EXACT OPTION PRICING

■ BINOMIAL OPTION PRICING

Previously we saw that

$$C = P + S - Ke^{-rT}$$

We could price a call if we knew the price of a put and we could price a put if we knew the price of a call. Now we would like to be able to price a put or a call using only the price of the underlying asset and the T-Bill rate.

To do this we have to make some assumptions about the way the stock price behaves.

<u>The assumption underlying the binomial model is</u>:

At each point in time, if you look ahead one "period", the stock price can take on one of two values, uS or dS, where S is the current stock price.

$$S \quad \begin{array}{l} uS \\ \\ dS \end{array}$$

Suppose the length of a binomial period is h where the units on h are fractions of a year. In order to preclude arbitrage it must be the case that

$$u > R > d.$$

where $R = e^{rh}$, the per period gross risk free rate.

Whether or not the binomial model describes the way option prices behave depends upon whether or not this model of the behavior of asset prices is a reasonable approximation to reality.

Example: Suppose S=50, u=2, d=.5 and R=1.25.

$$
S=50 \quad \begin{cases} uS=100 \\ dS=25 \end{cases}
$$

What is the value of a European call with K=50 one period prior to expiration?

STRATEGY FOR DETERMINING C:

- Determine the possible values that the call can have at expiration.

- Form a portfolio containing the underlying asset and a T-Bill which replicates the payoffs at expiration. The replicating portfolio is a synthetic call.

- Since, at expiration, the cash flows from the replication portfolio are the same as the cash flow from the call, the price of the call must be the same as the price of the replicating portfolio. If not, there is an arbitrage opportunity.

- The possible values for the call at expiration are:

$$
C \quad \begin{cases} C_u=\max[0,uS-K]=\max[0,50]=50 \\ C_d=\max[0,dS-K]=\max[0,-25]=0 \end{cases}
$$

- Form a portfolio which replicates the cash flows at expiration:

Define:

Δ = # of shares of the underlying asset in the replicating portfolio.

B = $'s in T-Bills

Note: + ===> buying or lending
- ===> shorting or borrowing

The payoff to a portfolio which contains Δ shares of stock and B dollars in T-Bills is:

$$\begin{array}{ll} \Delta uS + RB & \text{if } S_T = uS \\ \Delta dS + RB & \text{if } S_T = dS \end{array}$$

where $R = e^{rh}$.

The initial cost is :

$$\Delta S + B$$

We want to choose Δ and B such that

$$C_u = \Delta uS + RB \qquad (1)$$
$$C_d = \Delta dS + RB \qquad (2)$$

Note that we have two equations and two unknowns, so we always have a solution.

Solving for the two unknowns, Δ and B, we get

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$B = \frac{uC_d - dC_u}{R\,(u-d)}$$

In our example, S=50, K=50, u=2, R=1.25 and d=.5. Therefore

$$\Delta = \frac{(50-0\,)}{50(2-.5)} = 2/3$$

$$B = \frac{(2)(0)-(.5)(50)}{(1.25)(2-.5)} = -13.33$$

Check:

$$(2/3)100+(1.25)(-13.33) = 50$$

$$(2/3)25 +(1.25)(-13.33) = 0$$

Since the cash flow at expiration is exactly the same for the call and the replicating portfolio, the call and the portfolio must sell for the same price.

$$C = \Delta S + B = (2/3)50 - 13.33 = 20$$

Now we want to write the formula in a "nicer" way.

$$C = \Delta S + B$$

Substitute in for Δ and B

$$C = \left(\frac{C_u - C_d}{S(u-d)}\right)S + \frac{uC_d - dC_u}{R(u-d)}$$

Rearranging terms we get

$$C = \frac{\left(\frac{R-d}{u-d}\right)C_u + \left(\frac{u-R}{u-d}\right)C_d}{R}$$

Define p = (R-d)/(u-d) and (1-p) = (u-R)/(u-d)

The call price can be written as:

$$C = \frac{pC_u + (1-p)C_d}{R}$$

where $C_u = \max[0, uS-K]$
$C_d = \max[0, dS-K]$

What happens to the value of the call if:

1. K increases?

2. S increases?

3. r increases?

