Pricing Financial Futures

Basic Case

When pricing a futures contract, we will ignore marking-to-market. Empirically, marking-to-market does not significantly affect the price of the contract, i.e., price of a future and forward on the same asset appear to be roughly the same.

To price a future, we use the following procedure:

1. We create a risk free position using a futures contract and the underlying asset.

2. The futures price is found by equating the rate of return on the risk free position with the rate of return on a T-Bill of similar maturity.

To create a risk free **long position**, you buy the spot commodity and sell the future. This is referred to as a **cash and carry**. The spot commodity is then carried to the delivery date of the futures contract and delivered against the short futures position. The cash flows associated with this position are given below.

	t	T
buy the spot	$-P_t$	*
sell the future		$F_{t,T}$
Total	$-P_t$	$F_{t,T}$

*The spot commodity which was purchased at t is used at time T to cover the short futures position.

The cash flows resemble the cash flows for a T-Bill because you pay a fixed amount today and receive a known amount at a pre-specified day in the future. The gross rate of return on a cash and carry equal to

$$\frac{F_{t,T}}{P_t}$$

The gross rate of return on a T-Bill over the period t to T is $e^{r(T-t)}$. To rule out an arbitrage opportunity, the rate of return on the any risk free position must equal the rate of return on a T-Bill. Therefore the fundamental no arbitrage relationship is:

$$\frac{F_{t,T}}{P_t} = e^{r(T-t)}$$

or

$$F_{t,T} = e^{r(T-t)} P_t$$

The futures price is the future value of the spot price.

Example: Suppose the spot price of gold is $325/ounce and the six month T-Bill rate quoted in the WSJ is 6.1%. The continuously compounded interest rate is ln(1.061) = 5.92%. The futures price for gold with delivery in 3 months is:

$$\$325 * e^{.25*(.0592)} = \$329.85$$

Another way to think about this is to consider pre-paid forwards. Suppose you want to own 1 share of stock one year from now.

How much should you pay today for the stock which will delivered in one year?

The difference between owning the stock today and owning the stock in one year is that you will not receive any dividends and you have no control rights. If the stock doesn't pay dividends and you don't care about control rights, then buying the stock today with delivery in one year is equivalent to buying the stock today with immediate delivery. Therefore the payment today associated with the pre-paid forward should just be the current stock price, denoted P_0.

If instead of paying today, you pay for and receive the stock in one year, what should be your payment on the delivery date?

Everything is the same as before except that the payment has been moved forward one year. The payment should be the future value of the current stock price, $P_0 * e^{r(T-t)} = P_0 e^r$ since (T-t) =1.

Example: If the current stock price is P_0 = \$50/share and the continuously compounded interest rate is 5%, the forward payment should be

$$\$50 * e^{.05} = \$52.56.$$

To create a **risk free short position**, you do a reverse cash and carry, that is, you short the spot and buy the future. The cash flows associated with this position are given below.

	t	T
short the spot	P_t	*
buy the future		$-F_{t,T}$
Total	P_t	$-F_{t,T}$

*The asset which is delivered is used to cover the short position in the spot.

The rate of return on a reverse cash and carry is equal to

$$\frac{F_{t,T}}{P_t}$$

When there is no bid-ask spread, the rate of return on a risk free short position is the same as the rate of return on a risk free long position.

Just as before, in order to preclude arbitrage, the rate of return on the reverse cash and carry must be the same as the rate of return on a T-bill since they are both riskless positions.

Example: Suppose a futures contract is created where 1 share of IBM will be traded in one year at $F_{t,T}$=\$156/share. Suppose the current spot price is \$140/share and the continuously compounded T-Bill rate is 10%. Assume there are no dividends.

Is there an arbitrage opportunity?

The theoretical value for the future is

$$F_{t,T} = 140e^{.10*1} = 154.72$$

which is different from the market price. To take advantage of this you should:

1. sell the future since it is overvalued.
2. buy the spot to hedge the risk associated with the short future.
3. borrow (short a T-bill) to finance the purchase of the spot.

The cash flows associated with this position are given below:

	t	T
sell a future		156
buy the spot	-140	*
borrow (short a T-bill)	140	$-140*e^{.10*1}$ =-154.72
Total	0	1.28

*The spot is used to cover the short position in the future.

Calendar Spreads

To get the relationship between futures prices on the same asset but with different delivery dates, you use a forward cash and carry. This position entails buying the future with the near term delivery date and selling the future with the more distant delivery date. This is like a standard cash and carry with the near term future acting as the spot good. The cash flows associated with this position are given below. Let T1 < T2.

	t	T1	T2
buy contract 1		$-F_{t,T1}$	*
sell contract 2			$F_{t,T2}$
total		$-F_{t,T1}$	$F_{t,T2}$

*The asset which was purchased at T1 using the near term future is used to cover the short future with the T2 delivery date.

This position resembles a T-Bill from T1 to T2. The rate of return on this position is

$$\frac{F_{t,T2}}{F_{t,T1}}$$

Let the rate of return on a T-Bill over the period T1 to T2 be denoted $r_{T1,T2}$. There will not be an arbitrage opportunity if the rate of return on the calendar spread is equal to the rate of return on a T-Bill over the same period. Therefore the fundamental no arbitrage relationship is:

$$\frac{F_{t,T2}}{F_{t,T1}} = e^{r(T2-T1)}$$

or

$$F_{t,T2} = e^{r(T2-T1)}F_{t,T1}$$

Example: Suppose that there are two gold futures contracts, one with delivery on T1=June 15 and the other with delivery on T2=September 15. The price of the June contract is $450/ounce. The annual continuously compounded risk-free rate is 6%.

