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Maxmin Expected Utility over Savage Acts with a Set of Priors[1]

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This paper provides an axiomatic foundation for a maxmin expected utility over a set of priors (MMEU) decision rule in an environment where the elements of choice are Savage acts. This characterization complements the original axiomatization of MMEU developed in a lottery-acts (or Anscombe–Aumann) framework by I. Gilboa and D. Schmeidler (1989, *J. Math. Econ.* **18**, 141–153). MMEU preferences are of interest primarily because they provide a natural and tractable way of modeling decision makers who display an aversion to uncertainty or ambiguity. The novel axioms are formulated using standard sequence techniques, which allow cardinal properties of utility to be expressed directly through preferences. *Journal of Economic Literature* Classification Number: D81. © 2000 Academic Press

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1. INTRODUCTION

This paper provides an axiomatic foundation for a maxmin expected utility over a set of priors (MMEU) decision rule in an environment where the elements of choice are Savage [20] acts. This characterization complements the original axiomatization of MMEU developed in a lottery-acts (or Anscombe–Aumann [1]) framework by Gilboa and Schmeidler [12]. MMEU preferences are of interest primarily because they provide a natural and tractable way of modeling decision makers who display an aversion to uncertainty or ambiguity. A leading motivation for examining such preferences is the evidence described by Ellsberg [6] and many afterwards demonstrating that such aversion seems common and is incompatible with standard expected utility theory. A closely related representation that has also been used to capture uncertainty aversion is Choquet expected utility (CEU). CEU was first axiomatized in a lottery-acts framework (Schmeidler [21]) and later in settings with Savage acts (Gilboa [11], Wakker [22], Nakamura [18], Sarin and Wakker [19], and Chew and Karni [4]). In contrast, MMEU has never before been characterized in a setting with Savage acts.

A great attraction of settings with Savage acts is, of course, that no primitive notion of probabilities need be assumed. That probabilities nonetheless appear in a representation is then quite satisfying and provides a strong foundation. However, the interest in a Savage act characterization of CEU and MMEU is not only philosophical. A number of recent papers (for example, Ghirardato [8] and Sarin and Wakker [19]) point out that there may be real differences when using CEU preferences between a two-stage lottery-acts formulation and a one-stage Savage acts setting. Since MMEU is closely related to CEU, this suggests value in having a foundation for MMEU in both settings. CEU and MMEU over Savage acts also appear prominently in papers that examine randomization and uncertainty averse decision makers (Eichberger and Kelsey [5], Klibanoff [14]). The latter paper in particular suggests a privileged role for MMEU over CEU in modeling uncertainty aversion and randomization simultaneously in a Savage acts setting. Additionally, recent developments of alternative notions of uncertainty or ambiguity aversion have focused on the Savage acts setting (see Epstein [7] and Ghirardato and Marinacci [10]) and characterizing MMEU in such a setting is useful for the purposes of comparison (see Subsection 4.2 for a brief discussion).

The remainder of the paper presents a set of axioms and a representation theorem proving the equivalence between these axioms and an MMEU rule. The key axioms are a very weak version of act-independence (Gul [13], Chew and Karni [4]) and two axioms formulated using standard sequences, a measurement theory construction (see Krantz *et al.* [15]). Results in Nakamura [18] and Gilboa and Schmeidler [12] are useful for the proofs.

2. NOTATION AND FRAMEWORK

Ω is the *set of states*. A *state* in Ω is represented by ω. Σ is an algebra of subsets of Ω. *Events* are elements of Σ. $X = [m, M] \subset \mathbb{R}$, $m < M$ is the set of *prizes* or outcomes. A (Savage) *act* f is a function $f: \Omega \to X$. A *simple act* is an act with only finitely many distinct values. A simple act is Σ-measurable if $\{\omega \in \Omega \mid f(\omega) \in W\} \in \Sigma$ for all $W \subseteq X$. F is the set of all Σ-measurable acts defined as the closure in the supnorm of the set of all Σ-measurable simple acts. A set $G \subseteq F$ is *closed* if it is closed in the supnorm. A *constant act* f is one for which $f(\omega) = x$ for all $\omega \in \Omega$, for some $x \in X$; we denote this constant act by x^* or simply x when no confusion would result. F^* is the subset of F consisting of all constant acts; note that F^* can be identified with X. For any event $B \in \Sigma$ and $x, y \in X$, $x_B y$ denotes $f \in F$ such that $f(\omega) = x$ for $\omega \in B$ and $f(\omega) = y$ for $\omega \notin B$; such acts are referred to as B-measurable. The event $\Omega - B$ is denoted B^c. $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$ denote, respectively, the sets of all positive and all strictly positive integers. $\mathscr{P}$ is the set of all finitely additive probability measures $P: \Sigma \to [0, 1]$. *Weak preference* $\succcurlyeq$ is a binary relation on F, and the binary relations $\succ$ (*strict preference*) and $\sim$ (*indifference*) are derived in the usual way from $\succcurlyeq$. Finally, the expression $\{f, g\} \succcurlyeq h$ means $f \succcurlyeq h$ and $g \succcurlyeq h$, while $f \succcurlyeq \{g, h\}$ means $f \succcurlyeq g$ and $f \succcurlyeq h$.

Note that this environment is similar to that in Savage [20] with the difference that we impose more structure on the prize set (X). The important aspect of this structure is that X is connected and separable. This, together with a continuity assumption on preferences made below, generates a richness in preference equivalence classes that is heavily used in what follows. As a consequence of this richness in the prize set, we do not need to impose axioms that require the set of states (Ω) to be infinite as Savage does. In fact, other than the uninteresting case of Ω containing only one state, our axioms allow for Ω to be of any size, finite or infinite.

3. AXIOMS AND A REPRESENTATION THEOREM

3.1. *Axioms*

Axiom 1 (Weak order). $\succcurlyeq$ is complete and transitive.

DEFINITION 3.1. An *ordered null event* $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $x_B z \sim y_B z$ whenever $z \succcurlyeq y \succcurlyeq x$.

DEFINITION 3.2. An *ordered universal event* $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $z_B x \sim z_B y$ whenever $x \succcurlyeq y \succcurlyeq z$.

Hence, an event B is *ordered non-null* if there exist x, y and z in X with $z \succcurlyeq y \succcurlyeq x$ such that $x_B z \nsim y_B z$. Likewise, an event B is *ordered non-universal* if there exist x, y and z in X with $x \succcurlyeq y \succcurlyeq z$ such that $z_B x \nsim z_B y$. Intuitively, B ordered non-null implies that the prize given on B matters sometimes; and B ordered non-universal implies that the prize on B is not always the only thing that matters to the decision maker. Note that since we impose restrictions on the ordering of x, y and z, our definitions of null and universal (borrowed from Nakamura [18]) are weaker than the corresponding notions in Savage [20]. Section 4.1 contains a discussion of why the ordered notion is appropriate here.

The next axiom is structural and contains two parts.

Axiom 2 (Structure). (a) $x > y \Rightarrow x^* \succ y^*$.

(b) There exists an event $A \in \Sigma$ such that A is ordered non-null and ordered non-universal.

Regarding part (a), note that this will require preference over prizes to be increasing in the real number ordering. This is purely a simplifying assumption which allows the easy identification of the set of equivalence classes and eliminates the trivial case where preference is never strict. Excepting the trivial case, the richness and separability of the set of equivalence classes given by this assumption is already required by the continuity axiom (Axiom 3, below) together with the assumption that the set of prizes is connected and separable. Part (b) is needed to ensure the existence of meaningful cardinal (as opposed to ordinal) preferences. Without it, non-trivial trade-offs between prizes on different events may not exist. This is discussed in detail in Section 4.1.

The next axiom is a standard continuity assumption. Together with the structural assumptions on X (crucially that X is a connected and separable set) and the weak order axiom, it guarantees the existence of a real-valued representation of preferences and a certainty equivalent of any given act. It also implies that the utility function, u, in the representation is continuous.

Axiom 3 (Continuity). For all $f \in F$, the sets $M(f) = \{g \in F \mid g \succcurlyeq f\}$ and $W(f) = \{g \in F \mid f \succcurlyeq g\}$ are closed.

The following axiom is a monotonicity requirement. Specifically, part (a) requires that if, in every state, the prize that f gives is at least as good as the prize that g gives, then, overall, f must be at least as good as g. Part (b) requires this monotonicity to be strict on ordered non-null and ordered non-universal events (i.e., if f gives a prize strictly better than g on an ordered non-null event or on the complement of an ordered non-universal event and is weakly better than g elsewhere then, overall, f must be strictly better than g), subject to ordering conditions as in the definitions of ordered null and ordered universal. In this sense, each ordered non-null event (and complement of an ordered non-universal event) "matter" in determining overall preference.[2]

Axiom 4 (Monotonicity). (a) For all $f, g \in F$, if $f(\omega) \succcurlyeq g(\omega)$, for all $\omega \in \Omega$ then $f \succcurlyeq g$.

(b) If $B \in \Sigma$ is ordered non-null and $z \succcurlyeq x \succ y$, then $x_B z \succ y_B z$. If $B \in \Sigma$ is ordered non-universal and $x \succ y \succcurlyeq z$, then $z_B x \succ z_B y$.

The next axiom is a weakening of the act-independence axiom introduced in Gul [13]. As discussed there, act-independence is analogous to the independence axiom in the theory of expected utility over lotteries. We will refer to an act $f = x_A y$ as an *ordered A-measurable act* if $y \succcurlyeq x$.

Axiom 5 (Ordered A-Act-Independence).[3] Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ be such that $x_2 \succcurlyeq x_1$, $y_2 \succcurlyeq y_1$ and $z_2 \succcurlyeq z_1$. Let $f = x_{1A} x_2$, $g = y_{1A} y_2$ and $h = z_{1A} z_2$. Then,

(i) if $\{x_i, y_i\} \succcurlyeq z_i$ $(i = 1, 2)$ and $\begin{cases} f'(\omega) \sim h(\omega)_A f(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim h(\omega)_A g(\omega) & \text{for all } \omega \in \Omega \end{cases}$

then $[f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g']$;

(ii) if $z_i \succcurlyeq \{x_i, y_i\}$ $(i = 1, 2)$ and $\begin{cases} f'(\omega) \sim f(\omega)_A h(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim g(\omega)_A h(\omega) & \text{for all } \omega \in \Omega \end{cases}$

then $[f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g']$.

