determine the sign of $f(z)$ when z is close 0 and when it is close to 1. By the assumptions in the statement of the theorem, $0 < \phi'(0) < \phi'(-1) < \infty$ where the last inequality comes from the fact that ϕ' is continuous on $[-1,0]$ and thus bounded. Then $f(0) = f(1) = 0$. Therefore, the sign of $f(z)$ when z is close 0 and when it is close to 1 is determined by the sign of $f'(z)$ at 0 and 1 respectively. Differentiating f (and denoting the derivative of β_{π_1,π_0} with respect to z evaluated at $(z;y)$ by $\beta'_{\pi_1,\pi_0}(z;y)$) yields,

$$\begin{aligned}
f'(z) = & \frac{2\phi''[-\sum_{y\in\mathcal{X}}\pi_1(y)(1-\beta_{\pi_1,\pi_0}(z;y))^2]\sum_{y\in\mathcal{X}}\pi_1(y)(1-\beta_{\pi_1,\pi_0}(z;y))\beta'_{\pi_1,\pi_0}(z;y)}{\phi'[-(1-z)^2]} \\
& - \frac{2\phi'[-\sum_{y\in\mathcal{X}}\pi_1(y)(1-\beta_{\pi_1,\pi_0}(z;y))^2]\phi''[-(1-z)^2](1-z)}{(\phi'[-(1-z)^2])^2} \\
& + \frac{2\phi''[-\sum_{y\in\mathcal{X}}\pi_0(y)(\beta_{\pi_1,\pi_0}(z;y))^2]\sum_{y\in\mathcal{X}}\pi_0(y)(\beta_{\pi_1,\pi_0}(z;y))\beta'_{\pi_1,\pi_0}(z;y)}{\phi'(-z^2)} \\
& - \frac{2\phi''(-z^2)(z)\phi'[-\sum_{y\in\mathcal{X}}\pi_0(y)(\beta_{\pi_1,\pi_0}(z;y))^2]}{(\phi'(-z^2))^2}.
\end{aligned}$$

Thus,

$$f'(0) = 2\left(-\frac{\phi''(-1)}{\phi'(-1)}\right)\left[1-\sum_{y\in\mathcal{X}}\pi_1(y)\beta'_{\pi_1,\pi_0}(0;y)\right]+(0)\left(-\frac{\phi''(0)}{\phi'(0)}\right)\left[1-\sum_{y\in\mathcal{X}}\pi_0(y)\beta'_{\pi_1,\pi_0}(0;y)\right]$$

and

$$f'(1) = (0)\left(-\frac{\phi''(0)}{\phi'(0)}\right)\left[1-\sum_{y\in\mathcal{X}}\pi_1(y)\beta'_{\pi_1,\pi_0}(1;y)\right]+2\left(-\frac{\phi''(-1)}{\phi'(-1)}\right)\left[1-\sum_{y\in\mathcal{X}}\pi_0(y)\beta'_{\pi_1,\pi_0}(1;y)\right].$$

Since ϕ'' is negative and finite (since ϕ'' is continuous on a bounded interval), the coefficient of ambiguity aversion, $-\frac{\phi''}{\phi'}$, is everywhere positive and finite. This allows us to conclude that the sign of $f'(0)$ is the same as the sign of $1-\sum_{y\in\mathscr{X}}\pi_1(y)\beta'_{\pi_1,\pi_0}(0;y)$, while the sign of $f'(1)$ is the sign of $1-\sum_{y\in\mathscr{X}}\pi_0(y)\beta'_{\pi_1,\pi_0}(1;y)$. Differentiating $\beta_{\pi_1,\pi_0}(z;y)$ shows that $\beta'_{\pi_1,\pi_0}(0;y) = \frac{\pi_1(y)}{\pi_0(y)}/\frac{\pi_1(x)}{\pi_0(x)}$ and $\beta'_{\pi_1,\pi_0}(1;y) = \frac{\pi_1(x)}{\pi_0(x)}/\frac{\pi_1(y)}{\pi_0(y)}$. Thus $f'(0)<0$ and $f'(1)<0$ if and only if

$$\frac{1}{\sum_{y\in\mathcal{X}}\pi_0(y)\frac{\pi_0(y)}{\pi_1(y)}} < \frac{\pi_1(x)}{\pi_0(x)} < \sum_{y\in\mathcal{X}}\pi_1(y)\frac{\pi_1(y)}{\pi_0(y)}. \tag{14}$$