What should be the price of the September contract?

The price of the September contract should be

$$F_{t,T2} = (450)e^{.06*.25} = 456.80.$$

Leverage and Futures Contracts

A long futures position is equivalent to buying the underlying asset and financing 100% of the cost of the position by borrowing at the risk free rate. To see this consider the following two positions:

A. Buy a share of stock and borrow 100% of the cost at the risk free rate. Assume the position is closed out at time T.

	Today	T
Buy a share of stock	$- S_t$	S_T
Borrow	$+S_t$	$S_t e^{r(T-t)}$
Total	0	$S_T - S_t e^{r(T-t)}$

B. Buy a future on the stock

	Today	T
Buy a future	0	$S_T – F_{t,T} =$ $S_T - S_t e^{r(T-t)}$

Example:

A mutual fund with special expertise in fixed income offers a product know as the **S&P 500 Plus.** This product promises a return in excess of the S&P 500. How is this product constructed?

If the investor buys the S&P 500 Index, the cash flows will be:

	Today	T
Buy the S&P 500 Index	$-I_t$	I_T

Now suppose the investor gives the mutual fund I_t. The firm enters into the following positions:

	Today	T
Buys S&P 500 future	0	$I_T - F_{t,T} = I_T - I_t e^{r(T-t)}$
Invests in fixed income product	$-I_t$	$I_t e^{r^*(T-t)}$
Total	$-I_t$	$I_T + (I_t e^{r^*(T-t)} - I_t e^{r(T-t)})$

The return on the S&P 500 Plus will a return in excess of the return on the S&P 500 if the mutual fund can earn a return on the fixed income product, r* which reliably exceeds the risk free rate, r.

RELATIONSHIP BETWEEN $F_{t,T}$ AND THE EXPECTED FUTURE SPOT PRICE

Now we will investigate the relationship between the futures price and the expected spot price. For example, what would the price of a future for 1 share of IBM for delivery on 6-15-07 tell us about the spot price of IBM on 6-15-07. One possibility is that

$$F_{t,T} = E[P_T].$$

This means that the futures price is the market's best guess as to what the price of IBM will be on 6-15-07. Empirically, this is not the case and economically it doesn't make any sense. To see this consider the following:

The expected return on a long position in a IBM future is equal to

$$\frac{E[P_T] - F_{t,T}}{F_{t,T}}$$

If $F_{t,T} = E[P_T]$, the expected return would be zero! Since an IBM futures position is risky, this cannot hold. Investors will not hold a risky position if they expect to earn zero return.

To determine the relationship between $E[P_T]$ and $F_{t,T}$ we use the CAPM. First, we need to define some notation. Let

$E[r_{t,T}^{IBM}]$ = the expected return on IBM over the time period t to T

$E[r_{t,T}^{mkt}]$ = the expected return on the market over the time period t to T

$r_{t,T}$ = the riskfree rate over the time period t to T

β_{IBM} = the beta of IBM

The CAPM says that

$$E[r_{t\ T}^{IBM}] = r_{t,T} + \beta_{IBM}(E[r_{t\ T}^{mkt}] - r_{t,T}) \qquad (1)$$

In addition, we know that the expected price is equal to the expected rate of return times the current price, that is

$$E[P_T^{IBM}] = (1+E[r_{t\ T}^{IBM}])P_t^{IBM}. \qquad (2)$$

If we substitute (1) into (2), we get that

$$E[P_T^{IBM}] = (1+r_{t,T} + \beta_{IBM}(E[r_{t\ T}^{mkt}] - r_{t,T}))P_t^{IBM}.$$

This can be rewritten as

$$E[P_T^{IBM}] = (1+r_{t,T})P_t^{IBM} + \beta_{IBM}(E[r_{t\ T}^{mkt}] - r_{t,T})P_t^{IBM} \qquad (3)$$

Recall that using simple compounding the future prices is

$$F_{t\ T}^{IBM} = (1+r_{t,T})P_t^{IBM}.$$

This means that (3) can be written as

$$E[P_T^{IBM}] = F_{t\ T}^{IBM} + \beta_{IBM}(E[r_{t\ T}^{mkt}] - r_{t,T})P_t^{IBM}. \qquad (4)$$

Equation (4) tells us that the only time the futures price equals the expected spot price is when the beta is zero.

If beta is greater than zero, the expected spot exceeds the futures price. This implies that someone holding a long position in the future expects to earn a positive return. Since futures are zero-sum, the person holding the short position expects to earn a negative return. An investor is willing to hold a short position with a negative expected return because the return on the short position is negatively correlated with the market.