Suppose f, g and h are acts and B is an event. We will use the following terminology: If, for every state ω, $h(\omega)$ is indifferent to the act that gives $f(\omega)$ on B and $g(\omega)$ on B^c, then we say that h is a *statewise combination of f and g over the event B*. With this terminology, this last axiom reads: given ordered A-measurable acts f, g, and h, (case (i)) if f and g each dominate h and f' is a statewise combination of h and f over the event A and g' is a statewise combination of h and g over the event A, or (case (ii)) if h dominates both f and g and f' is a statewise combination of f and h over the event A and g' is a statewise combination of g and h over the event A, then f is at least as good as g if and only if f' is at least as good

[2] Since, in any state, preference between prizes is equated with preference between the corresponding constant acts, this axiom also implicitly rules out state *dependent* preferences.

[3] This is essentially a restriction of Chew and Karni's comonotonic act-independence axiom [4] to the event A and A-measurable acts. We choose the term ordered to distinguish this axiom from the one that is appropriate for the "reverse" ordering discussed in Section 4.

as g'. Observe that within each case, the statewise combinations required have the less preferred prizes on the event A (rather than on A^c). This is because we do not assume that A^c is ordered non-null or ordered non-universal. Therefore, the restriction on statewise combinations is needed to ensure that non-zero weight is placed on f and g.

As is shown in Lemma A.5, these first five axioms imply the existence of an expected utility representation for ordered A-measurable acts. That is, there is a strictly increasing and continuous function $u: X \to \mathbb{R}$ and a real number $\rho \in (0, 1)$ such that for all $x, y, v, w \in X$, if $y \succcurlyeq x$ and $w \succcurlyeq v$ then

$$x_A y \succcurlyeq v_A w \Leftrightarrow \rho u(x) + (1-\rho)\, u(y) \geqslant \rho u(v) + (1-\rho)\, u(w).$$

Moreover, u is unique up to positive affine transformations and ρ is unique. Observe that the decision maker attaches probability ρ to the event A when evaluating ordered A-measurable acts.

Ordered A-act-independence is the weakest version of act-independence for A-measurable acts that is compatible with MMEU. In fact, it is the weakest version that allows the standard sequences introduced below to meaningfully measure preferences.

How do preferences extend from ordered A-measurable acts to all acts? From the above, overall preferences must be a continuous, monotonic, weak order. Furthermore they must satisfy act-independence on ordered A-measurable acts. Where else must act-independence hold, and when it is violated, what form can the violation take? Different answers to these questions characterize different functional representations of preferences. If (in addition to the other axioms) act-independence is required to hold for all acts f, g, and h, expected utility preferences result.[4] This act-independence is incompatible, for example, with the Ellsberg Paradox. To characterize MMEU, act-independence must be required to hold on a much smaller set of acts. What is this set of acts? We know from Ghirardato *et al.* [9] that the MMEU functional must satisfy additivity in general only on sets of functions that are *affinely related*. This concept translates to acts in the following way, making use of the utility function u as above:

DEFINITION 3.3. Two acts f and g are *affinely related* if there exist $\alpha \geqslant 0$ and $\beta \in \mathbb{R}$ such that either $u(f(\omega)) = \alpha u(g(\omega)) + \beta$ for all $\omega \in \Omega$ or $u(g(\omega)) = \alpha u(f(\omega)) + \beta$ for all $\omega \in \Omega$.

In other words, f and g are affinely related if either f is a constant utility act or g is a constant utility act or there is a positive affine transformation

[4] For finite state spaces, this is essentially Gul's result [13], although it is closer to that of Chew and Karni [4], who, by using—as do we—results of Nakamura [18] are able to dispense with Gul's assumption of a symmetric ("half-half") event.

relating the utility f gives in each state with the utility g gives in each state.

Our next axiom expresses act-independence (or additivity) for affinely related acts directly through preferences (i.e., without referring to the u function). The axiom that guarantees this additivity in an Anscombe–Aumann framework is the certainty independence (C-independence) axiom of Gilboa and Schmeidler [12]:

C-*independence.* For any acts f and g, any constant act x, and any $\alpha \in (0, 1)$, if f', g' are such that

$$f' = \alpha f + (1 - \alpha) x$$

and

$$g' = \alpha g + (1 - \alpha) x$$

then,

$$f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g'.$$

Note that the convex combination operation is defined statewise and is well-defined since acts in their setting are functions from states to probability distributions over prizes. C-independence relaxes the independence axiom of Anscombe and Aumann [1] so that it is only required to hold when the third act, x, is a constant act. Since convex combinations are not meaningful in our setting, the approach we follow uses *standard sequences*.[5]

DEFINITION 3.4. Given $B \in \Sigma$ which is ordered non-null and ordered non-universal and a and $b \in X$ with $a \succ b$, we define a *standard sequence with respect to a and b using event B*, as a sequence $\{a_1, a_2, a_3, ...\}$ where $a_i \in X$, for which either (i) $a_i \succcurlyeq a$ and $a_B a_i \sim b_B a_{i+1}$ for all i such that a_i is not the last element in the sequence; or (ii) $b \succcurlyeq a_i$ and $a_{iB} b \sim a_{i+1B} a$ for all i such that a_i is not the last element in the sequence.

We use standard sequences as rulers with which to calibrate preferences. In standard sequences obeying (i) the elements are increasing ($a_{i+1} \succcurlyeq a_i$), while in standard sequences obeying (ii) the elements are decreasing ($a_i \succcurlyeq a_{i+1}$). Any consecutive elements in an increasing standard sequence are the same "distance" apart, in that the difference between receiving, say, a_{i+1} instead of a_i on the event B^c just compensates for the difference

[5] If we were willing to assume a stronger, non-ordered A-act-independence axiom in place of ordered A-act-independence (along with A^c ordered non-null and ordered non-universal) we could proceed to develop the remaining axioms for MMEU using statewise combinations (as defined above) rather than standard sequences. For such a development see Casadesus-Masanell *et al.* [2].

between receiving b instead of a on the event B. Similarly, in a decreasing standard sequence, receiving a_i instead of a_{i+1} on the event B just compensates for receiving b instead of a on B^c. Since axioms weak order, structure, continuity, monotonicity, and ordered A-act-independence guarantee an expected utility representation for ordered A-measurable acts, we can translate this notion of distance into utility terms by forming standard sequences using the event A. One of the contributions of this paper is the insight that, if an expected utility representation for ordered A-measurable acts exists, then standard sequences may be used to express any cardinal properties of utility directly in terms of preferences. In particular, it is not hard to see that in any increasing standard sequence using the event A,

$$u(a_{i+1}) - u(a_i) = \frac{\rho}{1-\rho} [u(a) - u(b)].$$

In any decreasing standard sequence using the event A,

$$u(a_i) - u(a_{i+1}) = \frac{1-\rho}{\rho} [u(a) - u(b)].$$

Hence, having an expected utility representation for ordered A-measurable acts is what lets us associate distance measured by standard sequences with distance measured by ratios of utility differences.[6]

We now state the needed independence axiom.

Axiom 6 (Constant-independence). Fix any $f, g, f', g' \in F$, $x \in F^*$, and $\alpha \in (0, 1)$. If, for each ω, there exist standard sequences $\{a_i^{f(\omega)}\}$ and $\{a_i^{g(\omega)}\}$ using event A, and strictly positive integers $i(\omega)$, $i'(\omega)$, $j(\omega)$, $j'(\omega)$, $k(\omega)$, and $l(\omega)$ with $i'(\omega) = \alpha k(\omega) + (1-\alpha)\, i(\omega)$ and $j'(\omega) = \alpha l(\omega) + (1-\alpha)\, j(\omega)$ such that

$$a_{i(\omega)}^{f(\omega)} \sim f(\omega), \qquad a_{i'(\omega)}^{f(\omega)} \sim f'(\omega),$$

$$a_{j(\omega)}^{g(\omega)} \sim g(\omega), \qquad a_{j'(\omega)}^{g(\omega)} \sim g'(\omega),$$

$$a_{k(\omega)}^{f(\omega)} \sim x \sim a_{l(\omega)}^{g(\omega)},$$

[6] For an excellent discussion of standard sequences and their use in both preference measurement and other types of measurement, see Krantz *et al.* [15].

then,

$$f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g'.$$

The easiest way to understand this axiom is by thinking about it in terms of utilities. Fix the acts and α as in the axiom. If standard sequences satisfying the conditions in the axiom exist then, for those standard sequences that are increasing, the utility distance between any two consecutive elements is $\Delta^I(\omega) \equiv \frac{\rho}{1-\rho}[u(a)-u(b)]$, (where $a \succ b$ are the prizes a standard sequence is with respect to and may vary from sequence to sequence). Therefore, for each such ω,

$$\begin{aligned} u(f'(\omega)) &= u(a^{f(\omega)}_{i'(\omega)}) = u(a^{f(\omega)}_1) + (i'(\omega)-1)\,\Delta^I(\omega) \\ &= u(a^{f(\omega)}_1) + (\alpha k(\omega) + (1-\alpha)\, i(\omega) - 1)\,\Delta^I(\omega) \\ &= \alpha(u(a^{f(\omega)}_1) + (k(\omega)-1)\,\Delta^I(\omega)) + (1-\alpha)(u(a^{f(\omega)}_1) \\ &\quad + (i(\omega)-1)\,\Delta^I(\omega)) \\ &= \alpha u(x) + (1-\alpha)\, u(f(\omega)). \end{aligned}$$

The equality on the second line follows from the fact that $i'(\omega) = \alpha k(\omega) + (1-\alpha)\, i(\omega)$. The last equality follows from $x \sim a^{f(\omega)}_{k(\omega)}$ and $f(\omega) \sim a^{f(\omega)}_{i(\omega)}$. For those sequences that are decreasing, the utility distance between any two consecutive elements is $\Delta^D(\omega) \equiv \frac{1-\rho}{\rho}[u(a)-u(b)]$. One can show in this case as well that $u(f'(\omega)) = \alpha u(x) + (1-\alpha)\, u(f(\omega))$. A similar analysis shows that $u(g'(\omega)) = \alpha u(x) + (1-\alpha)\, u(g(\omega))$. So any such f' and g' are a given convex combination of x with f and g. In this case, the axiom requires that the preference between f and g be the same as the preference between the convex combination. Conversely, if f' and g' are a given convex combination of x with f and g, then either standard sequences satisfying the conditions of the axiom exist, or these acts are the limit of a sequence of acts for which such standard sequences exist and the same conclusion obtains. Thus, this axiom is a standard sequence approach to implementing Gilboa and Schmeidler's C-Independence axiom [12].

