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Characterizing uncertainty aversion through preference for mixtures

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Abstract. Uncertainty aversion is often modelled as (strict) quasi-concavity of preferences over uncertain acts. A theory of uncertainty aversion may be characterized by the pairs of acts for which strict preference for a mixture between them is permitted. This paper provides such a characterization for two leading representations of uncertainty averse preferences; those of Schmeidler [24] (Choquet expected utility or CEU) and of Gilboa and Schmeidler [16] (maxmin expected utility with a non-unique prior or MMEU). This characterization clarifies the relation between the two theories.

1 Introduction

A large body of work has recently emerged in economics and decision theory with the goal of representing behavior in the face of subjective uncertainty that may violate the independence axiom of subjective expected utility theory. One branch of this literature, and the one that will be the focus below, considers preferences that may violate independence by displaying a preference for facing risk (or "objective" probabilities) as opposed to uncertainty. This preference is known as *uncertainty aversion*. One motivation for examining these preferences are the well-known problems posed by Ellsberg [10] and the

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huge experimental literature that has followed, in which many individuals behave as if they were uncertainty averse.

There are several ways that one could imagine defining uncertainty aversion. The definition that I will use here, and the one that has dominated the literature so far, is due to Schmeidler [24][1]. It states that for any two acts that an individual is indifferent between, a mixture over these two acts is at least as preferred as either act.[2] One may interpret this requirement as saying that the individual likes smoothing expected utility across states. This smoothing has the effect of making the outcome less subjective, and therefore such a mixing operation could be called "objectifying"[3]. Thus, in a natural sense, such an individual is displaying an aversion to uncertainty. An equivalent way of stating this characteristic is to say that preferences are quasi-concave ($f \succeq g$ and $\alpha \in (0,1)$ implies $\alpha f + (1-\alpha)g \succeq g$). In particular, observe that if $f \sim g$ then quasi-concavity allows $\alpha f + (1-\alpha)g \succ g$ while independence requires $\alpha f + (1-\alpha)g \sim g$.

From this viewpoint, a theory of uncertainty averse preferences may be characterized by the set of violations of independence in the direction of strict preference for mixture that it allows. The goal of this paper is to provide a characterization of this kind for two leading axiomatic theories of uncertainty aversion, the Choquet expected utility (CEU) theory of Schmeidler [24] and the maxmin expected utility (MMEU) theory of Gilboa and Schmeidler [16]. Such a characterization is useful not only from the point of view of theoretical understanding, but also as a guide to the design of experiments testing one theory of uncertainty aversion against another. For example, the results in Sect. 3 allow the easy identification of pairs of acts over which an MMEU decision maker may have a strict preference for mixture, while a CEU decision maker cannot. Furthermore, in the emerging literature applying these theories (e.g., Dow and Werlang [7], Klibanoff [17], Lo [19], Eichberger and Kelsey [9], Marinacci [21] on game theory; Wakker [25] on optimism and pessimism; Dow and Werlang [6], Chateauneuf et al. [3], Epstein and Wang [12] on financial markets; Mukerji [22] and Ghirardato [13] on contracting; and others) too often one theory or the other is adopted without much recognition of the ways in which the theories differ.

The next section introduces the CEU and MMEU theories and points out the known, yet frequently ignored, fact that under uncertainty aversion, MMEU is a strict generalization of CEU. Section 3 presents the main theorems characterizing the acts for which no convex combination is ever strictly preferred to both acts themselves under MMEU and under CEU respectively. Section 4 concludes.

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[1] Until the recent work of Epstein [11] and Ghirardato and Marinacci [15], Schmeidler's was essentially the *only* definition used in the literature.

[2] A mixture over two acts is formally defined in Sect. 2.1 below.

[3] I thank Mark Machina for suggesting this term.