Summarizing, we have shown that f is negative for values sufficiently close to 0 and positive for values sufficiently close to 1 if and only if (14) is satisfied. Therefore, it is exactly under these conditions that updating will be shaded downward compared to Bayesian updating for beliefs sufficiently close to 0 and shaded upward compared to Bayesian updating for beliefs sufficiently close to 1.

We now show that a neutral signal necessarily satisfies (14). Note that $\sum_{y\in\mathscr{X}} \pi_1(y)\frac{\pi_1(y)}{\pi_0(y)} \geq 1$ and $\sum_{y\in\mathscr{X}} \pi_0(y)\frac{\pi_0(y)}{\pi_1(y)} \geq 1$ because the strictly convex constrained minimization problem $\min_{w_1,...,w_{|\mathscr{X}|}} \sum_{i=1}^{|\mathscr{X}|} \frac{w_i^2}{v_i}$ subject to $\sum_{i=1}^{|\mathscr{X}|} w_i = 1$, assuming $\sum_{i=1}^{|\mathscr{X}|} v_i = 1$ and $v_i > 0$ for $i = 1, ..., |\mathscr{X}|$, has first order conditions equivalent to $\frac{w_i}{v_i}$ constant in i, thus the minimum is achieved at $\frac{1}{\sum_{i=1}^{|\mathscr{X}|} v_i} = 1$ with $w_i = \frac{v_i}{\sum_{i=1}^{|\mathscr{X}|} v_i} = v_i$. Moreover, since there exists at least one informative signal, i.e., $y \in \mathscr{X}$ such that $\frac{\pi_1(y)}{\pi_0(y)} \neq 1$, the unique minimum is not attained and so $\sum_{y\in\mathscr{X}} \pi_1(y)\frac{\pi_1(y)}{\pi_0(y)} > 1$ and $\sum_{y\in\mathscr{X}} \pi_0(y)\frac{\pi_0(y)}{\pi_1(y)} > 1$. Thus, (14) is always satisfied if $\frac{\pi_1(x)}{\pi_0(x)} = 1$ (i.e., if x is a neutral signal).

Finally, observe that if x is a neutral signal, then, since Bayesian updating would be flat, updating shaded downward implies updating is downward and updating shaded upward implies updating is upward, generating polarization. □

Remark 3. The theorem remains true if $\phi'(0) = 0$ and the requirements of the theorem are otherwise satisfied. This case requires an argument based on second-order comparisons. Intuitively, second-order differences that were previously masked may now become important in the limit because the zero creates unboundedly large ambiguity aversion (as measured by $-\frac{\phi''}{\phi'}$) near perfect predictions. Specifically, one can show that, for beliefs close to θ, a second-order comparison yields that the payoff following a neutral signal is larger than the expected payoff before seeing the signal. This drives the comparison of ex-ante versus interim hedging effects and generates the polarization. Moreover, in this case, the polarization result may be extended beyond neutral signals to all signals having a likelihood ratio lying in an interval containing 1.