The final axiom, uncertainty aversion, restricts the way that act-independence can be violated. Essentially it requires that the decision-maker weakly likes to smooth utilities across states of the world, since this leaves her less exposed to any uncertainty or ambiguity about the probability of various states. In an Anscombe–Aumann framework, Schmeidler's (see also Gilboa and Schmeidler [12]) uncertainty aversion axiom [21] states that any convex combination of two acts cannot be worse than both of the acts being combined. The following is a natural generalization of such an axiom to the Savage acts framework using a standard sequence approach:

Axiom 7 (Uncertainty aversion). Fix any f, g, $h \in F$, and $\alpha \in (0, 1)$. If $f \succcurlyeq g$ and, for each ω, there exist a standard sequence $\{a_i^\omega\}$ using event A, and strictly positive integers $i(\omega)$, $j(\omega)$, and $k(\omega)$ with $k(\omega) = \alpha i(\omega) + (1-\alpha)\, j(\omega)$ such that

$$a^\omega_{i(\omega)} \sim f(\omega),$$
$$a^\omega_{j(\omega)} \sim g(\omega),$$
$$a^\omega_{k(\omega)} \sim h(\omega),$$

then,

$$h \succcurlyeq g.$$

Again, the easiest way to understand this axiom is by thinking about it in terms of utilities. Here, the conditions of the axiom imply that $u(h(\omega)) = \alpha u(f(\omega)) + (1-\alpha)\, u(g(\omega))$ and, conversely, if $f \succcurlyeq g$ and h is such that for all ω, $u(h(\omega)) = \alpha u(g(\omega)) + (1-\alpha)\, u(f(\omega))$ then either the conditions of the axiom are satisfied or these acts are the limit of a sequence of acts for which such standard sequences exist. In either case, $h \succcurlyeq g$. Subsection 4.2 relates this type of uncertainty aversion axiom to recent alternatives suggested by Epstein [7] and Ghirardato and Marinacci [10].

3.2. *A Representation Theorem*

These seven axioms lead us to our main result:

THEOREM 3.5. *Let $\succcurlyeq$ be a binary relation on F. Then $\succcurlyeq$ satisfies the axioms weak order, structure, continuity, monotonicity, ordered A-act-independence, constant-independence and uncertainty aversion if and only if there exists a continuous and strictly increasing function $u: X \to \mathbb{R}$, and a non-empty, compact and convex set $\mathscr{C}$ of finitely additive probability measures on Σ such that*

$$[f \succcurlyeq g] \Leftrightarrow \left[\min_{P \in \mathscr{C}} \int u \circ f dP \geqslant \min_{P \in \mathscr{C}} \int u \circ g dP\right] \qquad \textit{for all} \quad f \textit{ and } g \in F.$$

Furthermore, there exists an event $A \in \Sigma$ for which $0 < \max_{P \in \mathscr{C}} P(A) < 1$. Moreover, u is unique up to positive affine transformations and the set $\mathscr{C}$ is unique.

Proof. See Subsection A.1 for the proof that the axioms are sufficient and see Subsection A.2 for the proof that the axioms are necessary. Uniqueness follows from Lemmas A.5 and A.12. ∎

Although the complete proof and relevant annotation is provided in the Appendix, we give a brief sketch of the sufficiency argument here. The first part of the argument, as mentioned above, uses the first five axioms to prove the existence of an expected utility representation for ordered A-measurable acts. The crucial step in this part of the proof is showing that several representational lemmas in Nakamura [18] can be applied in our setting. It is fairly straightforward that, given the structure axiom, the weak order, continuity, and monotonicity axioms guarantee the existence of a continuous, increasing, real-valued functional representation of preferences. The work is in showing that adding the ordered A-act-independence axiom ensures, at least for ordered A-measurable acts, this functional can be written in an expected utility form. This yields a utility function that we can use to give meaning to standard sequences. Applying this interpretation of standard sequences to the constant-independence and uncertainty aversion axioms, allows us to show (through some manipulation and heavy use of continuity) that any functional representation of preferences must satisfy the conditions of a fundamental lemma (variations of which have been proved by, for example, Gilboa and Schmeidler [12], Chateauneuf [3], and Marinacci [17]). This lemma proves the existence of the minimum expectation over a set of measures representation.

4. DISCUSSION

4.1. *Null and Universal Events*

In the above theory, we used the concepts of ordered null and ordered universal events. These concepts differ from the more familiar notions of null and universal as in Savage [20]. A natural question is why the ordered notion is appropriate in our setting and what occurs when an ordered non-null and ordered non-universal event does not exist. To address this issue formally, consider the following alternative definitions of null and universal (including ordered null and ordered universal as above):

DEFINITION 4.1. A *Savage null event* $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $x_B z \sim y_B z$. An *ordered null event* $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $x_B z \sim y_B z$ whenever $z \succcurlyeq y \succcurlyeq x$. A *reverse-ordered null event* $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $x_B z \sim y_B z$ whenever $x \succcurlyeq y \succcurlyeq z$.

DEFINITION 4.2. A *Savage universal event* $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $z_B x \sim z_B y$. An *ordered universal event* $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $z_B x \sim z_B y$ whenever $x \succcurlyeq y \succcurlyeq z$. A *reverse-ordered universal event* $B \in \Sigma$ is an event for which for all $x, y, z \in X$, $z_B x \sim z_B y$ whenever $z \succcurlyeq y \succcurlyeq x$.

Hence, an event B is *Savage non-null* if there exist x, y and z in X such that $x_B z \not\sim y_B z$, is *ordered non-null* if there exist x, y and z in X with $z \succcurlyeq y \succcurlyeq x$ such that $x_B z \not\sim y_B z$ and is *reverse-ordered non-null* if there exist x, y and z in X with $x \succcurlyeq y \succcurlyeq z$ such that $x_B z \not\sim y_B z$. Likewise, an event B is *Savage non-universal* if there exist x, y and z in X such that $z_B x \not\sim z_B y$, is *ordered non-universal* if there exist x, y and z in X with $x \succcurlyeq y \succcurlyeq z$ such that $z_B x \not\sim z_B y$ and is *reverse-ordered non-universal* if there exist x, y and z in X with $z \succcurlyeq y \succcurlyeq x$ such that $z_B x \not\sim z_B y$.

Since the Savage notions of non-null and non-universal are the most permissive, we begin by asking what can be said in the case where there does not exist an event that is Savage non-null and Savage non-universal. In such a case, an event will either always get zero weight or always get all the weight in determining preferences over acts. There will never be any trade-off between prizes on one event and prizes on another. Therefore, preferences will only be ordinally determined, and thus the utility function, u, in the representation will be determined only up to increasing transformations. There will be a unique probability measure that assigns weight 0 to all Savage null events and weight 1 to all Savage universal events.

To explore the remaining possibilities, assume that there exists a Savage non-null and Savage non-universal event A. Since A is Savage non-null, there exist x, y, z such that $x_A z \not\sim y_A z$. Let's consider the possible orderings of x, y and z:[7]

(i) If $\{x, y\} \succcurlyeq z$ then A is reverse-ordered non-null.

(ii) If $z \succcurlyeq \{x, y\}$ then A is ordered non-null.

(iii) If $x \succ z \succ y$ then, by monotonicity, part (a) and $x_A z \not\sim y_A z$, we have $x_A z \succ y_A z$. Also by monotonicity, part (a), $x_A z \succcurlyeq z_A z \succcurlyeq y_A z$, with at least one of the preference relations being strict. Hence, there are two cases:

Case 1. $x_A z \succ z_A z$. Here, A is reverse-ordered non-null.

Case 2. $z_A z \succ y_A z$. This implies A is ordered non-null.

Thus, A is either ordered or reverse-ordered non-null.

Since A is Savage non-universal, there exist x, y, z such that $z_A x \not\sim z_A y$. Again consider the possible orderings of x, y and z:

(i) If $\{x, y\} \succcurlyeq z$ then, A is ordered non-universal.

(ii) If $z \succcurlyeq \{x, y\}$ then, A is reverse-ordered non-universal.

[7] Without loss of generality, we may assume $x \succcurlyeq y$, as x and y play symmetric roles in all that follows.

(iii) If $x \succ z \succ y$ then, by monotonicity, part (a), $z_A x \succ z_A y$. Also by monotonicity, part (a), $z_A x \succcurlyeq z_A z \succcurlyeq z_A y$, with at least one of the preference relations being strict. Hence, there are two cases:

Case 1. $z_A x \succ z_A z$. Here, A is ordered non-universal.

Case 2. $z_A z \succ z_A y$. This implies A is reverse-ordered non-universal.

Thus, A is either ordered or reverse-ordered non-universal.

Therefore, assuming A Savage non-null and Savage non-universal implies that at least one of the following four possibilities must be true:

(1) A is ordered non-null and ordered non-universal.

(2) A is reverse-ordered non-null and reverse-ordered non-universal.

(3) A is ordered null and reverse-ordered universal.

(4) A is reverse-ordered null and ordered universal.

Case (1) is the one considered in the axiomatization in the main body of the paper. For A satisfying the conditions in case (2), it is easy to show that A^c must satisfy the conditions in case (1). Therefore, our axiomatization applies to this case as well, since A^c is the needed event.

Case (3) implies *all* weight is placed on the more preferred prize when evaluating A-measurable acts. Similarly, case (4) implies placing *all* weight on the less preferred prize when evaluating A-measurable acts. Therefore, in these cases there is no trade-off between any two events, and preferences are only ordinally determined.

In sum, if we wish to have an MMEU representation with meaningful cardinal preferences, then we need an event that is non-null and non-universal according to *either* the ordered or reverse-ordered notions. Note that in the context of expected utility, any Savage non-null and non-universal event must be both ordered and reverse-ordered non-null and non-universal.

4.2. *Relation with Alternative Uncertainty Aversion Axioms*

Our uncertainty aversion axiom is meant to be the analogue in a Savage setting of uncertainty aversion as proposed by Schmeidler [21] (also Gilboa and Schmeidler [12]) in an Anscombe–Aumann setting. The underlying intuition is that smoothing utility payoffs across states of the world can help hedge (reduce exposure to) uncertainty. Recently, Epstein [7] and Ghirardato and Marinacci [10] have proposed alternative notions of uncertainty aversion. The purpose of this section is to briefly describe these alternative notions and their relationship with preferences satisfying our axioms.

Both the Epstein and Ghirardato–Marinacci approaches start from a comparative notion describing when one preference relation is more uncertainty averse than another. Essentially, $\succcurlyeq_1$ is more uncertainty averse than $\succcurlyeq_2$ if:

(A) For all "unambiguous acts" $h \in F$ and all acts $f \in F$, both

$$h \succcurlyeq_2 f \Rightarrow h \succcurlyeq_1 f$$

and

$$h \succ_2 f \Rightarrow h \succ_1 f;$$

and

(B) $\succcurlyeq_1$ and $\succcurlyeq_2$ embody the "same" risk preferences.

