B Technical appendix: Analysis of the enforcement of $N > 2$ agents

In the main text we restrict attention to the case in which the enforcement agency oversees two agents, and has the resources to investigate just one of them. In Section 5 we discuss how our results would change if instead the enforcement agency oversaw $N > 2$ agents, while still possessing the resources to investigate just one agent. In this Appendix we establish several results (omitted from the main text) that are required for this generalization. These results also cover the case in which $N = 2$ but assumption (2) does not hold.

Most of our analysis is conducted in terms of the investigation probability function q. As we discussed in the main text, this function can be defined for N agents also. With N agents, the function q is still increasing, and has the same comparative statics with respect to precision h. The only other properties of q used in the analysis are that (A) when $N = 2$, it satisfies relation (1), and (B) it is convex over negative values and concave over positive values (see Lemma 4). To summarize, with respect to (A) we show that for $N > 2$ the equality (1) becomes an analogous inequality, which is still sufficient to imply the existence of a pure-strategy symmetric equilibrium; and with respect to (B), our main results all hold even when q does not possess these curvature properties, provided that a_M is close to 1/2.

B.1 Generalizing equality (1)

On a formal level, our analysis makes repeated use of Lemma 1. This result relies in turn on the inequality

$$p(1, a_M) - p(1, 1) \geq p(a_M, a_M) - p(a_M, 1). \qquad \text{(B-1)}$$

When $N = 2$, equation (1) implies that inequality (B-1) holds with equality for any investigation policy.

When $N > 2$ more work is needed to establish that (B-1) is satisfied. Now, an investigation policy defines a function $p : [0,1]^N \to [0,1]$, giving the probability that agent 1 is investigated as a function of an N-vector of action choices. Exactly as in the case of $N = 2$, it can be shown that the investigation policy "investigate the highest signal" maximizes $p(a, 0, \ldots, 0)$ and minimizes $p(a, 1, \ldots, 1)$. The proof of this result closely parallels that of Lemma 3 in the current paper, but requires considerable extra notation. The proof is included in an earlier working version of our paper, a copy of which is available upon request.

As such, "investigate the highest signal" is the investigation policy that maximizes the probability that no crime is an equilibrium, and minimizes the probability that severe crime is an equilibrium. Under this policy, for any pair of action choices a and a' the value $p(a', a, \ldots, a)$ depends only on the difference between a' and a. As before, we define a function q by $q(a' - a) = p(a', a, \ldots, a)$. In terms of q, inequality (B-1) is

$$q(1 - a_M) - q(0) \geq q(0) - q(a_M - 1). \tag{B-2}$$

Lemma B-1 *For any $N \geq 2$, inequality (B-2) holds.*

Given Lemma B-1, the equilibrium characterization is exactly as in the main text.

B.2 Dispensing with the curvature properties of q

In the main text we show that when $N = 2$ and assumption (2) holds, marginal penalties deliver an initial benefit under the crime wave selection rule, and an initial cost under the no crime wave selection rule when (4) holds. Below, we show that provided a_M is in the neighborhood of 1/2, this result extends to the general case in which $N \geq 2$ and assumption (2) is not required to hold.

Crime wave selection rule

From Proposition 4 and Figure 2, the conclusion that marginal benefits initially help under the crime wave selection rule requires

$$S > \frac{S a_M q(0)}{(1-a_M) q(0) + q(a_M - 1) a_M}.$$

Clearly this inequality is satisfied for all a_M that are not too much greater than 1/2.[24]

No crime wave selection rule

It is useful to rewrite each of the constraints (IC0-1), (IC0-M) and (ICM-1) as upper bounds on the taste-for-crime parameter λ:

$$\lambda \leq S q(1) \tag{IC0-1}$$

$$\lambda \leq \frac{s_M q(a_M)}{a_M} \tag{IC0-M}$$

$$\lambda \leq \frac{S q(1-a_M) - s_M q(0)}{1-a_M} \tag{ICM-1}$$

Conditions (IC0-M) and (ICM-1) are equivalent if and only if $s_M = \frac{S a_M q(1-a_M)}{(1-a_M) q(a_M) + a_M q(0)}$. For higher values of s_M, condition (ICM-1) implies (IC0-M).