2 Two models of uncertainty aversion

2.1 Notation and set-up

Throughout the paper, preferences are a binary relation, $\succeq$, on functions (acts) $f : S \rightarrow Y$ where S is a finite set of states of the world, X a set of prizes, and Y the set of all probability measures with finite support (lotteries) on X. Thus, in each state of the world an act yields a lottery over prizes (as in Anscombe and Aumann [1]). Lotteries are evaluated according to an affine utility function $u : Y \rightarrow \Re$. Denote the probability of prize $x \in X$ in state $s \in S$ induced by act f by $f(s)(x)$. For any $\alpha \in (0,1)$, define the α-mixture over f and g, $\alpha f + (1-\alpha)g$, by

$$(\alpha f + (1-\alpha)g)(s)(x) = \alpha f(s)(x) + (1-\alpha)g(s)(x), \quad \text{for all } x \in X,\ s \in S.$$

One interpretation of the mixture $\alpha f + (1-\alpha)g$ is a randomization over the acts f and g with probabilities α and $1-\alpha$ respectively. Under this interpretation a preference for mixtures implies a preference for randomization. Such a preference for randomization is controversial in the literature. An issue is whether randomization is a way of making mixtures feasible in particular settings.[4] The correctness and (in large part) interpretation of the analysis below is independent of one's position in this debate. The objects of study are acts, and mixtures are simply particular acts. If one does not accept the randomization interpretation, preference for mixtures may be read as simply a statement about preferences over pure acts whose utility payoffs happen to be related through convex combinations.

2.2 Two models

A leading representation of uncertainty averse preferences is the CEU representation axiomatized by Schmeidler [24]. Here preferences are represented by the Choquet integral of a utility function with respect to a capacity or non-additive measure. One of the properties which characterizes such preferences is *comonotonic independence*. Two acts, f and g, are said to be comonotonic if, for no pair of states of the world s and s', $f(s) \succ f(s')$ and $g(s') \succ g(s)$.[5] Preferences satisfy comonotonic independence if, for any acts f and g, $f \succeq g$ if and only if $\alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h$ for all $\alpha \in (0,1)$ and all h such that f, g, h are pairwise comonotonic. This is simply a restriction of the standard independence axiom (e.g., Anscombe and Aumann [1]) to pairwise comonotonic acts. From this axiom, the following is immediate:

[4] For two contrasting views of the impact, in the context of uncertainty aversion and randomization, of using a model with an Anscombe-Aumann-style mixture space of acts (as here) rather than Savage-style acts, see Eichberger and Kelsey [8] and Klibanoff [18].

[5] $f(s)$ should be understood as an act which gives the lottery that act f gives in state s no matter which state occurs.

Result 1. *Suppose that preferences satisfy comonotonic independence. Then for any comonotonic acts f and g, for each $\alpha \in (0,1)$, either $f \succeq \alpha f + (1-\alpha)g$ or $g \succeq \alpha f + (1-\alpha)g$ or both.*

Thus, strict preference for mixtures cannot occur with comonotonic acts. Notice that this observation derives from comonotonic independence *alone* and is in no way implied by uncertainty aversion per se.

Now consider a second common representation of preferences incorporating uncertainty aversion, namely the MMEU representation axiomatized in Gilboa and Schmeidler [16]. In this work, the axiom of comonotonic independence is replaced by an alternative axiom, denoted C-independence. C-independence requires the independence axiom to hold only when the act *h* used to form the mixtures gives the same expected utility in every state of the world.[6] Intuitively, acts which yield the same expected utility in every state leave no room for uncertainty about which state will occur to matter. C-independence is the assumption that mixing with such an act will not change either the way in which the decision maker perceives her uncertainty or the way in which she allows her attitude towards uncertainty to affect her preferences.

Gilboa and Schmeidler [16] showed that C-independence and the standard assumptions of weak order, continuity and monotonicity together with uncertainty aversion imply that preferences can be represented by the minimum expected utility of an act, where the minimum is taken over a closed, convex set of probability measures. Notice that an act which yields constant expected utility across states is comonotonic with any other act. In fact, comonotonic independence, weak order, continuity, and monotonicity imply C-independence. This means that, under the assumption of uncertainty aversion, any preferences that can be represented by CEU can also be represented in the MMEU framework (Schmeidler [23], [24]). The converse is not true, however, as the following example makes clear.

	s_1	s_2	s_3
f	1.5	2	3.5
g	0	2.1	4
h	0.75	2.05	3.75

Set of measures: $B = \left\{(p_1, p_2, p_3) \mid p_1 = p_3, \sum_{i=1}^{3} p_i = 1, 0 \leq p_i \leq 1\right\}$

Example 1

[6] Technically the axiom is more restrictive, requiring *h* to give the same lottery over outcomes in each state of the world, but together with the assumptions of weak order, continuity and monotonicity the axiom as described is implied. Note that the assumptions of weak order, continuity and monotonicity were also assumed in the Choquet expected utility theory of Schmeidler [24].