Thus, holding risk preferences constant, $\succcurlyeq_1$ is more uncertainty averse than $\succcurlyeq_2$ if 1 ranks "unambiguous acts" at least as high as 2 relative to general (and possibly ambiguous or uncertain) acts. This comparative notion can then be used to define an absolute notion of uncertainty aversion as follows: preferences ($\succcurlyeq$) are uncertainty averse if $\succcurlyeq$ are more uncertainty averse than some "uncertainty neutral" preferences.

Within this common framework, the two approaches differ as to the class of "unambiguous acts," the set of "uncertainty neutral" preferences, and what aspects of preference are included in risk preferences. Briefly, Epstein [7] assumes that the set of "unambiguous acts" are those acts measurable with respect to an exogenously specified and rich set of "unambiguous events." He also assumes that preference relations that are probabilistically sophisticated (as defined in Machina and Schmeidler [16]) are uncertainty neutral. Finally, he assumes that if two preference relations agree when restricted to the unambiguous acts then they satisfy condition (B). In contrast, Ghirardato and Marinacci [10] take the set of "unambiguous acts" in (A) to be the constant acts. Furthermore, they assume that the subjective expected utility preference relations are uncertainty neutral. Finally, in condition (B), they state conditions on standard sequences that require the two preference relations to have the same utility function (up to admissible transformations).

What can be said about the relation of these notions with the preferences described in this paper? Compared to Ghirardato and Marinacci [10], our notion of uncertainty aversion is stronger. Specifically, the preferences described in this paper are all uncertainty averse according to the Ghirardato–Marinacci definition. Furthermore, there are preferences that satisfy all of our axioms except for our uncertainty aversion axiom, yet are uncertainty averse according to Ghirardato and Marinacci. That this is the case should not be surprising; in both developments lack of ambiguity or uncertainty on some set of events is associated with the validity of the act-independence/sure-thing principle of subjective expected utility for preferences over acts measurable with respect to such events. The important difference that leads our notion to be stronger is that whereas Ghirardato and Marinacci examine only preference between pairs of acts where at least one act is unambiguous, our axiom imposes conditions on preference between certain pairs of ambiguous acts (one of which may be a "hedging" of the other) as well. Implicitly, the uncertainty aversion described in this paper is based on the idea that we sometimes are willing to say that one act is less ambiguous than another even when neither is completely unambiguous.

The relation with Epstein [7] is a bit more complicated because he considers all probabilistically sophisticated preferences (a class larger than expected utility) to be uncertainty neutral. Since some of the preferences considered in this paper that are uncertainty averse according to us are also probabilistically sophisticated, our notion does not imply his. For the reasons in the paragraph above, Epstein's notion does not imply our notion either, even in the presence of the other axioms. One thing that can be said, however, is that if the set of unambiguous events, U, underlying Epstein's definition happens to also be unambiguous in the sense that preferences over U-measurable acts admit an expected utility representation, then any MMEU preferences that are uncertainty averse as described in this paper will be uncertainty averse according to Epstein as well. (This last statement is shown as a theorem in Epstein [7].)

Overall then, our uncertainty aversion axiom should be thought of as embodying something more than definitions that are based purely on comparisons of the unambiguous with the possibly ambiguous. In particular, the idea that state-by-state combinations of a pair of possibly ambiguous acts may reduce ambiguity is fundamental to our axioms. Being uncertain is not being sure what the proper weights are for evaluating acts. Being uncertainty averse intuitively means that the decision maker worries that the "real weights" are going to be unfavorable for any given act. Since the unfavorable weights for each act may not be the same, combining two acts state-by-state limits the scope for varying weights to lead to unfavorable evaluations. A special case of this is when a constant act is generated as a state-by-state combination of two ambiguous acts (a perfect "hedge"). However, even when a constant or unambiguous act is not generated, the resulting act may still be a reduction in ambiguity. We think that this "something more" embodied in Schmeidler-type uncertainty aversion is quite reasonable.

One benefit of our uncertainty aversion formulation in the context of this paper is that a flexible and tractable representation emerges. It is an open question whether there exists a tractable representation of preferences characterized by all of our axioms excepting uncertainty aversion (or replacing uncertainty aversion with Ghirardato and Marinacci's weaker notion). If there is such a representation (even a less-than-tractable one) it currently lies beyond the reach of the authors.

APPENDIX A. PROOFS

A.1. *Sufficiency of the Axioms for the Representation Theorem*

Given any $f \in F$, denote its certainty equivalent by $m(f)$. That is, $m(f)$ is an element in F^* such that $m(f) \sim f$. For $x_1, x_2 \in X$, $B \in \Sigma$, define $m^B(x_1, x_2) \equiv m(x_{1B}x_2)$. The next lemma shows that a unique certainty equivalent exists for each act.

Lemma A.1. *For each $f \in F$, $m(f)$ exists and is unique.*

Proof. By Axiom 4 (monotonicity), part (a), the sets $M(f)_c := \{x \in X \mid x \succcurlyeq f\}$ and $W(f)_c := \{x \in X \mid f \succcurlyeq x\}$ are non-empty. By continuity, both of these sets are closed. By weak order, $M(f)_c \cup W(f)_c = X$. Therefore, since X is connected, $M(f)_c \cap W(f)_c \neq \emptyset$ and there must be at least one $x \in X$ such that $x \sim f$. Suppose there are two such prizes, $x_1 < x_2$. Then, by the structure axiom, part (a), and weak order, $x_2 \succ x_1 \sim f$, a contradiction.

Definition A.2. We say that $\succcurlyeq$ is *bounded* if, for each $f \in F$, there are $x, y \in X$ such that $x \succcurlyeq f \succcurlyeq y$.

Consider the following axioms adapted from Nakamura [18]:

A1. $\succcurlyeq$ on F is a bounded weak order.

A2. For $f \in F$, $B \in \Sigma$ and $x, y, z \in X$, if $y_B z \succcurlyeq f \succcurlyeq x_B z$ then $f \sim a_B z$ for some $a \in X$.

A3. If B is ordered non-null and $z \succcurlyeq x$ and $z \succcurlyeq y$, then $y \succcurlyeq x$ if and only if $y_B z \succcurlyeq x_B z$; if B is ordered non-universal and $x \succcurlyeq z$ and $y \succcurlyeq z$, then $y \succcurlyeq x$ if and only if $z_B y \succcurlyeq z_B x$.

A4. If $y \succcurlyeq x$ and $B \subseteq C \in \Sigma$ then $x_B y \succcurlyeq x_C y$.

A5. Every strictly bounded standard sequence is finite.

A6. If $x_2 \succcurlyeq x_1$ and $y_2 \succcurlyeq y_1$ with $y_i \succcurlyeq x_i$ for $i = 1, 2$, then

$$m^B(x_1, x_2)_B \, m^B(y_1, y_2) \sim m^B(x_1, y_1)_B \, m^B(x_2, y_2).$$

The next two lemmas show that our axioms imply Nakamura's A1–A5 and A6 with $B = A$. This allows us to use several lemmas from Nakamura [18] to show that an expected utility representation for ordered A-measurable acts exists.

LEMMA A.3. *Axioms weak order, structure, continuity, monotonicity, and ordered A-act-independence imply* A1–A5.

Proof. (Axioms $\Rightarrow$ A1) This is implied by weak order and monotonicity.

(Axioms $\Rightarrow$ A2) Consider $W(f)_B = \{b \in X \mid f \succcurlyeq b_B z\}$ and $M(f)_B = \{b \in X \mid b_B z \succcurlyeq f\}$. $W(f)_B$ and $M(f)_B$ are non-empty because $x \in W(f)_B$ and $y \in M(f)_B$ by assumption. We want to show that $W(f)_B$ is closed. To do this we will show that $X - W(f)_B$ is open, where $X - W(f)_B = \{b \in X \mid b_B z \succ f\}$. Let $W(f)^c = \{g \in F \mid g \succ f\}$. Let t be in $X - W(f)_B$. (If such t does not exist then $X - W(f)_B = \varnothing$ and $W(f)_B = [m, M]$ which is closed, so we are done.) Since $t \in X - W(f)_B$, we have that $t_B z \in W(f)^c$. By continuity, $W(f)^c$ is open. Hence $\exists \delta > 0$ such that $\forall g \in U_\delta(t_B z)$ we have that $g \in W(f)^c$, where $U_\delta(t_B z) = \{g \in F \mid \|g - t_B z\| < \delta\}$. Consider now the set $V_\delta(t_B z) = \{b \in X \mid \|b_B z - t_B z\| < \delta\}$. Note that $\{g \in F \mid g = b_B z$ for some $b \in V_\delta(t_B z)\} \subset U_\delta(t_B z)$. Since $\forall g \in U_\delta(t_B z)$ we have that $g \succ f$, we must have that $\forall b \in V_\delta(t_B z)$, $b_B z \succ f$, implying that $X - W(f)_B$ is open. Hence $W(f)_B$ is closed. Similarly $M(f)_B$ is closed. By weak order, $W(f)_B \cup M(f)_B = X$. Since X is connected, $W(f)_B \cap M(f)_B \neq \varnothing$. Therefore there exists $a \in X$ such that $a_B z \sim f$.

(Axioms $\Rightarrow$ A3) This follows from Axiom 4 (monotonicity).

(Axioms $\Rightarrow$ A4) This follows from Axiom 4 (monotonicity), part (a).

