

# Ellsberg behavior: a monotonicity consideration and its implications\*

Soheil Ghili<sup>†</sup> and Peter Klibanoff<sup>‡</sup>

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## Abstract

Consider a canonical problem in choice under uncertainty: choosing from a convex feasible set consisting of all (Anscombe-Aumann) mixtures of two acts  $f$  and  $g$ ,  $\{\alpha f + (1 - \alpha)g : \alpha \in [0, 1]\}$ . We propose a preference condition, *Monotonicity in Mixtures*, which says that clearly improving the act  $f$  (in the sense of weak dominance) makes putting more weight on  $f$  more desirable. We show that this property has strong implications for preferences exhibiting behavior as in the classic Ellsberg (1961) paradoxes. For example, we show that maxmin expected utility (MEU) preferences (Gilboa and Schmeidler 1989) satisfy Monotonicity in Mixtures if and only if they are expected utility preferences. Thus, for MEU, Monotonicity in Mixtures and Ellsberg behavior are incompatible. We extend this stark finding in several directions. Moreover, we demonstrate that the incompatibility is not between Monotonicity in Mixtures and Ellsberg behavior (or even global ambiguity aversion) *per se*. For example, in addition to deriving general implications of Monotonicity in Mixtures, we show that smooth ambiguity preferences (Klibanoff, Marinacci and Mukerji 2005) can satisfy both properties as long as they are not too ambiguity averse.

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<sup>†</sup>Cowles Foundation and School of Management, Yale University, New Haven CT USA. e-mail: soheil.ghili@yale.edu

<sup>‡</sup>Kellogg School of Management, Northwestern University, Evanston IL USA. e-mail: peterk@kellogg.northwestern.edu

# 1 Introduction

This paper explores the tension between two aspects of preferences. Consider the set of acts generated from all (Anscombe-Aumann) mixtures of two acts  $f$  and  $g$ :  $\{\alpha f + (1 - \alpha)g : \alpha \in [0, 1]\}$  and think of preferences over this set as inducing preferences over  $\alpha$ . As one varies the acts  $f$  and  $g$  under consideration, the resulting preferences over  $\alpha$  would be expected to change. One force we might expect to influence this change is the idea that making one of the acts (say,  $f$ ) more attractive makes higher weights on  $f$  more desirable. A conservative notion of “more attractive” is the notion of state-by-state (weak) dominance. We propose a preference condition, *Monotonicity in Mixtures*, which says that clearly improving the act  $f$  (in the sense of weak dominance) makes putting more weight on  $f$  more desirable.

For preferences as in Ellsberg’s (1961) classic paradoxes, there is another force that might influence preferences over  $\alpha$ . Acts corresponding to intermediate weights  $\alpha$  may have value as a hedge against ambiguity when  $f$  and  $g$  perform well under different distributions, as, for example, where  $f$  corresponds to winning a prize only if a red ball is drawn,  $g$  corresponds to winning only if a blue ball is drawn, the composition of red vs. blue balls is unknown, and  $\frac{1}{2}f + \frac{1}{2}g$  corresponds to a sure 50% chance of winning a prize. We define *Ellsberg Behavior* as preferences that, at least for some event, some prizes and some  $\alpha$ , strictly value this hedging.

Is this hedging influence on preferences over  $\alpha$  compatible with its role in responding to improvements? What are the implications of Monotonicity in Mixtures for preferences displaying Ellsberg behavior? We will show that the implications are stark under conditions applying to a broad class of models – in many cases (including MEU (Gilboa and Schmeidler, 1989), Choquet Expected Utility (Schmeidler, 1989),  $\alpha$ -MEU (Ghirardato, Maccheroni and Marinacci, 2004), Variational (Maccheroni, Marinacci and Rustichini, 2006) and more) Monotonicity in Mixtures and Ellsberg behavior are incompatible. They are not always incompatible, however, and we also provide some informative positive results on compatibility using the smooth ambiguity model (Klibanoff, Marinacci and Mukerji, 2005). For these preferences, Monotonicity in Mixtures is satisfied when relative ambiguity aversion is not too large. The idea that Monotonicity in Mixtures places upper bounds on the intensity

of ambiguity averse behavior is shown to hold more generally (Theorem 5).

Our positive results are closely related to work on comparative statics of portfolios of random variables (risky assets) under expected utility addressing the question of when any first-order stochastically dominant shift in the (conditional on any realization of the other assets) distribution of an asset will result in a risk-averse expected utility investor increasing that asset's share in the optimal portfolio (see Mitchell and Douglas (1997), Meyer and Ormiston (1994), Hadar and Seo (1990), Fishburn and Porter (1976)). Moreover, our results on incompatibilities could be applied back to that literature to yield results on comparative statics for various non-expected utility models of choice under risk. For example, any non-expected utility model relying on convex preferences with kinks (e.g., rank-dependent expected utility (Quiggin, 1982) with concave utility and probability transformation function) must sometimes lead a first-order stochastic improvement to reduce that asset's share in the optimal portfolio. Furthermore, our results imply that, in a more realistic setting where asset payoffs depend on events for which objective probabilities are not given, even risk neutral investors cannot be too ambiguity averse if such reductions in share are never to occur. Though all of our results are shown independently of the risk aversion (or lack thereof) of the individual, in the context of this portfolio application it is interesting to note that Fox, Rogers and Tversky (1996) find evidence of the combination of risk neutrality with sensitivity to ambiguity among professional options traders.

Another domain of insight from our results can be seen in Auster (2014, 2018), concerning bilateral trade under ambiguity about quality. Optimal offer behavior on the part of an ambiguity averse buyer derived there involves hedging-motivated mixing between a pooling price and a price that will be accepted only by a low quality seller. One comparative static Auster examines is what happens to the mixing weight as the buyer's valuation of the high quality seller's good increases. This corresponds to an improvement in the payoff to the pooling price in the sense of weak dominance. When the buyer has MEU preferences, in line with our result (Theorem 2) on incompatibility with Monotonicity of Mixtures, there are many cases where the optimal response is to offer the pooling price less often. Our results on the smooth ambiguity model (Theorems 3 and 4) explain why such a result could occur only with sufficiently strong ambiguity aversion.

This paper is organized as follows. In section 2, we describe the formal setting and notation. In section 3, we introduce the basic axioms on preferences that we maintain throughout. In section 4, we define Ellsberg Behavior. In section 5, we define Monotonicity in Mixtures. The main results of the paper, describing implications of Ellsberg Behavior and Monotonicity in Mixtures, are in Section 6. The final section briefly discusses some extensions, including an alternative to Monotonicity in Mixtures and some topics for further investigation.

## 2 Setting and Preliminaries

We operate within a standard Fishburn (1970)-style version of an Anscombe-Aumann (1963) setting. Let  $S$  be the finite set of states. An event  $E$  is a subset of  $S$ . Let  $Z$  be the set of prizes or outcomes.  $X$  is the set of all simple lotteries over prizes (i.e., the set of all finite-support probability distributions on  $Z$ ). Observe that  $X$  is a convex set with respect to the following mixture operation: for  $\alpha \in [0, 1]$ , and  $x, y \in X$ ,  $\alpha x + (1 - \alpha)y$  is the element of  $X$  defined, for all  $z \in Z$ , by

$$(\alpha x + (1 - \alpha)y)(z) \equiv \alpha x(z) + (1 - \alpha)y(z).$$

Acts are functions from  $S$  to  $X$ . Let  $\mathcal{F}$  denote the set of all acts. Acts are the objects of choice. Preferences will be defined by a binary relation  $\succsim$  over acts. The symmetric and asymmetric parts of  $\succeq$  are denoted by  $\sim$  and  $\succ$ , respectively. Mixtures over acts are defined through statewise mixing of the resulting lotteries: for  $\alpha \in [0, 1]$ , and  $f, g \in \mathcal{F}$ ,  $\alpha f + (1 - \alpha)g$  is the act defined, for all  $s \in S$ , by

$$(\alpha f + (1 - \alpha)g)(s) \equiv \alpha f(s) + (1 - \alpha)g(s).$$

For  $x, y \in X$  and an event  $E$ , let  $xEy$  denote the act  $f$  s.t.  $\forall s \in E, f(s) = x$  and  $\forall s \notin E, f(s) = y$ . Constant acts are those that give the same lottery in all states (i.e.,  $f(s) = f(s'), \forall s, s' \in S$ ). In a standard abuse of notation, we sometimes use  $x$  to denote the constant act giving  $x \in X$  in each state. An act  $f$  is an interior act if, for each state  $s$ ,

there exist  $\bar{x}(s), \underline{x}(s) \in X$  such that  $\bar{x}(s) \succ f(s) \succ \underline{x}(s)$ .

A set-function  $\rho : 2^S \rightarrow \mathbb{R}$  is a *capacity* if  $\rho(\emptyset) = 0$ ,  $\rho(S) = 1$ , and, for all  $A, B \subseteq S$  with  $A \subseteq B$ ,  $\rho(A) \leq \rho(B)$ .

### 3 Preferences

Throughout, we will restrict attention to  $\succsim$  satisfying a few standard axioms: Weak Order, State-by-state Monotonicity, Risk Independence and Archimedean Continuity. In our setting, these axioms define the MBA preferences of Cerreia-Vioglio et al. (2011) and are equivalent to assuming  $\succsim$  can be represented by

$$V((u(f(s)))_{s \in S}) \tag{3.1}$$

where  $u : X \rightarrow \mathbb{R}$  is a non-constant, affine utility function and  $V : u(X)^S \rightarrow \mathbb{R}$  is normalized, monotonic and sup-norm continuous. (Note:  $u(f(s)) \equiv \sum_z u(z)f(s)(z)$ )

**Axiom 1. Weak Order:**  $\succsim$  are non-trivial, complete and transitive.

**Axiom 2. State-by-state Monotonicity:** For all acts  $f, g$ , if  $f(s) \succsim g(s)$  for all  $s \in S$ , then  $f \succsim g$ .

**Axiom 3. Risk Independence:** For all lotteries  $x, y, z \in X$  and  $\alpha \in (0, 1]$ , if  $x \succ y$  then  $\alpha x + (1 - \alpha)z \succ \alpha y + (1 - \alpha)z$ .

**Axiom 4. Archimedean Continuity:** For all acts  $f, g, h$ , if  $f \succ g \succ h$  then there exist  $\alpha, \beta \in (0, 1)$  such that  $\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h$ .