Likewise, conditions (IC0-1) and (IC0-M) are equivalent if and only if $s_M = \frac{S a_M q(1)}{q(a_M)}$. For higher values of s_M, condition (IC0-1) implies (IC0-M), while for lower values (IC0-M) implies (IC0-1).

Observe that after substituting in for $\lambda_{NCW} = S q(1)$, the condition for marginal penalties to have an initial cost in the $N = 2$ case under the no crime wave selection rule — condition (4) — is equivalent to

$$\frac{S a_M q(1-a_M)}{(1-a_M) q(a_M) + a_M q(0)} < \frac{S a_M q(1)}{q(a_M)}.$$

[24] For the case of assumption (2) and $N = 2$, the inequality holds for all values of a_M: by footnote 13, $\frac{S a_M q(0)}{q(a_M - 1)} < S$, and from the main text, $\frac{S a_M q(0)}{(1-a_M) q(0) + q(a_M - 1) a_M}$ lies below $\frac{S a_M q(0)}{q(a_M - 1)}$.

Finally, we show that this condition implies that marginal penalties have an initial cost in the general N case under the no crime wave selection rule — exactly as in the $N = 2$ case.

First, observe that at $s_M = \frac{S a_M q(1)}{q(a_M)}$, condition (ICM-1) implies (IC0-1). Consequently, the same is true for higher s_M values. Hence when $s_M > \frac{S a_M q(1)}{q(a_M)}$, either (IC0-1) is satisfied, in which case (IC0-M) is also and no crime is the equilibrium outcome; or else (IC0-1) fails, in which case (ICM-1) fails also, implying that severe crime is the unique equilibrium outcome.

Second, consider any s_M above $\frac{S a_M q(1-a_M)}{(1-a_M)q(a_M)+a_M q(0)}$ but below $\frac{S a_M q(1)}{q(a_M)}$. If (IC0-M) holds then (IC0-1) holds also, implying that no crime is the equilibrium outcome. If instead (IC0-M) fails then (ICM-1) fails also, implying that severe crime is the unique equilibrium outcome.

Provided that $\frac{S a_M q(1-a_M)}{(1-a_M)q(a_M)+a_M q(0)} < S$, it follows that, as claimed, the adoption of marginal penalties has an initial cost whenever condition (4) holds. This inequality trivially holds in the neighborhood of $a_M = 1/2$.[25]

B.3 The effect of changing N

Finally, in Section 5 we claim that the ratio $q_N(x)/q_N(0)$ is decreasing (respectively, increasing) in the number of agents N if $x < 0$ (respectively, $x > 0$). That is, as N increases the probability of investigation decreases faster for an agent who commits a lesser crime, holding the actions of other agents fixed. Here, we establish this result:

Lemma B-2 *Suppose that the noise term ε is either normally distributed, or has a density function f such that $\frac{f(\varepsilon)}{f(x+\varepsilon)}$ is bounded. Then $q(x)/q(0)$ is a decreasing (increasing) function N when $x < 0$ ($x > 0$).*

[25] For the case of assumption (2) and $N = 2$, the inequality holds for all values of a_M: by footnote 14, $q(0) > a_M q(1-a_M)$, and so the denominator $(1-a_M)q(a_M) + a_M q(0)$ exceeds $a_M q(1-a_M)$.

B.4 Mathematical proofs

Proof of Lemma B-1: Let ξ be the highest realization of the $N-1$ signals $\varepsilon^2, \ldots, \varepsilon^N$. Let G and g denote the distribution and density function of ξ. Thus for any a,

$$q(a) = \int \Pr\left(a + \frac{\varepsilon}{h} \geq \frac{\xi}{h}\right) f(\varepsilon)\, d\varepsilon = \int G(ha + \varepsilon) f(\varepsilon)\, d\varepsilon,$$

and so

$$q'(a) = h \int g(ha + \varepsilon) f(\varepsilon)\, d\varepsilon = h \int g(\varepsilon) f(\varepsilon - ha)\, d\varepsilon.$$

