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Maxmin expected utility through statewise combinations

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Abstract

We provide an axiomatic foundation for a maxmin expected utility over a set of priors (MMEU) decision rule in an environment where the elements of choice are Savage acts. The key axioms are stated using statewise combinations as in Gul [Gul, F., 1992. Savage's theorem with a finite number of states. Journal of Economic Theory 57, 99–100].

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1. Introduction

This paper provides an axiomatic foundation for a maxmin expected utility over a set of priors (MMEU) decision rule in an environment where the elements of choice are Savage (1954) acts. This characterization complements the original axiomatization of MMEU developed in a lottery-acts (or Anscombe and Aumann, 1963) framework by Gilboa and Schmeidler (1989). MMEU preferences are of interest primarily because they provide a natural and tractable way of modeling decision makers who display an aversion to uncertainty or ambiguity. In Casadesus-Masanell et al. (1999), we characterized an MMEU rule over Savage acts using axioms stated in terms of standard sequences, a measurement theory construction (see Krantz et al., 1971). In contrast, the key axioms here involve statewise combinations of acts (as defined below). Statewise combinations have the advantage that they look much like the convex combinations used in Anscombe–Aumann and von Neumann–Morgenstern style theories. Thus, as in Gul (1992), they allow one to transfer much of the intuition from these settings to a setting with Savage acts. The disadvantage of this approach is that it is less

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general than the one in Casadesus-Masanell et al. (1999): the theory here will imply the existence of a binary partition of the state space over which subjective expected utility holds.

The remainder of the paper presents a set of axioms and a theorem proving the equivalence between these axioms and an MMEU rule. The novel axioms are weakenings of Gul (1992, Assumption 2) act-independence condition. Results in Nakamura (1990) and Gilboa and Schmeidler (1989) are useful for the proofs.

2. Notation and framework

Ω is the set of states. A state in Ω is represented by ω. Σ is an algebra of subsets of Ω. Events are elements of Σ. $X=[m,M]\subset\mathbb{R}$, $m<M$ is the set of prizes or outcomes. A (Savage) act f is a function $f:\Omega\to X$. A simple act is an act with only finitely many distinct values. A simple act is Σ-measurable if $\{\omega\in\Omega|f(\omega)\in W\}\in\Sigma$ for all $W\subseteq X$. F is the set of all Σ-measurable acts defined as the closure in the supnorm of all Σ-measurable simple acts. A set $G\subseteq F$ is closed if it is closed in the supnorm. A constant act f is one for which $f(\omega)=x$ for all $\omega\in\Omega$, for some $x\in X$; we denote this constant act by x^* or simply x when no confusion would result. F^* is the subset of F consisting of all constant acts. For any event $B\in\Sigma$ and x, $y\in X$, $x_B y$ denotes $f\in F$ such that $f(\omega)=x$ for $\omega\in B$ and $f(\omega)=y$ for $\omega\not\in B$; such acts are referred to as B-measurable. The event $\Omega-B$ is denoted B^c. For $f,g,h\in F$ and $B\in\Sigma$, if $h(\omega)\sim f(\omega)_B g(\omega)$ for all $\omega\in\Omega$ then h is a statewise combination of f and g over the event B. $\mathscr{P}$ is the set of all finitely additive probability measures $P:\Sigma\to[0,1]$. Finally, $\succeq$ is a binary relation on F. Note that this environment is similar to that in Savage (1954) with the difference that we impose more structure on the prize set (X). The important aspect of this structure is that X is connected and separable. This will allow the non-singleton set of states in our theory to be of any size, finite or infinite.

3. Axioms

Axiom 1. (Weak order) $\succeq$ is complete and transitive.

Definition 3.1. An event B is ordered non-null if there exist x, y and z in X with $x\preceq y\preceq z$ such that $x_B z\not\sim y_B z$. An event B is ordered non-universal if there exist x, y and z in X with $z\preceq y\preceq x$ such that $z_B x\not\sim z_B y$.

