Econometrica Supplementary Material

SUPPLEMENT TO "PERCEIVED AMBIGUITY AND RELEVANT MEASURES"

(*Econometrica*, Vol. 82, No. 5, September 2014, 1945–1978)

BY PETER KLIBANOFF, SUJOY MUKERJI, AND KYOUNGWON SEO

THIS SUPPLEMENT CONTAINS results identifying the relevant measures and implications of comparative ambiguity aversion for Continuous Symmetric versions of several additional models from the ambiguity literature: the extended MEU with contraction model (see, e.g., Gajdos, Hayashi, Tallon, and Vergnaud (2008), Gajdos, Tallon, and Vergnaud (2004), Kopylov (2008), Tapking (2004)), the vector expected utility model (see Siniscalchi (2009)), and the second-order Choquet representation (see Amarante (2009)) of invariant biseparable preferences (defined by Ghirardato, Maccheroni, and Marinacci (2004)). It concludes by describing a technique for identifying relevant measures in additional models.

APPENDIX D

D.1. *The Extended MEU With Contraction Model*

Motivated by the contraction representation of Gajdos et al. (2008, Theorem 6), this model has a functional form[13] that is a convex combination of MEU and expected utility.

Consider preferences having a representation of the form

$$\beta \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp + (1-\beta)\int u(f)\,dq,$$

where $D \subseteq \Delta(S)$ is finite, $q = \int \ell^\infty\,dm(\ell)$ for some $m \in \Delta(\Delta(S))$ such that $\operatorname{supp} m \subseteq D$, $0 < \beta < 1$, and u is a nonconstant vNM utility function. Call such preferences *i.i.d. extended MEU with contraction*.

THEOREM D.1: *For any i.i.d. extended MEU with contraction preference*, $R = D$.

The role of the finiteness restriction on D is to ensure Monotone Continuity of $\succsim^*$. Notice that any such preference also has a representation of the form $\min_{p\in\{\beta\ell^\infty+(1-\beta)q:\ell\in D\}} \int u(f)\,dp$, and therefore these preferences are a subset of the Continuous Symmetric MEU preferences.

[13]This functional form is much older than the derivation of it in Gajdos et al. (2008). It appears in Ellsberg (1961), and is acknowledged there as based upon a concept of Hodges and Lehmann (1952).

 DOI: 10.3982/ECTA9872

In the representation of Gajdos et al. (2008, Theorem 6), q is fully determined by the set over which the minimum is taken. It is a particular convex combination, known as the Steiner point, of the extreme points of that set. A property of the Steiner point of the set $\{\ell^\infty : \ell \in D\}$ is that $\operatorname{supp} m = D$. We do not require q to be the Steiner point, but do impose this support restriction in our comparative ambiguity aversion result below.

THEOREM D.2: *Let $\succsim_A$ and $\succsim_B$ be any two i.i.d. extended MEU with contraction preferences such that D_B is non-singleton*, $\operatorname{supp} m_A = D_A$, *and* $\operatorname{supp} m_B = D_B$. *Then, $\succsim_A$ is more ambiguity averse than $\succsim_B$ if and only if $\beta_A \geq \beta_B$, $D_A = D_B$, $\ell \in D_B$ implies $m_B(\ell) \geq \frac{1-\beta_A}{1-\beta_B} m_A(\ell)$ and (up to normalization) $u_A = u_B$.*

Under the conditions in the theorem, β reflects comparative ambiguity aversion and D is the set of relevant measures and is therefore related to perceived ambiguity. This offers an additional perspective on the model compared to Gajdos et al. (2008). In their setting, the sets D are objectively given and larger β reflects more imprecision aversion in the sense of stronger preference for having singleton sets.