Although all are standard, of these axioms Risk Independence is probably the most controversial. It rules out non-expected utility behavior over lotteries, and thus the departures from expected utility that are allowed by MBA preferences concern aggregation across states. In this sense, we restrict attention to preferences that may violate subjective expected utility but obey expected utility under ‘‘objective’’ risk. An advantage of doing so is that our analysis may be carried out in utility space, greatly facilitating our arguments.

## 4 Ellsberg Behavior

Motivated by Ellsberg’s two-color experiment we formalize “Ellsberg behavior” as follows:

**Axiom 5. Ellsberg Behavior:** *There exists an event  $E \subseteq S$ ,  $w, x, y \in X$  with  $w \succ x \succsim y$  and an  $\alpha \in (0, 1)$  such that*

$$\alpha wEy + (1 - \alpha)xEw \succ wEy \sim xEw \tag{4.1}$$

In the two-color experiment, for example, if red and black balls are treated symmetrically, taking  $E$  as the event a red ball is drawn from the unknown urn,  $w$  as \$100 for sure,  $x = y$  as \$0 for sure, and  $\alpha = \frac{1}{2}$  so that  $\alpha wEy + (1 - \alpha)xEw$  gives a 50% chance of \$100 and a 50% chance of \$0 no matter what color is drawn turns (4.1) into the typical Ellsberg pattern of preferring a 50-50 bet to a bet on either color from the unknown urn. Allowing  $x \neq y$ ,  $\alpha \neq 1/2$ , and flexibility in the choice of the event  $E$  is designed to accommodate possibilities including that not all events may be perceived as ambiguous, asymmetries in the perception of  $E$  versus  $E^c$ , and stake- and event-dependence in ambiguity attitudes.<sup>1</sup>

Subjective expected utility (SEU) preferences cannot satisfy Ellsberg Behavior. One way to see this is to observe that (4.1) is a direct violation of the Anscombe-Aumann Independence axiom: letting  $f \equiv wEy$  and  $g \equiv xEw$ , if  $f \sim g$ , then Independence implies  $\alpha f + (1 - \alpha)g \sim f \sim g$ .

Ellsberg Behavior is meant to be a fairly minimal and “local” condition. Under the assumption that Independence is violated somewhere, it is much weaker than common “global” properties appearing in the ambiguity literature such as Uncertainty Aversion (Schmeidler, 1989), Ambiguity Aversion (Epstein 1999, Ghirardato and Marinacci 2002) or Sure Expected Utility Diversification (Chateauneuf and Tallon, 2002). Its weakness makes our results showing that for a broad class of preferences there is a conflict between it and Monotonicity in Mixtures more powerful.

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<sup>1</sup>See e.g., Baillon and Placido (2017) and Abdellaoui et al. (2011) for experimental evidence of such dependence.

## 5 A Monotonicity Consideration: Monotonicity in Mixtures

The main novel property we introduce is the following:

**Axiom 6. Monotonicity in Mixtures:** For all acts  $f, f', g$  such that  $f'(s) \succsim f(s)$  for all  $s \in S$ , and all numbers  $\alpha', \alpha \in [0, 1]$  such that  $\alpha' \geq \alpha$ ,

$$\alpha' f + (1 - \alpha') g \underset{(\succsim)}{\succsim} \alpha f + (1 - \alpha) g \implies \alpha' f' + (1 - \alpha') g \underset{(\succsim)}{\succsim} \alpha f' + (1 - \alpha) g$$

Monotonicity in Mixtures says that if increasing the weight on  $f$  from  $\alpha$  to  $\alpha'$  is (strictly) good, then doing so for a dominating act  $f'$  is also (strictly) good. In other words, improving the act  $f$  via weak dominance makes putting more weight on it more desirable. Moving from  $\alpha$  to  $\alpha'$  is a shift away from  $g$  towards  $f$  (or  $f'$ ). In these terms, Monotonicity in Mixtures says that replacing  $f$  by a weakly dominating  $f'$  can only expand the shifts away from  $g$  that are desirable. Notice that both the weak and strict versions are needed to express these properties – without the strict version, one could have the increased weight on  $f$  being strictly valued but the same increase with  $f'$  being only indifferent. All subjective expected utility preferences satisfy Monotonicity in Mixtures.

An analogy with consumer theory can give further insight into Monotonicity in Mixtures. Consider the special case where  $g$  yields a fixed, positive utility level on an event  $A$  and zero utility elsewhere,  $f$  yields a fixed, positive utility level on an event  $B$  and zero utility elsewhere,  $A$  and  $B$  are disjoint, and  $f'$  strictly improves  $f$  only on  $B$  (and does so by a fixed amount of utility). One can then view preferences over mixtures between  $f$  and  $g$  as preferences over consumption bundles of two goods – utility in event  $A$ , and utility in event  $B$  – where the feasible bundles lie on the line segment in consumption space connecting the points  $(0, u(f(B)))$  and  $(u(g(A)), 0)$ . Replacing  $f$  by  $f'$  rotates this budget set outward, as utility in event  $B$  has effectively become cheaper. As depicted in Figure 5.1, Monotonicity in Mixtures implies that the optimal consumption of utility in event  $A$  cannot rise as a result of this price decrease on utility in event  $B$ . In the language of consumer theory, the substitution effect on consumption of utility in  $A$  of such a price change (non-positive) must be at least

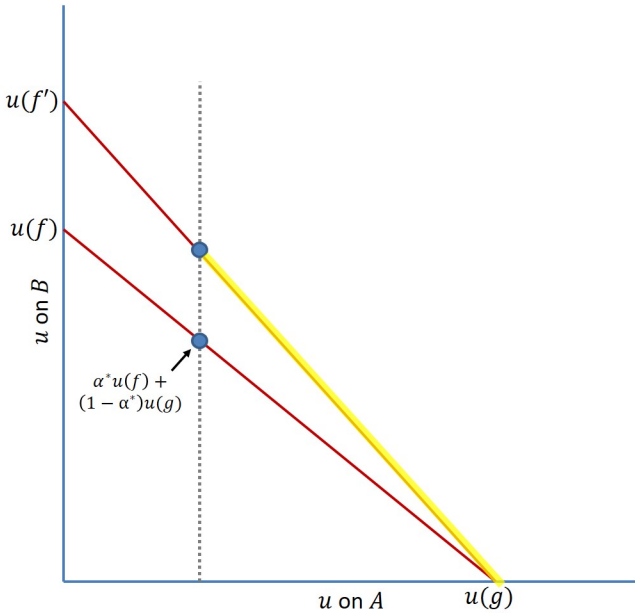


Figure 5.1: Monotonicity in Mixtures implies that if  $\alpha^*$  is the optimal mixture of  $f$  and  $g$  then the optimal mixture of  $f'$  and  $g$  cannot lie in the highlighted region.

as large in magnitude as the corresponding income effect (non-negative): Monotonicity in Mixtures implies that utilities in  $A$  and  $B$  must be gross substitutes. Observe that the linear indifference curves of subjective expected utility preferences imply a constant marginal rate of substitution in utility space and thus that utility in  $A$  and  $B$  are perfect substitutes.

## 6 Implications of Monotonicity in Mixtures

There is a potential conflict between Monotonicity in Mixtures and Ellsberg Behavior. For preferences exhibiting Ellsberg Behavior, intermediate weights  $\alpha$  may have value as a hedge against ambiguity when  $f$  and  $g$  perform well under different distributions over states. On the other hand, Monotonicity in Mixtures says that improving  $f$  (in the sense of weak dominance) makes putting more weight on  $f$  more desirable. Is the hedging role of  $\alpha$  under Ellsberg Behavior compatible with its role in responding to improvements? What are the implications of Monotonicity in Mixtures for some leading models of preferences displaying Ellsberg Behavior? These are questions to which we now turn.



## 6.1 Implications for MEU

We begin by considering a seminal model of ambiguity averse preferences: the Maxmin Expected Utility with Non-Unique Prior (MEU) model (Gilboa and Schmeidler, 1989). Each MEU preference can be represented by a functional of the following form:

$$\min_{p \in C} \sum_s u(f(s))p(s), \quad (6.1)$$

where  $u$  is a non-constant von Neumann-Morgenstern utility function and  $C$  is a non-empty, closed and convex set of probability measures over states.

Notice that when the set  $C$  contains only one probability measure, preferences are SEU. In all other cases, MEU preferences display Ellsberg Behavior. Formally:

**Proposition 1.** *An MEU preference displays Ellsberg Behavior if and only if the set of measures  $C$  is not a singleton.*

**Proof of Proposition 1.** Since Ellsberg Behavior is incompatible with SEU, it implies  $C$  is non-singleton. For the other direction, suppose  $C$  is non-singleton. Then there exists an event  $A$  s.t.  $\min_{p \in C} p(A) \neq \max_{p \in C} p(A)$ . Let  $p_1 \equiv \min_{p \in C} p(A)$  and  $p_2 \equiv \max_{p \in C} p(A)$  and note that  $0 \leq p_1 < p_2 \leq 1$ . By non-constancy of  $u$ , there are outcomes (i.e., degenerate lotteries)  $\bar{x}, \underline{x}$  such that  $u(\bar{x}) > u(\underline{x})$ . If  $\bar{x} \succ x \succsim \underline{x}$ , then (6.1) evaluates  $\bar{x}A\underline{x}$  as  $p_1 u(\bar{x}) + (1 - p_1)u(\underline{x})$  and  $xA\bar{x}$  as  $p_2 u(x) + (1 - p_2)u(\bar{x})$ . There are two cases to consider:

Case 1:  $p_1 + p_2 \geq 1$ . To show Ellsberg Behavior, let  $E = A$ ,  $w = \bar{x}$ ,  $y = \underline{x}$ ,  $x = \frac{p_1 + p_2 - 1}{p_2}w + \frac{1 - p_1}{p_2}y$  and  $\alpha = \frac{1 - p_1}{1 - p_1 + p_2}$ . Then  $w \succ x \succsim y$ , and (6.1) yields  $\alpha wEy + (1 - \alpha)xEw \succ wEy \sim xEw$  since

$$\begin{aligned} & \frac{1 - p_1}{1 - p_1 + p_2} u(\bar{x}) + \frac{p_2}{1 - p_1 + p_2} \left( \frac{p_1 + p_2 - 1}{p_2} u(\bar{x}) + \frac{1 - p_1}{p_2} u(\underline{x}) \right) \\ &= \frac{p_2}{1 - p_1 + p_2} u(\bar{x}) + \frac{1 - p_1}{1 - p_1 + p_2} u(\underline{x}) > p_1 u(\bar{x}) + (1 - p_1)u(\underline{x}) \\ &= p_2 \left( \frac{p_1 + p_2 - 1}{p_2} u(\bar{x}) + \frac{1 - p_1}{p_2} u(\underline{x}) \right) + (1 - p_2)u(\bar{x}) \\ &= p_2 u(x) + (1 - p_2)u(\bar{x}). \end{aligned}$$