(Axioms $\Rightarrow$ A5) Let $B \in \Sigma$ be ordered non-null and ordered non-universal and fix $a, b \in X$ such that $a \succ b$. Let $\{a_i\}$ be a strictly bounded standard sequence with respect to a and b using event B. Let's assume first that $\{a_i\}$ is such that $a_i \succcurlyeq a$ and $a_B a_i \sim b_B a_{i+1}$ for all i. Since B is ordered non-null and $a_{i+1} \succcurlyeq a \succ b$, monotonicity implies $a_B a_{i+1} \succ b_B a_{i+1} \sim a_B a_i$. Thus, $a_{i+1} \succ a_i$ for all i and $\{a_i\}$ is an increasing sequence. As $\{a_i\}$ is strictly bounded it must in particular be bounded above. Hence, if $\{a_i\}$ has infinitely many terms then it must converge. Towards a contradiction, assume $\{a_i\}$ has infinitely many terms. Let $a^* = \lim_{i \to \infty} a_i$. Consider $W(b_B a^*)$. Since for each i, $b_B a^* \succ b_B a_{i+1} \sim a_B a_i$, we have that for each i, $a_B a_i \in W(b_B a^*)$. Take limits to obtain $\lim_{i \to \infty} a_B a_i = a_B a^*$. By continuity, $a_B a^* \in W(b_B a^*)$ (i.e., $b_B a^* \succcurlyeq a_B a^*$). Now, B ordered non-null, $a \succ b$ and $a^* \succcurlyeq a$, together with the monotonicity axiom, implies $a_B a^* \succ b_B a^*$, which is a contradiction. So $\{a_i\}$ must have only a finite number of terms. The other case, where $\{a_i\}$ is such that $b \succcurlyeq a_i$ and $a_{iB} b \sim a_{i+1B} a$ for all i, follows from a similar argument using the fact that B is ordered non-universal and $\{a_i\}$ is decreasing and bounded below. Note that Nakamura's definition of a standard sequence also allows for decreasing

$\{a_i\}$ when $a_i \succcurlyeq a$ and increasing $\{a_i\}$ when $b \succcurlyeq a_i$. It is easy to adapt the arguments just given above to show that $\{a_i\}$ must be finite in these cases as well. ▮

LEMMA A.4. *Axioms weak order, structure, continuity, monotonicity, and ordered A-act-independence imply* A6 *with* $B = A$.

Proof. We divide the argument into two cases depending on preference between y_1 and $m^A(x_1, x_2)$. The argument in each case will require two applications of ordered A-act-independence.

Case 1. $y_1 \succcurlyeq m^A(x_1, x_2)$. Let f, g and $h \in F$ be such that

$$f = x_{1A} x_2$$
$$g = m^A(x_1, x_2)$$
$$h = y_{1A} y_2.$$

Consider f' and $g' \in F$ with, for all $\omega \in \Omega$,

$$f'(\omega) \sim f(\omega)_A\, h(\omega) = \begin{cases} x_{1A}\, y_1, & \omega \in A \\ x_{2A}\, y_2, & \omega \in A^c \end{cases}$$

$$g'(\omega) \sim g(\omega)_A\, h(\omega) = \begin{cases} m^A(x_1, x_2)_A\, y_1, & \omega \in A \\ m^A(x_1, x_2)_A\, y_2, & \omega \in A^c. \end{cases}$$

Since $f \sim g$, ordered A-act-independence implies $f' \sim g'$. That is,

$$m^A(x_1, y_1)_A\, m^A(x_2, y_2) \sim m^A(m^A(x_1, x_2), y_1)_A\, m^A(m^A(x_1, x_2), y_2)$$

Now, let $\hat{f}$, $\hat{g}$ and $\hat{h} \in F$ such that

$$\hat{f} = y_{1A}\, y_2$$
$$\hat{g} = m^A(y_1, y_2)$$
$$\hat{h} = m^A(x_1, x_2).$$

Consider $\hat{f}'$ and $\hat{g}' \in F$ with, for all $\omega \in \Omega$,

$$\hat{f}'(\omega) \sim \hat{h}(\omega)_A\, \hat{f}(\omega) = \begin{cases} m^A(x_1, x_2)_A\, y_1, & \omega \in A \\ m^A(x_1, x_2)_A\, y_2, & \omega \in A^c \end{cases}$$

$$\hat{g}'(\omega) \sim \hat{h}(\omega)_A\, \hat{g}(\omega) = m^A(x_1, x_2)_A\, m^A(y_1, y_2).$$

Since $\hat{f} \sim \hat{g}$, ordered A-act-independence implies $\hat{f}' \sim \hat{g}'$. Hence,

$$m^A(m^A(x_1, x_2), y_1)_A\, m^A(m^A(x_1, x_2), y_2) \sim m^A(x_1, x_2)_A\, m^A(y_1, y_2).$$

It follows that

$$m^A(x_1, x_2)_A\, m^A(y_1, y_2)$$

$$\sim m^A(x_1, y_1)_A\, m^A(x_2, y_2), \qquad \text{which is A6 with } B = A.$$

Case 2. $m^A(x_1, x_2) \succ y_1$. Note that $y_1 \succcurlyeq x_1$ together with $m^A(x_1, x_2) \succ y_1$ implies $x_2 \succcurlyeq y_1$. Also, $x_2 \succcurlyeq y_1$ together with $x_2 \succcurlyeq x_1$ implies $x_2 \succcurlyeq m^A(x_1, y_1)$.

Now, let f, g and $h \in F$ be such that

$$f = x_{1A}\, y_1$$

$$g = m^A(x_1, y_1)$$

$$h = x_{2A}\, y_2.$$

Consider f', $g' \in F$ with, for all $\omega \in \Omega$,

$$f'(\omega) \sim f(\omega)_A\, h(\omega) = \begin{cases} x_{1A} x_2, & \omega \in A \\ y_{1A}\, y_2, & \omega \in A^c \end{cases}$$

$$g'(\omega) \sim g(\omega)_A\, h(\omega) = \begin{cases} m^A(x_1, y_1)_A\, x_2, & \omega \in A \\ m^A(x_1, y_1)_A\, y_2, & \omega \in A^c. \end{cases}$$

Since $f \sim g$, ordered A-act-independence implies $f' \sim g'$. That is,

$$m^A(x_1, x_2)_A\, m^A(y_1, y_2) \sim m^A(m^A(x_1, y_1), x_2)_A\, m^A(m^A(x_1, y_1), y_2).$$

Finally, let $\hat{f}$, $\hat{g}$ and $\hat{h} \in F$ be such that

$$\hat{f} = x_{2A}\, y_2$$

$$\hat{g} = m^A(x_2, y_2)$$

$$\hat{h} = m^A(x_1, y_1).$$

Consider $\hat{f}'$, $\hat{g}' \in F$ with, for all $\omega \in \Omega$,

$$\hat{f}'(\omega) \sim \hat{h}(\omega)_A\, \hat{f}(\omega) = \begin{cases} m^A(x_1, y_1)_A\, x_2, & \omega \in A \\ m^A(x_1, y_1)_A\, y_2, & \omega \in A^c \end{cases}$$

$$\hat{g}'(\omega) \sim \hat{h}(\omega)_A\, \hat{g}(\omega) = m^A(x_1, y_1)_A\, m^A(x_2, y_2), \qquad \omega \in \Omega.$$

Since $\hat{f} \sim \hat{g}$, ordered A-act-independence implies $\hat{f}' \sim \hat{g}'$. Hence,

$$m^A(m^A(x_1, y_1), x_2)_A\, m^A(m^A(x_1, x_2), y_2) \sim m^A(x_1, y_1)_A\, m^A(x_2, y_2).$$

It follows that

$$m^A(x_1, x_2)_A\, m^A(y_1, y_2) \sim m^A(x_1, y_1)_A\, m^A(x_2, y_2), \qquad \text{which is A6 with } B = A.$$

Lemma A.5. *There is a strictly increasing, continuous function u: $X \to \mathbb{R}$ and a real number $\pi(A) \in (0, 1)$ such that for all x, y, v, $w \in X$, if $y \succcurlyeq x$ and $w \succcurlyeq v$ then*

$$x_A\, y \succcurlyeq v_A\, w \Leftrightarrow \pi(A)\, u(x) + (1 - \pi(A))\, u(y) \geqslant \pi(A)\, u(v) + (1 - \pi(A))\, u(w). \tag{A.1}$$

Moreover, u is unique up to positive affine transformations and $\pi(A)$ is unique.

Proof. Existence of a function u and a real number $\pi(A)$ satisfying all the conditions of the lemma other than strict monotonicity and continuity, follows from Lemmas A.3 and A.4 and Nakamura [18, Lemmas 1, 2, and 3].

To see that u is strictly increasing, assume $x > v$. By part (a) of the structure axiom, we have $x \succ v$. Apply the already proven part of the lemma to x, $y = x$, v, $w = v$ to obtain $u(x) > u(v)$. Continuity of u follows from the following argument: Since u is strictly increasing, the only discontinuities can be (an at most countable number of) jumps up. Therefore, limits from above and from below exist at each point in X. Suppose there is a jump of height $\delta > 0$ at $\hat{x} \in X$. Consider the case where $u(\hat{x}) = \lim_{y \to \hat{x}^+} u(y)$. By definition of δ, $u(\hat{x}) - \delta = \lim_{y \to \hat{x}^-} u(y)$. By (A.1), if $\hat{x} \succ y$, then $y_A\, \hat{x} \sim x$ only if $\pi(A)\, u(y) + (1 - \pi(A))\, u(\hat{x}) = u(x)$. Since $u(\hat{x}) - \delta = \lim_{y \to \hat{x}^-} u(y)$, for any $\varepsilon > 0$, there exists $\hat{y}$ such that $\hat{x} \succ \hat{y}$ and $u(\hat{y}) > u(\hat{x}) - \delta - \varepsilon$. Then, fixing $\varepsilon < [(1 - \pi(A))/\pi(A)]\, \delta$,

$$\begin{aligned} u(\hat{x}) &> \pi(A)\, u(\hat{y}) + (1 - \pi(A))\, u(\hat{x}) \\ &> \pi(A)(u(\hat{x}) - \delta - \varepsilon) + (1 - \pi(A))\, u(\hat{x}) \\ &= u(\hat{x}) - \pi(A)(\delta + \varepsilon) \\ &> u(\hat{x}) - \delta \\ &= \lim_{y \to \hat{x}^-} u(y). \end{aligned}$$

But this implies that $\hat{y}_A\, \hat{x}$ has no certainty equivalent, contradicting Lemma A.1. The case where $u(\hat{x}) = \lim_{y \to \hat{x}^-} u(y)$ or $\lim_{y \to \hat{x}^-} u(y) < u(\hat{x}) < \lim_{y \to \hat{x}^+} u(y)$ generate a contradiction by a similar argument. This shows u can have no jumps. ∎

Now we use this u function to show the implications, in utility terms, of the constant-independence and uncertainty aversion axioms. The following definition is useful.

Definition A.6. An act $f \in F$ is *interior* if $m < \inf_\omega f(\omega) \leqslant \sup_\omega f(\omega) < M$.