Proof. [Proof of Proposition 3] From Lemma 1, $\nu_{x^M} \gtreqless \mu$ if and only if

$$\sum_{y\in\mathcal{X}} \pi_1(y) \frac{\left(\frac{\pi_1(x^M)}{\pi_0(x^M)}\right)^{\frac{1}{\gamma}+2} - \frac{\pi_1(y)}{\pi_0(y)}}{[\alpha^*(x^M)\frac{\pi_1(y)}{\pi_0(y)} + (1-\alpha^*(x^M))\frac{\pi_1(x^M)}{\pi_0(x^M)}]^2} \gtreqless 0. \tag{15}$$

We consider the following exhaustive list of possibilities:

(i) $\left(\frac{\pi_1(x^M)}{\pi_0(x^M)}\right)^{\frac{1}{\gamma}+2} \geq \frac{\pi_1(x^H)}{\pi_0(x^H)}$. In this case, using $\frac{\pi_1(x^L)}{\pi_0(x^L)} < \frac{\pi_1(x^M)}{\pi_0(x^M)} < \frac{\pi_1(x^H)}{\pi_0(x^H)}$, the left-hand side of (15) is strictly positive, and therefore updating is always upward, so set $\tau(\gamma, \pi_0, \pi_1) = 0$. Note that a necessary condition for this case is that $\frac{\pi_1(x^M)}{\pi_0(x^M)} > 1$.

(ii) $\left(\frac{\pi_1(x^M)}{\pi_0(x^M)}\right)^{\frac{1}{\gamma}+2} \leq \frac{\pi_1(x^L)}{\pi_0(x^L)}$. In this case, using $\frac{\pi_1(x^L)}{\pi_0(x^L)} < \frac{\pi_1(x^M)}{\pi_0(x^M)} < \frac{\pi_1(x^H)}{\pi_0(x^H)}$, the left-hand side of (15) is strictly negative, and therefore updating is always downward, so set $\tau(\gamma, \pi_0, \pi_1) = 1$. Note that a necessary condition for this case is that $\frac{\pi_1(x^M)}{\pi_0(x^M)} < 1$.

(iii) $\frac{\pi_1(x^H)}{\pi_0(x^H)} > \left(\frac{\pi_1(x^M)}{\pi_0(x^M)}\right)^{\frac{1}{\gamma}+2} > \frac{\pi_1(x^L)}{\pi_0(x^L)}$. In this case, using $\frac{\pi_1(x^L)}{\pi_0(x^L)} < \frac{\pi_1(x^M)}{\pi_0(x^M)} < \frac{\pi_1(x^H)}{\pi_0(x^H)}$, in the left-hand side of (15), the term for $y = x^L$ is positive and has a denominator strictly decreasing in $\alpha^*(x^M)$, the term for $y = x^M$ is constant in $\alpha^*(x^M)$, and the term for $y = x^H$ is negative and has a denominator strictly increasing in $\alpha^*(x^M)$. Therefore the whole sum is strictly increasing in $\alpha^*(x^M)$ and thus can change signs at most once. Three sub-cases are relevant:

$(iii)(a)$ the left-hand side of (15) is non-negative when 0 is plugged in for $\alpha^*(x^M)$. In this case, updating is always upward, so set $\tau(\gamma, \pi_0, \pi_1) = 0$.

$(iii)(b)$ the left-hand side of (15) is non-positive when 1 is plugged in for $\alpha^*(x^M)$. In this case, updating is always downward, so set $\tau(\gamma, \pi_0, \pi_1) = 1$.

$(iii)(c)$ otherwise. In this case, continuity and strict increasingness of the left-hand side of (15) in $\alpha^*(x^M)$ implies there exists a unique solution for a in $(0, 1)$ to

$$\sum_{y \in \mathcal{X}} \pi_1(y) \frac{\left(\frac{\pi_1(x^M)}{\pi_0(x^M)}\right)^{\frac{1}{\gamma}+2} - \frac{\pi_1(y)}{\pi_0(y)}}{\left(a\frac{\pi_1(y)}{\pi_0(y)} + (1-a)\frac{\pi_1(x^M)}{\pi_0(x^M)}\right)^2} = 0.$$