In Example 1, an individual must choose over two possible pure acts, f and g, which give the expected utilities indicated above in the three possible states of the world. Observe that f and g are comonotonic. Suppose that the individual's preferences can be represented by minimum expected utility over the set of measures B (i.e., the set of all probability measures which assign equal weight to s_1 and s_3). Straightforward calculation shows that f and g each give a minimum expected utility of 2, while, for example, h, a half-half mixture between f and g, gives a minimum expected utility of 2.05. Therefore, this uncertainty averse individual will strictly prefer the mixture h, compared to either f or g. Since this violates comonotonic independence it shows that these preferences cannot be represented in the CEU framework, and also demonstrates that comonotonicity is not enough, in general, to guarantee that an uncertainty averse individual will not strictly prefer to objectify by mixing over acts. What is the right condition to guarantee no strict preference for mixtures in the MMEU representation? Is the comonotonicity condition a necessary as well as sufficient condition for no preference for mixing under CEU? The next section provides results to answer these questions.

3 Characterizing preference for mixtures

In examining when strict preference for mixtures is possible (or impossible) under the two theories, it is helpful to consider a previous result characterizing preference for mixtures under MMEU for a *specified* set of probability measures. While such results are of more interest in a setting where certain beliefs are focal (e.g., equilibrium beliefs in game theory), they will be used in proving the theorems to follow that apply to the whole domain of the respective theories.

Theorem 1. *(Klibanoff [17]) For any acts f and g such that $f \succeq g$, no mixture over these acts will be strictly preferred to either alone if and only if there exists some measure q in the set of measures such that q minimizes the expected utility of f over the set and such that the expected utility of f with respect to q is at least the expected utility of g with respect to q.*

For acts which the decision maker is indifferent between this simplifies to:

Corollary 1. *(Klibanoff [17]) For any acts f and g such that $f \sim g$, no mixture over these acts will be strictly preferred to either alone if and only if there exists some measure q in the set of measures such that q minimizes the expected utility of both f and g over the set of measures.*

Now we characterize the set of acts for which *no* MMEU decision maker would have a strict preference for a mixture. This result and the corresponding result for the CEU case are provided in the next two theorems.

Theorem 2. *Fix acts f and g. No convex combination of f and g will ever be strictly preferred to either alone (given MMEU preferences) if and only if (i) f*

weakly dominates g or vice-versa (i.e., $u(f(s)) \geq (\leq) u(g(s))$, for all $s \in S$.) **or** *(ii) there exists an $a \geq 0, b \in \Re$ such that either $u(g(s)) = au(f(s)) + b$ for all $s \in S$ or $u(f(s)) = au(g(s)) + b$ for all $s \in S$.*

Proof. The difficult direction is to show that no strict preference implies (i) or (ii). (The opposite direction is left for the reader to verify.) The key step in the proof is to show that the conditions for a convex combination to reduce uncertainty are equivalent to the existence of a pair of probability vectors satisfying a set of linear inequalities. This is done in the lemma below. The only task remaining is then to characterize existence. To do this I apply a well known result from the theory of linear inequalities, Motzkin's Theorem of the alternative (see e.g., Mangasarian [20]). The existence of a solution to the resulting alternative system is then (after a bit of rearrangement) shown to be equivalent to the conditions of the theorem.

Let the vector of utility payoffs to the act f be denoted $\mathscr{F}(\equiv \{u(f(s))\})$ and similarly for $\mathscr{G}$. The following lemma reduces the conditions for a convex combination of f and g to possibly reduce uncertainty to a question of existence of probabilities satisfying certain linear inequalities.

Lemma 1. *Fix $\mathscr{F}$ and $\mathscr{G}$. There exists a non-empty, closed, convex set of measures B for which some mixture of $\mathscr{F}$ and $\mathscr{G}$ is strictly preferred to either alone if and only if there exist probability vectors $p1$ and $p2$ satisfying:*

(i) $\mathscr{F} \cdot p2 - \mathscr{F} \cdot p1 > 0$

(ii) $\mathscr{F} \cdot p1 - \mathscr{G} \cdot p1 < 0$

(iii) $\mathscr{G} \cdot p1 - \mathscr{G} \cdot p2 > 0$

and

(iv) $\mathscr{G} \cdot p2 - \mathscr{F} \cdot p2 < 0$

Proof.