Note that since we impose restrictions on the ordering of x, y and z, our definitions of non-null and non-universal (borrowed from Nakamura, 1990) are weaker than the corresponding notions in Savage (1954). See Casadesus-Masanell et al. (1999) for an explanation of why these are the appropriate notions.

Axiom 2. (Structure) (a) $x>y\Rightarrow x^*>y^*$. (b) There exists an event $A\in\Sigma$ such that A and A^c are ordered non-null and ordered non-universal.

Part (a) is purely a simplifying assumption. The event A identified here is used in Axioms 5, 6 and 7 below.

Axiom 3. (Continuity) For all $f \in F$, the sets $M(f) = \{g \in F | g \succeq f\}$ and $W(f) = \{g \in F | f \succeq g\}$ are closed.

Axiom 4. (Monotonicity) (a) For all $f, g \in F$, if $f(\omega) \succeq g(\omega)$, for all $\omega \in \Omega$ then $f \succeq g$. (b) If $B \in \Sigma$ is ordered non-null and $z \succeq x \succ y$, then $x_B z \succ y_B z$. If $B \in \Sigma$ is ordered non-universal and $x \succ y \succeq z$, then $z_B x \succ z_B y$.

Observe that part (a) is weak monotonicity, and part (b) is strict monotonicity on ordered non-null and non-universal events.

Axiom 5. (A-act-independence) Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ and let $f = x_{1A} x_2$, $g = y_{1A} y_2$ and $h = z_{1A} z_2$. If $f', g' \in F$ are such that, for either $B = A$ or $B = A^c$, $f'(\omega) \sim h(\omega)_B f(\omega)$ and $g'(\omega) \sim h(\omega)_B g(\omega)$ for all $\omega \in \Omega$ then, $f \succeq g \Leftrightarrow f' \succeq g'$.

This axiom imposes act-independence (as introduced in Gul, 1992) only for A-measurable acts and the events A and A^c. In words, given A-measurable acts f, g and h and given the event $B = A$ or $B = A^c$, if f' is a statewise combination of h and f over the event B and g' is a statewise combination of h and g over the event B, then preference between f and g is the same as between f' and g'. As discussed in Gul (1992), act-independence is analogous to the independence axiom in the theory of expected utility over lotteries.

These first five axioms guarantee the existence of an expected utility representation for A-measurable acts. That is, there exists a strictly increasing and continuous function $u:X \to \mathbb{R}$ and $\rho \in (0,1)$ such that if $x, y, v, w \in X$ then

$$x_A y \succeq v_A w \Leftrightarrow \rho u(x) + (1-\rho)u(y) \geq \rho u(v) + (1-\rho)u(w).$$

Moreover, u is unique up to positive affine transformations and ρ is unique.

How do preferences extend from A-measurable acts to all acts? If act-independence is required to hold for all acts and non-null events, expected utility preferences result (see Gul, 1992; Chew and Karni, 1994). This act-independence is too strong for MMEU as it is incompatible, for example, with the Ellsberg Paradox (Ellsberg, 1961). We now develop the appropriate weakening of act-independence.

Definition 3.2. (Ghirardato et al., 1998) Two acts f and g are affinely related if there exist $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that either $u(f(\omega)) = \alpha u(g(\omega)) + \beta$ for all $\omega \in \Omega$ or $u(g(\omega)) = \alpha u(f(\omega)) + \beta$ for all $\omega \in \Omega$.

The key is the use of statewise combinations over the event A to form what will turn out to be sets of affinely related acts. This requires the following definitions:

Definition 3.3. A set $S \subset F$ contains all statewise combinations over the event A if $f_1, f_2 \in S$, and for all $\omega \in \Omega$, $f(\omega) \sim f_1(\omega)_A f_2(\omega)$ implies $f \in S$.