Pricing with Coupon Payments or Dividends

If the spot commodity is a stock or a bond then the investor will receive a dividend or coupon payment while carrying the spot commodity. To see how this will affect the futures price consider the following modified cash and carry:

t	t'	T
•sell the future •buy the spot	•receive the coupon or dividend •invest the payment in a T-bill	• receive the futures price • delivery the spot • receive proceeds From T-bill position

The cash flows associated with this position are given below:

	t	t'	T
buy spot	$-P_t$		0
sell future			$F_{t,T}$
receive dividend		D	
invest dividend		-D	$De^{r(T-t')}$
Total	$-P_t$	0	$F_{t,T}+ De^{r(T-t')}$

*The spot is used to cover the short futures position.

This position resembles a T-Bill. There will be no arbitrage opportunities if the rate of return on the cash and carry is the same as the rate of return on a T-Bill. That is,

$$(F_{t,T}+De^{r(T-t')})/P_t = e^{r(T-t)}$$

or

$$F_{t,T} = e^{r(T-t)} P_t - De^{r(T-t')}.$$

Note that in this case the futures price may or may not be greater than the spot price.

How do the dividend payments change our earlier discussion of pre-paid forwards? Suppose the dividend will be paid at time t'. The difference between owning the stock today and owning the stock in one year is that you will not receive any dividends. If the amount of the dividends is known, then the price you would be willing to pay is just the current price less the present value of the dividends which you will miss. Therefore the payment today associated with the pre-paid forward should just be the current stock price less the present value of the dividends, which equals $P_0 - De^{-r(t'-t)}$.

If instead of paying today, you pay for and receive the stock in one year, what should be your payment on the delivery date?

Everything is the same as before except that the payment has been moved forward one year. The payment should be:

$$[P_0 - De^{-r(t'-t)}]*e^{r(T-t)} = P_0\, e^{r(T-t)} - De^{-r(T-t')}.$$

How would we do this with continuous compounding?

To do this with continuous compounding, we need the dividend yield.

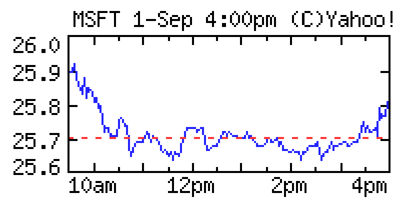
Aside on Dividend Yields

MICROSOFT CP (NasdaqGS:MSFT) Delayed quote data

Edit

After Hours: **25.80** **0.04 (0.15%)**

Last Trade:	**25.84**	Day's Range:	25.64 - 25.97
Trade Time:	Sep 1	52wk Range:	21.46 - 28.38
Change:	**0.14 (0.54%)**	Volume:	31,595,597
Prev Close:	25.70	Avg Vol (3m):	67,722,900
Open:	25.90	Market Cap:	N/A
Bid:	25.79 x 10000	P/E (ttm):	21.61
Ask:	25.80 x 400	EPS (ttm):	1.20
1y Target Est:	29.31	Div & Yield:	0.36 (1.40%)



1d 5d 3m 6m 1y 2y 5y max

Annual Report for MSFT

Example 1: Microsoft currently pays a $.09/share dividend quarterly. The annual dividend is $.09/share *4 = $.36. The current share price is $25.84/share. So the annual dividend yield is .36/25.84 = 1.40%. The relationship between the continuously compounded rate,∗ and the annually compounded rate, d is ∗ = ln(1+d), so the continuously compounded dividend yield in this case is ln(1.04) = 3.92%

In many cases in this class we will be treating dividends as if they are paid continuously. This is the standard approach in this area. This is a "modeling convention" which makes the formulas and intuition easier to understand and to work with.

If you buy one share stock which pays dividends continuously and you reinvest those dividends when they are paid, at the end of the period t to T you will have $1*e^{\delta(T-t)}$ shares where δ is the continuously compounded dividend yield.

<u>Example 2</u>: Suppose you buy one share of Microsoft and hold it for 2 years. The dividends are paid at continuously compounded rate of 3.92% and you reinvest all the dividends when they are paid, how many shares do you have at the end of two years?

You will have $e^{.0392*2}$ = 1.0816 shares.

<u>Example 3</u>: Suppose you want exactly 1 share of Microsoft one year from now. The dividends are paid at continuously compounded rate of 3.92% and you reinvest all the dividends when they are paid, how many shares should you buy today?

Let x be the number of shares. You want to choose x such that

$$Xe^{\delta(T-t)} = 1 \quad \rightarrow \quad X = e^{-\delta(T-t)}$$

Therefore in this case, you should buy X = $e^{-.0392*1}$= .9616 shares

The cash and carry with a continuous dividend yield is done slightly differently. At delivery you need to have one share of stock. So a time t you need buy just enough stock so that when you reinvest the dividends you have accumulated exactly one share by T. As we saw from the previous example, if you buy $1*e^{-\delta(T-t)}$ shares at time t, then at time T you will have

$$[1*(e^{-\delta(T-t)})](e^{\delta(T-t)}) = 1 \text{ share.}$$

The cash flows are:

	t	T
Buy the spot	$-P_t * 1*e^{-\delta(T-t)}$	
Sell the future		$F_{t,T}$
Total	$-P_t *1*e^{-\delta(T-t)}$	$F_{t,T}$

Since this is a risk-free transaction, it should earn the risk-free rate.

