Portfolio Risk and Return

The expected return on a portfolio is a weighted average of the individual returns with the portfolio proportions used as weights. The portfolio proportion of stock i is

$$w_i = \frac{\text{dollars spent on stock i}}{\text{total dollars invested in the portfolio}}$$

Example:

Stock	Expected Return	Price/ Share	# of Shares	$ Value	Portfolio Proportion
GE	21%	$40	100	$4000	$4000/$10,000 =.40
MSFT	18%	$60	100	$6000	$6000/$10,000=.60

$$ER = (.40)(.21) + (.60)(.18) = .193$$

Portfolio Risk and Return

In general the expected return is equal to

$$ER = \sum_{i=1}^{N} w_i r_i$$

Where w = stock i's portfolio weight
r_i = the expected return on stock i

Variance of a Portfolio

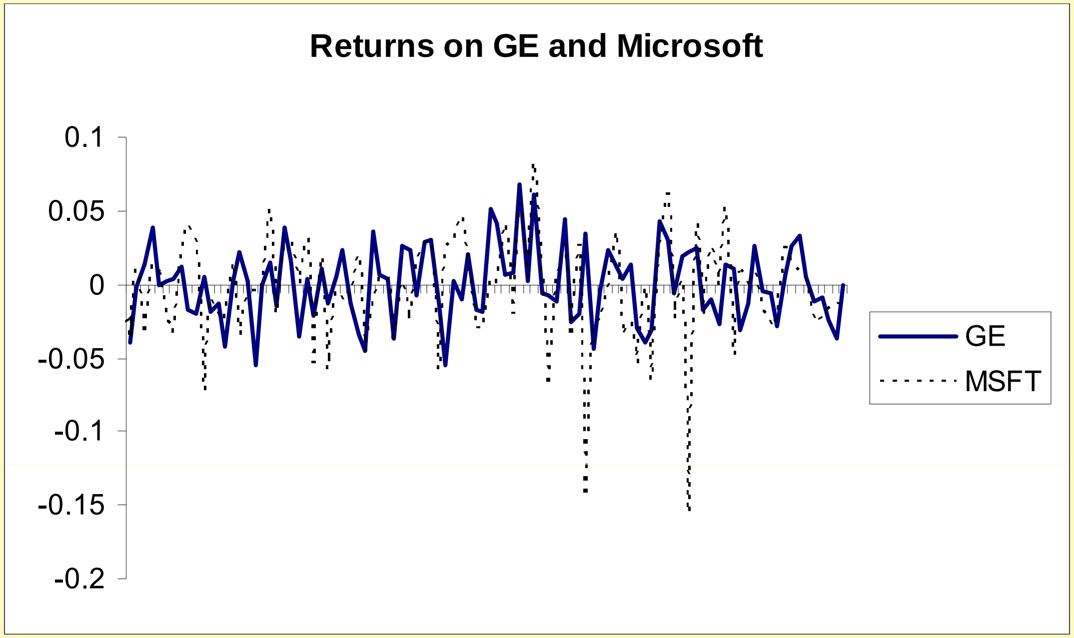
To compute the variance of a portfolio, we need the variance of the individual stocks and the portfolio proportions. In addition, we the **correlation** between the different stocks in the portfolio. Correlation measures how closely the returns on a pair of stocks moves together. Correlation, which is denoted ρ, can take on values between -1 and 1

Examples:

- $\rho = -1$
  - Given the return on one stock, the return on the other can be perfectly predicted and the stocks move inversely.
- $\rho = 0$
  - Knowing the return on one stock tells you nothing about the return on the other
- $\rho = 1$
  - Given the return on one stock, the return on the other can be perfectly predicted and the stocks move in the same direction.

Variance of a Portfolio

Example : Correlation between the return on GE and the return on MSFT



The correlation between the returns for GE and the returns for Microsoft is 34%.

Variance of a Portfolio

Two assets with

- variances σ_1 and σ_2
- portfolio weights w_1 and $w_2 = (1-w_1)$
- correlation ρ_{12}

The Variance of the portfolio is:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$

Variance of a Portfolio

Example:

	w	ER	Stdev
GE	.40	21%	18%
MSFT	.60	18%	15%

Correlation = 34%

$$\sigma_p^2 = (.40)^2(.18)^2 + (.60)^2(.15)^2 + 2(.40)(.60)(.18)(.15)(.34) = .01769$$

$$\sigma_p = \sqrt{\sigma_p^2} = \sqrt{.01769} = .133 = 13.3\%$$

Note that the standard deviation of the portfolio is lower than the standard deviation of the individual stocks!
This is **diversification**. Because the stocks are not perfectly correlated, the changes in returns sometimes offset each other which smoothes out the portfolio return.

Variance of a Portfolio

The formula for the variance of a portfolio when there are more than two assets is:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

The Optimal Allocation of Investment Dollars Between Two Assets

Given two assets with variances σ_1 and σ_2, respectively, and correlation ρ_{12}, what is the best way to allocate your funds between the two assets?

- Let's return to our previous example. Suppose we can allocate our dollars between GE and MSFT.
- Assume, as before, the standard deviation of GE is 18%, the standard deviation of MSFT is 15% and the correlation between the two is .34.
- We want to vary the portfolio proportions to determine which distribution is the best. Let w_{GE} and w_{MSFT} denote the portfolio proportions.
- Note that $w_{GE} + w_{MSFT} = 1$. This just means that you can't spend more than 100% of your wealth.