($\Leftarrow$) Suppose such $p1$ and $p2$ exist. Let B be the set of all convex combinations of $p1$ and $p2$. Either $f \succeq g$ or $g \succeq f$ or both. If $f \succeq g$ then by (i) and (ii), $p1$ is the only minimizer in B of the expected utility of f and the expected utility of f under $p1$, $\mathscr{F} \cdot p1$, is less than the expected utility of g under $p1$, $\mathscr{G} \cdot p1$. Therefore, by Theorem 1, there exists a mixture which is strictly preferred. If $g \succeq f$ then by (iii) and (iv) and Theorem 1 the same conclusion holds.

($\Rightarrow$) Suppose such a B exists. Consider the set $A_f = \{p \mid p \in \operatorname{argmin}_{p \in B} \mathscr{F} \cdot p\}$ and $A_g = \{p \mid p \in \operatorname{argmin}_{p \in B} \mathscr{G} \cdot p\}$. Consider $p1 \in A_f$ and $p2 \in A_g$. By definition of these sets we must have

(a) $\mathscr{F} \cdot p2 - \mathscr{F} \cdot p1 \geq 0$

and

(b) $\mathscr{G} \cdot p1 - \mathscr{G} \cdot p2 \geq 0.$

Suppose that (a) holds with equality for some such $p1$ and $p2$. Then if

$g \succeq f$, $\mathscr{G} \cdot p2 \geq \mathscr{F} \cdot p2$ which implies that the condition in Theorem 1 is satisfied and no mixture is strictly preferred. If $f \succ g$, then $\mathscr{F} \cdot p1 = \mathscr{F} \cdot p2 > \mathscr{G} \cdot p2$ and again appealing to Theorem 1 no mixture is strictly preferred. Similar arguments show that if (b) holds with equality for some such $p1$ and $p2$ then no mixture is strictly preferred. Thus for there to be a mixture that is strictly preferred it must be that for all $p1 \in A_f$ and $p2 \in A_g$,

(i) $\mathscr{F} \cdot p2 - \mathscr{F} \cdot p1 > 0$

and

(iii) $\mathscr{G} \cdot p1 - \mathscr{G} \cdot p2 > 0.$

Can it be that $\mathscr{F} \cdot p1 - \mathscr{G} \cdot p1 \geq 0$? This and (iii) would imply $\mathscr{F} \cdot p1 - \mathscr{G} \cdot p2 > 0$ which implies $f \succ g$ and thus by Theorem 1 and the hypothesis no mixture would be strictly preferred. Therefore,

(ii) $\mathscr{F} \cdot p1 - \mathscr{G} \cdot p1 < 0$

must hold. By an analogous argument,

(iv) $\mathscr{G} \cdot p2 - \mathscr{F} \cdot p2 < 0$

must hold as well and we are done. *QED*

Now that the lemma has been proved, the next step in proving the theorem is to combine conditions (i)–(iv) with the restrictions implied by the fact that $p1$ and $p2$ must be probability vectors. To this end, let n be the number of states in S. Then $\mathscr{F}, \mathscr{G}, p1$ and $p2$ are n-vectors. Let $\mathscr{F}$ and $\mathscr{G}$ be row vectors and $p1$ and $p2$ be column vectors. Let e be a row n-vector of 1's. Let

$$p = \begin{bmatrix} p1 \\ p2 \end{bmatrix}$$

and

$$A = \begin{bmatrix} -\mathscr{F} & \mathscr{F} \\ \mathscr{G} - \mathscr{F} & 0 \\ \mathscr{G} & -\mathscr{G} \\ 0 & \mathscr{F} - \mathscr{G} \end{bmatrix}.$$

Observe that (i)–(iv) is equivalent to $Ap > 0$. Furthermore the requirement that $p1$ and $p2$ be probabilities is equivalent to $p \geq 0$ and

$$\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{1}$$

Equivalently, we can replace the normalization (1) with the condition

$$[e \quad -e]p = 0$$

and the condition $p \geq 0$ with the equivalent

$$Ip \geq 0$$

where I is a $2n \times 2n$ identity matrix. To summarize, we would like to characterize when there exists a p such that