Since (16) holds with equality when $z = a$, using constant relative ambiguity aversion ($\phi'(z) = (-z)^\gamma$) and given the monotonicity of $\alpha^*(x^M)$ in μ, the associated threshold for μ may be found by substituting $z = a$ into (16) with equality and solving for $\mu = \tau(\gamma, \pi_0, \pi_1)$. Doing this yields

$$\frac{\tau(\gamma, \pi_0, \pi_1)}{1 - \tau(\gamma, \pi_0, \pi_1)} = \left(\frac{a}{1-a}\right)^{2\gamma+1}.$$

Therefore

$$\tau(\gamma, \pi_0, \pi_1) = \frac{a^{2\gamma+1}}{a^{2\gamma+1} + (1-a)^{2\gamma+1}}.$$

Collecting these results into an overall expression, the threshold is defined by:

$$\tau(\gamma, \pi_0, \pi_1) = \frac{b^{2\gamma+1}}{b^{2\gamma+1} + (1-b)^{2\gamma+1}},$$

where

$$b \equiv \begin{cases} 0 & \text{if} \quad S(0) \geq 0 \\ a & \text{if} \quad S(a) = 0 \text{ and } a \in (0,1) \\ 1 & \text{if} \quad S(1) \leq 0 \end{cases}$$

and

$$S(\lambda) \equiv \sum_{y \in \{x^L, x^M, x^H\}} \pi_1(y) \frac{\left(\frac{\pi_1(x^M)}{\pi_0(x^M)}\right)^{\frac{1}{\gamma}+2} - \frac{\pi_1(y)}{\pi_0(y)}}{(\lambda \frac{\pi_1(y)}{\pi_0(y)} + (1-\lambda)\frac{\pi_1(x^M)}{\pi_0(x^M)})^2}.$$

□

Proof. [Proof of Theorem 4] Polarization is equivalent to $\hat{\nu} \geq \hat{\eta}$ and $\check{\nu} \leq \check{\eta}$ with at least one inequality strict. If $\gamma = 0$, updating is Bayesian and polarization is impossible by Theorem 1, so set $\hat{\tau} = 1$ and $\check{\tau} = 0$. By Proposition 3, if $\gamma > 0$ then polarization occurs if and only if $\hat{\eta} \geq \tau(\hat{\gamma}, \pi_0, \pi_1)$ and $\check{\eta} \leq \tau(\check{\gamma}, \pi_0, \pi_1)$ with at least one inequality strict, where the τ function is the one defined in that result. □

Proof. [Proof of Corollary 1] From Proposition 3, $\hat{\tau} = \tau(\hat{\gamma}, \pi_0, \pi_1)$ and $\check{\tau} = \tau(\check{\gamma}, \pi_0, \pi_1)$. The rest is immediate from Theorem 4. □

Proof. [Proof of Corollary 2] From Proposition 3, such a threshold exists. Since $\pi_0(x^M) = \pi_1(x^M)$ implies $\pi_0(x^L) - \pi_1(x^L) = \pi_1(x^H) - \pi_0(x^H) > 0$, calculation shows that the relevant case in the proof of Proposition 3 is case $(iii)(c)$. Thus $\tau(\gamma, \pi_0, \pi_1) = \frac{a^{2\gamma+1}}{a^{2\gamma+1}+(1-a)^{2\gamma+1}} = \frac{1}{1+\left(\frac{1-a}{a}\right)^{2\gamma+1}}$ where $a \in (0,1)$ is the unique solution of $S(a)=0$. Simplifying yields

$$\frac{1-a}{a}=\sqrt{\frac{\pi_1\left(x^H\right)}{\pi_0\left(x^H\right)}\frac{\pi_1\left(x^L\right)}{\pi_0\left(x^L\right)}}.$$

□

B Further Results on the Direction of Updating

The next result combines Proposition 2 and equations (2) and (3) to show a general form relating fundamentals to the direction of updating.