The Optimal Allocation of Investment Dollars Between Two Assets

The following calculation shows what happens to the ER and StDev of the portfolio as we vary the fraction invested in GE and MSFT. The formulas used are:

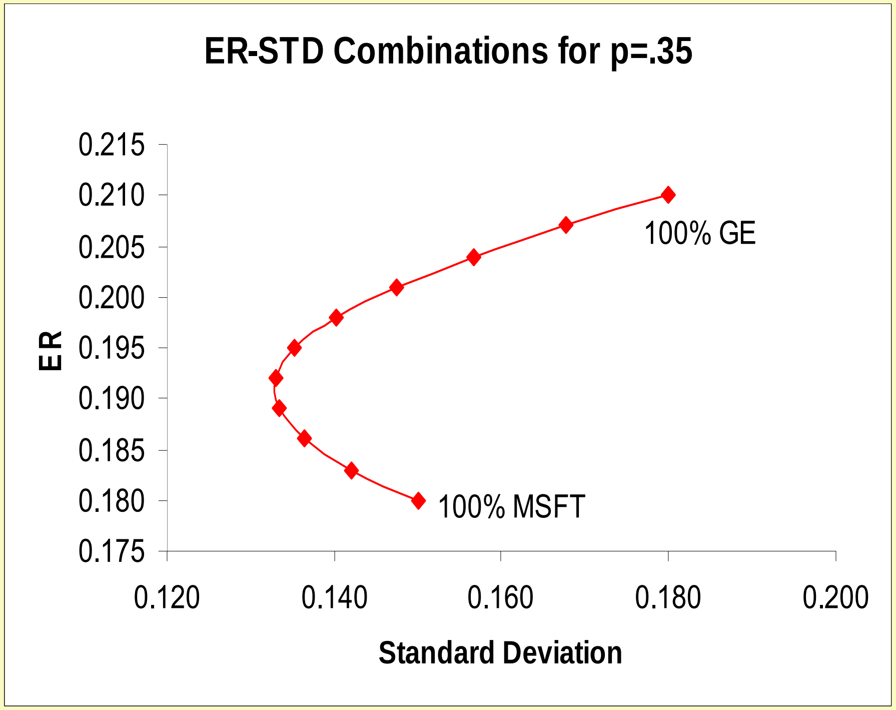
$$E(r_p) = w_{GE}E(r_{GE}) + (1 - w_{GE})E(r_{MSFT}) = w_{GE}.21 + (1 - w_{GE}).18$$

$$\sigma_p = \sqrt{w_{GE}^2\sigma_{GE}^2 + (1 - w_{GE})^2\sigma_{MSFT}^2 + 2w_{GE}(1 - w_{GE})\rho_{GE,MSFT}\sigma_{GE}\sigma_{MSFT}}$$

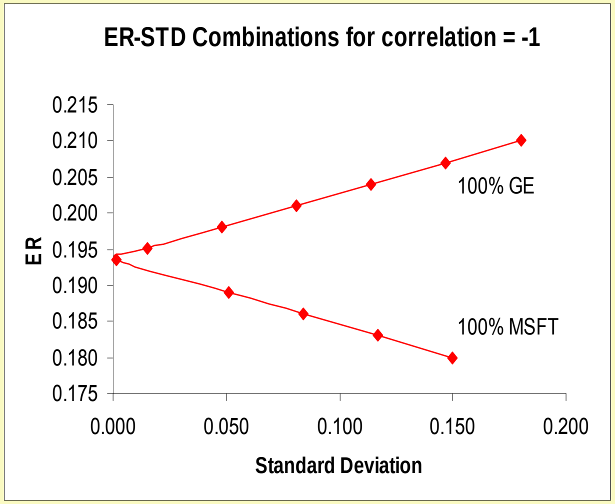
$$= \sqrt{w_{GE}^2(.18)^2 + (1 - w_{GE})^2.15^2 + 2w_{GE}(1 - w_{GE}) * .34 * .18 * .15}$$

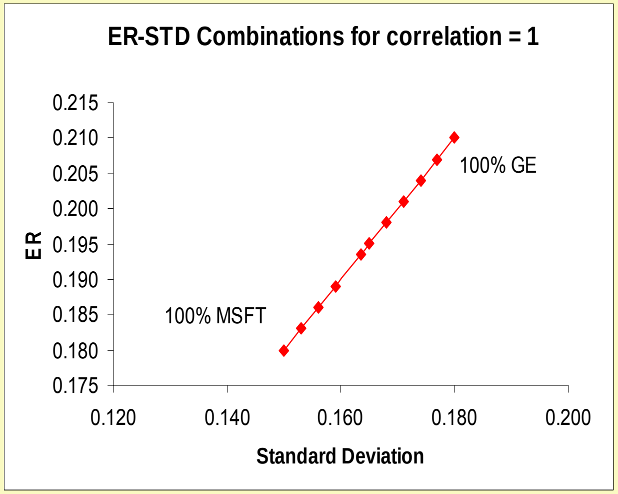
The Optimal Allocation of Investment Dollars Between Two Assets

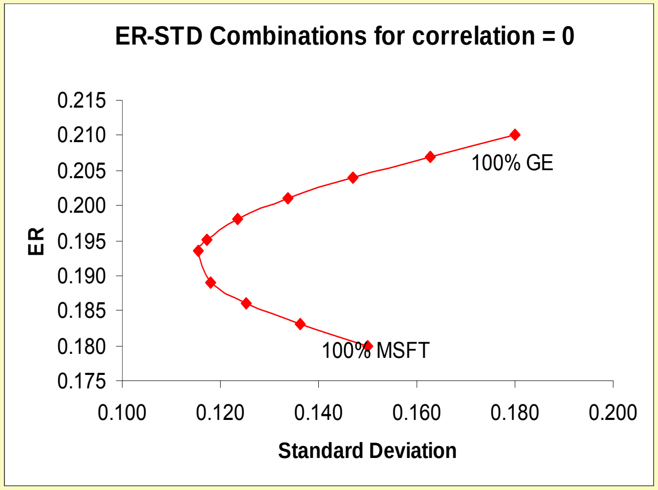
Fraction in GE	Fraction in MSFT	STD	ER
1.00	0.00	0.180	0.210
0.90	0.10	0.168	0.207
0.80	0.20	0.157	0.204
0.70	0.30	0.148	0.201
0.60	0.40	0.140	0.198
0.50	0.50	0.135	0.195
0.40	0.60	0.133	0.192
0.30	0.70	0.133	0.189
0.20	0.80	0.137	0.186
0.10	0.90	0.142	0.183
0.00	1.00	0.150	0.180



The Optimal Allocation of Investment Dollars Between Two Assets







Optimal Portfolios – Another Way to Think About Diversification

Suppose you have n stocks available. Does is make any difference if you put all your money in one stock or you invest 1/n in each stock?