$$\frac{F_{t,T}}{P_t e^{-\delta(T-t)}}$$

Rewriting this we get

$$F_{t,T} = e^{r(T-t)} P_t e^{-\delta(T-t)} = P_t e^{(r-\delta)(T-t)}$$

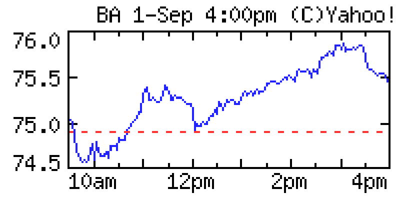
Example:

BOEING CO (NYSE:BA) Delayed quote data

Edit

After Hours: **75.55** **↑0.12 (0.16%)**

Last Trade:	**75.43**	Day's Range:	74.46 - 75.91
Trade Time:	Sep 1	52wk Range:	62.01 - 89.58
Change:	**↑0.53 (0.71%)**	Volume:	3,275,600
Prev Close:	74.90	Avg Vol (3m):	4,227,620
Open:	75.00	Market Cap:	N/A
Bid:	N/A	P/E (ttm):	29.78
Ask:	N/A	EPS (ttm):	2.53
1y Target Est:	91.58	Div & Yield:	1.20 (1.60%)



1d 5d 3m 6m 1y 2y 5y max

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Boeing's current share price is \$75.43/share and the continuously compounded dividend yield is ln(1.016) = 1.59%. If you wanted to create a synthetic 3 month T-bill using Boeing stock, the cash flows would be:

	t	T
Buy $1^*e^{-.0159^*.25}$ =.996 shares	-\$75.43/share*.996 shares = -\$75.13	
Sell the future		$F_{t,T}$
Total	-\$75.13	$F_{t,T}$

	Week ended Aug. 25, 2006
Treasury Bills (90 day) [c]	4.97
Commercial Paper (Finl. 90 day) [a]	5.27
Commercial Paper (Non-Finl. 90 day) [a]	5.22
Certs of Deposit (Resale 3 Month)	5.36
Certs of Deposit (Resale 6 Month)	5.43
Federal Funds (Overnight) [b]	5.24
Eurodollars (90 day) [b]	5.40
Treasury Bills (One Year) [c]	5.07
Treasury Notes (Two Year) [c]	4.87

The three month continuously compounded T-bill return is

$$\ln(1.0497) = 4.85\%.$$

Since this is a riskless transaction, the rate of return should be the riskless rate, i.e.,

$$\frac{F_{t,T}}{\$75.13} = e^{.0485*.25} = 1.0122$$

which implies that $F_{t,T}$ = (1.0122)*($ 75.13) = $76.046

Stock Index Futures

Definition: A stock index future is a futures contract whose payoff is based on the value of some index such as the Dow Jones Industrial Average or the S&P 500. These contracts are cash settled, that is, there is no physical delivery.

Index futures traded at the Chicago Mercantile Exchange (http://www.cme.com) include:

CME S&P 500
CME S&P 500 EOM Options
CME E-mini S&P 500
CME E-mini S&P 500 EOM Options
CME NASDAQ-100
CME E-mini NASDAQ-100
CME E-mini NASDAQ Biotechnology
CME E-mini NASDAQ Composite
CME E-mini MSCI EAFE
CME S&P MidCap 400
CME E-mini S&P MidCap 400
CME S&P SmallCap 600
CME Russell 2000
CME E-mini Russell 2000
CME E-mini Russell 1000
CME S&P 500/Citigroup Growth
CME S&P 500/Citigroup Value
CME SPCTR Futures
CME Nikkei 225
CME E-mini S&P Asia 50
CME Futures on ETFs

The S&P 500 Index

The S&P 500 Index (www. standardpoor.com) is defined as

$$I_t = \frac{\sum_{i=1}^{500} n_i P_i}{\text{constant}}$$

where

n_i = the number of shares outstanding of firm i
P_i = the current price per share of firm i

Note that $n_i P_i$ = market capitalization of firm i.

There are several things to note about the index:

1. The value of the index does not take into account dividends. This means that the return on the 500 stocks in the index is greater than the return on the index since dividends are not included in the value of the index.

2. A portfolio which mimics the index (excluding dividends) consists of shares in the 500 firms making up the index held in their market proportions.

Example:

Suppose the S&P 2 is made up of IBM and Exxon Mobil (XOM). IBM is currently selling for $81/share. The market capitalization of IBM is $123,892.9 million. Exxon Mobil is currently selling for $68.10 per share and the market capitalization is $404,851.6 million. Suppose the constant used to compute the index is 1,000 million. The value of the index is

$$I_t = [(\$123,892.9 \text{ million})+(\$404,851.6 \text{ million})]/1,000 \text{ million} = 528.74$$

IBM's fraction of the market capitalization of the S&P 2 is

$123,892.9 million/528,744.5 million = 23.43%.

Exxon Mobil's fraction of the market capitalization of the S&P 2 is

$404,851.6 million/528,744.5 million = 76.57%.

If I had $75,000 to invest and I wanted to mimic the S&P 2, I would invest 23.43% in IBM and 76.57% in Exxon Mobil which would be $17,572 and $57,428, respectively.

Therefore, I would buy

$17,572/$81 = 217shares of IBM

and

$57,428/$68.10 = 843 shares of Exxon.