Lemma A.7. *Let $f, g \in F$, $x \in F^*$ and $\alpha \in (0, 1)$. If*

(i) *$f' \in F$ is such that, for all $\omega \in \Omega$,*

$$u(f'(\omega)) = \alpha u(x) + (1-\alpha)\, u(f(\omega))$$

and

(ii) *$g' \in F$ is such that, for all $\omega \in \Omega$,*

$$u(g'(\omega)) = \alpha u(x) + (1-\alpha)\, u(g(\omega))$$

then

$$f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g'.$$

Proof. Normalize u such that $u(m)$ and $u(M)$ are rational. First consider the case where x is an interior act, $f(\omega)$ and $g(\omega)$ do not equal m for any ω, and α is rational. For each ω, we construct standard sequences $\{a_i^{f(\omega)}\}$ and $\{a_i^{g(\omega)}\}$ satisfying the conditions of the constant-independence axiom. Begin with $\{a_i^{f(\omega)}\}$. Set $\underline{r}(\omega) = \min\{f(\omega), x\}$ and $\bar{r}(\omega) = \max\{f(\omega), x\}$. If $\bar{r}(\omega) = \underline{r}(\omega)$ then $f'(\omega) = f(\omega) = x$. In this case, let $a_1^{f(\omega)} = \underline{r}(\omega)$ be the first element of a standard sequence with respect to $m + \underline{r}(\omega)$ and m using event A. Any such sequence satisfies the conditions in the axiom for this ω with $i(\omega) = i'(\omega) = k(\omega) = 1$ and any $\alpha \in (0, 1)$.

If, instead, $\bar{r}(\omega) > \underline{r}(\omega)$ then use the following argument. Assume $f(\omega) = \bar{r}(\omega)$ (the case where $x = \bar{r}(\omega)$ is similar). Define

$$\frac{k_1}{k_2} \equiv \alpha,$$

where $k_1, k_2 \in \mathbb{Z}_+$. This is possible since α was assumed to be rational. By definition of f',

$$\frac{u(f'(\omega)) - u(\underline{r}(\omega))}{u(\bar{r}(\omega)) - u(\underline{r}(\omega))} = \frac{\alpha u(x) + (1-\alpha)\, u(f(\omega)) - u(\underline{r}(\omega))}{u(\bar{r}(\omega)) - u(\underline{r}(\omega))}$$

$$= \alpha(0) + (1-\alpha)(1) = \frac{k_2 - k_1}{k_2}.$$

Pick $v(\omega) \in \mathbb{Z}_{++}$ large enough so that $(u(\bar{r}(\omega)) - u(\underline{r}(\omega)))/v(\omega)\, k_2 \leqslant (\pi(A)/(1-\pi(A)))(u(\underline{r}(\omega)) - u(m))$. The fact that u is continuous and increasing guarantees the existence of a $t(\omega)$ such that $(\pi(A)/(1-\pi(A)))(u(m+t(\omega)) - u(m)) = (u(\bar{r}(\omega)) - u(\underline{r}(\omega)))/v(\omega)\, k_2$.

Now, consider a standard sequence with respect to $m + t(\omega)$ and m using event A with first element $a_1^{f(\omega)} = \underline{r}(\omega)$. Define $\Delta^I(\omega) \equiv (\pi(A)/(1-\pi(A)))(u(m+t(\omega)) - u(m))$. From the calculations above and the definition of a standard sequence, we see that

$$u(f(\omega)) - u(\underline{r}(\omega)) = v(\omega)\, k_2 \Delta^I(\omega)$$

$$u(x) - u(\underline{r}(\omega)) = 0$$

$$u(f'(\omega)) - u(\underline{r}(\omega)) = v(\omega)(k_2 - k_1)\, \Delta^I(\omega).$$

Therefore,

$$a^{f(\omega)}_{v(\omega)\, k_2 + 1} \sim f(\omega)$$

$$a_1^{f(\omega)} \sim x$$

$$a^{f(\omega)}_{v(\omega)(k_2 - k_1) + 1} \sim f'(\omega).$$

Furthermore,

$$\alpha(1) + (1-\alpha)(v(\omega)\, k_2 + 1) = v(\omega)(k_2 - k_1) + 1.$$

This shows the constructed $\{a_i^{f(\omega)}\}$ satisfy the conditions in the constant-independence axiom. Similar arguments yield a satisfactory $\{a_i^{g(\omega)}\}$. To construct $\{a_i^{f(\omega)}\}$ and $\{a_i^{g(\omega)}\}$ for states in which $f(\omega)$ or $g(\omega)$ is m, use very similar arguments with decreasing standard sequences instead of increasing ones. Since the above arguments hold for any ω, the conditions in the constant-independence axiom are satisfied and therefore we have shown $f \succcurlyeq g \Leftrightarrow f' \succcurlyeq g'$, assuming α rational and x interior.

Now we will show that $f \succcurlyeq g$ implies $f' \succcurlyeq g'$ without assuming either rational α or x interior. Suppose $f \succcurlyeq g$. Let $\{\alpha^n\}$ be a sequence of rational numbers in $(0, 1)$ such that $\alpha^n \to \alpha$ and $\{x^n\}$ be a sequence of interior constant acts such that $x^n \to x$. Define sequences of acts $\{f'^n\}$ and $\{g'^n\}$ by $u(f'^n(\omega)) = \alpha^n u(x^n) + (1-\alpha^n)\, u(f(\omega))$ and $u(g'^n(\omega)) = \alpha^n u(x^n) + (1-\alpha^n)\, u(g(\omega))$ for all ω, n. By the argument above for rational α and interior x, $f \succcurlyeq g$ implies $f'^n \succcurlyeq g'^n$ for all n. Since $f'^n \to f'$ and $g' \to g'$ by construction, continuity of preferences requires that if $f'^n \succcurlyeq g'^n$ for all n then $f' \succcurlyeq g'$. Thus, $f \succcurlyeq g$ implies $f' \succcurlyeq g'$.

It remains to prove $f' \succcurlyeq g'$ implies $f \succcurlyeq g$ without the restrictions to rational α and x interior. We do this in three steps. First, we drop the rational α restriction but require f, g, and x to be interior; then we allow

f and g non-interior; and finally we allow x non-interior as well. The reason we do this in several steps is that the arguments at each stage use flexibility in constructing acts that would not necessarily be available if we dropped all restrictions at once.

Suppose first that f, g and x are interior acts. We will show that $f' \succcurlyeq g'$ implies $f \succcurlyeq g$. Suppose $f' \succcurlyeq g'$. Since f and g are interior, there exists an $\varepsilon > 0$ such that $m + 2\varepsilon \leqslant f(\omega) \leqslant M - 2\varepsilon$ and $m + 2\varepsilon \leqslant g(\omega) \leqslant M - 2\varepsilon$ for all $\omega \in \Omega$. By hypothesis, $f(\omega) = u^{-1}((u(f'(\omega)) - \alpha u(x))/(1-\alpha))$ and $g(\omega) = u^{-1}((u(g'(\omega)) - \alpha u(x))/(1-\alpha))$ for all $\omega \in \Omega$. By (uniform) continuity of u^{-1}: $[u(m), u(M)] \to [m, M]$, there exists $\delta_1 > 0$ such that if $|z - y| < \delta_1$ then $|u^{-1}(z) - u^{-1}(y)| < \varepsilon$. Similarly, for each $\alpha \in (0, 1)$ there exists a $\delta_2(\alpha) > 0$ such that if $|\alpha_1 - \alpha| < \delta_2(\alpha)$ then $|(u(z) - \alpha_1 u(y))/(1-\alpha_1) - (u(z) - \alpha u(y))/(1-\alpha)| < \delta_1$ for any $z, y \in X$. So, for any $\hat{\alpha} \in (0, 1)$, if $|\hat{\alpha} - \alpha| < \delta_2(\alpha)$ then $\hat{f}$ and $\hat{g}$ defined by $\hat{f}(\omega) = u^{-1}((u(f'(\omega)) - \hat{\alpha} u(x))/(1-\hat{\alpha}))$ and $\hat{g}(\omega) = u^{-1}((u(g'(\omega)) - \hat{\alpha} u(x))/(1-\hat{\alpha}))$ are well-defined interior acts. Fix a sequence $\{\alpha^n\}$ such that $\alpha^n \to \alpha$, $|\alpha^n - \alpha| < \delta_2(\alpha)$ and α^n is rational for all n. Using the formulas $f^n(\omega) = u^{-1}((u(f'(\omega)) - \alpha^n u(x))/(1-\alpha^n))$ and $g^n(\omega) = u^{-1}((u(g'(\omega)) - \alpha^n u(x))/(1-\alpha^n))$ for all $\omega \in \Omega$, generate sequences of acts $\{f^n\}$ and $\{g^n\}$ such that $f^n \to f$ and $g^n \to g$. Now, since $u(f'(\omega)) = \alpha^n u(x) + (1-\alpha^n)\, u(f^n(\omega))$ and $u(g'(\omega)) = \alpha^n u(x) + (1-\alpha^n)\, u(g^n(\omega))$ for all ω and n, $f' \succcurlyeq g' \Rightarrow f^n \succcurlyeq g^n$ for all n by the argument for rational α. Continuity then yields $f \succcurlyeq g$.

Next we allow for f and g non-interior, but maintain x interior. We proceed by proving the contrapositive, $g \succ f$ implies $g' \succ f'$. Fix $g \succ f$ where f and g are not necessarily interior. Fix $\alpha \in (0, 1)$. Define f' and g' by $u(f'(\omega)) = \alpha u(x) + (1-\alpha)\, u(f(\omega))$ and $u(g'(\omega)) = \alpha u(x) + (1-\alpha)\, u(g(\omega))$ for all $\omega \in \Omega$. Since $g \succ f$, by continuity there exists $\varepsilon > 0$ such that for all f'' with $\sup_\omega |f''(\omega) - f(\omega)| < \varepsilon$ and $\sup_\omega |g''(\omega) - g(\omega)| < \varepsilon$, $g'' \succ f''$. By construction, there exists β with $0 < \beta < \alpha$ such that the acts f'' and g'' defined by $u(f''(\omega)) = \beta u(x) + (1-\beta)\, u(f(\omega))$ and $u(g''(\omega)) = \beta u(x) + (1-\beta)\, u(g(\omega))$ for all $\omega \in \Omega$ satisfy $\sup_\omega |f''(\omega) - f(\omega)| < \varepsilon$ and $\sup_\omega |g''(\omega) - g(\omega)| < \varepsilon$. Observe that since x is an interior act, f'' and g'' are also interior. Now,

$$
\begin{aligned}
u(f'(\omega)) &= \gamma u(x) + (1-\gamma)\, u(f''(\omega)) \\
&= \gamma u(x) + (1-\gamma)(\beta u(x) + (1-\beta)\, u(f(\omega))) \\
&= (\gamma + (1-\gamma)\,\beta)\, u(x) + (1-\gamma)(1-\beta)\, u(f(\omega)).
\end{aligned}
$$

Therefore, $\gamma + (1-\gamma)\,\beta = \alpha$, which implies that $\gamma = \frac{\alpha - \beta}{1-\beta} \in (0, 1)$. Since $g'' \succ f''$, f'', g'' are interior acts, and $u(f'(\omega)) = \gamma u(x) + (1-\gamma)\, u(f''(\omega))$ and $u(g'(\omega)) = \gamma u(x) + (1-\gamma)\, u(g''(\omega))$ for all ω, $g' \succ f'$.