Case 2:  $p_1 + p_2 \leq 1$ . To show Ellsberg Behavior, let  $E = A^c$ ,  $w = \bar{x}$ ,  $y = \underline{x}$ ,  $x = \frac{1-p_1-p_2}{1-p_1}w + \frac{p_2}{1-p_1}y$  and  $\alpha = \frac{p_2}{1-p_1+p_2}$ . Then  $w \succ x \succsim y$ , and (6.1) yields  $\alpha wEy + (1-\alpha)xEw \succ wEy \sim xEw$  since

$$\begin{aligned} & \frac{p_2}{1-p_1+p_2}u(\bar{x}) + \frac{1-p_1}{1-p_1+p_2}\left(\frac{1-p_1-p_2}{1-p_1}u(\bar{x}) + \frac{p_2}{1-p_1}u(\underline{x})\right) \\ &= \frac{1-p_1}{1-p_1+p_2}u(\bar{x}) + \frac{p_2}{1-p_1+p_2}u(\underline{x}) > (1-p_2)u(\bar{x}) + p_2u(\underline{x}) \\ &= (1-p_1)\left(\frac{1-p_1-p_2}{1-p_1}u(\bar{x}) + \frac{p_2}{1-p_1}u(\underline{x})\right) + p_1u(\bar{x}) \\ &= (1-p_1)u(x) + p_1u(\bar{x}). \end{aligned}$$

This completes the proof. □

Which MEU preferences satisfy Monotonicity in Mixtures?

**Proposition 2.** *An MEU preference satisfies Monotonicity in Mixtures if and only if the set of measures  $C$  is a singleton.*

**Proof of Proposition 2.** If  $C$  is a singleton, MEU preferences are SEU preferences, and therefore satisfy Monotonicity in Mixtures. Suppose  $C$  is non-singleton. By Proposition 1, these preferences display Ellsberg Behavior. The conclusion that Monotonicity in Mixtures is violated then follows as a special case of Theorem 2 in Section 6.3, since MEU preferences satisfy (6.7) with capacity  $\rho(A) \equiv \min_{p \in C} p(A)$  for all  $A \subseteq S$ . □

These results reveal that for the MEU model, Monotonicity in Mixtures and Ellsberg Behavior are incompatible. In particular, MEU preferences satisfy Monotonicity in Mixtures if and only if they are SEU preferences.

Next we present a result (Theorem 1) showing that this incompatibility extends well beyond MEU. The proof of this result provides a constructive argument (with associated graphical intuition) showing how Monotonicity in Mixtures is violated in the presence of Ellsberg Behavior generated by kinks in preferences. As a further application (Theorem 2), we will see that the same stark incompatibility found under MEU applies to all *c-linearly biseparable* (Ghirardato and Marinacci, 2001) preferences, a large class that includes not only MEU, but Choquet Expected Utility (Schmeidler, 1989),  $\alpha$ -Maxmin Expected Utility and more.

## 6.2 Implications for Ellsberg Behavior generated by kinks

Our result will be presented in terms of the geometry of preferences restricted to two-dimensional utility spaces. To state it, we first need to define sets of acts generating such spaces (Definition 1) and associated better-than sets (Definition 2).

**Definition 1.** For disjoint, non-empty events  $A, B \subseteq S$  and act  $h$ , let  $\mathcal{F}^{A,B,h}$  denote the set of acts  $f$  such that  $f(s) \sim f(t)$  for all  $s, t \in A$ ,  $f(s) \sim f(t)$  for all  $s, t \in B$  and  $f(s) = h(s)$  for all  $s \notin A \cup B$ .

**Definition 2.** Let  $\succsim$  be represented by  $V((u(f(s)))_{s \in S})$  as in (3.1). For any disjoint, non-empty events  $A, B \subseteq S$  and any act  $k \in \mathcal{F}^{A,B,h}$ , define  $G(k) \equiv \{(u(f(A)), u(f(B))) : f \in \mathcal{F}^{A,B,h} \text{ and } f \succsim k\}$ .

The theorem says that the existence of distinct lines “locally” supporting (i.e., supporting within a rectangle) the better-than set at any given point in such a two-dimensional utility space generates a violation of Monotonicity in Mixtures.

**Theorem 1.** *Let  $\succsim$  be represented by  $V((u(f(s)))_{s \in S})$  as in (3.1).*

*Fix disjoint, non-empty events  $A, B \subseteq S$  and interior act  $h \in \mathcal{F}^{A,B,h}$ .*

*If there exist a rectangle  $R \subseteq u(X) \times u(X)$  containing  $(u(h(A)), u(h(B)))$  in its interior and two distinct lines in  $u(X) \times u(X)$  that intersect  $G(h) \cap R$  only at  $(u(h(A)), u(h(B)))$ , then  $\succsim$  violates Monotonicity in Mixtures.*

Theorem 1 implies that any MBA preferences that use kinks (in at least some indifference curve in utility space) as their method of generating Ellsberg Behavior necessarily conflict with Monotonicity in Mixtures. This follows because such kinks allow there to exist the distinct lines on which the kink point is a “local” optimum (i.e., optimal among points on the line within the rectangular neighborhood) that the theorem relies on. The consumer theory analogy mentioned when discussing Monotonicity in Mixtures in Section 5 offers useful insight. Recall that Monotonicity in Mixtures implies that utility on  $A$  and  $B$  must be gross substitutes. The fact that budget lines with multiple slopes can locally support an indifference curve at a kink implies the absence of substitution effects there, implying a violation of gross substitutes.

**Proof of Theorem 1.** Fix  $V, u, A, B$  and  $h$  as in the statement of the theorem. Suppose that  $R$  is a rectangle in  $u(X) \times u(X)$  containing  $(u(h(A)), u(h(B)))$  in its interior, and  $l_1$  and  $l_2$  are distinct lines in  $u(X) \times u(X)$  that intersect  $G(h) \cap R$  only at  $(u(h(A)), u(h(B)))$ . Graphically, we can represent acts in  $\mathcal{F}^{A,B,h}$  as points in a two-dimensional Cartesian coordinate system with the vertical coordinate representing the utility level the act delivers in event  $B$ , and the horizontal coordinate representing the utility level the act delivers in event  $A$ . Monotonicity of  $V$  implies that all points  $(a, b) \in u(X) \times u(X)$  such that  $a \geq u(h(A))$  and  $b \geq u(h(B))$  lie in  $G(h)$ . Therefore, since  $l_1$  and  $l_2$  intersect  $G(h) \cap R$  only at  $(u(h(A)), u(h(B)))$ , it follows that both  $l_1$  and  $l_2$  must have negative and finite slopes.

The main part of our argument constructing a violation of Monotonicity in Mixtures will assume that  $l_1$  and  $l_2$  intersect both the left and bottom sides of the rectangle  $R$ . However, for a given  $R$  as above, this need not hold (see Figure 6.1 for an illustration). Therefore, before turning to the main construction, we show that the existence of an  $R$  as in the theorem implies the existence of a (possibly smaller) rectangle  $R' \subseteq R$  containing  $(u(h(A)), u(h(B)))$  in its interior and such that  $l_1$  and  $l_2$  intersect both the left and bottom sides of  $R'$  and intersect  $G(h) \cap R'$  only at  $(u(h(A)), u(h(B)))$ . Observe that, starting from the given  $R$ , by moving the bottom side upwards (but not all the way to  $u(h(B))$ ) and/or the left side rightwards (but not all the way to  $u(h(A))$ ), we can ensure that  $l_1$  and  $l_2$  intersect both these left and bottom sides. The resulting rectangle,  $R'$ , still contains  $(u(h(A)), u(h(B)))$  in its interior, and, since  $G(h) \cap R' \subseteq G(h) \cap R$ ,  $l_1$  and  $l_2$  intersect  $G(h) \cap R'$  only at  $(u(h(A)), u(h(B)))$ . Thus, it is without loss of generality to assume, as we do for the remainder of this proof, that  $l_1$  and  $l_2$  intersect both the left and bottom sides of the rectangle  $R$ .

We now construct a violation of Monotonicity in Mixtures. Since  $l_1$  and  $l_2$  are distinct and contain a common point, their slopes must differ. Without loss of generality, let  $l_1$  have the steeper slope. Therefore, for points with horizontal coordinate below  $u(h(A))$ ,  $l_1$  lies above  $l_2$ , while for points with horizontal coordinate above  $u(h(A))$ ,  $l_2$  lies above  $l_1$ . Fix acts  $f, g, f', g' \in \mathcal{F}^{A,B,h}$  such that  $(u(f(A)), u(f(B)))$  is the point where  $l_2$  intersects the left side of  $R$ ,  $(u(g(A)), u(g(B)))$  is the point where  $l_1$  intersects the bottom side of  $R$ ,  $(u(f'(A)), u(f'(B)))$  is the point where  $l_1$  intersects the left side of  $R$  and  $(u(g'(A)), u(g'(B)))$