Definition 3.4. Fix $f \in F$. We define $S_0^f = \{f\} \cup F^*$. Let $\bar{S}^f \supseteq S_0^f$ be the smallest closed set containing all statewise combinations over the event A.

It can be shown that given an act f, the set $\bar{S}^f$ may be constructed by the following iterative method: at each step $i = 1,2,3,...$ we produce a set

$$S_i^f = \{f^i \in F : f^i(\omega) \sim f_1^{i-1}(\omega)_A f_2^{i-1}(\omega), \text{ for all } \omega \in \Omega \text{ where } f_1^{i-1}, f_2^{i-1} \in S_{i-1}^f\}.$$

Finally, $\bar{S}^f$ is the closure of $\cup_{i=1}^{\infty} S_i^f \subseteq F$. Observe that S_1^f consists of statewise combinations over the event A of either f or a constant act with either f or a constant act. By the expected utility representation for A-measurable acts, we end up with either a constant utility act or a positive affine transformation of the state-by-state utility of f. Either way the resulting acts are affinely related to f. To form S_2^f, we take any two acts in S_1^f and combine them statewise over A. Since both of these acts are affinely related to the act f, the resulting act will also be affinely related to f. This argument plus continuity shows that $\bar{S}^f$ consists only of acts that are affinely related to f. In fact, if f is not a constant act, the set $\bar{S}^f$ contains *all* acts that are affinely related to f.

Axiom 6. ($\bar{S}$-act-independence) For any $f, g, h \in F$ such that there exist $l, k \in \{f,g\}$ for which $f, h \in \bar{S}^l$ and $g, h \in \bar{S}^k$, if $f', g' \in F$ are such that, for either $B = A$ or $B = A^c$, $f'(\omega) \sim h(\omega)_B f(\omega)$ and $g'(\omega) \sim h(\omega)_B g(\omega)$ for all $\omega \in \Omega$ then, $f \succeq g \Leftrightarrow f' \succeq g'$.

Using the representation for A-measurable acts and the definition of $\bar{S}^f$, it can be shown that the axiom applies only if: (1) h is a constant act; or (2) h is not a constant act, but f, g and h are pairwise affinely related.

It is useful to compare $\bar{S}$-act-independence to the certainty-independence (C-independence) axiom of Gilboa and Schmeidler (1989):

(C-independence) For any acts f and g, any constant act h, and any $\alpha \in (0,1)$, if f', g' are such that, $f' = \alpha f + (1-\alpha)h$ and $g' = \alpha g + (1-\alpha)h$ then, $f \succeq g \Leftrightarrow f' \succeq g'$.

Note that the convex combination operation is defined statewise and is well-defined since acts in their setting are functions from states to probability distributions over prizes. C-independence relaxes the independence axiom of Anscombe and Aumann (1963) so that it is only required to hold when the third act, h, is a constant act.

$\bar{S}$-act-independence and C-independence are quite similar in form, with two salient differences. First, statewise combinations over A or A^c replace convex combinations. Second, as pointed out in possibility (2) above, $\bar{S}$-act-independence applies to some h which are not constant acts. In fact, the first difference leads to the second one. In an Anscombe–Aumann framework, all probabilities in the unit interval are available. Consequently, to express the fact that preference is preserved by homogeneous transformations (of utility), one need only consider convex combinations of the act in question and a constant act. In contrast, the only probabilities that are available through statewise combinations over A are those of A and A^c. For example, if the revealed probability of A happens to be 1/3 and we want to show that multiplying the utility of a pair of acts by 1/2 preserves the preference ordering between them, we cannot construct the '1/2-acts' through statewise combinations over A without taking combinations of two non-constant acts. This is why we cannot restrict h to be a constant act in the $\bar{S}$-act-independence axiom.

The final axiom, act-uncertainty aversion, restricts the way that act-independence can be violated. It requires that the decision-maker weakly likes to smooth utilities across states of the world, since this leaves her less exposed to any uncertainty or ambiguity about the probability of various states. Specifically it modifies Gilboa and Schmeidler (1989) uncertainty aversion axiom by replacing convex combinations with statewise combinations over A.