REMARK D.1: Observe that the inequality relating m_A and m_B is automatically satisfied if $q_A = q_B$. Equality of the q's is, for example, implied if the q's were Steiner points as in Gajdos et al. (2008). A step in the proof of Theorem D.2 is to show that $\succsim_A$ and $\succsim_B$ are strictly monotonic for bets on non-null limiting frequency events. This provides a set of Continuous Symmetric MEU preferences for which the conclusions of Theorem 3.4 hold.[14]

D.2. *The Vector Expected Utility (VEU) Model*

Next, we turn to a version of the VEU model of Siniscalchi (2009). Consider preferences having a representation of the form

$$\int u(f)\,dp + A\left(\left(\int \zeta_i u(f)\,dp\right)_{1\leq i\leq n}\right),$$

where (i) u is a nonconstant vNM utility function, (ii) $p \in \Delta(\Omega)$, (iii) $\zeta = (\zeta_1, \dots, \zeta_n)$ is a bounded, measurable, vector-valued function on Ω into $\mathbb{R}^n$ such that, for each i, $\int \zeta_i\,dp = 0$, (iv) $A(0) = 0$, and $A(a) = A(-a)$ for all $a \in \mathbb{R}^n$, and (v) the whole functional is weakly monotonic. Call such preferences *VEU*.

[14]More generally, any MEU preference with a set of exchangeable measures such that each measure has the same set of i.i.d. measures in its support will be strictly monotonic for bets on non-null limiting frequency events.

Consider also preferences that are VEU and have a VEU representation that additionally satisfies (vi) n is finite, (vii) $p = \int \ell^\infty \, dm(\ell)$ for some $m \in \Delta(\Delta(S))$, (viii) for each i, for all $\pi \in \Pi$, $\zeta_i(\omega) = \zeta_i(\pi\omega)$, p-almost-everywhere, and (ix) A is Lipschitz continuous. Call such preferences *i.i.d. VEU*.

THEOREM D.3: *For any i.i.d. VEU preference*, $R = \operatorname{supp} m$.

Thus, for such preferences, the relevant measures, R, are those $\ell \in \Delta(S)$ given weight by p. It is interesting to observe that these are the same relevant measures as for the expected utility preference represented by $\int u(f)\,dp$. Note that the symmetry conditions on p and the ζ_i are imposed to ensure Event Symmetry, while n finite and Lipschitz continuity of A are imposed to ensure Monotone Continuity of $\succsim^*$. The remaining conditions are standard for the VEU model.

In characterizing comparative ambiguity aversion, it turns out that Continuous Symmetry is not required and our result applies to VEU preferences with differentiable A:

THEOREM D.4: *Let $\succsim_A$ and $\succsim_B$ be any two VEU preferences such that A_A and A_B are Fréchet differentiable. Then, $\succsim_A$ is more ambiguity averse than $\succsim_B$ if and only if $p_A = p_B$, $A_A((\int \zeta_i^A u(f)\,dp_A)_{1\le i\le n}) \le A_B((\int \zeta_i^B u(f)\,dp_B)_{1\le i\le n})$ for all $f \in \mathcal{F}$, and (up to normalization) $u_A = u_B$.*

Compared to Siniscalchi's result on comparative ambiguity aversion in VEU (Siniscalchi (2009, Proposition 4)), this theorem derives equality of p as an implication rather than assuming it. The differentiability assumption is what allows this.

D.3. *The Second-Order Choquet Model of Invariant Biseparable Preferences*

As shown by Amarante (2009), the Invariant Biseparable preferences defined in Ghirardato, Maccheroni, and Marinacci (2004) may be represented by a Choquet integral of expected utilities. These preferences generalize both the MEU model of Gilboa and Schmeidler (1989) and the Choquet Expected Utility model of Schmeidler (1989). Here we consider a Continuous Symmetric version, where the expected utilities are calculated with respect to i.i.d. measures.

Some notation and definitions are necessary in order to formally describe the representation of such preferences. Let v be a capacity mapping subsets of $\Delta(S)$ to $[0,1]$. An event E is v-non-null if there is an event E' such that $v(E \cup E') > v(E')$. The support of v, denoted $\operatorname{supp} v$, is defined to be the set of elements $\ell \in \Delta(S)$ such that any open set containing ℓ is v-non-null.

Consider preferences having a representation of the form

$$\int \int u(f)\,d\ell^\infty \,dv(\ell),$$

where u is a nonconstant vNM utility function, v is a capacity on $\Delta(S)$ with finite support, and the outer integral is taken in the sense of Choquet. Call such preferences *i.i.d. second-order Choquet*.

THEOREM D.5: *For any i.i.d. second-order Choquet preference*, $R = \operatorname{supp} v$.

Therefore, the relevant measures are exactly the measures in the support of the representing capacity v. Our next result shows that an everywhere lower v characterizes more ambiguity aversion.