What happens to the value of the call if the spread between u and d increases?

Example: Suppose S=50, K=50, u=2, d=.5 and R=1.25

	uS=100		C_u=50
S=50		C=?	
	dS=25		C_d=0

$$p = (1.25-.5)/(2-.5)=.5 \implies C = \frac{.5(50)+.5(0)}{1.25} = 20$$

Suppose everything is the same except the volatility increases. In particular let u=2.25 and d=.25.

	uS=112.50		C_u=62.50
S=50		C=?	
	dS=12.50		C_d=0

$$p=(1.25-.25)/(2.25-.25)=.5 \implies C=\frac{.5(62.50)+.5(0)}{1.25} = 25$$

Increasing the volatility increases the value of the call option. Why?

■ Note that the option pricing formula does not require that we know anything about:

- ■ investors' attitudes toward risk

or

- ■ how likely it is that the stock will go to uS or dS.

Does this mean that changes in investors' attitude toward risk or changes in investors' feelings about the prospects for a stock are not reflected in the option price?

■ Delta:

Recall that $C = \Delta S + B$. Delta, denoted Δ, gives the sensitivity of the call price to a change in the stock price, i.e. if the stock goes up by $1, the call increases in price by approximately Δ. Delta is a measure of risk exposure and it also tells you how you can hedge the risk.

■ Interpretation of p:

p is equal to the probability the stock goes to uS in a world where everyone is risk neutral. To see this consider the following:

	uS with prob. q		
S		E(return)	$= \dfrac{q(uS)+(1-q)(dS)}{S}$
	dS with prob. 1-q		$= qu+(1-q)d$

If everyone is risk neutral, no asset would earn a return for risk since no one cares about risk. Therefore the return on all assets would be the risk-free rate. In particular, the return on the stock would be the risk-free rate.

$$R = qu+(1-q)d$$

Solving for q gives: q=R-d/u-d=p

Given that we do not live in a risk neutral world, why is this of interest? This is of interest because if you know the distribution of stock returns, then you can use the distribution to price the options in a "risk neutral" way, i.e. the option price equals the discounted expected value of the payoffs at expiration discounted at the risk free rate.

■ PRICING A CALL TWO PERIODS FROM EXPIRATION

Suppose K=50, S=50, u=2, d=.5 and R=1.25

		$u^2S=200$
	$uS=100$	
$S=50$		$udS=50$
	$dS=25$	
		$d^2S=12.50$

The strategy for pricing the call is the same as before. Start with the value of the call at expiration and work back.

If you are one period from expiration and the stock price is uS. The problem looks like

	$u^2S=200$
$uS=100$	
	$udS=50$

	$C_{uu}=\max[0,u^2S-K]=150$
$C_u=?$	
	$C_{ud}=\max[0,udS-K]=0$

This is just a one period problem which we already know how to solve. In particular, p=.5

$$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{R} = \frac{(.5)(150) + (.5)(0)}{(1.25)} = 60$$

Remember implicit in using this formula is the idea that you can construct a replicating portfolio and set the price of the call equal to the price of the replicating portfolio. That is

$$C = \Delta S + B$$

where one periods from expiration Δ and B are equal to:

$$\Delta_u = \frac{150-0}{100(2-.5)} = 1 \qquad B_u = \frac{(2)(0)-(.5)(150)}{(1.25)(2-.5)} = -40$$

If you are one period from expiration and the stock price is dS, then

$$C_d = \frac{pC_{ud} + (1-p)C_{dd}}{R} = \frac{(.5)(0) + (.5)(0)}{(1.25)} = 0$$

$$\Delta_u = 0 \qquad B = 0$$

Now go back two periods from expiration. Looking a step ahead we see:

	uS=100		$C_u = \frac{pC_{uu} + (1-p)C_{ud}}{R} = 60$
S=50		C=?	
	dS=25		$C_d = \frac{pC_{ud} + (1-p)C_{dd}}{R} = 0$