If you invest in all the stocks, you'll receive the average expected return.

$$\sum_{i=1}^{n}\left(\frac{1}{n}\right)ER = \overline{ER}$$

Optimal Portfolios – Another Way to Think About Diversification

However the variance can be quite different depending on the correlation between the stocks. Recall the formula for the variance is

$$\sigma_p^2 = \frac{1}{n}\sum_{i=1}^{N}\frac{1}{n}\sigma_i^2 + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{1}{n^2}Cov(r_i,r_j)$$

where $Cov(r_i,r_j) = \rho_{ij}\,\sigma_i\,\sigma_j$

There are n variance terms and n(n-1) covariance terms. The average variance and covariance are given by:

$$\bar{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{N}\sigma_i^2$$

$$\overline{Cov} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}Cov(r_i,r_j)$$

Given this, we can write the variance as

$$\sigma_p^2 = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\overline{Cov}$$

Optimal Portfolios – Another Way to Think About Diversification

$$\sigma_p^2 = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\overline{Cov}$$

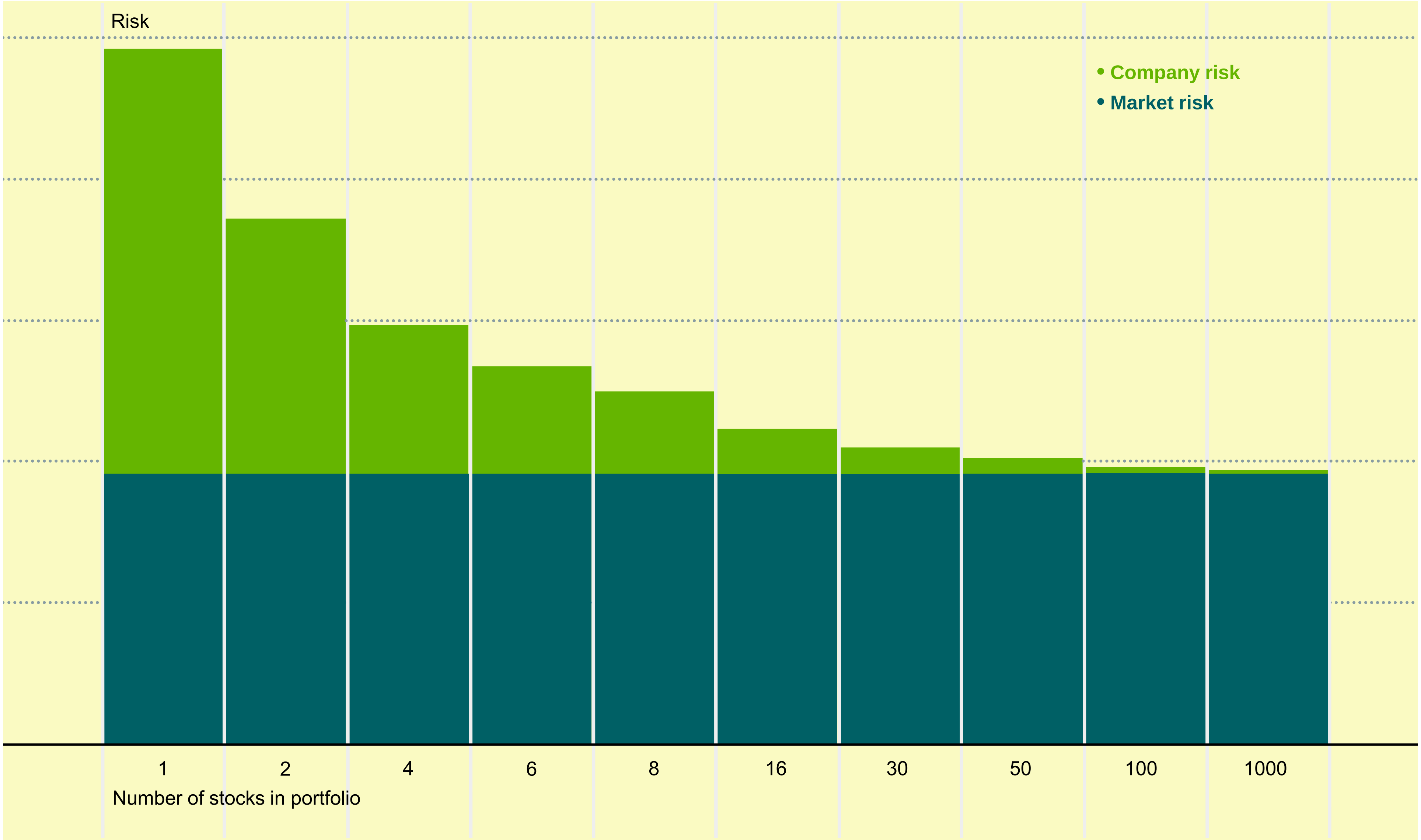
If is easy to see that as n gets bigger and bigger:

1. The variance of the portfolio gets closer and closer to the average covariance of the portfolio.

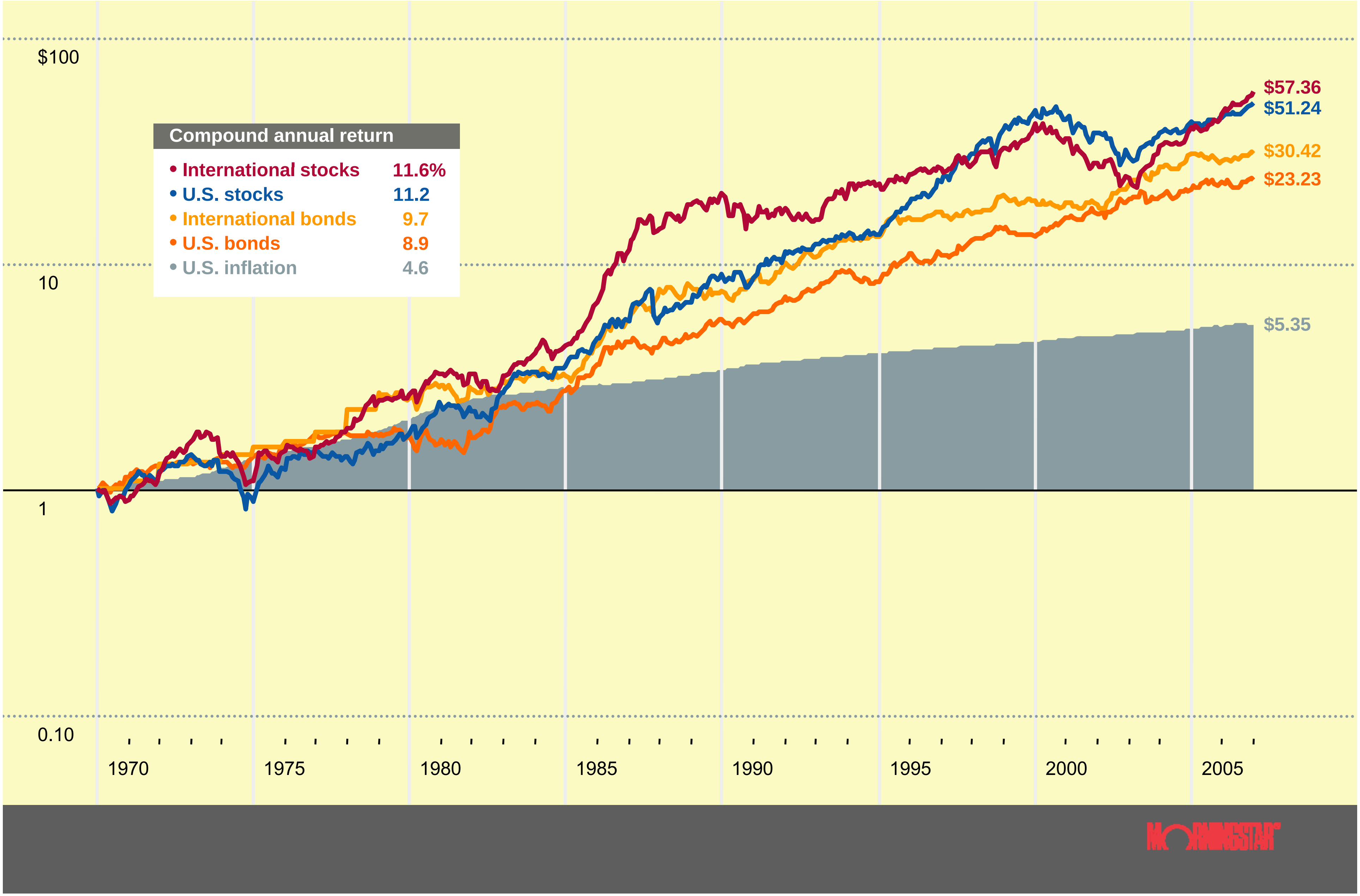
2. The average variance of the individual assets doesn’t affect the variance of the portfolio.