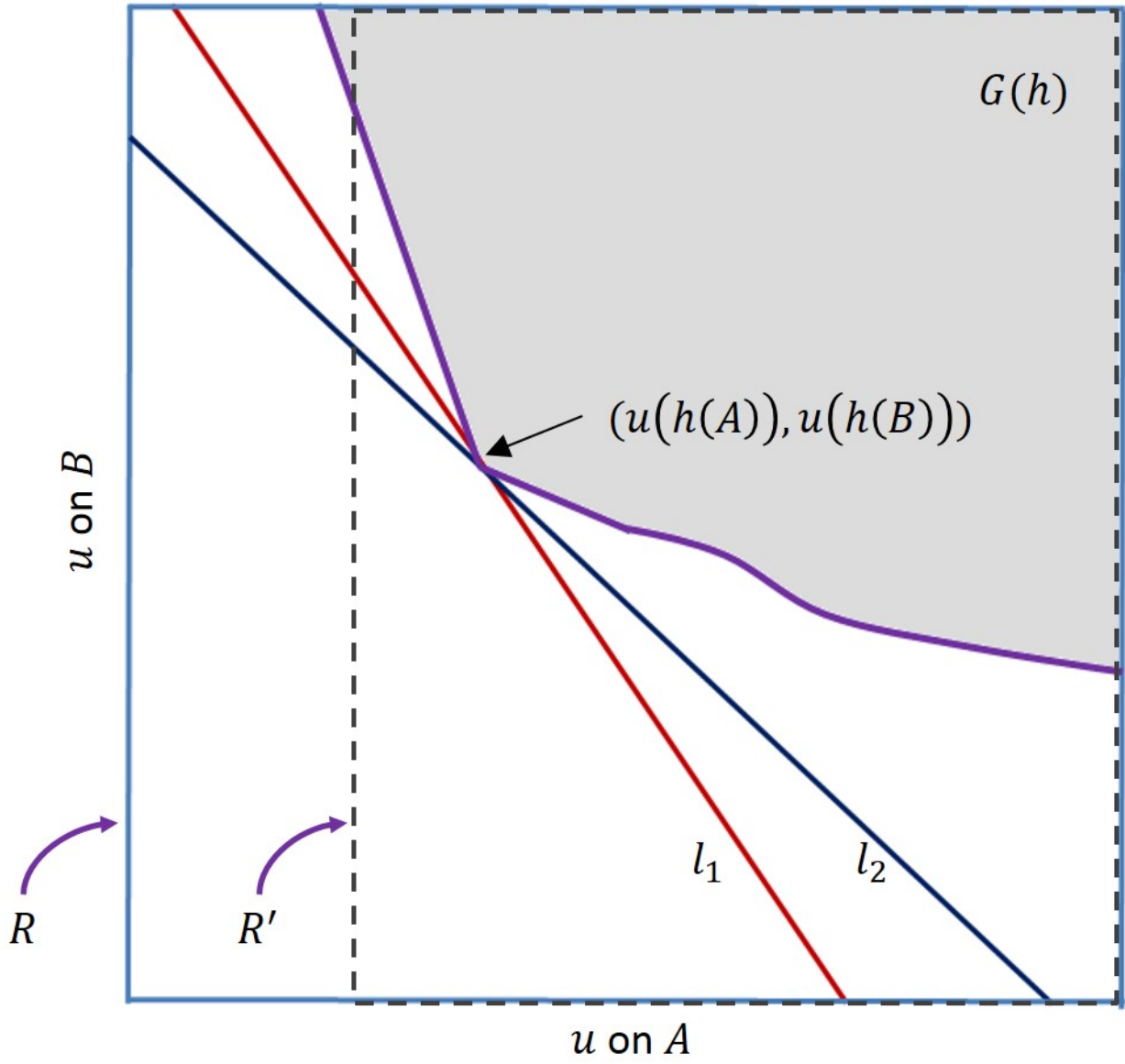


Figure 6.1: Shrinking rectangle  $R$  to get a smaller  $R'$  that lines  $l_1$  and  $l_2$  intersect on the left and bottom.

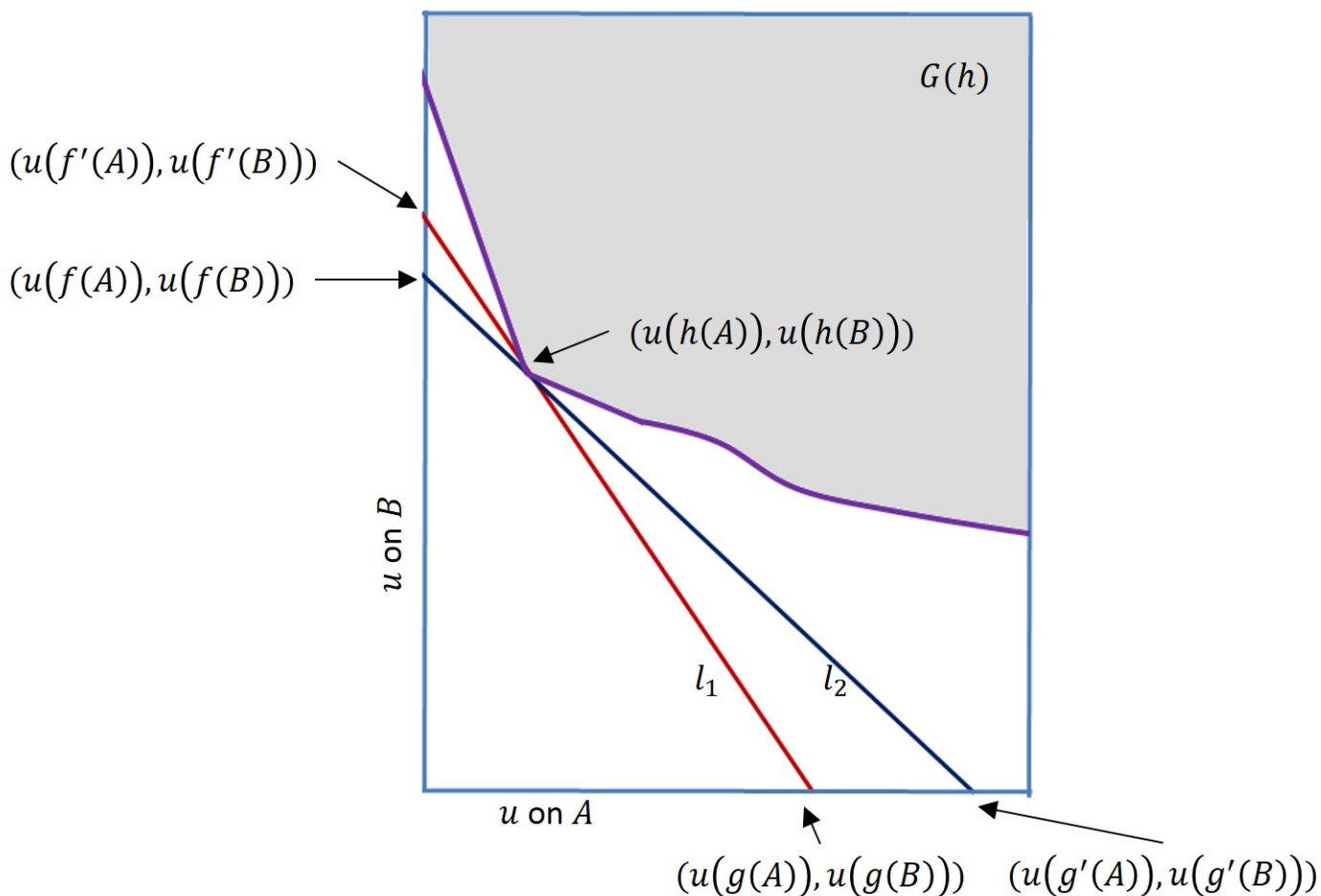


Figure 6.2: Choosing the acts  $f, g, f', g'$

is the point where  $l_2$  intersects the bottom side of  $R$  (See Figure 6.2). Observe that  $f'$  weakly dominates  $f$  (with strict dominance only on  $B$ ), and  $g'$  weakly dominates  $g$  (with strict dominance only on  $A$ ). Further observe that the points on  $l_1$  contained in  $R$  correspond to the acts  $\lambda f' + (1 - \lambda)g$ ,  $\lambda \in [0, 1]$  and the points on  $l_2$  contained in  $R$  correspond to the acts  $\lambda g' + (1 - \lambda)f$ ,  $\lambda \in [0, 1]$ . Let  $\lambda_1, \lambda_2 \in (0, 1)$  be the unique numbers such that  $\lambda_1 f' + (1 - \lambda_1)g$  and  $\lambda_2 g' + (1 - \lambda_2)f$  each correspond to  $(u(h(A)), u(h(B)))$ . It follows that  $\lambda_1 u(f'(A)) + (1 - \lambda_1)u(g(A)) = \lambda_1 u(f(A)) + (1 - \lambda_1)u(g(A)) = \lambda_2 u(g'(A)) + (1 - \lambda_2)u(f(A))$ , and thus

$$\lambda_2 < 1 - \lambda_1. \quad (6.2)$$

Since  $l_1$  and  $l_2$  intersect  $G(h) \cap R$  only at  $(u(h(A)), u(h(B)))$ ,

$$\lambda_1 f' + (1 - \lambda_1)g \succ \lambda f' + (1 - \lambda)g \text{ for all } \lambda \neq \lambda_1 \quad (6.3)$$

and

$$\lambda_2 g' + (1 - \lambda_2)f \succ \lambda g' + (1 - \lambda)f \text{ for all } \lambda \neq \lambda_2. \quad (6.4)$$

Because  $f'$  weakly dominates  $f$ , given (6.3), Monotonicity in Mixtures implies

$$\lambda_1 f + (1 - \lambda_1)g \succ \lambda f + (1 - \lambda)g \text{ for all } \lambda > \lambda_1. \quad (6.5)$$

Because  $g'$  weakly dominates  $g$ , given (6.5), Monotonicity in Mixtures implies

$$(1 - \lambda_1)g' + \lambda_1 f \succ \lambda g' + (1 - \lambda)f \text{ for all } \lambda < 1 - \lambda_1. \quad (6.6)$$

By (6.2), (6.6) contradicts (6.4). Therefore Monotonicity in Mixtures must be violated (and the violation occurs when applying the axiom to either the acts  $f', f, g$  or the acts  $g', g, f$ ). (See also Figure 6.3, which illustrates this contradiction graphically. The yellow highlighted segments correspond to the acts on the right-hand sides of (6.5) and (6.6) respectively, which are strictly worse than the left-hand side acts corresponding to the lower endpoints of the highlighted segments).  $\square$