THEOREM D.6: *Let* $\succsim_A$ *and* $\succsim_B$ *be any two i.i.d. second-order Choquet preferences.* $\succsim_A$ *is more ambiguity averse than* $\succsim_B$ *if and only if* $v_B \geq v_A$ *and* (*up to normalization*) $u_A = u_B$.

Observe that if $v_A(\{\ell\}) > 0$ for all $\ell \in \operatorname{supp} v_A$ and $v_B(\Delta(S) \setminus \{\ell\}) < 1$ for all $\ell \in \operatorname{supp} v_B$, then $v_B \geq v_A$ implies $\operatorname{supp} v_A = \operatorname{supp} v_B$. Thus, these conditions are sufficient for R to be unaffected by increases and decreases in ambiguity aversion.

D.4. *A Technique for Further Applications*

Theorems 4.5 and B.5 provide a way to leverage the fact that there are extant characterizations of the set C for a variety of models as a step toward identifying R in Continuous Symmetric instances of such models. Given an explicit characterization of the set C for a continuous symmetric preference, Theorem 4.5 shows how to determine the relevant measures. For example, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) characterize the set C for the variational preferences of Maccheroni, Marinacci, and Rustichini (2006) as the closure of the set of probability measures assigned finite values by the cost function c in the variational representation. Therefore, for Continuous Symmetric variational preferences, by Theorem 4.5, all of the measures in this closure are exchangeable measures and R is determined by looking at the marginals of i.i.d. measures appearing in the supports. See Cerreia-Vioglio et al. (2011) and Ghirardato and Siniscalchi (2010) for characterizations of C for a variety of models.

D.5. *Proofs of Results in the Supplemental Material*

D.5.1. *Proof of Theorem D.1*

Suppose $\succsim$ is an i.i.d. extended MEU with contraction preference. We first show that $D \subseteq R$. Suppose $\hat{\ell} \in D$ and fix any $L \in \mathcal{O}_{\hat{\ell}}$. Consider $f = 1_{\Psi^{-1}(L)}$ and $g = 1_{\emptyset}$ and observe that $\int f \, d\ell^\infty = \int g \, d\ell^\infty$ for all $\ell \in \Delta(S) \setminus L$. Observe that $\int u(f) \, d\ell^\infty > \int u(g) \, d\ell^\infty$ for all $\ell \in L$, while $\int u(f) \, d\ell^\infty \geq \int u(g) \, d\ell^\infty$ for all $\ell \in D$ and thus also $\int u(f) \, dq \geq \int u(g) \, dq$. Therefore, if $q(\Psi^{-1}(L)) > 0$, $f \succ g$ and $\hat{\ell}$ is relevant. If $q(\Psi^{-1}(L)) = 0$, consider instead $f = \frac{1}{2} 1_{\Psi^{-1}(L)} + \frac{1}{2} 1_{\Psi^{-1}(\Delta(S) \setminus L)}$

and $g = \frac{1}{2}1_\emptyset + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\setminus L)}$ and observe that $\int f\,d\ell^\infty = \int g\,d\ell^\infty$ for all $\ell \in \Delta(S)\setminus L$, while $\min_{\ell\in D}\int u(f)\,d\ell^\infty = \frac{1}{2}u(x^*) + \frac{1}{2}u(x_*) > u(x_*) = \min_{\ell\in D}\int u(g)\,d\ell^\infty$ so that $f \succ g$ and again $\hat{\ell}$ is relevant.

We now show that $\succsim$ satisfies Continuous Symmetry. Since W is a real-valued representation, $\succsim$ satisfies Weak Order. Since W is sup norm continuous, $\succsim$ satisfies Mixture Continuity. All the remaining axioms will be shown by way of Theorem 4.5, as we now demonstrate that $\succsim^*$ may be represented as in (4.3). Suppose $\int u(f)\,dp \geq \int u(g)\,dp$ for all $p \in co\{\beta\ell^\infty + (1-\beta)q : \ell \in D\}$. Fix any $\lambda \in [0,1]$ and acts $f, g, h \in \mathcal{F}$, and let $\hat{\ell}^\infty \in \arg\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(\lambda f + (1-\lambda)h)\,dp$. Then