I can think of the scale of my portfolio in terms of the number of units of the index. In this example, one "unit" of the index consists of

$$\frac{528.74*.2343}{81} = 1.53 \text{ shares of IBM}$$

and

$$\frac{528.74*.7657}{68.10} = 5.945 \text{ shares of Exxon Mobil}$$

and has a current market value of $528.74. The market value of this portfolio will always equal the current value of the index. If I have $75000 to spend, I can buy 75000/528.74 = 141.85 units of the index.

The spot commodity underlying the S&P 500 future is defined to be 250 units of the index. $F_{t,T}$ is the futures price for 1 unit of the index. This is analogous to a wheat future where the spot commodity underlying the contract is 5000 bushels of wheat and the price is quoted on a per bushel basis. There is also S&P 500 mini future which is defined as 50 units of the index.

How Gains and Loses Are Computed on S&P 500 Index Futures

Recall that index futures are cash settled, that is there is no physical delivery.

Suppose the futures price on an S&P 500 contract is $F_{t,T}$. If the value of the index at the maturity of the contract is I_T then the buyer's profit is

$$Profit_{buyer} = 250(I_T-F_{t,T}).$$

The buyer earns (loses) $250 for every point the index is above (below) the futures price. The sellers profit is

$$Profit_{seller} = 250(F_{t,T}-I_T).$$

The seller earns (loses) $250 for every point the index is below (above) the futures price.

For example, if the futures price $F_{t,T}$=1260 and I_T=1250,

- the buyer loses 250(1250-1260)=-$2500
- the seller earns 250(1260-1250)=$2500.

Pricing Stock Index Futures

To price a stock index future, you follow the usual procedure, i.e., make a synthetic T-Bill with the future and the underlying commodity and then equate the rates of return on the synthetic and real T-bills.

With continuous compounding the cash and carry looks like:

	t	T
Buy $250e^{-\delta (T-t)}$ units of the index	$-250 * e^{-\delta(T-t)} * I_t$	$250 * I_T$
Sell the future		$250*(F_{t,T}-I_T)$
Total	$-250 * e^{-\delta(T-t)} * I_t$	$250 * F_{t,T}$

Since there is no physical delivery, you sell your portfolio instead of delivering it against the short position in the future.

Since this is a risk-free transaction, it should earn the risk-free rate:

$$\frac{250\ F_{t,T}}{250 * I_t e^{-\delta(T-t)}} = e^{r(T-t)}$$

which implies

$$F_{t,T} = I_t e^{(r-\delta)(T-t)}$$

The formula for a calendar spread is:

$$F_{t,T2} = F_{t,T1} e^{(r-\delta)(T2-T1)}$$

In August 2004 the S&P and Nasdaq futures prices were:

S

Data retrieved at 08/11/04 11:54:31 • All quotes are in exchange local time • Data provided by FutureSource

Contract	Month	Last	Change	Open	High	Low	Volume	OpenInt	Exch	Date	Time
E-Mini S&P 500	Sep '04	1071.25	-5.00	1076.25	1076.25	1065.25	418183e	617668	CMEE	08/11/04	11:44:22
E-Mini S&P 500	Dec '04	1071.00	-5.25	1073.50	1075.00	1066.75	174e	48378	CMEE	08/11/04	11:30:51

Nasdaq 100 Comp. - cme

Data retrieved at 08/11/04 11:55:57 • All quotes are in exchange local time • Data provided by FutureSource

Contract	Month	Last	Change	Open	High	Low	Volume	OpenInt	Exch	Date	Time
Nasdaq 100	Sep '04	1318.00	-18.50	1336.50	1337.00	1310.50	9989	66656	CME	08/11/04	11:42:37
Nasdaq 100	Dec '04	1341.50y					2	2323	CME	08/10/04	15:23:39
Nasdaq 100	Mar '05	1346.50y						3	CME	08/10/04	15:23:40
Nasdaq 100	Index	1317.26	-29.45	1319.70	1323.85	1309.50			CME	08/11/04	11:45:44

In early September 2006, the prices were:

SP 500 - Mini - cme

Data retrieved at Sep 04 21:36:18 GMT • All quotes are in Greenwich Mean Time • Data provided by eSignal

Contract	Month	Last	Chg	Open	High	Low	Volume	OpenInt	Exchange	Date	Time
E-Mini S&P 500	Sep '06	**1316.25**	**3.75**	1311.75	1316.75	1311.50	551526	1492123	CMEE	09/04/06	15:29:57
E-Mini S&P 500	Dec '06	**1327.50**	**3.50**	1322.25	1327.75	1322.25	14495	99647	CMEE	09/04/06	15:22:40

Nasdaq 100 Comp. - cme

Data retrieved at Sep 04 20:28:16 GMT • All quotes are in Greenwich Mean Time • Data provided by eSignal

Contract	Month	Last	Chg	Open	High	Low	Volume	OpenInt	Exchange	Date	Time
Nasdaq 100	Sep '06	**1595.00**	**4.50**	1589.00	1596.00	1589.00	4256	56039	CME	09/04/06	15:26:39
Nasdaq 100	Dec '06	**1609.25**	**7.50**	1603.75	1616.00	1603.75	80	227	CME	09/01/06	20:21:03
Nasdaq 100	Mar '07	**1628.25**	**7.50**	1628.25	1628.25	1628.25	0	0	CME	09/01/06	20:21:03

Example: Suppose the current value of the S&P 500 index is 1400, the continuously compounded interest rate is 6% and the continuously compounded dividend yield on the index is 2%.