### 6.3 Implications for c-linearly biseparable preferences

Ghirardato and Marinacci (2001) define and axiomatize a broad class of preferences they call c-linearly biseparable. This class includes, among others, the well-known MEU, Choquet Expected Utility (Schmeidler, 1989), and  $\alpha$ -MEU (where preference is represented by a convex combination of MEU and max-max EU) models. In terms of numerical representation, a key property satisfied by any c-linearly biseparable preference is that there is a unique capacity  $\rho$  such that

$$W(xEy) \equiv u(x)\rho(E) + u(y)(1 - \rho(E)) \quad (6.7)$$

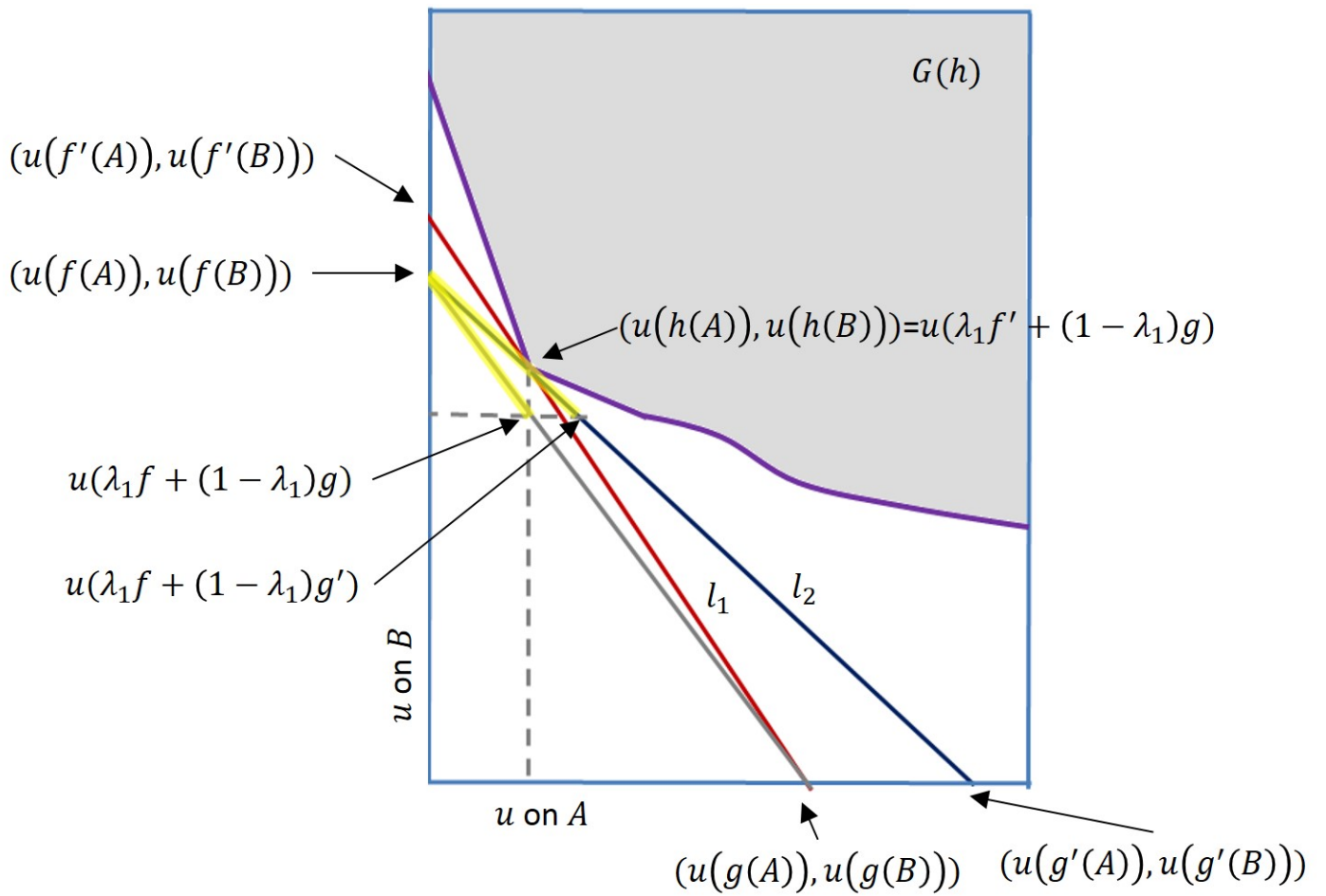


Figure 6.3: Violation of Monotonicity in Mixtures.



represents  $\succsim$  over acts of the form  $xEy$  for all  $E \subseteq S$ ,  $x, y \in X$  with  $x \succsim y$ . The next result shows that Ellsberg Behavior and Monotonicity in Mixtures are incompatible for any such preferences. The proof works by showing that (6.7) together with Ellsberg Behavior imply the existence of a kinked configuration that can be used to apply Theorem 1 to show that Monotonicity in Mixtures is violated.

**Theorem 2.** *All  $c$ -linearly biseparable preferences displaying Ellsberg Behavior must violate Monotonicity in Mixtures.*

**Proof of Theorem 2:** By Ellsberg Behavior, there exists an  $E \subseteq S$ ,  $w, x, y \in X$  with  $w \succ x \succsim y$  and an  $\alpha \in (0, 1)$  such that  $\alpha wEy + (1 - \alpha)xEw \succ wEy \sim xEw$ . By applying (6.7), we now show that, for this event  $E$ ,  $\rho(E) + \rho(E^c) < 1$ . Since  $wEy \sim xEw$ , (6.7) implies

$$u(w)\rho(E) + u(y)(1 - \rho(E)) = u(w)\rho(E^c) + u(x)(1 - \rho(E^c)).$$

Thus,

$$\rho(E) + \rho(E^c) = \frac{2u(w) - u(x) - u(y)}{u(w) - u(x)}\rho(E) - \frac{u(x) - u(y)}{u(w) - u(x)}, \quad (6.8)$$

and, equivalently,

$$\rho(E) + \rho(E^c) = \frac{2u(w) - u(x) - u(y)}{u(w) - u(y)}\rho(E^c) + \frac{u(x) - u(y)}{u(w) - u(y)}. \quad (6.9)$$

There are two cases to consider:

Case 1:  $\alpha w + (1 - \alpha)x \succsim \alpha y + (1 - \alpha)w$ . By Ellsberg Behavior and (6.7),  $(\alpha u(w) + (1 - \alpha)u(x))\rho(E) + (\alpha u(y) + (1 - \alpha)u(w))(1 - \rho(E)) > u(w)\rho(E) + u(y)(1 - \rho(E))$ . Thus,

$$\rho(E) < \frac{u(w) - u(y)}{2u(w) - u(x) - u(y)}.$$

Together with (6.8), this implies

$$\rho(E) + \rho(E^c) < \frac{u(w) - u(y)}{u(w) - u(x)} - \frac{u(x) - u(y)}{u(w) - u(x)} = 1,$$

as desired.

Case 2:  $\alpha y + (1 - \alpha)w \succ \alpha w + (1 - \alpha)x$ . By Ellsberg Behavior and (6.7),  $(\alpha u(w) + (1 -$

$\alpha)u(x))(1 - \rho(E^c)) + (\alpha u(y) + (1 - \alpha)u(w))(\rho(E^c)) > u(w)\rho(E^c) + u(x)(1 - \rho(E^c))$ . Thus,

$$\rho(E^c) < \frac{u(w) - u(x)}{2u(w) - u(x) - u(y)}.$$

Together with (6.9), this again implies

$$\rho(E) + \rho(E^c) < \frac{u(w) - u(x)}{u(w) - u(y)} + \frac{u(x) - u(y)}{u(w) - u(y)} = 1,$$

as desired.

Since for this  $E$ ,  $\rho(E) + \rho(E^c) < 1$ , we can now use Theorem 1: Apply Theorem 1, with  $A = E$ ,  $B = E^c$ ,  $R = u(X) \times u(X)$ ,  $h$  an interior constant act, and lines through  $(u(h(E)), u(h(E^c)))$  with slopes  $\frac{1}{4}(-\frac{1-\rho(E)}{\rho(E)}) + \frac{3}{4}(-\frac{\rho(E^c)}{1-\rho(E^c)})$  and  $\frac{3}{4}(-\frac{1-\rho(E)}{\rho(E)}) + \frac{1}{4}(-\frac{\rho(E^c)}{1-\rho(E^c)})$  respectively (if  $\rho(E) = 0$ , replace  $-\frac{1-\rho(E)}{\rho(E)}$  by any finite number  $n$  such that  $n < -\frac{\rho(E^c)}{1-\rho(E^c)}$ ) to conclude that Monotonicity in Mixtures is violated.  $\square$

## 6.4 Implications for the smooth ambiguity model

Are there preferences that can both satisfy Monotonicity in Mixtures and exhibit Ellsberg Behavior? In this section, we show that the answer is yes. To do so, we consider the smooth ambiguity model (Klibanoff, Marinacci and Mukerji, 2005). In our setting, each smooth ambiguity preference can be represented by a functional of the following form:

$$\int \phi \left( \sum_s u(f(s))p(s) \right) d\mu(p), \quad (6.10)$$

where  $u$  is a non-constant von Neumann-Morgenstern utility function,  $\phi$  is a continuous and strictly increasing function and  $\mu$  is a countably additive probability measure over probability measures over states. For some results in this section we further assume that  $\phi$  is twice continuously differentiable on the interior of  $u(X) = [0, \infty)$  with  $\phi' > 0$  and  $\phi'' < 0$ . Such preferences exhibit Ellsberg Behavior if and only if  $\mu$  has a non-singleton support (Proposition 3). We provide an upper bound on the coefficient of ambiguity aversion,  $-\frac{\phi''(x)}{\phi'(x)}$ , that is sufficient and, if  $\mu$  is unrestricted, necessary for such preferences to satisfy Monotonicity in Mixtures (Theorems 3 and 4). Thus, when applied to smooth ambiguity preferences,

Monotonicity in Mixtures is compatible with ambiguity aversion as long as the aversion isn't too strong. From these results we also see that Monotonicity in Mixtures provides a different categorization of ambiguity averse preferences than the first-order versus second-order ambiguity aversion (very roughly, kinked versus smooth) categorization explored in Lang (2017). While our Theorem 1 implies that no first-order ambiguity averse preferences satisfy Monotonicity in Mixtures, Theorems 3 and 4 in this section show that some second-order ambiguity averse preferences satisfy Monotonicity in Mixtures while others do not.

