ARBITRAGE RESTRICTIONS

Arbitrage restrictions are restrictions on the possible prices of puts and calls. They are called arbitrage restrictions because if these restrictions are violated, an investor can earn an arbitrage profit.

An arbitrage profit is the profit earned on a portfolio with the following two characteristics:

1. self-financing
2. riskless

What rate of return should a portfolio with these characteristics earn?

These restrictions are very general. In particular, they do not require we know any thing about investor risk preferences or the stochastic behavior of stock prices.

PUT-CALL PARITY - The relationship between the values of European puts and calls with the same expiration date, same exercise price and same underlying asset.

- Previously we saw that

```
                          looks like
1 written call  =============>  1 written put
1 long share
```

and

```
                          looks like
1 long put      ==============> 1 long call
1 long share
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This suggests that we can make a "synthetic" put from a call and the stock and a "synthetic" call from a put and the stock. By synthetic, I mean exactly replicate the cash flows.

The cash flow from a long call is:

	Today	AT EXPIRATION $S_T \leq K$	$S_T > K$
1 LONG CALL	-C	0	S_T - K

Suppose we try to recreate the cash flows at expiration with a put and the stock

	Today	AT EXPIRATION $S_T \leq K$	$S_T > K$
1 LONG PUT	-P	K-S_T	0
1 LONG SHARE	-S	S_T	S_T
TOTAL	-P-S	K	S_T

The cash flows don't match! We have K too much at expiration.

Now try borrowing PV(K) to get the cash flows to match.

		AT EXPIRATION	
	Today	S_T K	$S_T > K$
1 LONG PUT	-P	$K-S_T$	0
1 LONG SHARE	-S	S_T	S_T
BORROW PV(K)	Ke^{-rT}	-K	-K
TOTAL	$-P-S+Ke^{-rT}$	0	S_T-K

*Always think of borrowing as shorting a T-Bill and lending as buying a T-Bill.

Now the cash flows are the same.

Now I have two portfolios:

- 1 long call

- 1 long put
  1 long share ==========> SYNTHETIC CALL
  KR^{-T} borrowed

Both give <u>exactly</u> the same payoff regardless of the value of S_T. Therefore, they must sell for the same price.

$$\mathbf{C = P + S - Ke^{-rT}}$$

This is called an arbitrage relationship because if this relationship doesn't hold there is an arbitrage opportunity.

Example: Suppose

K=40
S=40
r=.10
T=.5 (6 months)
C=4.50
P=2.41

In this case, put-call parity is violated since

$$C = 4.50 > P + S - e^{-rT} = 2.41 + 40 - (40)e^{-.10*.5} = 4.36$$

The real call is selling for more than the synthetic call. This means there is an arbitrage opportunity.

KEY RULE FOR ANY ARBITRAGE:

BUY LOW (i.e., buy the relatively underpriced asset)
SELL HIGH (i.e., sell the relatively overpriced asset)

IN THIS CASE, SELL THE REAL CALL AND BUY THE SYNTHETIC CALL!

Aside:

A positive sign implies a long position --> buying or lending
A negative sign implies a short position --> shorting or borrowing

The cash flows associated with this strategy are given below:

		AT EXPIRATION	
	Today	$S_T \leq K$	$S_T > K$
1 SHORT CALL	C= 4.50	0	$K-S_T=40-S_T$
1 LONG PUT	-P=-2.41	$K-S_T=40-S_T$	0
1 LONG SHARE	-S=-40	S_T	S_T
BORROW Ke^{-rT}	$Ke^{-rT}=38.04$	-K=-40	-K=-40
TOTAL	$C-P-S-Ke^{-rT}=.14$	0	0

Suppose $C < P + S - Ke^{-rT}$. What is the arbitrage?

Put-Call Parity gives a "recipe" for making a synthetic call.

How would you make a synthetic put?

How would you make a synthetic share?

How would you make a synthetic T-Bill?

PUT-CALL PARITY FOR OPTIONS ON STOCKS WHICH PAY DIVIDENDS:

How do dividends change things?

Let δ be the continuously compounded dividend yield.

REAL CALL

	Today	AT EXPIRATION $S_T \leq K$	$S_T > K$
1 LONG CALL	-C	0	S_T - K

SYNTHETIC CALL

	Today	AT EXPIRATION $S_T \leq K$	$S_T > K$
1 LONG PUT	-P	K-S_T	0
LONG PV of 1 SHARE	$-Se^{-\delta T}$	S_T	S_T
BORROW Ke^{-rT}	Ke^{-rT}	-K	-K
TOTAL	$-P + Se^{-\delta T} + Ke^{-rT}$	0	S_T-K

Since the payoff is the same as the payoff for the call, put-call parity becomes

$$\mathbf{C = P + Se^{-\delta T} - Ke^{-rT}}$$

where T is the time to expiration, δ is the continuous dividend yield and r is the continuously compounded risk-free rate.

PUT-CALL PARITY FOR OPTIONS ON CURRENCIES