Axiom 7. (Act-uncertainty aversion) For all $f, g, f' \in F$, if $f \sim g$ and $f'(\omega) \sim f(\omega)_A g(\omega)$ for all $\omega \in \Omega$ then, $f' \succeq f$.

See Casadesus-Masanell et al. (1999) for a discussion relating this type of uncertainty aversion axiom to recent alternatives suggested by Epstein (1999) and Ghirardato and Marinacci (1998).

4. A representation theorem

Theorem 4.1. *Let $\succeq$ be a binary relation on F. Then $\succeq$ satisfies Axioms 1–7 if and only if there exists a continuous and strictly increasing function $u:X \to \mathbb{R}$, and a non-empty, compact and convex set $\mathscr{C}$ of finitely additive probability measures on Σ such that*

$$[f \succeq g] \Leftrightarrow \left[\min_{P \in \mathscr{C}} \int u \circ f dP \geq \min_{P \in \mathscr{C}} \int u \circ g dP \right] \text{ for all } f \text{ and } g \in F.$$

Furthermore, there exists an event $A \in \Sigma$ and a $\rho \in (0,1)$ such that $P \in \mathscr{C}$ implies $P(A) = \rho$. Moreover, u is unique up to positive affine transformations and the set $\mathscr{C}$ is unique.

Proof. We sketch sufficiency; necessity is omitted. Axioms 1–5 imply versions of the appropriate axioms of Nakamura (1990) using events A and A^c. Apply Nakamura's Theorem 1 to yield an expected utility representation for A-measurable acts. The continuity axiom and structure axiom part (a) guarantee that u is continuous and strictly increasing. Let ρ be the probability of A. Let $K = u(X)$. We normalize u such that $K = [-2,2]$.

We now construct a functional $J:F \to \mathbb{R}$ that represents preferences. For any constant act $f = x \in X$ we define $J(f) = u(x)$. For general acts $f \in F$, let $J(f) = u(c)$, where c is the certainty equivalent of f. Continuity ensures that c exists.

Let B be the space of bounded (in the supnorm), Σ-measurable, real valued functions on Ω. For $\gamma \in \mathbb{R}$, we denote by γ^* the element of B that assigns γ to every ω. Let $B(K)$ be the subset of functions in B with values in K. Observe that for $f \in F$, $u \circ f \in B(K)$, and for $d \in B(K)$ there exists $f \in F$ such that $u \circ f = d$.

Define the functional $I:B(K) \to \mathbb{R}$ by $I(u \circ f) = J(f)$. Since J represents preferences, it is clear that I does as well. It can be shown that an extension of I to B satisfies the following properties:

(i) $I(1^*) = 1$;
(ii) (I is monotonic) For all a, $b \in B$, $a \geq b$ implies $I(a) \geq I(b)$;
(iii) (I is homogeneous of degree 1) For all $b \in B$, $\alpha \geq 0$, $I(\alpha b) = \alpha I(b)$;
(iv) (I is C-independent) For all $b \in B$, $\gamma \in \mathbb{R}$, $I(b + \gamma^*) = I(b) + I(\gamma^*)$; and,
(v) (I is superadditive) For all a, $b \in B$, $I(a + b) \geq I(a) + I(b)$.

Properties (i) and (ii) on $B(K)$ are immediate. The proof of (iii) on $B(K)$ uses $\bar{S}$-act-independence heavily. Then extend I to all of B by homogeneity. This preserves homogeneity and monotonicity. Now (iv) and (v) can be shown for the extension of I by arguments similar to those in Gilboa and Schmeidler (1989).

Properties (i)–(v) imply the existence of the minimum expectation over a set of measures representation. This is a fundamental lemma variations of which have been proved by, for example, Gilboa and Schmeidler (1989), Chateauneuf (1991) and Marinacci (1997). □

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