3. If ρ=0, then the risk of the portfolio goes to zero

Stock Diversification



Global Investing
1970–2006

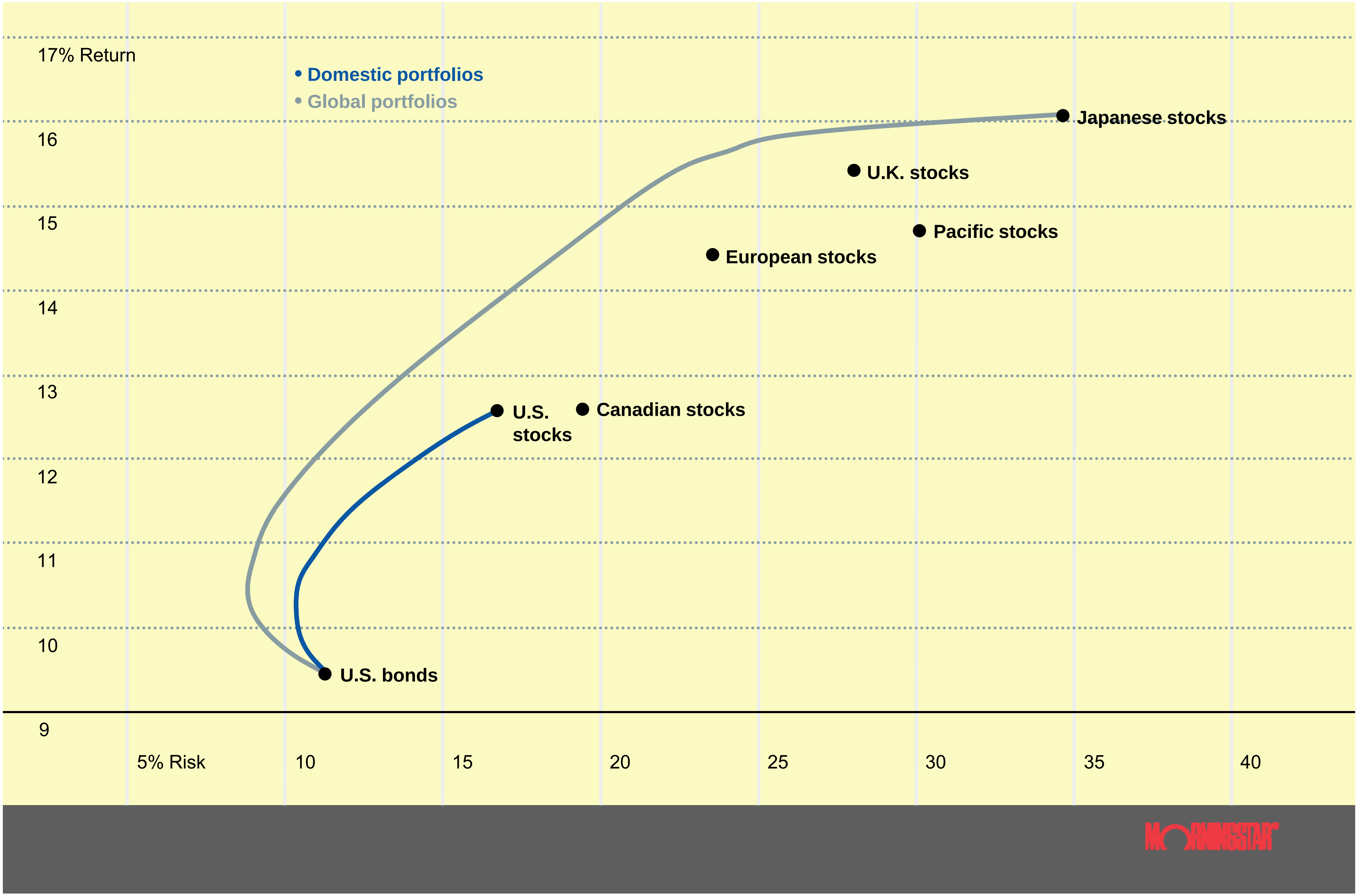


Correlations by Region

1988–2006

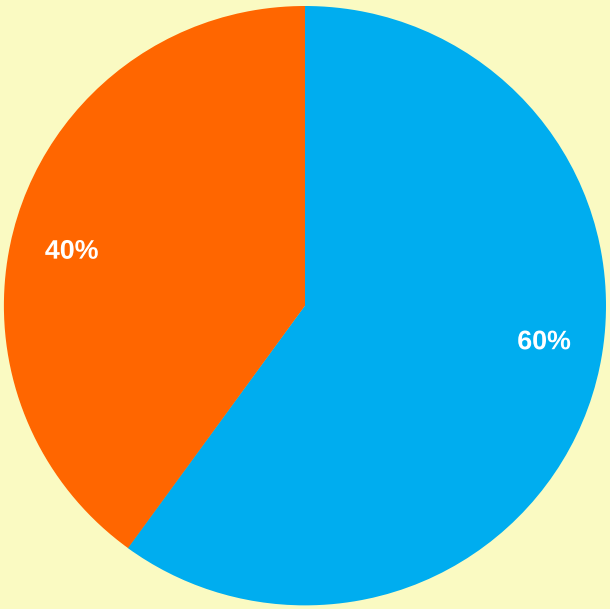
	Asia	Pacific	Europe	U.K.	Japan	Canada	U.S.
Asia	1.00	0.95	0.53	0.49	0.41	0.61	0.57
Pacific	0.95	1.00	0.56	0.55	0.42	0.66	0.60
Europe	0.53	0.56	1.00	0.76	0.45	0.62	0.68
U.K.	0.49	0.55	0.76	1.00	0.46	0.53	0.64
Japan	0.41	0.42	0.45	0.46	1.00	0.40	0.35
Canada	0.61	0.66	0.62	0.53	0.40	1.00	0.74
U.S.	0.57	0.60	0.68	0.64	0.35	0.74	1.00

International Enhances Domestic Portfolios
1970–2006



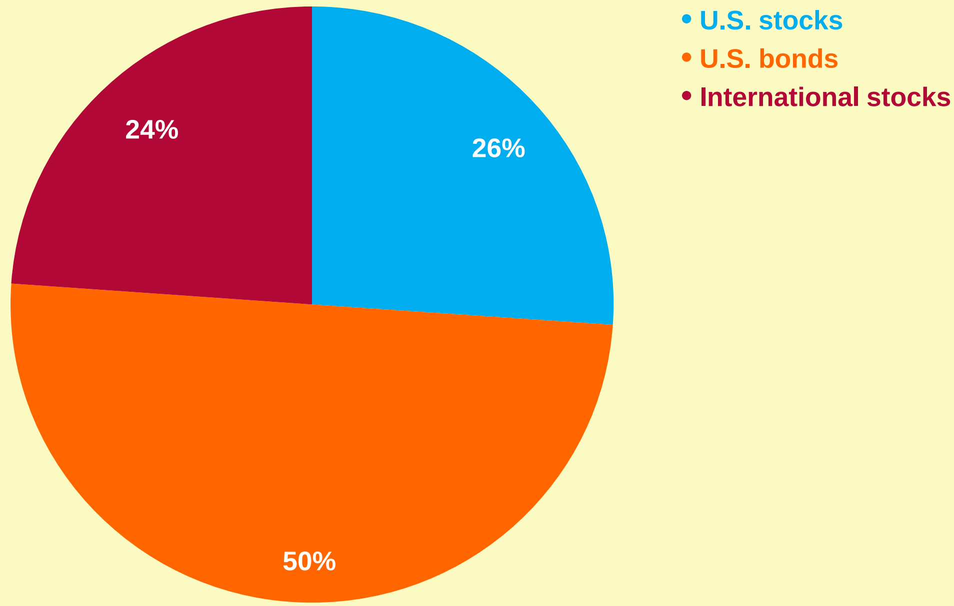
Domestic Versus Global
1970–2006

Domestic portfolio



Average return	10.3%
Risk	10.9%

Global portfolio



Average return	10.3%
Risk	9.4%

Optimal Portfolios

Now we want to make the problem progressively more realistic and see what the implications are. The generalizations we want to consider are:

1. Short Sales
2. Multiple Assets
3. Risk free Asset
4. Constraints on Borrowing

Surprisingly, making the problem more realistic makes the solution simpler!

Optimal Portfolios – Short Sales

A short position entails borrowing the stock, selling at the prevailing market price and the replacing the borrowed share at a later date.

