■ **PRICING CALLS ON DIVIDEND PAYING STOCKS**:

Assume that you know the value of all dividends between now and expiration.

■ European calls

Assume the stock price behaves in the usual way:

uS
S
dS

where

u and d are <u>ex-dividend</u> stock price movements

θ is the gross dividend yield over one period (i.e. $\theta = e^{\delta T}$). This means that if you start with 1 share and you reinvest the dividends, you will have θ shares one period from now.

The call price tree then becomes:

$C_u = \max[0, uS-K]$
C
$C_d = \max[0, dS-K]$

To price the call one period from expiration, we replicate the payoffs to the call with the stock

$$\Delta\theta\, uS + RB = C_u$$
$$\Delta\theta\, dS + RB = C_d$$

Note that if you have Δ shares initially, you will have $\Delta\theta$ shares one period later due to the dividend.

Solving we get:

$$\Delta = \frac{C_u - C_d}{\theta S(u-d)} \qquad B = \frac{uC_d - dC_u}{R(u-d)}$$

To preclude arbitrage, the price of the call must be the same as the price of the replicating portfolio, so

$$C = \Delta S + B = \frac{p_{DIV} C_u + (1 - p_{DIV}) C_d}{R}$$

where

$$p_{DIV} = \frac{\frac{R}{q} - d}{u - d}$$

■ American Calls on Dividend Paying Stocks

For an American call the value one period from expiration is just

$$C_{AM} = \max\left[S-K, \frac{p_{DIV}C_u + (1-p_{DIV})C_d}{R}\right]$$

Example:

S =45, K=40, u=1.2, d=1/1.2, θ=1.03 and R=1.025

$$p_{DIV} = \frac{\frac{1.025}{1.03} - \frac{1}{1.2}}{1.2 - \frac{1}{1.2}} = .44$$

		u^2S = 64.80
	uS=54	
S=45		udS = 45
	dS=37.50	
		d^2S = 31.25

C_{uu} = 24.80

C_u=max(54-40=14, $\frac{(.44)(24.80)+(.56)(5)}{(1.025)} = 13.37$) = 14

EXERCISE EARLY

C=max(45-40=5, $\frac{(.44)(14)+(.56)(2.146)}{(1.025)} = 7.18$)=7.18

C_{ud} = 5

DON'T EXERCISE EARLY

C_u=max(37.50-40=-2.50, $\frac{(.44)(5)+(.56)(0)}{(1.025)} = 2.146$) = 2.1446

DON'T EXERCISE EARLY

C_{dd} = 0

BINOMIAL PRICING FOR OPTIONS ON CURRENCIES

Let S be the price of the foreign currency in dollars (i.e $1.73 per British pound)
Let R_f be the per period gross foreign risk-free rate = $e^{r_f h}$
Let R_d be the per period gross domestic risk-free rate = e^{rh}

Assume the currency price behaves in the usual way:

$$S \quad \begin{matrix} uS \\ \\ dS \end{matrix}$$

The foreign interest rate acts like a dividend yield. If you start with Δ units of the currency, you will have $R_f\Delta$ units of the currency one period from now.

The call price tree is the same as before:

$$C \quad \begin{matrix} C_u = \max[0, uS-K] \\ \\ C_d = \max[0, dS-K] \end{matrix}$$

To price the call one period from expiration, we replicate the payoffs to the call.

$$\Delta R_f uS + R_d B = C_u$$
$$\Delta R_f dS + R_d B = C_d$$

Solving we get

$$\Delta = \frac{C_u - C_d}{R_f S(u-d)} \qquad B = \frac{uC_d - dC_u}{R_d(u-d)}$$

To preclude arbitrage, the price of the call must be the same as the price of the replicating portfolio, so

$$C = \Delta S + B = \frac{p_{curr}C_u + (1 - p_{curr})C_d}{R}$$

where

$$p_{curr} = \frac{\frac{R_d}{R_f} - d}{u - d}$$

This is like the binomial with dividends where $\theta = R_f$

For an American call the value one period from expiration is just

$$C_{AM} = \max[S-K, \frac{p_{cur}C_u + (1-p_{cur})C_d}{R}]$$

Example:

S =1.65, K=.85, u=1.5, d=.5, R_d=1.05 and R_f=1.10

	uS=2.475		C_u =max[0, uS-K] = 1.625
S=1.65		C =	
	dS=.825		C_d= max[0,dS-K] = 0

p_{cur} = (1.05/1.1 - .5)/(1.5 - .5) = .4545

C_{EUR} = [(.4545)(1.625)]/(1.05) =.703

C_{AM} = max[S-K, C_{EUR}] = max[.80, .703] = .80 ⇒ Exercise

BINOMIAL PRICING FOR OPTIONS ON FUTURES

Assume the futures price behaves in the following way:

$$
F \rightarrow \begin{cases} uF \\ dF \end{cases}
$$

The call price tree then becomes:

$$
C \rightarrow \begin{cases} C_u = \max[0, uF\text{-}K] \\ C_d = \max[0, dF\text{-}K] \end{cases}
$$

To price the call one period from expiration, we replicate the payoffs to the call with the future and a T-Bill. Note that the payoff to the one future at expiration is (uF-F) or (dF-F). This differs from a stock which pays off uS or dS. We want to choose Δ and B such that:

$$
\Delta(uF-F) + RB = C_u
$$
$$
\Delta(dF-F) + RB = C_d
$$

Solving we get

$$
\Delta = \frac{C_u - C_d}{F(u-d)} \qquad B = \frac{(1-d)C_u + (u-1)C_d}{R(u-d)}
$$

To preclude arbitrage, the price of the call must be the same as the price of the replicating portfolio. Note that the cost of delta futures is zero, so

$$
C = \Delta S + B = \frac{p_{future} C_u + (1 - p_{future})C_d}{R}
$$

where

$$
p_{future} = \frac{1-d}{u-d}
$$

This is like the binomial with dividends with θ=R.
For an American call the value one period from expiration is just

$$C_{AM} = \max[S-K, \frac{p_{future}C_u + (1-p_{future})C_d}{R}]$$

Example:

F =200, K=40, u=1.5, d=.5 and R=1.10

	uF=300		C_u =max[0, uF-K] = 260
F=200		C =	
	dF=100		C_d= max[0,dF-K] = 60

p_{future} = (1 - .5)/(1.5 - .5) = .5

C_{EUR} = [(.5)(260) + (.5)(60)]/(1.10) = 145.45

C_{AM} = max[S-K, C_{EUR}] = max[160, 145.45] = 160 $\Rightarrow$ Exercise

Note that an American call on a future will be exercised early if it is sufficiently deep in the money.

■ PRICING PUTS

Puts are priced using the same approach as with calls.

	uS			$P_u = \max[0, K-uS]$
S		=====>	P	
	dS			$P_d = \max[0, K-dS]$

We want to price a put by replicating the payoffs and then setting the price of the put equal to the price of the replicating portfolio.

	$\Delta uS + RB = P_u$		
$\Delta S + B$		====>	$\Delta = \dfrac{P_u - P_d}{S(u-d)}$
	$\Delta dS + RB = P_d$		$B = \dfrac{uP_d - dP_u}{R(u-d)}$

Since the replicating portfolio has the same payoffs as the put, the two portfolios must sell for the same price:

$$P = \Delta S + B = \frac{pP_u + (1-p)P_d}{R}$$

where

$$p = \frac{R-d}{u-d}$$

■ Early Exercise with American Puts:

Recall that early exercise is always a possibility with an American put. Therefore you must always check whether or not early exercise is optimal.

<u>Example</u>: Suppose S=50, u=2, d=.5, R=1.25 and K=60

$u^2S=200$

$uS=100$

$S=50$ $\qquad$ $udS=50$

$dS=25$

$d^2S=12.50$

$P_{uu} = 0$

$$P_u = \max\left[60-100=-40, \frac{(.5)(0)+(.5)(10)}{1.25}=4\right]=4$$

$$P=\max\left[60-50=10, \frac{(.5)(4)+(.5)(35)}{1.25}=15.60\right]=15.60 \qquad P_{ud} = 10$$

$$P_d=\max\left[60-25=35, \frac{(.5)(10)+(.5)(47.50)}{1.25}=23\right]=35$$

$P_{dd} = 47.50$

■ ELASTICITY

Recall that

1. $\Delta = \frac{C_u - C_d}{uS - dS} =$ change in the call price given a $1 change in the stock price

2. for a call

$$0 \leq \Delta \leq 1$$

Define the elasticity as

$$\Omega_{call} = \frac{(C_u - C_d)/C}{(uS - dS)/S} = \frac{\text{\% change in the call price}}{\text{\% change in the stock price}}$$

$$= \text{\% change in the call price given 1\% change in the stock price}$$

If you rearrange terms in the expression above we get

$$\Omega_{call} = \Delta S/C$$

<u>Example</u>: S=80, K=80, R=1.1, u=1.5 and d=.5 ===> p=.6

	uS=120	C_u=40
S=80	C=21.81	
	dS=40	C_d= 0

$\Delta = (C_u - C_d)/S(u-d) = .5$

$\Omega = .5(80)/(21.81) = 1.83$

The delta says that I get 50% of the "action" of 1 share when I buy 1 call. The cost of a call relative to the share is C/S = 21.81/80 = 27%. Therefore I get 50% of the action for 27% of the cost. The ratio of action to percentage cost is the elasticity. That is