1. What is the future price for a contract which matures in one year?

$$F_{t,T} = 1400e^{(.06-.02)*1} = 1457.135$$

2. Suppose that traders think that the S&P 500 is overvalued. Since futures have lower transaction's costs, traders sell S&P 500 futures instead of the individual stocks making up the S&P 500. As a result of this selling pressure, the future price for the one year contract drops to 1420 but the index is still 1400. What should you do?

The future is undervalued relative to the index. Therefore you should buy the future, sell the index and invest the proceeds in a T-Bill, i.e., you should do a reverse cash and carry arbitrage. The cashflows associated with this arbitrage are given below.

	t	T
short the index	$1400*250e^{-.02}=$ $1400*245$	$-(250)I_T$
buy the future		$(250)(I_T-1420)$
lend the proceeds from the short sale	$-1400*245$	$1400*245e^{.06}$
Total	0	$245*1400*e^{.06}-250*1420$ $=9209.94$

This is an example of what is referred to as index arbitrage which is a common type of program trading. This type of trading is a way for information to be transmitted from the futures market to the equity market.

Using Stock Index Futures

Hedging Portfolios

Hedging involves taking a position in an asset whose price moves inversely with the price of the asset whose value you wish to hedge. When one does this, gains on one asset offset losses on the other asset and the value of the total position stays roughly the same. How well the hedge works depends on the correlation between the prices of the two assets.

Perfect Hedging

In a perfect hedge all risk associated with future price changes is eliminated. This means that future revenues are known with certainty.

Example: **Market Timing**

Stock index futures enable portfolio managers to move between equities and T-bills in a transactions cost minimizing way. Previously, we saw that we could make a synthetic T-Bill by buying the index and shorting a future, i.e.,

$$\text{Index} - \text{Future} = \text{T-bills}. \qquad (1)$$

A minus sign indicates a short position and a plus sign indicates a long position. By rearranging the terms in the above expression we get

$$\text{Future} + \text{T-Bill} = \text{Index}. \qquad (2)$$

Equation (1) suggests that if a portfolio manager is long an S&P 500 portfolio and she would like to switch into T-bills, she could keep her portfolio and short an S&P 500 future and receive the same cash flows as a long position in a T-Bill.

Equation (2) suggests that if a portfolio manager is long T-bills and she would like to switch to the S&P 500, she could keep the T-bills and buy a S&P 500 future.

Example: Suppose a portfolio manager has a 100 million-dollar S&P 500 portfolio. The current value of the index is 1000 and the continuously compounded interest rate is 6%. The dividend yield on the index is 2%. She thinks that over the next three months T-bills will out perform equities, so she would like to switch to T-bills. She could sell the portfolio and buy T-bills or she could keep the portfolio and sell an S&P 500 future. The cash flows associated with these two choices are given below.

I. Sell portfolio and buy T-bills

	t	T
Sell portfolio	100M	
Buy T-bills	-100M	$(100M)e^{.06*.25}$=101.5M
Total	0	101.5M

Note that:

i. The futures price is $F_{t,T} = I_t e^{(.06-.02)*.25}$ =1010.05

ii. $100M is 100M/1000 = 100,000 units of the index.

iii. Since the dividend yield is 2%, at the end of three months, the portfolio will contain 100,000 * $e^{.02*.25}$ = 100,501 units of the index

iii. Since each contract represents 250 units of the index you should sell 100,501/250 = 402 contracts.

II. Keep equities and sell a future

	t	T
Keep equities		100,501* I_T
Sell 402 futures contracts		(402)(250)(1010.05-I_T)
Total	0	101.5M

Suppose a manager has 100M in T-bills and would like to hold equities for the next three months. She could sell the T-bills and buy an S&P 500 portfolio or she could keep the T-bills and buy a future. The cash flows are given below.

I. Sell T-bills and buy equity

	t	T
Sell T-bills	100M	
Buy equities	-100M	$(100{,}000)\ e^{.02*.25}\ I_T$*
Total	0	$(100{,}000)\ e^{.02*.25}\ I_T$ $=100{,}500 I_T$

*Recall that $100M is equal to 100,000 units of the index

II. Keep T-bills and buy future

	t	T
Keep T-bills		$(100M)e^{.06*.25} = 101.5M$
Buy 402 futures contracts		$(402)(250)(I_T\text{-}1010.05)$ $= (100{,}500)(I_T - I_t e^{(.06-.02)*.25})$
Total	0	$100{,}500 * I_T$

Imperfect Hedge

Suppose you want to hedge a portfolio with an S&P 500 Index future. Your portfolio is not perfectly correlated with the S&P 500. What should be the size of your position?