Example:
Suppose the current price of IBM is $100/share and you short 100 shares. How well you did on this transaction depends on the price of IBM on the day you replace the shares. Consider the following possibilities:

Possible Replacement Prices	$80	$120
Profit	($100-$80)*100= $2000	($100-$120)*100= -$2000

Optimal Portfolios – Short Sales

In our portfolio analysis short sales have a negative portfolio weight because instead of paying money we receive money when we enter into the transaction.

Example: Suppose I have $1000 to invest and there are two stocks to invest in, A and B. Suppose the first thing I do is short $500 worth of A. Now I have $1500 to invest in B. There for my portfolio weight are:

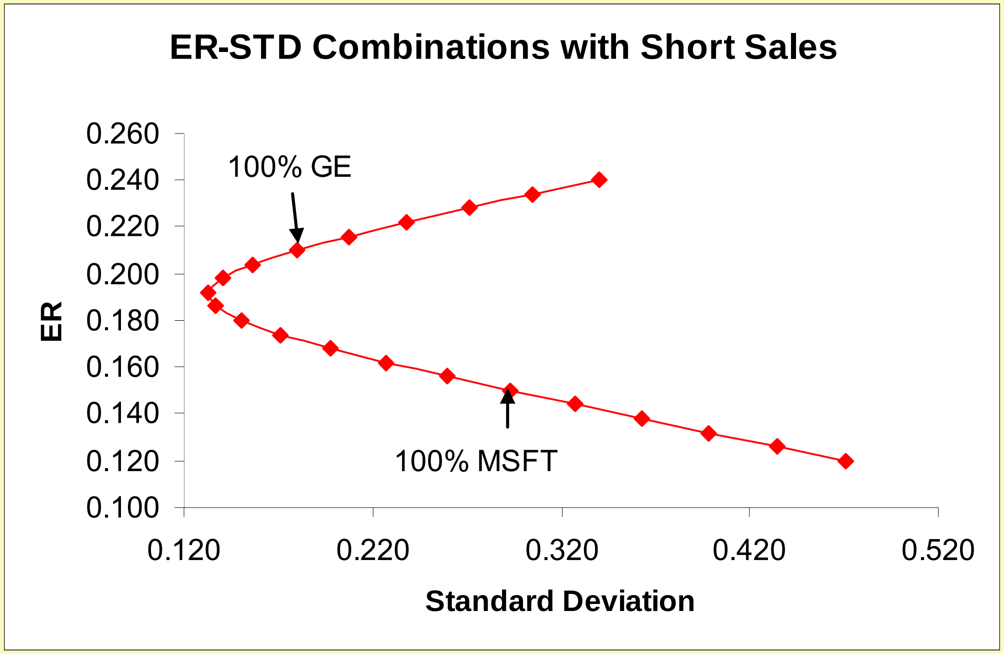
w_A = - $500/$1000 = -.5

w_B = $1500/$1000 = 1.5

The sign on the weight indicates whether I am long or short. The absolute value indicates how big my position in relative to the amount of money I have to invest

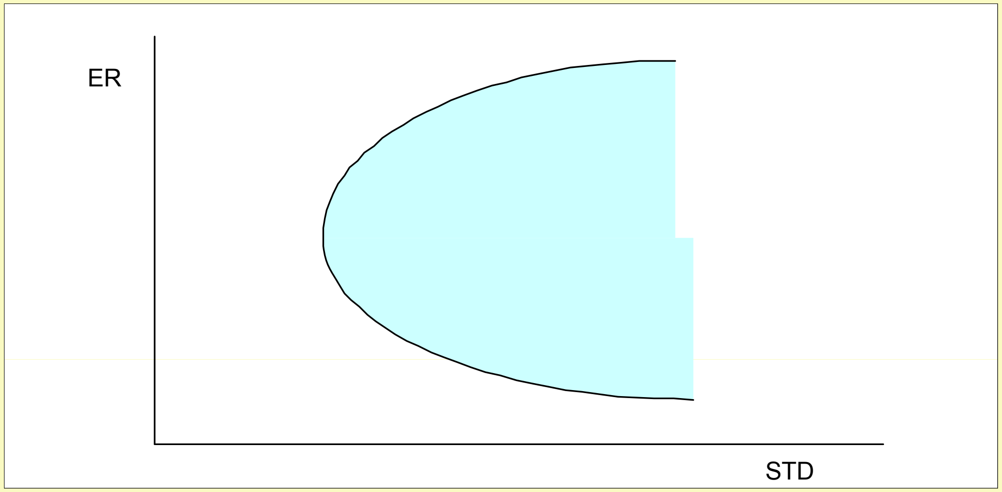
Optimal Portfolios – Short Sales

Shorting expands our investment possibilities because more extreme positions can be taken.



Optimal Portfolios – Multiple Assets

If we consider portfolios with more than two assets the set of expected return and standard deviation combinations looks like:



Optimal Portfolios – Risk Free Asset

Suppose that we can invest in both risky assets and a riskfree asset such as a T-bill. This turns out to have a very dramatic effect on our portfolio problem.

First consider an investment problem where we are going to divide our invest dollars between a risky asset and a risk free asset.

Suppose: r_f = 5% σ_f = 0.

r_A = 15% σ_A = 25%

The correlation between risk free asset and any risky asset is zero.

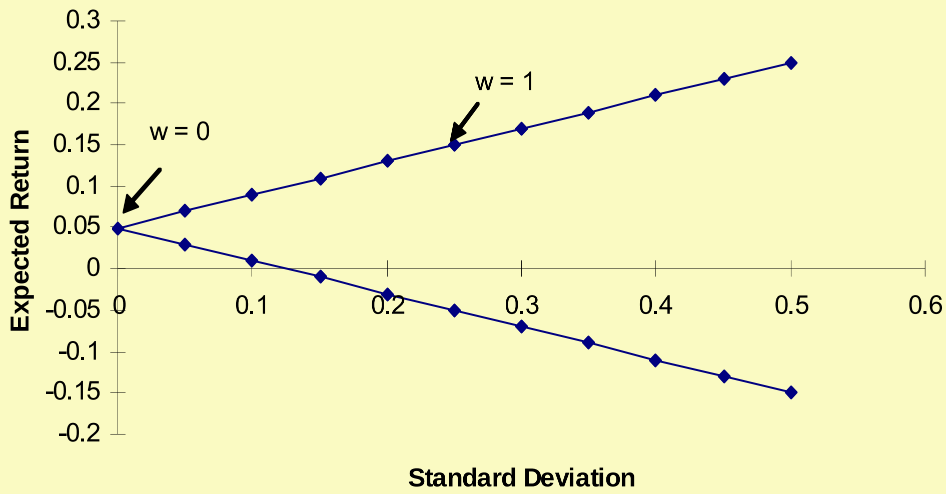
Let w denote the portfolio proportion in the risky asset and (1-w) is the amount invested in the risk free asset.

