Risk sharing example: Pooling Investments

A friend and I have initial wealth W_a and W_b; r_a and r_b are our coefficients of absolute risk aversion.

We own portfolios which have a expected returns of I_a and I_b, and variances $Var(I_a)$ and $Var(I_b)$, respectively..

Let ? be a cash transfer from me (indexed as a) to my friend (indexed as b).

Q: How do we efficiently split these streams of income?

To be determined:

How do we split the returns from each investment?
Will there be side payments between us? What will they be?

Let a be my share of I_a, let ß be my share of I_b.

Let ? be a cash transfer from me (indexed as a) to my friend (indexed as b).

My utility/CEQ: W_a - ? + a I_a + ß I_b + ½ r_a var(aI_a) + ½ r_a var ($ßI_b$)
$$= W_a - ? + a\ I_a + ß\ I_b + ½\ r_a\ [\ a^2\ var(I_a) + ß^2\ var\ (I_b)\ ]$$

Friend's utility/CEQ: W_b + ? + (1-a) I_a + (1-ß) I_b + ½ r_b [$(1-a)^2$ var(I_a) + $(1-ß)^2$ var(I_b)]

Total value: $W_a + W_b + I_a + I_b + Prem_a + Prem_b$, where

$$Prem_a \ ' \ 1/2\ var(I_a)\ [r_a a^2\ \%\ r_b(1\&a)^2]$$
$$Prem_b \ ' \ 1/2\ var(I_b)\ [r_a ß^2\ \%\ r_b(1\&ß)^2]$$

Note:

C   ? drops out: transfer does not affect efficiency

C   a, ß only enter through risk premia

Choose a, ß to maximize total value:

$$\frac{\partial TV}{\partial a} \; = \; r_a a \; var(I_a) \; \& \; r_b(1\&a) \; var(I_a) \; = \; 0$$

$$\frac{a}{1\&a} \; = \; \frac{r_b}{r_a}$$

$$\frac{\partial TV}{\partial ß} \; = \; 0 \quad 6 \quad \frac{ß}{1\&ß} \; = \; \frac{r_b}{r_a}$$

Therefore,

$$a \; = \; ß \; = \; \frac{r_b/r_a}{1 \; \% \; r_b/r_a}$$

Efficiency dictates:

C   Higher r_b/r_a (more risk-tolerant I am relative to my friend), more of the share I should take.

C   Share does not depend on the riskiness of the stream.

C   Does not depend on who initially owns which stream.

C   Risk-neutral person bears entire risk.

C   *Efficiency gains arise because pooling allows us to allocate risk so that it imposes least costs.*

Individual rationality dictates:

C   ? must be chosen such that each of us does at least as well as we would if we did not pool our investments.

C   range of possible values that satisfy individual rationality.