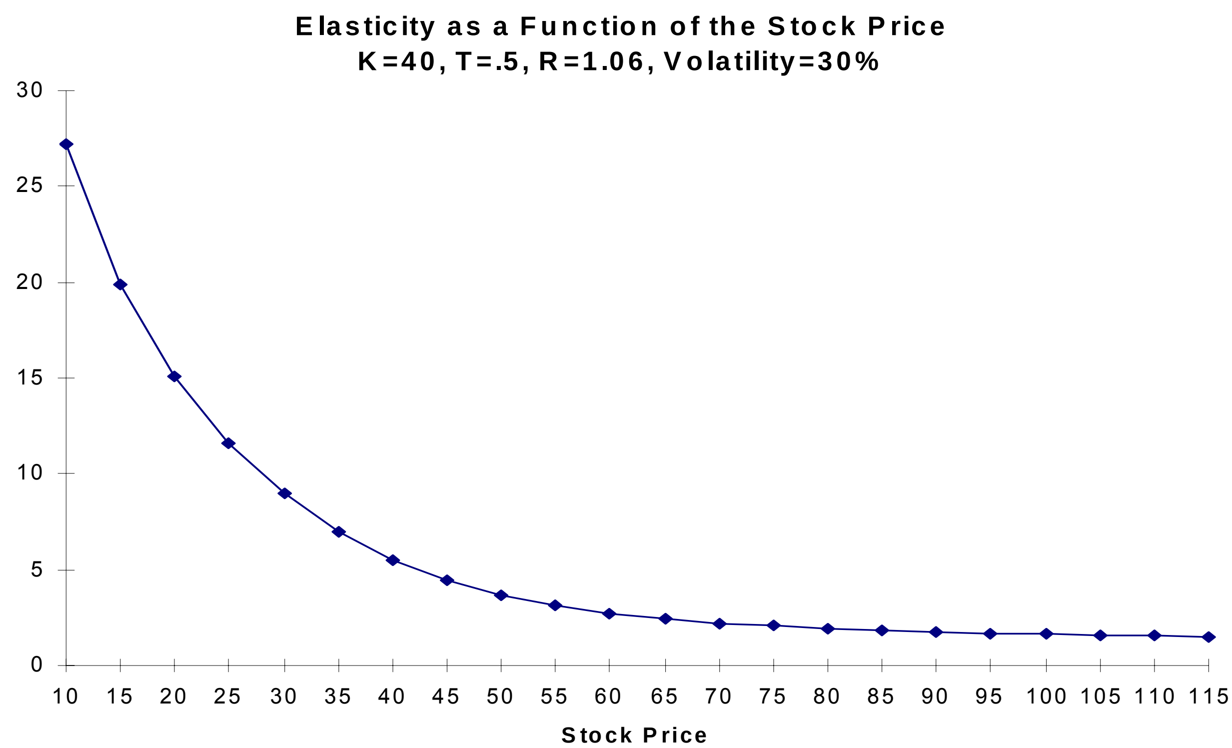
$$\Omega = \Delta/(C/S)$$

The elasticity tells you how much "action" you get from a call relative to a stock as a percentage of money invested.