**Proposition 3.** *Preferences represented by the smooth ambiguity model as in (6.10) with  $\phi$  strictly increasing and strictly concave exhibit Ellsberg Behavior if and only if  $\mu$  has a non-singleton support.*

**Proof of Proposition 3:** If  $\mu$  has only one measure in its support, then (6.10) reduces to a strictly increasing transformation of an SEU preference and thus cannot exhibit Ellsberg Behavior. For the other direction, suppose that  $\mu$  has a non-singleton support. Then there exists an event  $A$  s.t.  $\min_{p \in \text{supp}(\mu)} p(A) < \int p(A) d\mu(p) < \max_{p \in \text{supp}(\mu)} p(A)$ . By non-constancy of  $u$ , there are outcomes (i.e., degenerate lotteries)  $\bar{x}, \underline{x}$  such that  $u(\bar{x}) > u(\underline{x})$ . There are two cases to consider:

Case 1:  $\bar{x}A\underline{x} \succsim \underline{x}A\bar{x}$ . To show Ellsberg Behavior, let  $E = A$ ,  $w = \bar{x}$ ,  $y = \underline{x}$ ,  $\alpha = \frac{1}{2}$ , and  $x$  be the lottery  $\lambda\bar{x} + (1 - \lambda)\underline{x}$  such that  $\bar{x}A\underline{x} \sim \underline{x}A\bar{x}$ . Then  $w \succ x \succ y$ , and (6.10) yields  $\alpha wEy + (1 - \alpha)xEw \succ wEy \sim xEw$  since strict concavity of  $\phi$  implies

$$\begin{aligned} & \int \phi \left( \frac{1}{2}u(\bar{x})p(A) + \frac{1}{2}u(\underline{x})(1 - p(A)) + \frac{1}{2}u(x)p(A) + \frac{1}{2}u(\bar{x})(1 - p(A)) \right) d\mu(p) \\ & > \int \left( \frac{1}{2}\phi(u(\bar{x})p(A) + u(\underline{x})(1 - p(A))) + \frac{1}{2}\phi(u(x)p(A) + u(\bar{x})(1 - p(A))) \right) d\mu(p). \end{aligned}$$

Case 2:  $\bar{x}A^c\underline{x} \succsim \underline{x}A^c\bar{x}$ . To show Ellsberg Behavior, let  $E = A^c$ ,  $w = \bar{x}$ ,  $y = \underline{x}$ ,  $\alpha = \frac{1}{2}$ , and  $x$  be the lottery  $\lambda\bar{x} + (1 - \lambda)\underline{x}$  such that  $\bar{x}A^c\underline{x} \sim \underline{x}A^c\bar{x}$ . Then  $w \succ x \succ y$ , and (6.10)

yields  $\alpha wEy + (1 - \alpha)xEw \succ wEy \sim xEw$  since strict concavity of  $\phi$  implies

$$\begin{aligned} & \int \phi \left( \frac{1}{2}u(\underline{x})p(A) + \frac{1}{2}u(\bar{x})(1 - p(A)) + \frac{1}{2}u(\bar{x})p(A) + \frac{1}{2}u(x)(1 - p(A)) \right) d\mu(p) \\ & > \int \left( \frac{1}{2}\phi(u(\underline{x})p(A) + u(\bar{x})(1 - p(A))) + \frac{1}{2}\phi(u(\bar{x})p(A) + u(x)(1 - p(A))) \right) d\mu(p). \end{aligned}$$

□

**Theorem 3.** *Preferences represented by the smooth ambiguity model as in (6.10) with  $\phi$  twice continuously differentiable on the interior of  $u(X) = [0, \infty)$ ,  $\phi' > 0$  and  $\phi'' < 0$  satisfy Monotonicity in Mixtures if  $\phi$  is everywhere at most as concave as natural log:*

$$-\frac{\phi''(a)}{\phi'(a)} \leq \frac{1}{a}, \text{ for all } a > 0.$$

**Proof of Theorem 3:** For  $\alpha \in [0, 1]$ ,  $v \in u(X)^S$  and act  $g \in \mathcal{F}$ , define

$$W^g(\alpha, v) \equiv \int \phi \left( \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha)u(g(\hat{s})))p(\hat{s}) \right) d\mu(p). \quad (6.11)$$

If each of the cross-partial derivatives with respect to  $\alpha$  and the  $s^{th}$  component of  $v$ ,  $W_{\alpha v(s)}^g(\alpha, v)$ , are non-negative for all  $\alpha \in (0, 1)$  and strictly positive  $v$ , then by e.g., Theorem 2 from Milgrom and Roberts (1990),  $W^g$  is supermodular with respect to  $\alpha$  and any component  $s$  of  $v$ . This is sufficient for Monotonicity in Mixtures, since it implies that the increased state-by-state utility from improving  $f$  to a weakly dominating  $f'$  can only increase the desirability of increasing the mixing weight from  $\alpha$  to  $\alpha'$ . We now show that when  $-\frac{\phi''(a)}{\phi'(a)} \leq \frac{1}{a}$ , for all  $a > 0$  these cross-partials are indeed non-negative. By differentiating, we obtain

$$W_{\alpha}^g(\alpha, v) = \int \left( \sum_{\hat{s}} (v(\hat{s}) - u(g(\hat{s})))p(\hat{s}) \right) \phi' \left( \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha)u(g(\hat{s})))p(\hat{s}) \right) d\mu(p)$$

and

$$\begin{aligned}
& W_{\alpha v(s)}^g(\alpha, v) \tag{6.12} \\
&= \int \left[ p(s) \phi' \left( \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha) u(g(\hat{s}))) p(\hat{s}) \right) \right. \\
&\quad \left. + \alpha p(s) \left( \sum_{\hat{s}} (v(\hat{s}) - u(g(\hat{s}))) p(\hat{s}) \right) \phi'' \left( \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha) u(g(\hat{s}))) p(\hat{s}) \right) \right] d\mu(p).
\end{aligned}$$

From  $-\frac{\phi''(a)}{\phi'(a)} \leq \frac{1}{a}$  with  $a = \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha) u(g(\hat{s}))) p(\hat{s}) > 0$ , we obtain that, for all  $p$ ,

$$\begin{aligned}
& \phi' \left( \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha) u(g(\hat{s}))) p(\hat{s}) \right) \tag{6.13} \\
&\geq -\phi'' \left( \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha) u(g(\hat{s}))) p(\hat{s}) \right) \left( \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha) u(g(\hat{s}))) p(\hat{s}) \right).
\end{aligned}$$

Substituting (6.13) into (6.12) yields,

$$W_{\alpha v(s)}^g(\alpha, v) \geq \int \left[ p(s) \phi'' \left( \sum_{\hat{s}} (\alpha v(\hat{s}) + (1 - \alpha) u(g(\hat{s}))) p(\hat{s}) \right) \left( -\sum_{\hat{s}} u(g(\hat{s})) p(\hat{s}) \right) \right] d\mu(p) \geq 0$$

as claimed.  $\square$

**Theorem 4.** *For  $\phi$  twice continuously differentiable on the interior of  $u(X) = [0, \infty)$  with  $\phi' > 0$  and  $\phi'' \leq 0$ , if  $-\frac{\phi''(a)}{\phi'(a)} > \frac{1}{a}$ , for some  $a > 0$ , then there exists a measure  $\mu$  such that preferences represented by (6.10) violate Monotonicity in Mixtures.*

What is the reason for the role of  $\mu$  in Theorem 4? Intuitively, the extent to which higher ambiguity aversion manifests itself in behavior depends on how much ambiguity the individual perceives herself as facing, with more ambiguity leading to stronger effects of any given ambiguity aversion. This suggests that a  $\mu$  reflecting an extremely ambiguous event should be a good candidate for generating the required violation as soon as ambiguity aversion exceeds the tightest possible bound. The proof uses this strategy, constructing the violation using bets on an event assigned only probabilities 1 and 0 by measures in the support of  $\mu$ , with at least some measures assigning each.<sup>2</sup>

<sup>2</sup>According to Jewitt and Mukerji (2017), no event is more ambiguous.

**Proof of Theorem 4:** By assumption,  $\exists a > 0$  s.t.  $-\frac{\phi''(a)}{\phi'(a)} > \frac{1}{a}$  which, since  $\phi'(a) > 0$ , implies  $\phi'(a) + a\phi''(a) < 0$ .

We construct a  $\mu$  that will generate a violation of Monotonicity in Mixtures. Let  $\mu(p_1) = \mu(p_2) = \frac{1}{2}$ , where  $p_1$  and  $p_2$  are probability measures on  $S$  such that  $\exists E \subset S$  with  $p_1(E) = 1$  and  $p_2(E) = 0$ .

Consider the following acts  $f$  and  $g$ :  $f = xEz$  and  $g = zEx$  where  $u(x) = 2a$  and  $u(z) = 0$ .

By concavity of  $\phi$ ,  $\phi(a) \geq \frac{1}{2}\phi(\alpha 2a) + \frac{1}{2}\phi((1-\alpha)2a)$ , which implies that for all  $\alpha \in [0, 1]$ ,

$$\frac{1}{2}f + \frac{1}{2}g \succsim \alpha f + (1-\alpha)g \quad (6.14)$$

Define  $\bar{W}(\alpha, v) \equiv W^{zEx}(\alpha, vE0)$  where  $W^{zEx}(\alpha, vE0)$  is defined as in (6.11) in the proof of Theorem 3 with  $g = zEx$  and  $vE0$  is the vector representing the state-by-state utility of the act  $yEz$  where  $u(y) = v$ . Substituting our constructed  $\mu$ , we get

$$\bar{W}(\alpha, v) = \frac{1}{2}[\phi(\alpha v) + \phi((1-\alpha)2a)].$$

Differentiating w.r.t.  $\alpha$  yields

$$\bar{W}_\alpha(\alpha, v) = \frac{1}{2}[v\phi'(\alpha v) - 2a\phi'((1-\alpha)2a)] \quad (6.15)$$

Further differentiating w.r.t.  $v$ , we get

$$\bar{W}_{\alpha v}(\alpha, v) = \frac{1}{2}[\phi'(\alpha v) + \alpha v\phi''(\alpha v)] \quad (6.16)$$

Now, observe that  $\bar{W}_{\alpha v}(\alpha, v)$  is negative when  $\alpha = \frac{1}{2}$  and  $v = 2a$ :

$$\bar{W}_{\alpha v}(\frac{1}{2}, 2a) = \frac{1}{2}[\phi'(a) + a\phi''(a)] < 0. \quad (6.17)$$

From (6.15),  $\bar{W}_\alpha(\frac{1}{2}, 2a) = 0$ , and thus, by (6.17), there exists a  $b > 2a$  such that

$$\bar{W}_\alpha(\frac{1}{2}, b) < 0. \quad (6.18)$$

Therefore, there exists an  $\hat{\alpha} < \frac{1}{2}$  such that

$$\overline{W}\left(\frac{1}{2}, b\right) < \overline{W}(\hat{\alpha}, b). \quad (6.19)$$

Letting  $f' = yEz$  and setting  $v = b$ , we see that  $f'$  weakly dominates  $f$ . However, while from (6.14)  $\frac{1}{2}f + \frac{1}{2}g \succsim \hat{\alpha}f + (1 - \hat{\alpha})g$ , (6.19) implies  $\hat{\alpha}f' + (1 - \hat{\alpha})g \succ \frac{1}{2}f' + \frac{1}{2}g$ , violating Monotonicity in Mixtures.  $\square$

These results on Monotonicity in Mixtures in the context of the smooth ambiguity model are closely related to work on comparative statics of portfolios of random variables (risky assets) under expected utility. A strand of that literature addresses the question of when any first-order stochastic dominant shift in the (conditional on any realization of the other assets) distribution of an asset will result in a risk-averse expected utility investor increasing that asset's share in the optimal portfolio. The answer is when utility is everywhere at most as concave as natural log (equivalently,  $xu'(x)$  increasing, absolute risk aversion at any  $x \leq \frac{1}{x}$ , or relative risk aversion everywhere at most 1). Fishburn and Porter (1976) showed this for the special case of one risky and one safe asset. Hadar and Seo (1990) extended this result to any two assets with independently distributed returns. Meyer and Ormiston (1994) extended this to the case where returns across assets may be dependent (in which case the shift being to the conditional distribution becomes important). Finally, Mitchell and Douglas (1997) established the result for the case of  $n$  (possibly dependent) assets.