(a) $Ap > 0$

(b) $Ip \geq 0$

and

(c) $[e \quad -e]p = 0.$

By Motzkin's Theorem of the alternative (Mangasarian [20]), either (a), (b) and (c) has a solution p or

$$(*) \quad \left\langle \begin{matrix} A'y_1 + I'y_3 + [e \quad -e]'y_4 = 0 \\ y_1 > 0, y_3 \geq 0 \end{matrix} \right\rangle$$

has a solution y_1, y_3, y_4, but never both. (Note that $y_1 > 0$ means that each element of y_1 is greater than or equal to zero with at least one element strictly positive. $y_3 \geq 0$ means almost the same thing except that it allows all elements to be zero.)

All that remains is to rewrite system $(*)$ to get an interpretable condition (namely the one in the theorem.) First notice that since the elements of y_3 are all non-negative, $(*)$ has a solution if and only if

$$(**) \quad \left\langle \begin{matrix} A'y_1 + [e \quad -e]'y_4 \leq 0 \\ y_1 > 0 \end{matrix} \right\rangle$$

has a solution y_1, y_4. Adding up the inequalities determined by the first line of $(**)$ yields

$$(\mathscr{G} - \mathscr{F})'(y_{21}^1 - y_{41}^1) \leq 0,$$

where

$$y_1 = \begin{bmatrix} y_{11}^1 \\ y_{21}^1 \\ y_{31}^1 \\ y_{41}^1 \end{bmatrix}.$$

This implies that either $y_{21}^1 = y_{41}^1$ or one of f and g is weakly dominated by the other. So, a solution to $(*)$ exists if and only if either weak dominance between f and g holds or $(**)$ is satisfied with $y_{21}^1 = y_{41}^1$. Imposing the latter restriction and disaggregating the inequality in $(**)$ we obtain the system

$$\left\langle \begin{matrix} \mathscr{G}'(y_{21}^1 + y_{31}^1) - \mathscr{F}'(y_{11}^1 + y_{21}^1) + ey_4 \leq 0 \\ -\mathscr{G}'(y_{21}^1 + y_{31}^1) + \mathscr{F}'(y_{11}^1 + y_{21}^1) - ey_4 \leq 0 \\ y_1 > 0 \end{matrix} \right\rangle$$

which is equivalent to

$$(***) \quad \left\langle \begin{matrix} \mathscr{G}'(y_{21}^1 + y_{31}^1) - \mathscr{F}'(y_{11}^1 + y_{21}^1) + ey_4 = 0 \\ y_1 > 0 \end{matrix} \right\rangle.$$

Observe that without loss of generality y_{21}^1 can be set to zero since it can be incorporated into y_{11}^1 and y_{31}^1. Now, suppose that one of y_{11}^1 or y_{31}^1 is zero. Then a solution will exist if and only if either g or f or both are constant utility acts. Finally, consider the remaining case where both y_{11}^1 and y_{31}^1 are positive. Here a solution exists if and only if there exists an $\alpha > 0, \beta > 0$, and y_4 such that

$$\alpha\mathscr{G}' - \beta\mathscr{F}' + ey_4 = 0.$$

This last condition is equivalent to

$$\mathscr{G}' = a\mathscr{F}' + be', \quad \text{for some } a > 0,\ b \in \Re.$$

Now note that the case where $a = 0$ corresponds to the cases where g is a constant act. If only f is a constant act, simply reverse the roles of f and g and again set $a = 0$.

Pulling the different possibilities together, we have that a solution to $(*)$ exists if and only if either f and g are ordered by weak dominance or

$$\mathscr{G}' = a\mathscr{F}' + be', \quad \text{for some } a \geq 0,\ b \in \Re$$

or,

$$\mathscr{F}' = a\mathscr{G}' + be', \quad \text{for some } a \geq 0,\ b \in \Re.$$

Our application of Motzkin's Theorem now yields the desired conclusion.

QED

The analogue for CEU is given in the next theorem. Note that this result is related to the prior work of Bassanezi and Greco [2] who show that the Choquet integral is additive for all capacities if and only if the functions being integrated are comonotonic.

Theorem 3. *Fix acts f and g. No convex combination of f and g will ever be strictly preferred to either alone (given CEU preferences) if and only if (i) f weakly dominates g or vice-versa (i.e., $u(f(s)) \geq (\leq) u(g(s))$, for all $s \in S$.)* **or** *(ii) f and g are comonotonic.*

Proof.