Let S be the price of the foreign currency in dollars (i.e., $1.73 per British pound)
Let r_f be the foreign risk-free rate
Let r_d be the domestic risk-free rate

Consider two portfolios:
<u>REAL CALL</u>

		AT EXPIRATION	
	Today	$S_T \leq K$	$S_T > K$
1 LONG CALL	-C	0	S_T - K

<u>SYNTHETIC CALL</u>

		At Expiration	
	Today	$S_T \leq K$	$S_T > K$
Buy 1 long put	-P	K-S_T	0
Buy exp(-r_ft) of the foreign currency and invest at r_f	$-Se^{-rfT}$	S_T	S_T
Borrow K*exp(-r_dt)	Ke^{-rT}	-K	-K
Total	$-P-Se^{-rfT}+Ke^{-rT}$	0	S_T-K

Therefore

$$\mathbf{C - P = Se^{-rfT} - Ke^{-rT}}$$

This is like put-call parity on a stock which pays a dividend but with $\delta = r_f$.

PUT-CALL PARITY FOR OPTIONS ON FUTURES

Let S_T = spot price at time T
F = Futures price

Consider two portfolios:

		At Delivery	
	Today	$S_T \leq F$	$S_T > F$
1 Long Future	0	S_T-F	S_T-F

		At Expiration	
	Today	$S_T \leq K$	$S_T > K$
1 Long Call with Strike Price K	-C	0	S_T-K
1 Short Put with Strike Price K	P	S_T-K	0
Lend PV(K-F)	$-(K-F)e^{-rT}$	K-F	K-F
Total	$-C+P-(K-F)e^{-rT}$	S_T-F	S_T-F

Since the two portfolios give the same payoffs at expiration, they must sell for the same price. The initial cost of the future is zero, so

$$C = P - (K-F)e^{-rT} = P + (F - K)e^{-rT}$$

If K=F, then C = P
If K>F, then C < P
If K<F, then C > P

This is like put-call parity on a stock which pays a dividend but with δ=r.

OTHER ARBITRAGE RESTRICTIONS:

We will use the same approach as before (i.e., riskless arbitrage) to derive several other important pricing relationships.

- **$S \geq C \geq \max[0, S\text{-}K]$**

Suppose C > S. What should you do to earn an arbitrage profit?

Sell the call and use the proceeds to buy the stock. You now have a covered short position in a call. You will receive C-S>0 up-front and at expiration you will have \$K if $S_T > K$ and $S_T > 0$ if $S_T \geq K$.

- **The value of an American call must be <u>strictly</u> greater than S-K (i.e., the value of immediate exercise) at any time other the expiration date or just before an ex-dividend date.**

We know from previous restriction that $C \geq S\text{-}K$. Therefore this restriction is violated if at anytime other than expiration or an ex-dividend day, C = S-K.

If C = S-K, then C is relatively <u>underpriced</u>. A portfolio consisting of 1 share and \$K short in T-bills is relatively <u>overpriced</u>.

	TODAY	SOMETIME PRIOR TO EX-DIVIDEND DATE
BUY CALL	-C	EXERCISE AND RECEIVE S-K
SHORT STOCK	S_t	COVER SHORT -S
BUY T-BILLS	-K	K+INT
TOTAL	S_t-K-C=0	INT

The previous restriction implies that a call should <u>never</u> be exercised at any time other than the expiration date or just before an ex-dividend date.

One of the implications of this is that an American call option on a stock which pays no dividend prior to expiration should never be exercised early.

What is the relationship between the value an American call and a European call on a stock which pays no dividend prior to expiration?

INTUITION:

- First consider the case where the stock pays no dividend prior to expiration. What are the costs and benefits of early exercise?

<u>COSTS</u>

1. Pay $K sooner and lose interest on K
2. Lose the right to change your mind later

<u>BENEFITS</u>

1. None

Therefore never exercise early

- Now consider the case where the stock pays a dividend prior to expiration. What are the costs and benefits of early exercise?

<u>COSTS</u>

1. Pay $K sooner and lose interest on K
2. Lose the right to change your mind later

<u>BENEFITS</u>

1. Receive the dividend

You <u>may</u> want to exercise early

What determines whether or not you want to exercise early?

1. How deep in or out of the money the option is (i.e., S-K)

2. r

3. T

4. volatility

5. size of the dividend

- RESTRICTIONS ARISING FROM THE STRIKE PRICE

- **$C(K_1) > C(K_2)$ where $K_1 < K_2$**

If this relationship is violated (i.e., $C(K_1) < C(K_2)$), what is the arbitrage?

1. sell $C(K_2)$
2. buy $C(K_1)$

<u>Example</u>: Suppose K_1=40 and K_2=50, C(40)=2.78 and C(50)=3.00

		AT EXPIRATION		
	TODAY	S_T<40	40≤S_T≤50	S_T>50
BUY C(40)	-2.68	0	S_T-40	S_T-40
SELL C(50)	3.00	0	0	50-S_T
TOTAL	.32	0	S_T-40 > 0	10

- **$K_2 - K_1 \geq C(K_1) - C(K_2)$**