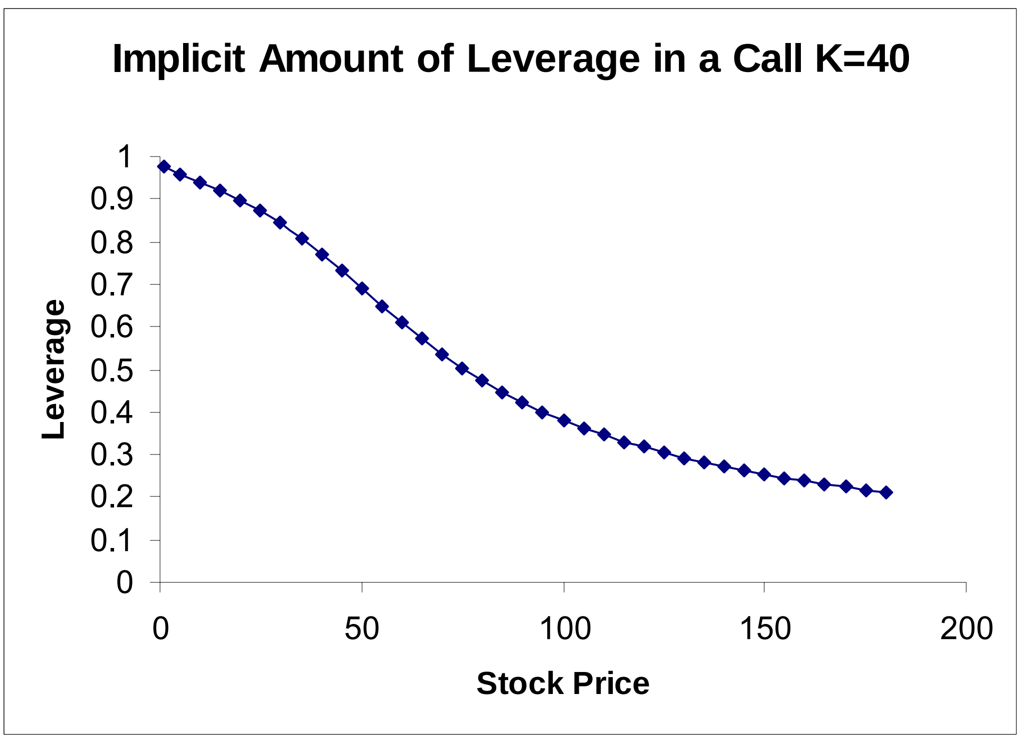
It can be shown that $\Omega_{call} \geq 1$

To understand this, recall put-call parity

$$C = P + (S - KR^{-T})$$
$$= \text{(insurance)} + \text{(levered position in the stock)}$$



	K=40, r=.05, T=1, v=.3, div=0			
	Delta	Call	B	% Leverage
1	0.0000	0.0000	0.0000	0.9759
10	0.0000	0.0000	-0.0001	0.9400
20	0.0231	0.0470	-0.4147	0.8983
30	0.2603	1.2269	-6.5835	0.8429
40	0.6243	5.6925	-19.2776	0.7720
50	0.8555	13.2310	-29.5458	0.6907
60	0.9524	22.3505	-34.7913	0.6089
70	0.9854	32.0703	-36.9110	0.5351
80	0.9957	41.9861	-37.6695	0.4729
90	0.9987	51.9613	-37.9249	0.4219
100	0.9996	61.9540	-38.0086	0.3802
110	0.9999	71.9518	-38.0358	0.3458
120	1.0000	81.9511	-38.0447	0.3171
130	1.0000	91.9509	-38.0477	0.2927
140	1.0000	101.9509	-38.0487	0.2718
150	1.0000	111.9508	-38.0490	0.2537
160	1.0000	121.9508	-38.0491	0.2378



When you have leverage, you get more action per dollar invested.
Put Elasticity:

$$\Omega_{put} = \frac{(P_u-P_d)/P}{uS-dS)/S} = \Delta_{put}S/P \leq 0$$

Note that the put elasticity is not necessarily less than -1.

Example: S=80, K=80, R=1.1, u=1.5 and d=.5 ===> p=.6

	uS=120		P_u=0
S=80		P=14.54	
	dS=40		P_d=40

$\Delta = (P_u-P_d)/S(u-d) = -.5$

This can be interpreted as meaning it takes -.5 shares to replicate the "action" of 1 put or it takes 2 puts to replicate the "action" of 1 share

If I wanted to get the action associated with owning one share I could either

1. buy one share for $80

2. short two puts for (2)(14.54)=-29.08

The elasticity is :

$\Omega = \Delta S/P = -2.75$

Expected Return and Beta of an Option

The expected return on the underlying asset is

$$ER(Stock) = q\frac{uS}{S} + (1-q)\frac{dS}{S}$$

and the expected return on a call is

$$ER(call) = q\frac{C_u}{C} + (1-q)\frac{C_d}{C}$$

Recall that Δ and B were defined such that

$$\Delta uS + RB = C_u \qquad (1)$$

$$\Delta dS + RB = C_d \qquad (2)$$

If you multiple (1) by q and (2) by (1-q), add the two equations together and rearrange terms, you get

$$\Delta S*ER(Stock) + RB = C+ER(Call) \qquad (3)$$

Note that since C = ΔS+B, we can write B as C-ΔS, substitute into (3) and rearrange terms to get

$$\Delta S*[ER(Stock)-R] = C*[ER(Call)-R]$$

Or

$$ER(Call) – R = (\Delta S/C)[ER(Stock) –R] = \Omega[ER(Stock)-R] \qquad (4)$$

Using the CAPM we can find the beta of a call.