Finally we allow x non-interior as well and prove the contrapositive, $g \succ f$ implies $g' \succ f'$. We show that there exist interior acts f'', $g'' \in F$, x', $x'' \in F^*$, and numbers γ, $\beta \in (0, 1)$ such that,

$$u(f''(\omega)) = \gamma u(x'') + (1-\gamma)\, u(f(\omega)) \qquad \text{for all } \omega, \tag{A.2}$$

$$u(g''(\omega)) = \gamma u(x'') + (1-\gamma)\, u(g(\omega)) \qquad \text{for all } \omega, \tag{A.3}$$

$$u(f''(\omega)) = \beta u(x') + (1-\beta)\, u(f'(\omega)) \qquad \text{for all } \omega, \tag{A.4}$$

$$u(g''(\omega)) = \beta u(x') + (1-\beta)\, u(g'(\omega)) \qquad \text{for all } \omega. \tag{A.5}$$

Take $f, g, f', g' \in F$, $x \in F^*$, $x' \in F^*$ an interior act, and α, $\beta \in (0, 1)$ as fixed. We show that there exist appropriate f'', g'', x'' interior acts and $\gamma \in (0, 1)$. Define f'' as the act that satisfies $u(f''(\omega)) = \beta u(x') + (1-\beta)\, u(f'(\omega))$ for all ω. This implies that $u(f''(\omega)) = \beta u(x') + (1-\beta)(\alpha u(x) + (1-\alpha)\, u(f(\omega)))$ for all ω. Construct g'' similarly.

Then $(1-\gamma) = (1-\beta)(1-\alpha)$, if and only if $\gamma = \beta + \alpha(1-\beta)$, which is in $(0, 1)$. Also $\gamma u(x'') = (\beta + \alpha(1-\beta))\, u(x'') = \beta u(x') + \alpha(1-\beta)\, u(x)$, which implies, $u(x'') = (\beta u(x') + \alpha(1-\beta)\, u(x))/(\beta + \alpha(1-\beta)) \in (u(m), u(M))$. Furthermore, since x'' is an interior act, so are f'' and g''. Assume $g \succ f$. Then, (A.2) and (A.3) imply $g'' \succ f''$ by the previous argument.

Now, suppose $f' \succcurlyeq g'$. Then, (A.4) and (A.5) imply $f'' \succcurlyeq g''$, which is a contradiction. Therefore we must have $g' \succ f'$. This completes the proof of the lemma. ∎

LEMMA A.8. *Let f, $g \in F$ and $\alpha \in (0, 1)$. Suppose $f \succcurlyeq g$. If $h \in F$ is such that, for all $\omega \in \Omega$,*

$$u(h(\omega)) = \alpha u(f(\omega)) + (1-\alpha)\, u(g(\omega))$$

then

$$h \succcurlyeq g.$$

Proof. By arguments mimicking those in the proof of Lemma A.7, the hypotheses of the uncertainty aversion axiom are satisfied for such f, g, $h \in F$, $\alpha \in (0, 1)$ if f and g are interior and α is rational. If f or g are not interior, then construct sequences of interior acts with limits f and g by taking state-by-state mixtures with an interior constant act, where the weight on the constant act goes to zero as n increases. By Lemma A.7, this mixing preserves the preference between f and g. Finally, let $\alpha^n \to \alpha$, where α^n rational for all n, and use these interior sequences to form the corresponding $\{h^n\}$. The axiom applied to each n in the sequence plus continuity implies $h \succcurlyeq g$. ∎

We next construct a real-valued representation of preferences over acts by fixing u and assigning each act the utility of its certainty equivalent.

LEMMA A.9. *Given a* $u: X \to \mathbb{R}$ *from Lemma* A.5, *there is a unique* $J: F \to \mathbb{R}$ *such that*:

(i) *for all* f *and* $g \in F$, $f \succcurlyeq g$ *if and only if* $J(f) \geqslant J(g)$;

(ii) *for any constant act* $f = x \in X$, $J(f) = u(x)$.

Proof. For constant acts, we uniquely define $J(\cdot)$ by (ii). For general acts $f \in F$, let $J(f) = u(m(f))$. Clearly, $J(\cdot)$ satisfies (i) and is unique. ▮

Remark 1. For any $x, y \in X$ such that $y \succcurlyeq x$,

$$J(x_A\, y) = \pi(A)\, u(x) + (1 - \pi(A))\, u(y),$$

where $\pi(A)$ is given by Lemma A.5.

Proof. Let u be the utility function used in the construction of J. By Lemma A.5, $x_A\, y \sim m^A(x, y)$ implies $u(m^A(x, y)) = \pi(A)\, u(x) + (1 - \pi(A))\, u(y)$. Note that J has been constructed so that $J(x_A\, y) = u(m^A(x, y))$. Hence, $J(x_A\, y) = \pi(A)\, u(x) + (1 - \pi(A))\, u(y)$. ▮

Let $K = u(X)$. Since u is continuous, K is a closed interval in $\mathbb{R}$. We normalize u such that $K = [-2, 2]$. Let B be the space of bounded (in the sup-norm), Σ-measurable, real valued functions on Ω. For $\gamma \in \mathbb{R}$, we denote by γ^* the element of B that assigns γ to every ω. Let $B(K)$ be the subset of functions in B with values in K. Observe that for $f \in F$, $u \circ f \in B(K)$, and for $d \in B(K)$ there exists $f \in F$ such that $u \circ f = d$. Now we use this observation to construct a functional on $B(K)$ that represents preferences.

DEFINITION A.10. For $f \in F$ we define the functional $I: B(K) \to \mathbb{R}$ by $I(u \circ f) = J(f)$.

Since J represents preferences, it is clear that I does as well. The next lemma shows that I satisfies several important properties, and that these properties may be preserved when extending I from $B(K)$ to all of B.

LEMMA A.11. $I: B(K) \to \mathbb{R}$ *may be extended to all of* B *in such a way that*:

(i) $I(1^*) = 1$;

(ii) (*I is monotonic*) *For all* $a, b \in B$, $a \geqslant b$ *implies* $I(a) \geqslant I(b)$;

(iii) (*I is homogeneous of degree* 1) *For all* $b \in B$, $\alpha \geqslant 0$, $I(\alpha b) = \alpha I(b)$;

(iv) (*I is* C-*independent*) *For all* $b \in B$, $\gamma \in \mathbb{R}$, $I(b+\gamma^*) = I(b) + I(\gamma^*)$; *and*,

(v) (*I is superadditive*) *For all* $a, b \in B$, $I(a+b) \geqslant I(a) + I(b)$.

Proof. First note that there exists $x \in X$ such that $u(x) = 1$. By construction then, $I(1^*) = J(x^*) = u(x) = 1$. Also, monotonicity of I on $B(K)$ follows directly from the monotonicity axiom. We will now show that I is homogeneous of degree 1 on $B(K)$.

It suffices to prove homogeneity for $\alpha \in [0, 1]$, as $\alpha > 1$ then follows by considering the reciprocal. First note that there exists $z \in X$ such that $u(z) = 0$. Suppose for a, $b \in B(K)$, $a = \alpha b$ for some $\alpha \in (0, 1)$. (The cases $\alpha = 0$ and $\alpha = 1$ are trivial.) Let f, $g \in F$ be such that $u \circ f = a$ and $u \circ g = b$. Then for all $\omega \in \Omega$, $u(f(\omega)) = \alpha u(g(\omega)) + (1-\alpha)\, u(z)$. Now let $y \in X$ be such that $y \sim g$. Also let $x \in X$ be such that $u(x) = \alpha u(y) + (1-\alpha)\, u(z)$. By Lemma A.7, $g \sim y$ implies $f \sim x$. Thus, $I(a) = I(u \circ f) = u(x) = \alpha u(y) = \alpha I(u \circ g) = \alpha I(b)$.

This shows that I is homogeneous of degree 1 on $B(K)$. Next, we extend I to all of B by homogeneity. Such an extension preserves homogeneity and monotonicity. It remains to be shown that I is C-independent and superadditive.

We now demonstrate C-independence of I. Consider $a \in B$ and $\gamma \in \mathbb{R}$. By homogeneity, we may assume without loss of generality that $\max(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)})\, a \in B(K)$ and $\max(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)})\, \gamma^* \in B(K)$. Note that by the structure of $B(K)$ (in particular the fact that K is an interval around 0), it follows that $\frac{1}{1-\pi(A)}\, a \in B(K)$ and $\frac{1}{\pi(A)}\, \gamma^* \in B(K)$. Define $\beta = I(\frac{1}{1-\pi(A)}\, a)$. By homogeneity, $\beta = \frac{1}{1-\pi(A)}\, I(a)$. Let $f \in F$ be such that $u \circ f = \frac{1}{1-\pi(A)}\, a$. Let y, $z \in X$ satisfy $u(y) = \beta$ and $u(z) = \frac{1}{\pi(A)}\, \gamma$. By construction of I, $J(f) = \beta$ and $J(y) = u(y) = \beta$, implying $f \sim y$. Now, let $g' \in F^*$ be the constant act such that, for all $\omega \in \Omega$,

$$u(g'(\omega)) = \pi(A)\, u(z) + (1-\pi(A))\, u(y).$$

Thus, $u(g'(\omega)) = \gamma + (1-\pi(A))\, \beta = I(\gamma^*) + I(a)$.

Now, let $f' \in F$ be an act such that, for all $\omega \in \Omega$,

$$u(f'(\omega)) = \pi(A)\, u(z) + (1-\pi(A))\, u(f(\omega)).$$

By Lemma A.7 and the previously noted fact that $f \sim y$, we have $f' \sim g'$. Therefore, $I(a+\gamma^*) = J(f') = J(g') = I(a) + I(\gamma^*)$ and I is C-Independent.

Finally, we show that I is superadditive. Consider a, $b \in B$. As above, by homogeneity we may assume without loss of generality that $\max(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)})\, a \in B(K)$ and $\max(\frac{1}{1-\pi(A)}, \frac{1}{\pi(A)})\, b \in B(K)$. Specifically, this implies $\frac{1}{1-\pi(A)}\, a \in B(K)$ and $\frac{1}{\pi(A)}\, b \in B(K)$. Let acts f, $g \in F$ be such that $u \circ f = \frac{1}{\pi(A)}\, b$

and $u \circ g = \frac{1}{1-\pi(A)} a$. The argument proceeds by considering the possible orderings of $I(\frac{1}{1-\pi(A)} a)$ and $I(\frac{1}{\pi(A)} b)$.