$$ER = w * .15 + (1 - w) * .05$$

$$STD = \sqrt{w^2 * .25^2} = |w| * .25$$

Optimal Portfolios – Risk Free Asset

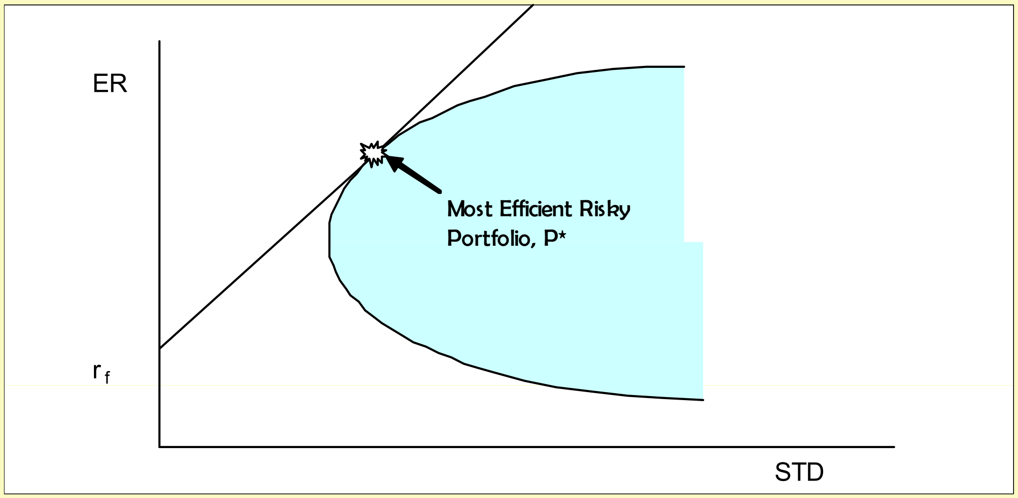
The combinations of expected return and standard deviation as we vary w look like:



We can see that the expected return/standard deviation combinations available if we have a portfolio of a risky asset and a risk free asset lie on a straight line.

Optimal Portfolios – Risk Free Asset

Now lets combine our multiple risky asset analysis with the risk free asset analysis.



What is P*?

Suppose for simplicity we assume that:

1. All investors have access to the same set of investments
2. All investors have the same assessment of the prospects for the investments

If this was the case then all investors would be faced with the same portfolio problem. If this was approximately true then,

All efficient portfolios are some combination of a position in the risk free asset and a position in the efficient risky portfolio, P*.

Where on that line a particular investor wants to be depends on their risk aversion.

What is P*?

If some security is not in P*, then no one would want to hold this asset. The security's price will fall (hence the return will rise) to the point where the security becomes more desirable to investors. In the end prices must adjust so that every asset is held (i.e. supply = demand). This implies two very important things:

1. P* must be the **market portfolio**. The market portfolio is the portfolio which contains every asset with portfolio proportion

$$w_i^* = \frac{\text{market value of } i}{\text{total value of the market}}$$

w_i is referred to as the **market proportion of security i**.

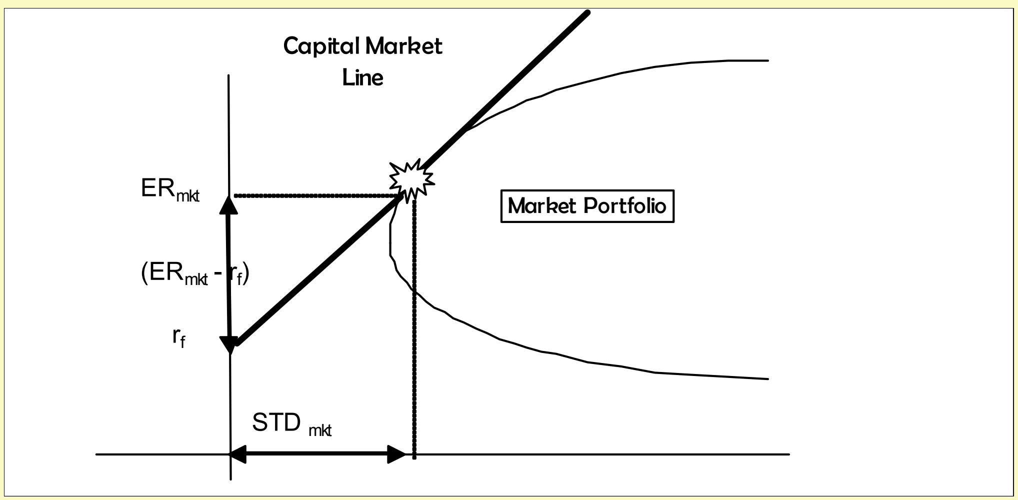
Example: Suppose that the IBM is selling for $100/share and there are 5 million shares outstanding. In addition suppose that the total value of the market is $100 billion. The market proportion of IBM is

$$w_{IBM} = \frac{(\$100/\text{share} * 5{,}000{,}000)}{\$100 \text{ Billion}} = .005$$

Capital Market Line

Our analysis tells us that an efficient portfolio is some combination of the market portfolio and the risk free security. The line depicting the combinations of risk and return are available using this type of investment is referred to as the **Capital Market Line** (CML). This provides a "reality check" on the realistic combinations of risk and return.