$$\begin{aligned} W\big(\lambda f + (1-\lambda)h\big) &= \int u\big(\lambda f + (1-\lambda)h\big)\,d\big(\beta\hat{\ell}^\infty + (1-\beta)q\big) \\ &\geq \int u\big(\lambda g + (1-\lambda)h\big)\,d\big(\beta\hat{\ell}^\infty + (1-\beta)q\big) \\ &\geq W\big(\lambda g + (1-\lambda)h\big), \end{aligned}$$

so that $f \succsim^* g$. Going the other direction, suppose $f \succsim^* g$ and that there exists a $\hat{p} \in co\{\beta\ell^\infty + (1-\beta)q : \ell \in D\}$ such that $\int u(f)\,d\hat{p} < \int u(g)\,d\hat{p}$. This implies that there exists an $\hat{\ell} \in D$ such that $\int u(f)\,d(\beta\hat{\ell}^\infty + (1-\beta)q) < \int u(g)\,d(\beta\hat{\ell}^\infty + (1-\beta)q)$. Let $\hat{h} = 1_{\Psi^{-1}(D\setminus\hat{\ell})}$. Choose $\hat{\lambda} \in (0,1)$ small enough to satisfy

$$\begin{aligned} &(1-\hat{\lambda})\big(u\big(x^*\big) - u(x_*)\big) \\ &\quad > \hat{\lambda}\max\bigg[\int u(f)\,d\hat{\ell}^\infty - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(f)\,dp, \\ &\qquad \int u(g)\,d\hat{\ell}^\infty - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(g)\,dp\bigg]. \end{aligned}$$

Then,

$$\begin{aligned} &\min_{p\in\{\ell^\infty:\ell\in D,\ell\neq\hat{\ell}\}}\int u\big(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\big)\,dp \\ &\quad \geq \hat{\lambda}\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(f)\,dp + (1-\hat{\lambda})u\big(x^*\big) \\ &\quad > \hat{\lambda}\int u(f)\,d\hat{\ell}^\infty + (1-\hat{\lambda})u(x_*) \\ &\quad = \int u\big(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\big)\,d\hat{\ell}^\infty, \end{aligned}$$

which implies $\hat{\ell}^\infty \in \arg\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(\hat{\lambda} f + (1-\hat{\lambda})\hat{h})\,dp$.

Similarly, $\hat{\ell}^\infty \in \arg\min_{p\in\{\ell^\infty:\ell\in D\}} \int u(\hat{\lambda}g+(1-\hat{\lambda})\hat{h})\,dp$. Thus,

$$\begin{aligned}\min_{p\in\{\ell^\infty:\ell\in D\}}\int u\big(\hat{\lambda}f+(1-\hat{\lambda})\hat{h}\big)\,dp &= \int u\big(\hat{\lambda}f+(1-\hat{\lambda})\hat{h}\big)\,d\hat{\ell}^\infty \\ &< \int u\big(\hat{\lambda}g+(1-\hat{\lambda})\hat{h}\big)\,d\hat{\ell}^\infty \\ &= \min_{p\in\{\ell^\infty:\ell\in D\}}\int u\big(\hat{\lambda}g+(1-\hat{\lambda})\hat{h}\big)\,dp.\end{aligned}$$

Therefore, as $\beta>0$,

$$\begin{aligned}W\big(\hat{\lambda}f+(1-\hat{\lambda})\hat{h}\big) &= \int u\big(\hat{\lambda}f+(1-\hat{\lambda})\hat{h}\big)\,d\big(\beta\hat{\ell}^\infty+(1-\beta)q\big) \\ &< \int u\big(\hat{\lambda}g+(1-\hat{\lambda})\hat{h}\big)\,d\big(\beta\hat{\ell}^\infty+(1-\beta)q\big) \\ &= W\big(\lambda g+(1-\lambda)h\big),\end{aligned}$$

contradicting $f \succsim^* g$. Summarizing, we have shown that

$$\begin{aligned}& f\succsim^* g \quad \text{if and only if} \quad \int u(f)\,dp \ge \int u(g)\,dp \\ &\quad \text{for all } p\in co\big\{\beta\ell^\infty+(1-\beta)q:\ell\in D\big\}.\end{aligned}$$

Therefore, applying Theorem 4.5 and noting that $co\{\beta\ell^\infty+(1-\beta)q:\ell\in D\}$ is weak* compact because D is finite, $\succsim$ represented by $W(f)$ satisfies Continuous Symmetry.