Proposition 4. *The posterior ν_x is above/equal to/below the prior μ if and only if the fundamentals $(\mu, \phi, \pi_1, \pi_0)$ are such that*

$$\frac{z}{1-z}\frac{\phi'[-z^2]}{\phi'[-(1-z)^2]} \gtreqless \frac{\mu}{1-\mu}, \tag{16}$$

for the unique $z \in (0,1)$ solving

$$\frac{z}{1-z}\frac{\phi'\left[-z^2\sum_{y\in\mathcal{X}}\frac{\pi_0(y)\left(\frac{\pi_1(y)}{\pi_0(y)}\right)^2}{\left(z\frac{\pi_1(y)}{\pi_0(y)}+(1-z)\frac{\pi_1(x)}{\pi_0(x)}\right)^2}\right]}{\phi'\left[-(1-z)^2\sum_{y\in\mathcal{X}}\frac{\pi_1(y)\left(\frac{\pi_1(x)}{\pi_0(x)}\right)^2}{\left(z\frac{\pi_1(y)}{\pi_0(y)}+(1-z)\frac{\pi_1(x)}{\pi_0(x)}\right)^2}\right]} = \frac{\mu}{1-\mu}\frac{\pi_1(x)}{\pi_0(x)}. \tag{17}$$

Proof. Substituting (3) into (4) and rearranging yields

$$\frac{\alpha^*(x)}{1-\alpha^*(x)}\frac{\phi'[-\alpha^*(x)^2]}{\phi'[-(1-\alpha^*(x))^2]} \gtreqless \frac{\mu}{1-\mu}.$$

From (2), we obtain for all $y \in \mathscr{X}$,

$$\alpha^*(y)=\frac{\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}}{\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}+(1-\alpha^*(x))\frac{\pi_1(x)}{\pi_0(x)}}.$$

Using this together with (3), $\alpha^*(x)$ is the unique solution to

$$\frac{\alpha^*(x)}{1-\alpha^*(x)} \frac{\phi'\left[-\alpha^*(x)^2 \sum_{y\in\mathcal{X}} \frac{\pi_0(y)\left(\frac{\pi_1(y)}{\pi_0(y)}\right)^2}{\left(\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}+(1-\alpha^*(x))\frac{\pi_1(x)}{\pi_0(x)}\right)^2}\right]}{\phi'\left[-(1-\alpha^*(x))^2 \sum_{y\in\mathcal{X}} \frac{\pi_1(y)\left(\frac{\pi_1(x)}{\pi_0(x)}\right)^2}{\left(\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}+(1-\alpha^*(x))\frac{\pi_1(x)}{\pi_0(x)}\right)^2}\right]}$$
$$= \frac{\mu}{1-\mu}\frac{\pi_1(x)}{\pi_0(x)}.$$

□

In interpreting inequality (16), it is important to realize that z is an increasing function of beliefs μ (as follows from the argument used in proving part (i) of Proposition 1 with z playing the role of $\alpha^*(x)$). In fact, (17) combines (2) and (3). This implies that $z = \alpha^*(x)$, the optimal prediction given the observation x. From (17), in the case of ambiguity neutrality (ϕ affine) $\frac{z}{1-z}$ is simply a multiple of $\frac{\mu}{1-\mu}$ so that updating is either always upward (if $\frac{\pi_1(x)}{\pi_0(x)} \geq 1$) or always downward (if $\frac{\pi_1(x)}{\pi_0(x)} \leq 1$). Similarly, we see that under ambiguity aversion, $\frac{z}{1-z}$ is generally a non-linear function of $\frac{\mu}{1-\mu}$ (reflecting the balancing of the desire to hedge with the likelihood based motivation from the ambiguity neutral case) which creates the possibility that inequality (16) may change direction as beliefs μ change. In general, the regions where it goes one way and where it goes the other may be very complex. We now offer a characterization of when updating follows a threshold rule so that 16 changes direction at most once.