The condition that utility is at most as concave as natural log also appears in the literature on general equilibrium with additively separable utilities, where it has been identified as leading demand for contingent goods to have the gross substitutes property and to existence of a unique equilibrium (see Dana (2001) for a survey). Returning to the consumer theory analogy, for additively separable utilities over consumption goods, natural log utility leads income and substitution effects to just offset each other, further suggesting a connection with Monotonicity in Mixtures.

## 6.5 More general implications: Bounds on slopes in utility space at different points

Our result (Theorem 1) ruling out Ellsberg Behavior-generating kinks shows how multiple slopes “locally” supporting (i.e., supporting within an appropriate rectangle) the better-than set at the *same* point in two-dimensional utility space generates a violation of Monotonicity in Mixtures. One lesson from Theorem 4, however, is that such kinks are not necessary to generate conflict with Monotonicity in Mixtures. Here we show that arguments similar to those in Theorem 1 can be used, even under smoothness, to provide implications of Monotonicity in Mixtures for how slopes of the “local” supports at *distinct* points relate. Specifically, our next result (Theorem 5) shows that Monotonicity in Mixtures bounds how different the slope of a “local” support of the better-than set at one point in utility space can be from the slope of such a support at another point. Monotonicity in Mixtures is necessarily violated whenever these slopes change “too fast”.

The specific form of the bound derived in Theorem 5 delivers useful insights. The bound is on the ratio of these slopes and becomes tighter as the points are translated up by adding positive constants (i.e., moved parallel to the 45 degree line). When utility is unbounded above, so that these positive constants may be taken to infinity, this ratio bound converges to one. Satisfying this bound therefore implies that absolute ambiguity aversion eventually disappears in the sense that for any bounded rectangle of points, translating this rectangle up by adding large enough positive constants makes any convexity of indifference curves within the translated rectangle arbitrarily uniformly small. In particular, if preferences are convex (i.e., satisfy Schmeidler’s Uncertainty Aversion), then the bound implies that preferences eventually approach expected utility (and thus ambiguity neutrality) in any such region of large enough stakes. Thus one implication of Theorem 5 is that any Variational preferences (Maccheroni, Marinacci and Rustichini, 2006) (which are the constant absolute ambiguity averse preferences satisfying Uncertainty Aversion (Grant and Polak, 2013)) that display Ellsberg Behavior must violate Monotonicity in Mixtures.<sup>3</sup>

The statement of the theorem makes use of the following definition, giving conditions

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<sup>3</sup>As some Variational preferences have no kinks, this could not have been established using our earlier result, Theorem 1.



for a rectangle and two support lines in two-dimensional utility space to form a *support configuration* (as illustrated in Figure 6.4).

**Definition 3.** Let  $\succsim$  be represented by  $V((u(f(s)))_{s \in S})$  as in (3.1). Fix any disjoint, non-empty events  $A, B \subseteq S$  and interior acts  $h, k \in \mathcal{F}^{A, B, h}$  such that  $k(A) \succsim h(A)$  and  $h(B) \succsim k(B)$ . Given a rectangle  $R \subseteq u(X) \times u(X)$  containing  $(u(h(A)), u(h(B)))$  and  $(u(k(A)), u(k(B)))$  in its interior and two distinct lines  $l_h$  and  $l_k$  in  $u(X) \times u(X)$ ,  $(R, l_h, l_k)$  is a *support configuration* if

- (i)  $l_h$  intersects  $G(h) \cap R$  only at  $(u(h(A)), u(h(B)))$  and has an intersection with the left edge of rectangle  $R, \{(q(A), q(B)) \in R : q(A) = \min_{r \in R} r(A)\}$ ,
- (ii)  $l_k$  intersects  $G(k) \cap R$  only at  $(u(k(A)), u(k(B)))$  and has an intersection with the bottom edge of rectangle  $R, \{(q(A), q(B)) \in R : q(B) = \min_{r \in R} r(B)\}$ ,
- (iii) the line,  $l_3$ , passing through  $(u(k(A)), u(h(B)))$  and the intersection of  $l_h$  with the left edge of rectangle  $R$ , has an intersection with the bottom edge of rectangle  $R$ .

The definition requires that the rectangle  $R$  is small enough so that, within  $R$ , the lines  $l_h$  and  $l_k$  intersect the upper sets  $G(h)$  and  $G(k)$  only at  $u(h)$  and  $u(k)$ , respectively, and large enough so that the lines  $l_h$  (respectively,  $l_k$  and  $l_3$ ) intersect the left (respectively, bottom) side of  $R$ . The theorem gives a bound that Monotonicity in Mixtures imposes on the ratio of the slopes of the lines  $l_h$  and  $l_k$  in any support configuration.

**Theorem 5.** Let  $\succsim$  be represented by  $V((u(f(s)))_{s \in S})$  as in (3.1).

Fix any disjoint, non-empty events  $A, B \subseteq S$  and interior acts  $h, k \in \mathcal{F}^{A, B, h}$  such that  $k(A) \succsim h(A)$  and  $h(B) \succsim k(B)$ .

If  $\succsim$  satisfies Monotonicity in Mixtures, then

$$\frac{\text{Slope}(l_h)}{\text{Slope}(l_k)} \leq \frac{u(k(A)) - \min_{r \in R} r(A)}{u(h(A)) - \min_{r \in R} r(A)} \times \frac{u(h(B)) - \min_{r \in R} r(B)}{u(k(B)) - \min_{r \in R} r(B)}, \quad (6.20)$$

for all support configurations  $(R, l_h, l_k)$ .

When  $V$  is differentiable in the utility values, (6.20) may be written entirely in terms

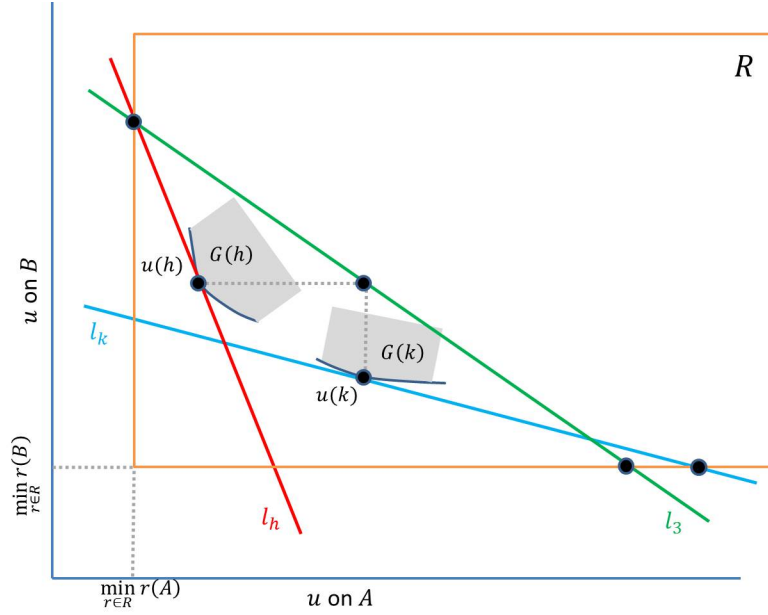


Figure 6.4: A support configuration.

of  $V$  and  $u$ , since

$$\text{Slope}(l_h) = -\frac{V_A}{V_B} \Big|_{u(h)}$$

and

$$\text{Slope}(l_k) = -\frac{V_A}{V_B} \Big|_{u(k)}$$

where for  $E \subseteq S$ ,  $V_E$  denotes the derivative of  $V$  with respect to the utility attained on the event  $E$ .

It is worth noting that if preferences are uncertainty averse (i.e., convex) when restricted to acts in  $\mathcal{F}^{A,B,h}$  and  $u(X) = [0, \infty)$ , then support configurations will always exist and the only rectangles  $R$  that need be considered are those with  $\min_{r \in R} r(A) = \min_{r \in R} r(B) = 0$ . In this case, the right-hand side of the bound (6.20) is unchanged as utilities are multiplicatively scaled up. In this sense, Monotonicity in Mixtures requires that such preferences not display too much relative ambiguity aversion anywhere. Note that the bound will be satisfied with equality at all points for preferences that are Cobb-Douglas in utility space when restricted to  $\mathcal{F}^{A,B,h}$  (i.e., preferences representable on  $\mathcal{F}^{A,B,h}$  by  $u(f(A))^\gamma u(f(B))^{1-\gamma}$  or, equivalently,  $\gamma \ln(u(f(A))) + (1 - \gamma) \ln(u(f(B)))$  for some  $\gamma \in (0, 1)$ ). Thus the bound can be viewed as saying that preferences (on these two-dimensional slices) must be every-

where at most as convex as Cobb-Douglas. This connection of Monotonicity in Mixtures with Cobb-Douglas for uncertainty averse preferences is a more general manifestation of the connection with natural log (and the corresponding connection with the gross substitutes property) we found for uncertainty averse smooth ambiguity preferences in Section 6.4. In particular, with the  $\mu$  used in the proof of Theorem 4,  $\phi = \ln$  yields Cobb-Douglas restricted to  $\mathcal{F}^{E,E^c,h}$  and the smooth ambiguity preferences restricted to  $\mathcal{F}^{E,E^c,h}$  are at most as convex as Cobb-Douglas if and only if  $\phi$  is at most as concave as natural log.