To establish (B-2) it suffices to show that $q'(a) \geq q'(-a)$. For this, it suffices to show that

$$g(\varepsilon) f(\varepsilon - ha) \geq g(\varepsilon - ha) f(\varepsilon),$$

for which in turn it suffices to show that

$$(\ln g)' \geq (\ln f)'.$$

In general, $G(x) = F(x)^{N-1}$ and so $g(x) = (N-1) F(x)^{N-2} f(x)$. Since certainly $(\ln F)' > 0$, the result follows. ∎

Proof of Lemma B-2: For expositional ease we prove the result for $h = 1$. The general case is identical. Clearly $q(0) = 1/N$, while for any $x \in \Re$

$$q(x) = \int_{-\infty}^{\infty} F(x + \varepsilon)^{N-1} f(\varepsilon)\, d\varepsilon.$$

Thus

$$\frac{q(x)}{q(0)} = \int_{-\infty}^{\infty} N F(x + \varepsilon)^{N-1} f(x + \varepsilon) \frac{f(\varepsilon)}{f(x + \varepsilon)} d\varepsilon.$$

Integration by parts gives

$$\frac{q(x)}{q(0)} = \left[F(x + \varepsilon)^N \frac{f(\varepsilon)}{f(x + \varepsilon)}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F(x + \varepsilon)^N \frac{\partial}{\partial \varepsilon} \left(\frac{f(\varepsilon)}{f(x + \varepsilon)}\right) d\varepsilon.$$

For $x < 0$ the likelihood ratio $\frac{f(\varepsilon)}{f(x+\varepsilon)}$ is decreasing in ε. To evaluate the first term in the expression above, we need to evaluate

$$\lim_{\varepsilon \to -\infty} F(x+\varepsilon)^N \frac{f(\varepsilon)}{f(x+\varepsilon)}.$$

If $\frac{f(\varepsilon)}{f(x+\varepsilon)}$ is bounded above, this term is clearly zero. Otherwise, further conditions are required. L'Hôpital's rule gives

$$\lim_{\varepsilon \to -\infty} F(x+\varepsilon)^N \frac{f(\varepsilon)}{f(x+\varepsilon)} = \lim_{\varepsilon \to -\infty} \frac{f(x+\varepsilon) NF(x+\varepsilon)^{N-1}}{\frac{\partial}{\partial \varepsilon}\left(\frac{f(x+\varepsilon)}{f(\varepsilon)}\right)},$$

provided the righthand side exists. When the noise term is distributed normally,

$$\frac{f(x+\varepsilon)}{\frac{\partial}{\partial \varepsilon}\left(\frac{f(x+\varepsilon)}{f(\varepsilon)}\right)} = \frac{\exp\left(-\frac{1}{2\sigma^2}(x+\varepsilon)^2\right)}{\frac{\partial}{\partial \varepsilon}\exp\left(-\frac{1}{2\sigma^2}(x^2+2x\varepsilon)\right)} = \frac{\exp\left(-\frac{1}{2\sigma^2}\left((x+\varepsilon)^2-(x^2+2x\varepsilon)\right)\right)}{-\frac{x}{\sigma^2}} \to 0 \text{ as } \varepsilon \to -\infty.$$

So provided either $\frac{f(\varepsilon)}{f(x+\varepsilon)}$ is bounded above, or $\lim_{\varepsilon \to -\infty} \frac{f(x+\varepsilon)}{\frac{\partial}{\partial \varepsilon}\left(\frac{f(x+\varepsilon)}{f(\varepsilon)}\right)} = 0$,

$$\frac{q(x)}{q(0)} = \lim_{\varepsilon \to \infty} \frac{f(\varepsilon)}{f(x+\varepsilon)} - \int_{-\infty}^{\infty} F(x+\varepsilon)^N \frac{\partial}{\partial \varepsilon}\left(\frac{f(\varepsilon)}{f(x+\varepsilon)}\right) d\varepsilon.$$

For $x < 0$ the term $\frac{\partial}{\partial \varepsilon}\left(\frac{f(\varepsilon)}{f(x+\varepsilon)}\right)$ is everywhere negative, since f is log concave and so $\ln f(\varepsilon) - \ln f(x+\varepsilon)$ is a decreasing function of ε. It follows that the ratio $q(x)/q(0)$ is decreasing in N for $x < 0$. ∎