($\Leftarrow$) It is straightforward that (i) implies the weakly dominant act will be at least as good as any mixture. Result 1 stated earlier says that (ii) implies no mixture strictly preferred.

($\Rightarrow$) We will show that Not ((i) or (ii)) implies there exists a mixture that may be strictly preferred to both f and g. Not ((i) or (ii)) implies f, g not comonotonic and no weak dominance between them. Since the two acts are not comonotonic, there exist states $s_f, s_g \in S$ such that $f(s_f) \succ f(s_g)$ and $g(s_g) \succ g(s_f)$. Consider the restriction of f and g to $\{s_f, s_g\}$. There are two possible cases:

Case I. Neither restricted act weakly dominates the other. In this case, without loss of generality assume that $f(s_f) \succeq g(s_g) \succ g(s_f) \succeq f(s_g)$. Consider the capacity v such that $v(\{s_f, s_g\}) = 1$ and $v(A) = 0$ for any set A such that $\{s_f, s_g\} \nsubseteq A$. Relative to this capacity, we can calculate the Choquet expected utility (CEU) of f and g: $\text{CEU}(f) = u(f(s_g))$ and $\text{CEU}(g) = u(g(s_f))$. By continuity of preferences, there exists an $\alpha^* \in (0,1)$ such that

$$\alpha^* g(s_g) + (1-\alpha^*) f(s_g) \succ g(s_f).$$

Taking the CEU of this convex combination with respect to v yields,

$$\begin{aligned}&\mathrm{CEU}(\alpha^* g + (1-\alpha^*) f)\\ &\quad = \min[\alpha^* u(g(s_f)) + (1-\alpha^*) u(f(s_f)), \alpha^* u(g(s_g)) + (1-\alpha^*) u(f(s_g))]\\ &\quad > u(g(s_f)) \geq u(f(s_g)).\end{aligned}$$

Therefore $\alpha^* g + (1-\alpha^*) f \succ f$ and $\alpha^* g + (1-\alpha^*) f \succ g$ for this v.

This proves the claim for the case where neither restricted act weakly dominates the other. Now we examine the remaining possibility:

Case II. One restricted act weakly dominates the other. Without loss of generality assume $f(s_f) \succ f(s_g) \succeq g(s_g) \succ g(s_f)$. Since, over the whole space, S, we assumed neither act weakly dominates the other, there must exist an $s' \in S$ such that $g(s') \succ f(s')$.

There are several possibilities. First, suppose that $g(s') \succeq g(s_g)$. Then $g(s') \succ f(s')$ implies $f(s_g) \succ f(s')$ so that f and g are not comonotonic on $\{s_g, s'\}$ and Case I applies to the restriction of f and g to $\{s_g, s'\}$.

Another possible ordering of the states by g is $g(s_g) \succ g(s') \succeq g(s_f)$. Here $g(s') \succ f(s')$ implies $f(s_f) \succ f(s')$ and Case I applies to the restriction of f and g to $\{s_f, s'\}$.

Finally, assume (the only remaining possibility) that $g(s_f) \succ g(s') \succ f(s')$. Consider the capacity v such that $v(\{s_f, s_g, s'\}) = 1$, $v(\{s_f, s_g\}) = k$, and for all other sets v assigns the lowest nonnegative value consistent with monotonicity of the capacity. Choose $k \in (0,1)$ to satisfy

$$ku(f(s_g)) + (1-k)u(f(s')) = ku(g(s_f)) + (1-k)u(g(s')).$$

Such a k exists under our ordering assumptions. Using the capacity v,

$$\mathrm{CEU}(f) = ku(f(s_g)) + (1-k)u(f(s')),$$

and

$$\mathrm{CEU}(g) = ku(g(s_f)) + (1-k)u(g(s')).$$

Thus for this capacity v and utility u, $f \sim g$. Now, using the fact that we can represent the CEU preferences under v as the maxmin expected utility over the set of probability measures that are in the core of v (i.e., $\{p \mid p(s_f) + p(s_g) \geq k, p(s_f) + p(s_g) + p(s') = 1\}$) (Schmeidler [23], [24]), we can apply Corollary 1 to show that some convex combination will be strictly preferred to both f and g.