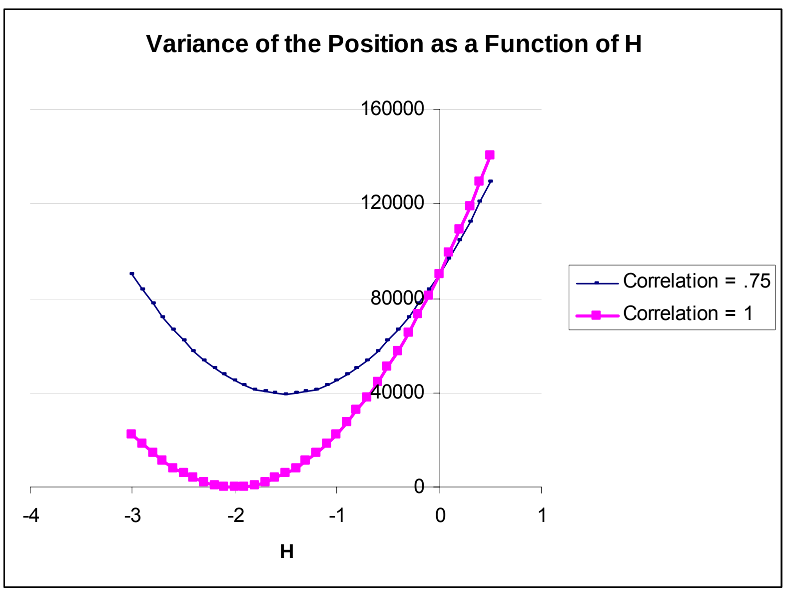
Let r_p = return on your portfolio
I_p = value of your portfolio
H = number of futures contracts
N = notational amount of the futures contract
$r_{S\&P} - r_f$ = return on the futures contract

The return on the hedged position is:

$$r_P I_p + H * N * (r_{S\&P} - r_f)$$

The variance of this position is[1]:

$$\sigma_P^2 I_P^2 + H^2 N^2 \sigma_{S\&P}^2 + 2 I_P H N \, Cov(r_P, r_{S\&P})$$



[1] *Recall the formula for the variance of the sum (aX + bY) where a and b are constants and X and Y are random variables is:

$$Var(aX+bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$$

The variance is minimized at the point where the slope is zero. This holds when [2]:

$$H = -\frac{I_P}{N}\frac{Cov(r_P, r_{S\&P})}{\sigma^2_{S\&P}} = -\frac{I_P}{N}\beta_P$$

Substituting the value of H back into the variance equation we get:

$$\sigma_P^2 I_P^2 + \left(\frac{I_P}{N}\right)^2 \beta_P^2 * N^2 \sigma_{S\&P}^2 - 2 * \left(\frac{I_P}{N}\right)\beta_P * N * cov(r_P, r_{S\&P}) = \sigma_P^2 I_P^2 (1-\rho^2)$$

If the correlation is perfect (ρ = ± 1) then the risk is completely hedged.

If the correlation is zero then H is zero and there is no reduction in the risk of the portfolio.

[2] H is found by taking the derivative of the variance with respect to H and setting the derivative equal to zero.

Currency Futures

Currencies are an asset, just like a stock or a bond. It is assumed that while an investor holds foreign currency, he/she puts the money in foreign T-bills which earn the foreign risk free rate, denoted r_f.

To price a currency future we proceed as before using a cash and carry analysis assuming there is a payout associated with the asset.

We will assume that the interest on the foreign T-bill is earned continuously at the continuously compounded rate r_f.

```
 |-------------------------earn interest continuously-----------------------|
 t                                                                          T
 |--------------------------------------------------------------------------|
sell the future                                              future expires
buy the currency                                             and delivery occurs
```

In a cash and carry, an investor buys the spot good today in order to have it available to deliver against the futures position on the delivery date. When investing in a T-bill, the size of the position grows at the rate of interest. Therefore, to have one unit of foreign currency at delivery, one needs to buy the present value of one unit today, i.e., the investor should buy
$1^*e^{-r_f(T-t)}$ units of the currency.

<u>Example</u>: If the British riskfree rate was 6% and you wanted to have 1 British pound in 6 months, you would buy $e^{-.06^*.5}$=.97 pounds today.

Let P_t be the exchange rate, that is, the cost of one unit of the foreign currency in terms of the domestic currency. If an investor was buying British pounds with dollars, P_t would be the dollar cost of one pound. At the current dollar pound exchange rate P_t is equal to approximately \$1.80.

The cash flows associated with the cash and carry are given below:

	t	T
buy spot	$-P_t{*}e^{-r_f[T-t]}$	*
sell future		$F_{t,T}$
Total	$-P_t{*}e^{-r_f[T-t]}$	$F_{t,T}$

*The spot position has grown from $e^{-r_f t}$ units of the currency to $(e^{-r_f t})*(e^{r_f t})$ =1 unit and this unit is used to cover the short futures position.

This position resembles a T-Bill. Since the investment (i.e., the purchase of the foreign currency) was made in the domestic currency, the investment should earn the domestic risk free rate, denoted r_d.

$$\frac{F_{t,T}}{P_t e^{-r_f(T-t)}} = e^{r_d(T-t)} \Rightarrow F_{t,T} = P_t e^{-r_f(T-t)} e^{r_d(T-t)} = P_t e^{(r_d - r_f)(T-t)}$$

Note that in this case, if the $r_d > r_f$, the futures price is greater than the spot price and if the $r_d < r_f$, the futures price is less than the spot price.