Proposition 5. *There is a threshold rule for updating μ after observing x if* *and only if*

$$\frac{\phi'[-z^2]}{\phi'[-(1-z)^2]}\frac{\pi_1(x)}{\pi_0(x)} - \frac{\phi'\left[-z^2\sum_{y\in\mathcal{X}}\frac{\pi_0(y)\left(\frac{\pi_1(y)}{\pi_0(y)}\right)^2}{\left(z\frac{\pi_1(y)}{\pi_0(y)}+(1-z)\frac{\pi_1(x)}{\pi_0(x)}\right)^2}\right]}{\phi'\left[-(1-z)^2\sum_{y\in\mathcal{X}}\frac{\pi_1(y)\left(\frac{\pi_1(x)}{\pi_0(x)}\right)^2}{\left(z\frac{\pi_1(y)}{\pi_0(y)}+(1-z)\frac{\pi_1(x)}{\pi_0(x)}\right)^2}\right]} \tag{18}$$

as a function of z has at most one zero in $(0,1)$ and, if a zero exists, (18) is increasing at that zero.

Proof. The result follows by combining the definition of a threshold updating rule with the characterization of the direction of updating given by Proposition 4. □

Finally, we present a lemma showing how inequality (4), which identifies the direction of updating after observing a signal, simplifies under the assumption of constant relative ambiguity aversion. In proving Theorem 4, we use this inequality to help establish and calculate the threshold rule.

Lemma 1. *With constant relative ambiguity aversion $\gamma > 0$, the posterior ν_x is above/equal to/below the prior μ if and only if*

$$\sum_{y\in\mathcal{X}}\pi_1(y)\frac{\left(\frac{\pi_1(x)}{\pi_0(x)}\right)^{\frac{1}{\gamma}+2}-\frac{\pi_1(y)}{\pi_0(y)}}{\left(\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}+(1-\alpha^*(x))\frac{\pi_1(x)}{\pi_0(x)}\right)^2} \gtreqless 0. \tag{19}$$

Proof. From inequality (16) and equation (17), $\nu_x \gtreqless \mu$ if and only if

$$\frac{\phi'[-(\alpha^*(x))^2]}{\phi'[-(1-\alpha^*(x))^2]}\frac{\pi_1(x)}{\pi_0(x)} \gtreqless \frac{\phi'\left[-\alpha^*(x)^2\sum_{y\in\mathcal{X}}\frac{\pi_0(y)\left(\frac{\pi_1(y)}{\pi_0(y)}\right)^2}{\left(\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}+(1-\alpha^*(x))\frac{\pi_1(x)}{\pi_0(x)}\right)^2}\right]}{\phi'\left[-(1-\alpha^*(x))^2\sum_{y\in\mathcal{X}}\frac{\pi_1(y)\left(\frac{\pi_1(x)}{\pi_0(x)}\right)^2}{\left(\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}+(1-\alpha^*(x))\frac{\pi_1(x)}{\pi_0(x)}\right)^2}\right]}. \tag{20}$$

Under constant relative ambiguity aversion, $\phi'(z) = (-z)^\gamma$ and therefore (20) is equivalent to

$$\left(\frac{\pi_1(x)}{\pi_0(x)}\right)^{\frac{1}{\gamma}} \gtreqless \frac{\sum_{y\in\mathcal{X}} \frac{\pi_0(y)\left(\frac{\pi_1(y)}{\pi_0(y)}\right)^2}{\left(\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}+(1-\alpha^*(x))\frac{\pi_1(x)}{\pi_0(x)}\right)^2}}{\sum_{y\in\mathcal{X}} \frac{\pi_1(y)\left(\frac{\pi_1(x)}{\pi_0(x)}\right)^2}{\left(\alpha^*(x)\frac{\pi_1(y)}{\pi_0(y)}+(1-\alpha^*(x))\frac{\pi_1(x)}{\pi_0(x)}\right)^2}}.$$

Simplifying yields inequality (19). □