Since $\succsim$ is Continuous Symmetric and every measure in D is relevant, Theorem 3.3 implies $R=\overline{D}$. Since D is finite, $\overline{D}=D$.

D.5.2. *Proof of Theorem D.2*

Consider the "only if" direction. That $u_A=u_B$ up to normalization is equivalent to the two preferences agreeing on constant acts, a necessary condition for $\succsim_A$ is more ambiguity averse than $\succsim_B$.

Let $W_i(f)=\beta_i\min_{p\in\{\ell^\infty:\ell\in D_i\}}\int u_i(f)\,dp+(1-\beta_i)\int u_i(f)\,dq_i$ with $q_i=\int \ell^\infty\,dm_i(\ell)$ for $i=A,B$.

By Theorem D.1, $D=R$ for such preferences. Applying Theorem 3.4, we obtain $D_B=D_A$ since $\beta_A,\beta_B<1$ implies $\succsim_A$ and $\succsim_B$ are strictly monotonic for bets on non-null limiting frequency events. (To see the latter, let $\Psi^{-1}(L)$ be a non-null event, $x,y,z\in X$, and $x\succ y$. Then, $L\cap D\neq\emptyset$ and $\int u(x\Psi^{-1}(L)z)\,dq>\int u(y\Psi^{-1}(L)z)\,dq$. Since $\beta<1$, $x\Psi^{-1}(L)z\succ y\Psi^{-1}(L)z$.)

Consider the act $1_{\Psi^{-1}(\ell)}$ for some $\ell \in D_A$. Since $D_A = D_B$ is non-singleton, this act is evaluated as $W_A(1_{\Psi^{-1}(\ell)}) = (1-\beta_A)m_A(\ell)$ and $W_B(1_{\Psi^{-1}(\ell)}) = (1-\beta_B)m_B(\ell)$, respectively. Since $W_A = W_B$ on constant acts, $[f \succsim_A x \Rightarrow f \succsim_B x$ for all $x \in X$ and $f \in F]$ implies $[W_B(f) \geq W_A(f)$ for all $f \in F]$. Therefore

$$(1-\beta_B)m_B(\ell) \geq (1-\beta_A)m_A(\ell) \quad \text{for all } \ell \in D_A. \tag{D.1}$$

Summing (D.1) over $\ell \in D_A$ yields $\beta_A \geq \beta_B$. For all $\ell \in D_B$, (D.1) yields $m_B(\ell) \geq \frac{1-\beta_A}{1-\beta_B} m_A(\ell)$.

Turn to the "if" direction. For all acts f and measures $q = \int \ell^\infty \, dm(\ell)$ for some $m \in \Delta(\Delta(S))$ such that $\operatorname{supp} m \subseteq D$, $\min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp \leq \int u(f)\, dq$, so that increasing β can only lower the valuation of an act. Observe that W evaluates f by taking a convex combination of the $\int u(f)\, d\ell^\infty$ values. The weights in this convex combination are $(1-\beta)m(\ell)$ for all but some ℓ yielding the lowest of the values, which is assigned weight $(1-\beta)m(\ell)+\beta$. The conditions on m_A and m_B ensure that $(1-\beta_B)m_B(\ell) \geq (1-\beta_A)m_A(\ell)$, so that each ℓ yielding a value other than the minimum is assigned weakly more weight by $W_B(f)$ than by $W_A(f)$. Therefore, $W_B(f) \geq W_A(f)$ with equality for constant acts, so that $f \succsim_A x \Rightarrow f \succsim_B x$ for all $x \in X$ and $f \in F$.