Pound Comp. - cme

Data retrieved at 08/11/04 11:49:50 • All quotes are in exchange local time • Data provided by FutureSource

Contract	Month	Last	Change	Open	High	Low	Volume	OpenInt	Exch	Date	Time
British Pound	Sep '04	1.8228	-0.0014	1.8209	1.8250	1.8196	13633	68789	IMM	08/11/04	11:38:04
British Pound	Dec '04	1.8060	-0.0036	1.8066	1.8100	1.8060	275	819	IMM	08/11/04	10:38:02
British Pound	Mar '05	1.7966y						6	IMM	08/10/04	16:17:58
British Pound	Jun '05	1.7836y							IMM	08/10/04	16:17:58
British Pound	Sep '05	1.7706y						1	IMM	08/10/04	16:17:58
British Pound	Dec '05	1.7576y							IMM	08/10/04	16:17:58
British Pound	Index	1.8273	0.0035	1.8287	1.8287	1.8265			IMM	08/11/04	11:01:09

Yen Comp. - cme

Data retrieved at 08/11/04 11:51:40 • All quotes are in exchange local time • Data provided by FutureSource

Contract	Month	Last	Change	Open	High	Low	Volume	OpenInt	Exch	Date	Time
Japanese Yen	Sep '04	0.9037	0.0053	0.8996	0.9055	0.8987	17200	99590	IMM	08/11/04	11:40:55
Japanese Yen	Dec '04	0.9075	0.0049	0.9053	0.9083	0.9045	334	10830	IMM	08/11/04	11:17:21
Japanese Yen	Mar '05	0.9077y						10	IMM	08/10/04	16:17:44
Japanese Yen	Jun '05	0.9137y						3	IMM	08/10/04	16:17:44
Japanese Yen	Sep '05	0.9207y							IMM	08/10/04	16:17:45
Japanese Yen	Dec '05	0.9281y							IMM	08/10/04	16:17:45
Japanese Yen	Index	0.9031	0.0039	0.9007	0.9031	0.8977			IMM	08/11/04	11:01:09

Pricing with Transactions Costs

There are a number of transactions costs associated with cash and carry positions. There are bid-ask spreads on both the future and the spot good and there are differential borrowing and lending rates. In this section, we consider how these transactions costs affect futures pricing.

Notation:

$F^{A}_{t,T}$ = the futures ask price

$F^{B}_{t,T}$ = the futures bid price

P^{A}_{t} = the spot ask price

P^{B}_{t} = the spot bid price

$r^{B}_{t,T}$ = borrowing rate

$r^{L}_{t,T}$ = lending rate

Note that you **buy at the ask price** and you **sell at the bid price**.

Once again the futures prices are determined by a no arbitrage condition. In order to determine what these conditions are, consider a "cashless" cash and carry and a reverse "cashless" cash and carry.

"Cashless" Cash and Carry

	t	T
buy the spot	$-P_t^A$	
sell the future		$F_{t,T}^B$
borrow the spot price	P_t^A	$-P_t^A e^{r_B(T-t)}$
total	0	$F_{t,T}^B - P_t^A e^{r_B(T-t)}$

If $F_{t,T}^B - e^{r_B(T-t)} P_t^A > 0$, then there is an arbitrage opportunity. Therefore to eliminate this type of arbitrage,

$$F_{t,T}^B \leq e^{r_B(T-t)} P_t^A.$$

Reverse "Cashless" Cash and Carry

	t	T
sell spot	P^B_t	
buy future		$-F^A_{t,T}$
buy T-Bill with the proceeds from the short	$-P^B_t$	$P^B_t\, e^{r_L(T-t)}$
total	0	$-F^A_{t,T}+e^{r_L(T-t)})P^B_t$

If $-F^A_{t,T}+e^{r_L(T-t)}\, P^B_t>0$ then there is an arbitrage opportunity. To eliminate this type of arbitrage opportunity it must be the case that

$$F^A_{t,T} \leq e^{r_L(T-t)}\, P^B_t.$$

If the bid-ask spread for futures is approximately zero then the restrictions on the futures price become

$$e^{r_B(T-t)}\, P^A_t > F_{t,T} > e^{r_L(T-t)}\, P^B_t.$$

<u>Example</u>: Consider the following prices:

Bid on IBM	139.90	Ask on IBM	140.10
Bid on IBM future	154.90	Ask on IBM future	155.90
Bid on 1 yr T-Bill	908678	Ask on 1 yr T-Bill	909504

T-Bills have a face value of $1,000,000.

The lending rate is the rate if we buy a T-bill at the ask. This rate is

$$\$1{,}000{,}000/\$909{,}504 = 1.0995 = 1 + r_{lend}$$

The borrowing rate is the rate if we short a T-bill at the bid. This rate is

$$\$1{,}000{,}000/\$908{,}678 = 1.1005 = 1 + r_{borrow}$$

If we buy the share of IBM at the ask, borrow by shorting a T-bill and sell the future at the bid, our cash flows are:

	t	T
buy the spot at the ask	-140.10	
sell the future at the bid		154.90
borrow at the borrowing rate	140.10	-(1.1005)(140.10) = -154.18
Total	0	.72 **ARBITRAGE**

If we short the share of IBM at the bid, lend the proceeds by buying a T-bill and buy the future at the ask, our cash flows will be:

	t	T
short the spot at the bid	-139.90	
buy the future at the ask		- 155.90
lend at the lending rate	139.90	-(1.0995)(139.90)
Total	0	-$2.08 **NO ARBITRAGE**