**Proof of Theorem 5:** Fix  $V, u, A, B, h, k$  and a support configuration  $(R, l_h, l_k)$  as in the statement of the theorem. Monotonicity of  $V$  implies that, for any  $f \in \mathcal{F}^{A,B,h}$  all points  $(a, b) \in u(X) \times u(X)$  such that  $a \geq u(f(A))$  and  $b \geq u(f(B))$  lie in  $G(f)$ . Therefore, since  $l_h$  intersects  $G(h) \cap R$  only at  $(u(h(A)), u(h(B)))$  and  $l_k$  intersects  $G(k) \cap R$  only at  $(u(k(A)), u(k(B)))$ , it follows that both  $l_h$  and  $l_k$  must have negative and finite slopes, as does  $l_3$  by construction. Name the acts in  $\mathcal{F}^{A,B,h}$  corresponding to the intersections of  $l_h, l_k, l_3$  with the edges of  $R$  as specified in (i),(ii) and (iii) of the definition of support configuration by  $i, j$  and  $m$  respectively. (See Figure 6.5)

Observe that the slope of  $l_3$  is the following:

$$\begin{aligned} \text{Slope}(l_3) &= -\frac{u(i(B)) - u(h(B))}{u(k(A)) - u(i(A))} = -\frac{u(i(B)) - u(h(B))}{u(h(A)) - u(i(A))} \times \frac{u(h(A)) - u(i(A))}{u(k(A)) - u(i(A))} \\ &= \text{Slope}(l_h) \times \frac{u(h(A)) - u(i(A))}{u(k(A)) - u(i(A))}. \end{aligned} \quad (6.21)$$

Now let  $l_4$  be the line passing through  $m$  and  $k$ . The slope of  $l_4$  is:

$$\begin{aligned} \text{Slope}(l_4) &= -\frac{u(k(B)) - u(j(B))}{u(m(A)) - u(k(A))} = -\frac{u(h(B)) - u(j(B))}{u(m(A)) - u(k(A))} \times \frac{u(k(B)) - u(j(B))}{u(h(B)) - u(j(B))} \\ &= \text{Slope}(l_3) \times \frac{u(k(B)) - u(j(B))}{u(h(B)) - u(j(B))}. \end{aligned} \quad (6.22)$$

Combining (6.21) and (6.22) yields,

$$\text{Slope}(l_4) = \text{Slope}(l_h) \times \frac{u(h(A)) - u(i(A))}{u(k(A)) - u(i(A))} \times \frac{u(k(B)) - u(j(B))}{u(h(B)) - u(j(B))}. \quad (6.23)$$

If  $\succsim$  satisfies Monotonicity in Mixtures, then, since by construction  $h$  is the unique

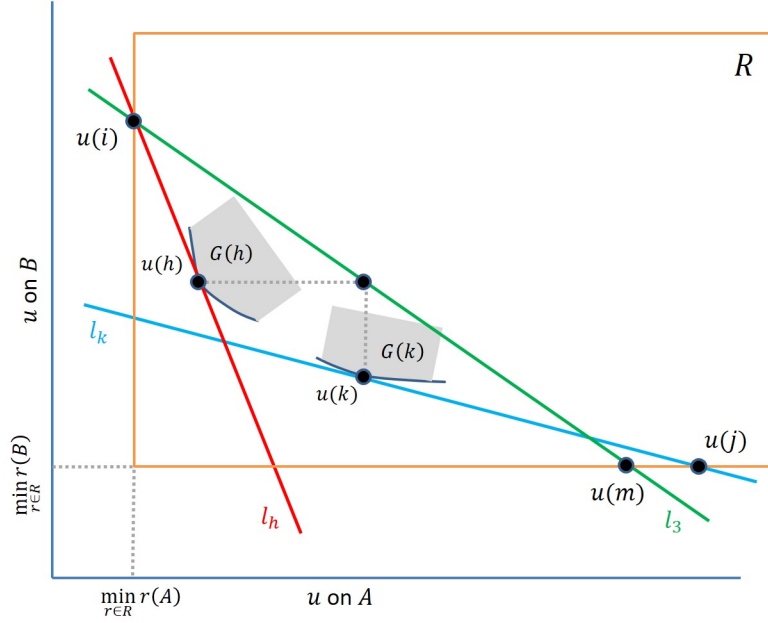


Figure 6.5:  $i, j$  and  $m$

optimum on  $l_h \cap R$ , it must be that all optima on  $l_3 \cap R$  have vertical coordinates weakly below  $u(h(B))$ . Applying Monotonicity in Mixtures a second time yields that, since all optima on  $l_3 \cap R$  have vertical coordinates weakly below  $u(h(B))$ , all optima on  $l_4 \cap R$  have vertical coordinates weakly below  $u(k(B))$ . From this, we will now show that if  $Slope(l_4) < Slope(l_k)$  (equivalently,  $u(m)$  is to the left of  $u(j)$ ) then Monotonicity in Mixtures must be violated (Figure 6.6). To see this, suppose to the contrary that  $Slope(l_4) < Slope(l_k)$  and Monotonicity in Mixtures is satisfied. Since  $k$  is uniquely optimal on  $l_k \cap R$  (by the assumptions of the theorem),  $V$  is monotonic and  $Slope(l_4) < Slope(l_k)$ ,  $k \succ f$  for all  $(u(f(A)), u(f(B))) \in l_4 \cap R$  having vertical coordinate strictly below  $u(k(B))$  (as each such  $(u(f(A)), u(f(B)))$  is weakly dominated by some point on  $l_k \cap R$  that is strictly worse than  $(u(k(A)), u(k(B)))$ ). Thus, since we showed above that Monotonicity in Mixtures implies that all optima on  $l_4 \cap R$  have vertical coordinates weakly below  $u(k(B))$ , it follows that  $k$  is strictly optimal on  $l_4 \cap R$ . However,  $k, R, l_k$  and  $l_4$  now satisfy the conditions of Theorem

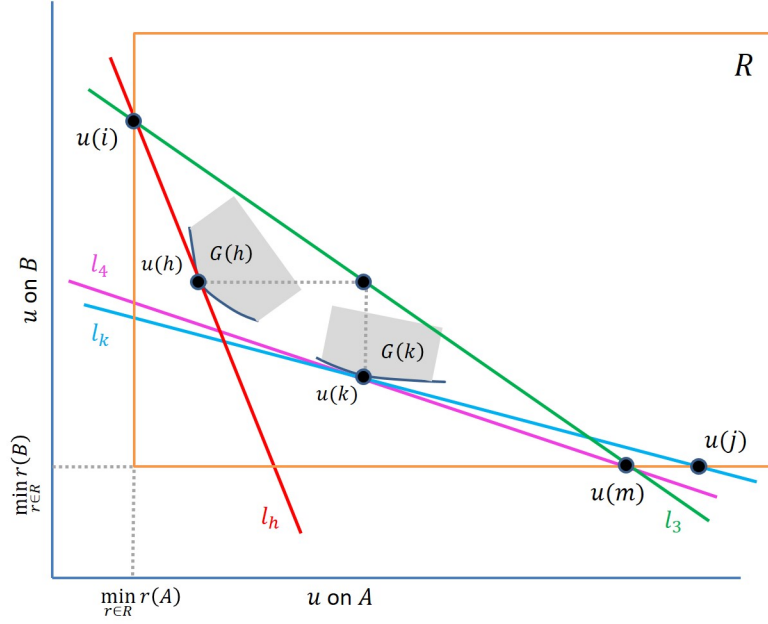


Figure 6.6: Violation of the slope bound (6.20)

1 and therefore imply Monotonicity in Mixtures is violated, a contradiction. Thus, we have shown that Monotonicity in Mixtures implies

$$\frac{\text{Slope}(l_4)}{\text{Slope}(l_k)} \leq 1. \quad (6.24)$$

Applying (6.23) yields

$$\frac{\text{Slope}(l_4)}{\text{Slope}(l_k)} = \frac{\text{Slope}(l_h)}{\text{Slope}(l_k)} \times \frac{u(h(A)) - u(i(A))}{u(k(A)) - u(i(A))} \times \frac{u(k(B)) - u(j(B))}{u(h(B)) - u(j(B))}.$$

Therefore, Monotonicity in Mixtures implies

$$\frac{\text{Slope}(l_h)}{\text{Slope}(l_k)} \leq \frac{u(k(A)) - u(i(A))}{u(h(A)) - u(i(A))} \times \frac{u(h(B)) - u(j(B))}{u(k(B)) - u(j(B))}. \quad (6.25)$$

Since  $u(i(A)) = \min_{r \in R} r(A)$  and  $u(j(B)) = \min_{r \in R} r(B)$ , (6.25) is the inequality (6.20).  $\square$

## 7 Extensions

### 7.1 Monotonicity in Optimal Mixtures

The following is a weakening of Monotonicity in Mixtures to have a more directly comparative static flavor:

**Axiom 7. *Monotonicity in Optimal Mixtures:*** For all acts  $f, f', g$  such that  $f'(s) \succsim f(s)$  for all  $s \in S$ ,  $\alpha^* f + (1 - \alpha^*)g$  optimal in  $\{\alpha f + (1 - \alpha)g : \alpha \in [0, 1]\}$  implies there exists an  $\alpha' \geq \alpha^*$  such that  $\alpha' f' + (1 - \alpha')g$  is optimal in  $\{\alpha f' + (1 - \alpha)g : \alpha \in [0, 1]\}$ .

It is immediate that Monotonicity in Mixtures implies Monotonicity in Optimal Mixtures, so this is potentially a weakening of Monotonicity in Mixtures. However, by inspection of our earlier proofs, one can verify that all of our results providing conditions under which Monotonicity in Mixtures is violated also show violations of Monotonicity in Optimal Mixtures. Thus, this is an alternative condition we could have used in our analysis. Note also that the analogy with the gross substitutes property from consumer theory made in several places in the paper relied only on Monotonicity in Optimal Mixtures.

### 7.2 Further directions

In addition to the smooth ambiguity model, there are other models of preferences capable of displaying Ellsberg Behavior in the literature while sometimes also satisfying Monotonicity in Mixtures. For example, the Confidence preference model of Chateauneuf and Faro (2009), which imposes constant relative ambiguity aversion, can do so in some instances. As of yet, we have not been able to find nice conditions under which this occurs, but we think it would be interesting to do so.

Ellsberg Behavior is, as was mentioned, much weaker than typical global ambiguity aversion/uncertainty aversion conditions in the literature. Mukerji and Tallon (2003) (see also Higashi et al., 2008) have an axiom A1 that is also meant to be a weak/local condition, and it may be interesting to explore the relationship between A1 and Ellsberg Behavior.

In applications using specific preferences that our results show violate Monotonicity in

Mixtures, it may be interesting to explore exactly which types of improvements do or do not generate such violations. For instance, the violation of Monotonicity in Mixtures used in the proof of Theorem 4 involves an improvement in act  $f$  that makes  $f'$  in a natural sense more ambiguous than  $f$ . One can verify that one would not obtain a violation of Monotonicity in Mixture if  $f'$  were constructed by improving the payoff of  $f$  on event  $E^c$  which would make  $f'$  less ambiguous. Future research might try to build this intuition into a formal characterization (possibly tailored to specific preference models) of which improvements of  $f$  would generate violations of Monotonicity in Mixtures. This would link to, for example, comparative statics of portfolio choice under risk and ambiguity.

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