To summarize, in each of the possible cases where Not ((i) or (ii)) holds the above has shown that there exists a convex combination that may be strictly preferred to both f and g. *QED*

To facilitate a comparison with Theorem 2 the following corollary is provided:

Corollary 2. *Fix acts f and g. No convex combination of f and g will ever be strictly preferred to either alone (given CEU preferences) if and only if (i) f weakly dominates g or vice-versa (i.e.,* $u(f(s)) \geq (\leq) u(g(s))$*, for all* $s \in S$*.)* **or** *(ii) there exists an act h and weakly increasing functions w and x on* $\Re$ *such that, for all* $s \in S$, $u(f(s)) = w(u(h(s)))$ *and* $u(g(s)) = x(u(h(s)))$.

Proof. By Denneberg [5, Proposition 4.5], two functions $d, e : S \to \Re$ are comonotonic if and only if there exists a function $z : S \to \Re$ and weakly increasing functions w, x on $\Re$ such that $d = w(z)$ and $e = x(z)$. Let $d = u \circ f$, $e = u \circ g$, and $z = u \circ h$ and the result follows from theorem 3 and the fact that f and g are comonotonic if and only if $u \circ f$ and $u \circ g$ are. *QED*

To see how this result compares to Theorem 2, observe that if we require w and x to be affine then condition (ii) of the corollary is equivalent to condition (ii) of Theorem 2. While CEU prevents strict preference for mixture for acts that are weakly increasing tranformations of the same utility payoffs, MMEU does so only if the transformations are affine. Intuitively, this says that MMEU decision makers may care about the cardinal properties of the distribution of utilities across states when evaluating whether one act is more uncertain than another, while CEU individuals must consider distributions of utilities that (roughly) order states the same way as representing equivalent levels of uncertainty.

Remark. As the results above concern strict preference for mixture, the reader may wonder whether this addresses all the relevant possibilities for strict quasi-concavity of the preferences. Specifically, can there exist acts f and g satisfying (i) or (ii) of the appropriate theorem above such that indifference curves over mixtures are strictly quasi-concave, yet no mixture is strictly preferred? It is easily seen that the answer may be yes only if (ii) is violated. To see this note that if (ii) is satisfied then for any MMEU preferences the same probability measure will be used to evaluate all mixtures, generating linear indifference curves. Conversely, if (ii) is violated then $u(f)$ and $u(g)$ are not related by a positive affine transformation and therefore order probability measures distinctly. Given one minimizing measure for f and another for g, it follows that the measure used to evaluate $\alpha f + (1 - \alpha) g$ must generate more than the minimum expected utility level for one of the two acts, producing strict quasi-concavity of preferences. Arguments similar to the ones above could be used to show this more formally and demonstrate it for the Choquet case as well.[7] There is then no essential loss in limiting our analysis, as we have, to preference for mixtures. Furthermore, by examining the preference for mixtures case, we see that only weak dominance limits the extent of the quasi-concavity permitted by a violation of (ii).[8]

[7] See Ghirardato et al. [14] for elaboration.

[8] An alternative reason for interest in preference for mixtures per se is that such preferences correspond precisely to violations of the analogue for uncertain acts of the *betweeness* property for preferences under risk (e.g., Dekel [4]).

Thus we have a characterization of the uncertainty aversion (as expressed through strict preference for mixtures) that the two theories allow. It is hoped that this will further understanding of what distinguishes these two representations.

4 Conclusion

Theories of uncertainty aversion may differ in the circumstances under which they allow violations of independence, and in particular strict preference for mixtures. This paper has provided a characterization of those acts which may never admit such a strict preference for the two leading representations of uncertainty aversion: maxmin expected utility and Choquet expected utility. The fact that these characterizations are substantially different has implications for empirical testing of the theories as well as for those trying to apply one or the other model and wondering what the consequences of the modelling choice are. Fundamentally, CEU decision makers view uncertainty in terms of (roughly) how states are ordered by an act's utility payoffs. Given a set of acts which all induce the same ordering, a CEU decision maker acts exactly like an expected utility (and thus uncertainty neutral) decision maker. MMEU decision makers, in contrast, may view uncertainty not only in terms of ordering of states, but also in terms of how much better the payoff is in one state as opposed to another. MMEU allows the decision maker to be averse to such cardinal variations across states even among acts that order states in the same way.

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