Case 1. $I(\frac{1}{1-\pi(A)} a) = I(\frac{1}{\pi(A)} b)$. Then $f \sim g$. Define the act f' by, for all $\omega \in \Omega$,

$$u(f'(\omega)) = (1-\pi(A))\, u(g(\omega)) + \pi(A)\, u(f(\omega)).$$

Thus, $u \circ f' = \pi(A)(u \circ f) + (1-\pi(A))(u \circ g) = b + a$ and $J(f') = I(u \circ f') = I(a+b)$. By Lemma A.8, we have that $f' \succcurlyeq f$. Therefore, $I(a+b) = J(f') \geqslant J(f) = \frac{1}{\pi(A)} I(b) = (\frac{1-\pi(A)+\pi(A)}{\pi(A)})\, I(b) = (\frac{1-\pi(A)}{\pi(A)})\, I(b) + I(b) = I(a) + I(b)$, since $I(a) = (1-\pi(A))\, I(\frac{1}{\pi(A)} b)$.

Case 2. $I(\frac{1}{\pi(A)} b) > I(\frac{1}{1-\pi(A)} a)$. Let $\gamma = I(\frac{1}{\pi(A)} b) - I(\frac{1}{1-\pi(A)} a) > 0$. Let $\frac{1}{1-\pi(A)} c = \frac{1}{1-\pi(A)} a + \gamma^*$. By C-independence of I, $I(\frac{1}{1-\pi(A)} c) = I(\frac{1}{1-\pi(A)} a) + \gamma = I(\frac{1}{\pi(A)} b)$. By Case 1, $I(c+b) \geqslant I(c) + I(b)$. But $I(c+b) = I(a + (1-\pi(A))\, \gamma^* + b) = I(a+b) + (1-\pi(A))\, \gamma$ by C-independence. Similarly, $I(c) = I(a + (1-\pi(A))\, \gamma^*) = I(a) + (1-\pi(A))\, \gamma$. Thus, $I(a+b) + (1-\pi(A))\, \gamma = I(c+b) \geqslant I(c) + I(b) = I(a) + I(b) + (1-\pi(A))\, \gamma$. Thus, $I(a+b) \geqslant I(a) + I(b)$.

The third and final case, where $I(\frac{1}{\pi(A)} b) < I(\frac{1}{1-\pi(A)} a)$, is proved similarly. This shows that I is superadditive and completes the proof of the lemma. ∎

The importance of Lemma A.11 is made clear by the next result which states that such an I may be written as the minimum expectation over a compact and convex set of finitely additive probability measures.

LEMMA A.12. *Let I: $B \to \mathbb{R}$ be a functional satisfying*:

(i) $I(1^*) = 1$;

(ii) $I(a) \geqslant I(b)$ *if* $a \geqslant b$ *for all* $a, b \in B$;

(iii) $I(a+b) \geqslant I(a) + I(b)$ *for all* $a, b \in B$;

(iv) $I(\alpha a + \beta 1^*) = \alpha I(a) + \beta$ *for all* $a \in B$, $\alpha \geqslant 0$ *and* $\beta \in \mathbb{R}$.

Then there exists a unique convex and w*-*compact set* $\mathscr{C} \subseteq \mathscr{P}$ *such that*

$$I(a) = \min_{P \in \mathscr{C}} \int a\, dP \qquad \textit{for all} \quad a \in B.$$

Proof. See Gilboa and Schmeidler [12, Lemma 3.5] and the argument for uniqueness in the proof of their Theorem 1. ∎

Observe that (i)–(v) in Lemma A.11 imply that I satisfies (i)–(iv) of Lemma A.12. Therefore, we may represent $\succcurlyeq$ on F by $J(f) = I(u \circ f) = \min_{P \in \mathscr{C}} \int u \circ f\, dP$ with $\mathscr{C}$ unique, convex and w*-compact and u strictly increasing, continuous and unique up to positive affine transformations. This representation together with the representation (A.1) in Lemma A.5 imply that $\max_{P \in \mathscr{C}} P(A) = \pi(A)$ and $0 < \max_{P \in \mathscr{C}} P(A) < 1$. This proves sufficiency of the axioms in Theorem 3.5.

A.2. *Necessity of the Axioms*

The two lemmas below demonstrate that the representation in Theorem 3.5 must satisfy ordered A-act-independence and constant-independence. The proof that the representation implies the remaining axioms is straightforward and thus is omitted.

LEMMA A.13. *The representation in Theorem* 3.5 $\Rightarrow$ *Axiom* 5 (*ordered* A-*act-independence*).

Proof. Let x_1, x_2, y_1, y_2, z_1 and $z_2 \in X$ be such that $x_2 \succcurlyeq x_1$, $y_2 \succcurlyeq y_1$ and $z_2 \succcurlyeq z_1$. Let $f = x_{1A} x_2$, $g = y_{1A} y_2$ and $h = z_{1A} z_2$.

Case (i). Suppose

$$\{x_i, y_i\} \succcurlyeq z_i \ (i=1,2) \qquad \text{and} \qquad \begin{cases} f'(\omega) \sim h(\omega)_A\, f(\omega) & \text{for all } \omega \in \Omega, \\ g'(\omega) \sim h(\omega)_A\, g(\omega) & \text{for all } \omega \in \Omega. \end{cases}$$

Let $\alpha = \max_{P \in \mathscr{C}} P(A)$. Since the weights are chosen from the set $\mathscr{C}$ to minimize the expected utility,

$$\begin{aligned} & f' \succcurlyeq g' \\ \text{iff} \quad & \alpha u(m^A(z_1, x_1)) + (1-\alpha)\, u(m^A(z_2, x_2)) \\ & \quad \geqslant \alpha u(m^A(z_1, y_1)) + (1-\alpha)\, u(m^A(z_2, y_2)) \\ \text{iff} \quad & \alpha[\alpha u(z_1) + (1-\alpha)\, u(x_1)] + (1-\alpha)[\alpha u(z_2) + (1-\alpha)\, u(x_2)] \\ & \quad \geqslant \alpha[\alpha u(z_1) + (1-\alpha)\, u(y_1)] + (1-\alpha)[\alpha u(z_2) + (1-\alpha)\, u(y_2)] \\ \text{iff} \quad & \alpha u(x_1) + (1-\alpha)\, u(x_2) \geqslant \alpha u(y_1) + (1-\alpha)\, u(y_2) \\ \text{iff} \quad & f \succcurlyeq g. \end{aligned}$$

Case (ii). Suppose

$$z_i \succcurlyeq \{x_i, y_i\} \ (i=1,2) \qquad \text{and} \qquad \begin{cases} f'(\omega) \sim f(\omega)_A\, h(\omega) & \text{for all } \omega \in \Omega \\ g'(\omega) \sim g(\omega)_A\, h(\omega) & \text{for all } \omega \in \Omega. \end{cases}$$

A similar argument applies. ∎

LEMMA A.14. *The representation in Theorem* 3.5 $\Rightarrow$ *Axiom* 6 (*constant-independence*).

Proof. Consider f, g, f', g', x, and α as in the axiom.

Suppose, for each ω, there exist standard sequences using event A, $\{a_i^{f(\omega)}\}$ and $\{a_i^{g(\omega)}\}$, satisfying

$$a_{i(\omega)}^{f(\omega)} \sim f(\omega)$$

$$a_{i'(\omega)}^{f(\omega)} \sim f'(\omega)$$

$$a_{j(\omega)}^{g(\omega)} \sim g(\omega)$$

$$a_{j'(\omega)}^{g(\omega)} \sim g'(\omega)$$

$$a_{k(\omega)}^{f(\omega)} \sim x \sim a_{l(\omega)}^{g(\omega)}$$

for some positive integers $i(\omega)$, $i'(\omega)$, $j(\omega)$, $j'(\omega)$, $k(\omega)$, and $l(\omega)$, where

$$i'(\omega) = \alpha k(\omega) + (1-\alpha)\, i(\omega)$$

and

$$j'(\omega) = \alpha l(\omega) + (1-\alpha)\, j(\omega).$$

Let $\rho \equiv \max_{P \in \mathscr{C}} P(A)$. For any standard sequence with respect to a and b using event A, if the sequence is increasing the utility distance between consecutive elements is $\Delta^I \equiv \frac{\rho}{1-\rho}(u(a) - u(b))$. If the sequence is decreasing this distance is $\Delta^D \equiv \frac{1-\rho}{\rho}(u(a) - u(b))$. So, for example, if $\{a_i^{f(\omega)}\}$ is increasing for some ω, then for that state,

$$u(f(\omega)) = u(a_1^{f(\omega)}) + (i(\omega) - 1)\, \Delta^I,$$

$$u(f'(\omega)) = u(a_1^{f(\omega)}) + (i'(\omega) - 1)\, \Delta^I,$$

$$u(x) = u(a_1^{f(\omega)}) + (k(\omega) - 1)\, \Delta^I.$$

Using the fact that $i'(\omega) = \alpha k(\omega) + (1-\alpha)\, i(\omega)$, we have that

$$u(f'(\omega)) = \alpha u(x) + (1-\alpha)\, u(f(\omega)).$$

A similar calculation using Δ^D yields the same conclusion if $\{a_i^{f(\omega)}\}$ is decreasing. Furthermore, the same arguments using $\{a_i^{g(\omega)}\}$ allow us to show $u(g'(\omega)) = \alpha u(x) + (1-\alpha)\, u(g(\omega))$.

Note now that x a constant act implies that $\int u \circ x dP = \int u \circ x dP'$ for all $P' \in \mathscr{C}$. Now,

$$
\begin{aligned}
& f' \succcurlyeq g' \\
\text{iff} \quad & \min_{P \in \mathscr{C}} \int u \circ f' dP \geqslant \min_{P \in \mathscr{C}} \int u \circ g' dP \\
\text{iff} \quad & \min_{P \in \mathscr{C}} \int ((1-\alpha)(u \circ f) + \alpha(u \circ x))\, dP \\
& \quad \geqslant \min_{P \in \mathscr{C}} \int ((1-\alpha)(u \circ g) + \alpha(u \circ x))\, dP \\
\text{iff} \quad & \min_{P \in \mathscr{C}} \int (1-\alpha)(u \circ f)\, dP \geqslant \min_{P \in \mathscr{C}} \int (1-\alpha)(u \circ g)\, dP
\end{aligned}
$$

iff $f \succcurlyeq g$. This proves constant-independence. ∎

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