$$ER^{e}_{portfolio} = r_f + \frac{(ER_{mkt} - r_f)}{\sigma_{mkt}} \sigma^{e}_{portfolio}$$

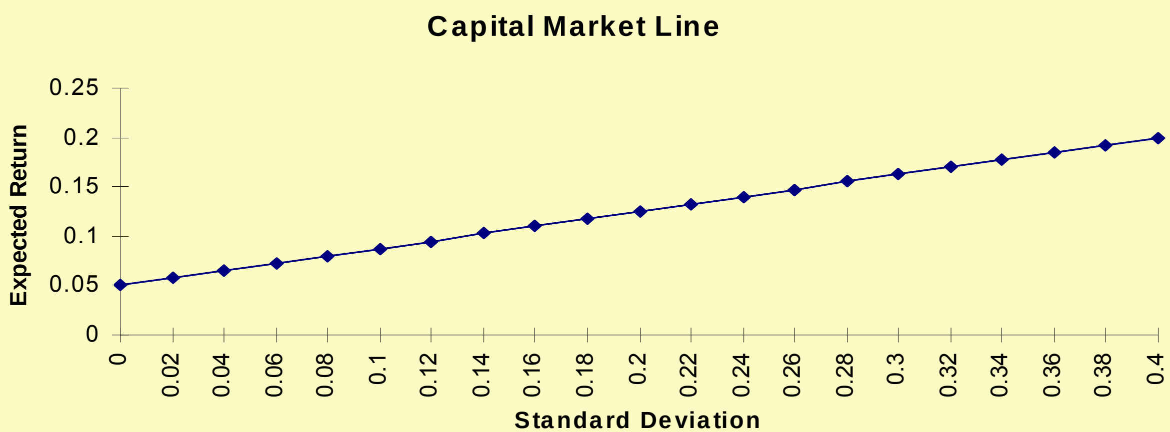


Capital Market Line

Example: Suppose that the return on a risk free security such as a T-Bill is 5%, expected return on the market is 13% and the standard deviation for the market is 20%. The slope of the capital market line is:

$$\frac{.13 - .05}{.20} = .4$$

The capital market line is depicted below.



What Else Does [P*=Market Portfolio] Imply?

Prices and hence expected returns must be just right in order for investors to be willing to hold every security in its market proportion. The formula for the expected return such that this is the case is:

$$ER_i = r_f + \beta_i(ER_{mkt} - r_f)$$

where

$$\beta_i = \frac{cov(r_i, r_{mkt})}{var(r_{mkt})}$$

Note:

1. The beta of the market is 1 because cov(x,x) = var(x)
2. The beta of a riskfree asset in 0.
3. The beta of a portfolio is equal to the weighted average of the betas of the stocks in the portfolio

$$\beta_{portfolio} = \sum w_i \beta_i$$

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What Else Does [P*=Market Portfolio] Imply?

AMOUNT OF RISK — REWARD PER UNIT OF RISK

$$ER_i = r_f + \beta_i (ER_{mkt} - r_f)$$

TIME VALUE — RISK

This equation has a very important interpretation. Investors earn a return for

1. Time value at a rate r_f
2. Risk at rate $\beta_i (ER_{mkt} – r_f)$
   1. β_i measures the “amount of risk” where $\beta_i = \frac{cov(r_i, r_{mkt})}{var(r_{mkt})}$
   2. β_i measures the “market risk” of an asset rather than it’s total risk.
   3. $(ER_{mkt} – rf)$ measures the return per unit of risk.

Beta

There are a number of sources for firm betas. Some examples of beta are given below. The source is *Value Line.*

Firm	Beta
Com Ed	.75
Texaco	.7
General Motors	1.1
Ford	1.15
Apple	1.3
Microsoft	1.3
Merrill Lynch	1.9

Note that these are equity betas. The asset betas will tend to be lower. How much lower depends on the capital structure.

Beta

<u>Example</u>: Suppose a firm is:

- 35% debt and 65% equity
- beta of the equity is 1.2. If the debt is high grade, the beta of the debt will be approximately zero.

Suppose you buy all the debt and equity of a firm. You now own all the assets of the firm. The beta of a portfolio which contains all the debt and all the equity is

$$\beta_{portfolio} = .35*0 + .65*1.2 = .78$$

Since this portfolio contains all the assets of the firm, the beta of the this portfolio equals the beta of the assets.

Note: as the amount of debt increases, the beta of the assets stays the same but the beta of equity goes up. This is because there is two sources of risk for equity, the risk of the assets and the risk associated with the leverage.

Aside on risk of leverage:

<u>Example</u>: Suppose you buy a house for \$500,000. Next year the house will be worth either \$600,000 or \$400,000 with equal probability.

If you pay cash for the house (no borrowing), your return will be either be

$$\frac{\$400{,}000 - \$500{,}000}{\$500{,}000} = -20\%$$

or

$$\frac{\$600{,}000 - \$500{,}000}{\$500{,}000} = +20\%$$

The expected return is

$$(1/2) * (-20\%) + (1/2) * (+20\%) = 0$$

The standard deviation is

$$\sqrt{.5 * (.2 - 0)^2 + .5 * (-.2 - 0)^2} = .2$$

$$\frac{\$600{,}000 - \$500{,}000}{\$500{,}000} = +20\%$$

Aside on risk of leverage

Suppose you invest $500,000 as before and in addition you borrow $500,000 and you buy two houses. Assume for simplicity that the interest rate is zero i.e. you only have to repay the principal. Then you return will be either

$$\frac{(\$800{,}000 - \$500{,}000) - \$500{,}000}{\$500{,}000} = -40\%$$

or

$$\frac{(\$1{,}200{,}000 - \$500{,}000) - \$500{,}000}{\$500{,}000} = +40\%$$

The expected return is

$$(1/2) * (-40\%) + (1/2) * (+40\%) = 0$$

The standard deviation is:

$$\sqrt{.5 * (-.4 - 0)^2 + .5 * (.4 - 0)^2} = .4$$