D.5.3. *Proof of Theorem D.3*

First, we show $\operatorname{supp} m \subseteq R$. Suppose $\hat{\ell} \in \operatorname{supp} m$ and fix any $L \in \mathcal{O}_{\hat{\ell}}$. Take $x_1, x_2, x_3 \in X$ such that $x_2 \sim \frac{1}{2}x_1 + \frac{1}{2}x_3$ and $x_1 \succ x_3$. Define two acts f and g by

$$f(\omega) = \begin{cases} x_1, & \text{if } \Psi(\omega) \in L, \\ x_2, & \text{otherwise} \end{cases} \quad \text{and} \quad g(\omega) = \begin{cases} x_3, & \text{if } \Psi(\omega) \in L, \\ x_2, & \text{otherwise.} \end{cases}$$

Since $\int f \, d\ell^\infty = \int g \, d\ell^\infty$ for all $\ell \in \Delta(S) \setminus L$, it suffices to show that $f \nsim g$. For each $i = 1, \dots, n$,

$$\begin{aligned}
\int \zeta_i u(f)\, dp &= \int_{\Psi^{-1}(L)} \zeta_i u(x_1)\, dp + \int_{\Omega \setminus \Psi^{-1}(L)} \zeta_i u(x_2)\, dp \\
&= \int_{\Psi^{-1}(L)} \zeta_i \big[u(x_1) - u(x_2)\big]\, dp + \int_{\Omega} \zeta_i u(x_2)\, dp \\
&= \int_{\Psi^{-1}(L)} \zeta_i \big[u(x_1) - u(x_2)\big]\, dp \\
&= \int_{\Psi^{-1}(L)} \zeta_i \big[u(x_2) - u(x_3)\big]\, dp \\
&= -\int \zeta_i u(g)\, dp.
\end{aligned}$$

The third equality follows because $\int \zeta_i \, dp = 0$, and the fourth comes from $x_2 \sim \frac{1}{2}x_1 + \frac{1}{2}x_3$. Because $A(a) = A(-a)$,

$$A\left(\left(\int \zeta_i u(f)\, dp\right)_{1\le i\le n}\right) = A\left(\left(\int \zeta_i u(g)\, dp\right)_{1\le i\le n}\right).$$

Then,

$$\begin{aligned}&\int u(f)\, dp + A\left(\left(\int \zeta_i u(f)\, dp\right)_{1\le i\le n}\right)\\&\quad \neq \int u(g)\, dp + A\left(\left(\int \zeta_i u(g)\, dp\right)_{1\le i\le n}\right)\end{aligned}$$

because $\int u(f)\, dp > \int u(g)\, dp$ by the fact that $m(L) > 0$ and $x_1 \succ x_3$. Thus, $f \not\sim g$ and each measure in $\operatorname{supp} m$ is relevant.

Next, we show that $\succsim$ satisfies Continuous Symmetry. The form assumed for p and the symmetry property assumed for each ζ_i ensure that Event Symmetry is satisfied. The other properties in Symmetry along with Mixture Continuity follow directly from the properties of VEU (see Siniscalchi (2009)). To see Monotone Continuity of $\succsim^*$, observe that $x' \succsim^* xA_kx''$ if and only if, for all $\alpha \in [0, 1]$ and $h \in \mathcal{F}$,

$$\begin{aligned}&\alpha u(x') + A\left(\left((1-\alpha)\int u(h)\zeta_i \, dp\right)_{1\le i\le n}\right)\\&\quad \ge \alpha\big(p(A_k)u(x) + \big(1 - p(A_k)\big)u(x'')\big)\\&\qquad + A\Bigg(\Bigg(\alpha\left[u(x)\int_{A_k} \zeta_i \, dp + u(x'')\int_{A_k^c} \zeta_i \, dp\right]\\&\qquad + (1-\alpha)\int u(h)\zeta_i \, dp\Bigg)_{1\le i\le n}\Bigg).\end{aligned}$$

Since p is countably additive and ζ_i is bounded and measurable, $A_k \searrow \emptyset$ implies $p(A_k) \to 0$ and $\int_{A_k} \zeta_i \, dp \to 0$ and $\int_{A_k^c} \zeta_i \, dp \to \int_{S^\infty} \zeta_i \, dp = 0$. Therefore, since n is finite and A is Lipschitz continuous, there exists a k such that A_k is small enough so that $x' \succsim^* xA_kx''$. This proves Monotone Continuity of $\succsim^*$.

Because $\succsim$ is Continuous Symmetric and every measure in $\operatorname{supp} m$ is relevant, we can apply Theorem 3.3 to conclude $R = \operatorname{supp} m$.