The CAPM says that

$$ER(Stock) - R = \beta^*[ER_{mkt\,-}R]$$

Substituting into (4) we get

$$ER(Call) - R = \Omega[ER(Stock)-R] = \Omega\, {}^*\beta^*[ER_{mkt\,-}R]$$

Therefore

$$\beta_{call\,=}\ \Omega\beta_{\ stock}$$

The Relationship Between DCF and Option Pricing

Consider the following information

S	=	100	R	=	1.05
u	=	1.20	K	=	100
d	=	.833	$E(R_S)$	=	1.15

Let q denote the actual (as opposed to the risk neutral) probability that the stock will increase by u next period.

If 1.15 is the actual expected return then by definition :

$$[q*uS + (1-q)*dS]/S = E(R_S)$$

and which implies that:

$$q = [E(R_S) - d]/[u-s] = [1.15 - .8333]/[1.2 - .8333] = .8636.$$

Given these assumptions, the stock price tree is:

			172.80	(probability $= q^3 = .6441$)
		144		
	120		120	(probability $= 3q^2(1-q) = .3052$)
S=100		100		
	83.333		83.333	(probability $= 3q(1-q)^2 = .0482$)
		69.444		
			57.87	(probability $= (1-q)^3 = .0025$)

Expected stock price is:

$$E(S) = 172.80*.6441 + 120*.3052 + 83.333*.0482 + 57.87*.0025 = 152.086$$

A naive person who knew DCF analysis but not option pricing might try to value a call option on this stock by computing E(S) -K, where E(S) is the expected stock price and K is the strike price on the option and then discounting E(S)-K by the stock's discount rate.

This would produce an option value equal to:

$$[152.086 - 100]/[1.15]^3 = 34.25$$

This is NOT correct. Why?

The main problem stems from the fact that options are levered positions and the amount leverage varies over the life of the option. Therefore the discount rate is different depending upon where you are in the "tree".

Now we will show that a correctly done DCF analysis will yield a price which is exactly the same as the binomial price. By DCF analysis I mean a valuation method which entails calculating the expected cash flows using the true probabilities and then discounting the expected cash flows at a risk-adjusted rate. Recall that the binomial pricing formula can be interpreted as a valuation method which entails calculating the expected cash flows using the risk neutral probabilities and then discounting at the risk-free rate.

Why don't we see DCF analyses of options? As we will see that this approach is much more cumbersome, which way no one ever does it this way.

Computing an option price using a DCF approach:

1. Compute the true probability of the stock going up which is given by:

$$q = [E(R_S) - d]/ [u-d]$$

2. Compute the expected payoff to the call next period using the true probabilities

3. Compute the appropriate discount rate.

The discount rate is clearly not the discount rate for the stock since the option is riskier due to the leverage.

We know that a call is equivalent to a portfolio containing Δ units of the stock and B dollars in T-bills. Therefore the required rate of return for the call is the same as the required rate of return for the replicating portfolio.

The expected return for the replicating portfolio is given by the weighted average of the return on the stock and the return on the T-bill, i.e.,

$$\gamma = \frac{\Delta S}{\Delta S + B} E(R_S) + \frac{B}{\Delta S + B} R$$

where $\Delta = (C_u - C_d)/S(u-d)$ and $B = (uC_d - dC_u)/R(u-d)$

4. Therefore the DCF option price is :

$$C = \frac{qC_u + (1-q)C_d}{\gamma}$$

Define: $E(C_+)$ = The expected payoff for the option one period ahead

			Price = 72.80 S=172.80
		Price = 48.765 $E(C_+)$= 65.60 S=144 Δ=1, B=-95.223 γ = 1.3452	
	Price=31.84 S=120 $E(C_+)$= 43.65 Δ=.8527, B=-70.485 γ = 1.371		Price = 20 S=120
Price = 20.354 $E(C_+)$=28.36 S=100 Δ=.6951, B=-49.156 γ=1.3915		Price= 11.2556 $E(C_+)$= 17.27 S=100 Δ=.5454, B=-43.284 γ = 1.5346	
	Price=6.33 $E(C_+)$ = 9.7203 S=83.333 Δ=.3683, B=-24.36 γ = 1.5348		Price = 0 S=83.333
		Price = 0 S=69.444 $E(C_+)$ = 0 Δ=0, B=0 γ = N/A	
			Price = 0 S=57.870