<u>Example</u>: Suppose K_1=40 and K_2=50, $C(K_1)$=12.00 and C(50)=.50.

Since K_2-K_1=10 < $C(K_1) - C(K_2)$=11.50. The restriction is violated.

$C(K_1) - C(K_2)$ is bigger than it should be, so $C(K_1)$ is overvalued relative to $C(K_2)$.

What are the trades you should undertake to take advantage of this?

		AT EXPIRATION		
	TODAY	S_T<40	40≤S_T≤50	S_T>50
SELL C(40)	12.00	0	40-S_T	40-S_T
BUY C(50)	-.50	0	0	S_T-50
LEND	-11.50	11.50+INT	11.50+INT	11.50+INT
TOTAL	.00	11.50+INT	51.50-S_T+INT	1.50+INT

Note that this strategy will work for both American and European call options.

- **Let $(K_3 - K_2)/(K_3 - K_1) = \lambda$ then $C(K_2) \leq \lambda C(K_1) + (1-\lambda)C(K_3)$**

This is called the <u>convexity condition</u>.

<u>Example</u>: Suppose K_1=40, K_2=50 and K_3=60 ===> λ = .5

The arbitrage restriction says

$$C_2 \leq .5C_1 + .5C_3$$

This can be rewritten as

$$2C_2 \leq C_1 + C_3$$

Two C_2's should sell for less than 1 C_1 and 1 C_3.

Example: Suppose K_1=40, K_2=50 and K_3=70 ===> λ = 2/3

The arbitrage restriction says

$$C_2 \leq (2/3)C_1 + (1/3)C_3$$

This can be rewritten as

$$3C_2 \leq 2C_1 + C_3$$

Three C_2's should sell for less than 2 C_1's and 1 C_3.

Example: Suppose C(40)=2.50, C(50)=2.00 and C(70)=.50.

Convexity is violated since λ = (70-50)/(70-40) = 2/3 and

$$3x(2.00)=6 > 2x(2.50)+ 1x(.50)=5.50$$

How can you take advantage of this?

- What is relatively overpriced?
- What is relatively underpriced?

The cash flows are given below:

		At Expiration			
	Today	S_T<40	40≤S_T<50	50≤S_T<70	S_T>70
BUY 2 C(40)	-5.00	0	2(S_T-40)	2(S_T-40)	2(S_T-40)
SELL 3 C(50)	6.00	0	0	3(50-S_T)	3(50-S_T)
BUY 1 C(70)	-.50	0	0	0	(S_T-70)
TOTAL	.50	0	2(S_T-40)	70-S_T	0

Example: Suppose

1. Current stock price is $40
2. $r=15\%$
3. The January expiration is 3 months from now
4. The stock pays no dividends

Consider the following call prices:

K\T	JAN	APR	JUL
35	1	6	14
40	2	5	7
45	4	3	5

Which arbitrage restrictions are violated?

- RESTRICTIONS ARISING FROM THE TIME TO EXPIRATION

- **$C(T_2) \geq C(T_1)$ where $T_2 \geq T_1$**

Arbitrage Restrictions for Puts

- **$K \geq P \geq \max[0, K-S]$**

The most you could ever earn from a put is K (this occurs if the stock price equals zero), so you should never pay more than K.

Since you never have to exercise the put, the price must always be greater than or equal to zero.

The put price for an American put must exceed the value of immediate exercise or else there is an arbitrage opportunity.

- Optimal Exercise

Intuition:

- First consider the case where there are no dividends:

COSTS:

1. lose the right to change your mind later

BENEFITS:

1. receive the strike price sooner

At every point in time there is both a cost and a benefit. Therefore, early exercise is always a possibility.

• Now consider the case where there are dividends:

<u>COSTS</u>:
1. lose the right to change your mind later
2. the put becomes more profitable if you wait until after the stock goes ex-dividend.

<u>BENEFITS</u>:

1. receive the strike price sooner

What determines whether or not you want to exercise early?

1. How deep in or out of the money the option is (i.e., S-K)

2. r

3. T

4. volatility

5. size of the dividend

- RESTRICTIONS ARISING FROM THE STRIKE PRICE
- **$P(K_2) \geq P(K_1)$ if $K_2 \geq K_1$**
- **$K_2 - K_1 \geq P(K_2) - P(K_1)$**
- **$P(K_2) \geq \lambda P(K_1) + (1-\lambda)P(K_3)$ where $\lambda=(K_3-K_2)/(K_3-K_1)$**

<u>EXTENSIONS OF PUT-CALL PARITY</u>

- **Put-Call Parity for Non-Dividend Paying American Options**

$$\mathbf{C - S + K \geq P \geq C - S + Ke^{-rT}}$$

Note:

- The RHS is just the standard put-call parity relationship for European options. Since we know that an American option is always worth at as much as a European option this is not a surprise.

- If $C - S + K < P$, there is an arbitrage opportunity. What is it?

- This relationship implies that the maximum difference between the value of an American put and a European put is

$$(C - S + K) -(C - S + Ke^{-rT}) = Ke^{-rT}(e^{rT} - 1)=\text{interest on PV(K)}$$

- **Put-Call Parity for Dividend Paying American Options**

$$C - (Se^{--\delta T}) + K \geq P \geq C - S + Ke^{-rT}$$