D.5.4. *Proof of Theorem D.4*

Consider the "only if" direction. Suppose $\succsim_A$ is more ambiguity averse than $\succsim_B$. Since the two preferences agree on constant acts, $u_A = u_B$ up to normalization. Choose a common normalization and let $u = u_A = u_B$.

Note that if A is differentiable and $A(a) = A(-a)$ for all $a \in \mathbb{R}^n$, $DA(0) = 0$. This can be seen from $DA(a) = -DA(-a)$ for all $a \in \mathbb{R}^n$ and setting $a = 0$. Let $I_k(u(f)) \equiv \int u(f)\, dp_k + A_k((\int \zeta_i^k u(f)\, dp_k)_{1\le i\le n})$ for $k = A, B$. Then,

$$DI_A(c)(\varphi) = \int \varphi\, dp_A + DA_A\big((0)_{1\le i\le n}\big)\left(\left(\int \zeta_i^A \varphi\, dp_A\right)_{1\le i\le n}\right)$$

$$= \int \varphi\, dp_A.$$

Similarly, $DI_B(c)(\varphi) = \int \varphi\, dp_B$. Then, Lemma C.1 implies $p_A = p_B$. The rest of the result including the "if" direction follows directly from Proposition 4 of Siniscalchi (2009).

D.5.5. *Proof of Theorem D.5*

Suppose $\succsim$ is an i.i.d. second-order Choquet preference. We first show that $\operatorname{supp} v \subseteq R$. Suppose $\hat{\ell} \in \operatorname{supp} v$ and take $L \in \mathcal{O}_{\hat{\ell}}$. By definition, there is $L' \subseteq \Delta(S)$ such that $v(L \cup L') > v(L')$. Consider $f = 1_{\Psi^{-1}(L\cup L')}$ and $g = 1_{\Psi^{-1}(L')}$. Then, $\int f\, d\ell^\infty = \int g\, d\ell^\infty$ for all $\ell \in \Delta(S) \setminus L$. Compute $\int\int u(f)\, d\ell^\infty\, dv(\ell) = v(L\cup L') > v(L') = \int\int u(g)\, d\ell^\infty\, dv(\ell)$. Thus, each $\hat{\ell} \in \operatorname{supp} v$ is relevant.

We now show that $\succsim$ satisfies Continuous Symmetry. Monotone Continuity of $\succsim^*$ and Mixture Continuity follow by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2013, Theorem 7) and the fact that a capacity with finite support is continuous. That the other axioms are satisfied is straightforward.

We have shown that every measure in $\operatorname{supp} v$ is relevant and $\succsim$ satisfies Continuous Symmetry. Thus we can apply Theorem 3.3 to conclude $R = \operatorname{supp} v$.

D.5.6. *Proof of Theorem D.6*

Consider the "if" direction first. Let $u = u_A = u_B$ without loss of generality. It is a property of the Choquet integral that $v_B \ge v_A$ implies

$$\int\int u(f)\, d\ell^\infty\, dv_B(\ell) \ge \int\int u(f)\, d\ell^\infty\, dv_A(\ell)$$

for all $f \in F$ with equality for constant acts (see, e.g., Denneberg (1994, Propositions 5.1 and 5.2)). This implies $\succsim_A$ is more ambiguity averse than $\succsim_B$.

Now turn to the "only if" direction. That $u_A = u_B$ up to normalization is equivalent to the two preferences agreeing on constant acts, a necessary condition for $\succsim_A$ is more ambiguity averse than $\succsim_B$. Suppose, to the contrary, that $\succsim_A$ is more ambiguity averse than $\succsim_B$ but $v_B(L') < v_A(L')$ for some $L' \subseteq \Delta(S)$. Then $1_{\Psi^{-1}(L')} \sim_A v_A(L'^* + (1 - v_A(L'))x_* \succ_B 1_{\Psi^{-1}(L')}$, contradicting $\succsim_A$ is more ambiguity averse than $\succsim_B$.

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Dept. of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, Evanston, IL 60208, U.S.A.; peterk@kellogg.northwestern.edu,

Dept. of Economics, University of Oxford, Oxford, OX1 3UQ, U.K.; sujoy.mukerji@economics.ox.ac.uk,

and

Graduate School of Finance, Korea Advanced Institute of Science and Technology, Seoul, South Korea; kseo@kaist.ac.kr.

Manuscript received March, 2011; final revision received May, 2014.