This looks like a one period problem. Using the one period formula we get

p=.5

$$C = \frac{pC_u + (1-p)C_d}{R} = \frac{(.5)(60)+(.5)(0)}{1.25} = 24$$

Two periods from expiration Δ and B are equal to:

$$\Delta = \frac{C_u - C_d}{S(u-d)} = \frac{60}{50(2-.5)} = .8 \qquad B = \frac{uC_d - dC_u}{R(u-d)} = \frac{-(.5)(60)}{(1.25)(2-.5)} = -16$$

and therefore

$$C = (.8)(50)-16=24$$

The general formula for pricing a call two periods from expiration is found by substituting in for C_u and C_d :

$$C = \frac{p^2C_{uu} + 2(1-p)pC_{ud} + (1-p)^2C_{dd}}{R^2}$$

To Summarize:

C_{uu}=150

(Δ=1, B=-40)

C_u = 60

$$= \frac{pC_{uu} + (1-p)C_{ud}}{R}$$

(Δ=.8, B=-16)

C = 24

$$= \frac{pC_u + (1-p)C_d}{R}$$

$$= \frac{p^2C_{uu} + 2p(1-p)C_{ud} + (1-p)^2C_{dd}}{R^2}$$

= ΔS + B

C_{ud}=0

(Δ=0, B=0)

C_u = 0

$$= \frac{pC_{ud} + (1-p)C_{dd}}{R}$$

C_{dd}=0

■ Arbitraging a Mispriced Call Three Periods From Expiration

To determine whether or not an option is mispriced the "theoretical value" must be computed.

Suppose S=80, K=80, u=1.5, d=.5 and R=1.1. This implies that p=.6.

			$u^3S=270$
		$u^2S=180$	
	$uS=120$		$u^2dS=90$
$S=80$		$ud^2S=60$	
	$dS=40$		$ud^2S=30$
		$d^2S=20$	
			$d^3S=10$

			$C_{uuu}=190$
		$C_{uu}=107.27$ (1, -72.72)	
	$C_u=60.49$ (.849,-41.33)		$C_{uud}=10$
$C=34.07$ (.719,-23.44)		$C_{ud}=5.45$ (.167,-4.54)	
	$C_d=2.97$ (.136,-2.48)		$C_{udd}=0$
		$C_{dd}=0$ (0, 0)	
			$C_{ddd}=0$

The numbers in parentheses are Δ and B respectively.

Now suppose the call is selling for \$36 three periods prior to expiration. We know the price should be \$34.07. What should we do?

1. Sell the real call
2. Buy the synthetic call =====> buy .719 shares = Δ
   short 23.44 in T-Bills = B

■ What do you get up front?

$$C - \Delta S + B = 36 - 34.07 = 1.93$$

■Suppose the stock price is 120 two periods from expiration. If you close out your position at that point what will you get?

■ If the call price equals the "theoretical value", \$60.49, you will receive

$$\Delta uS - e^{rh} B - C_u = (.719)(120)-(1.1)(-23.44)-(60.49) = 0$$

■ If the call price is less than 60.49, closing out the position will yield more than zero since it is cheaper to by back the call.

■ If the call price is more than 60.49, closing out the position will yield less than zero! What should we do now?

KEEP THE ARBITRAGE GOING

In particular, this means that you should continue to hold a short position in the real call and a long position in the synthetic call.

Notice that the "synthetic call" part of your portfolio is now

$$\Delta = .719 \qquad\qquad B = (1.1)(-23.44) = -25.78$$

One period from now this part of the portfolio will be worth

$$(.719)(180)+(1.1)(-25.78) = 101.06 \qquad \text{if } S=180$$
$$(.719)(\ 60)+(1.1)(-25.78) = 14.78 \qquad \text{if } S=\ 60$$

This is no longer replicating the call. The initial Δ and B only guarantee replication for one period.

To continue replicating the call the synthetic position must be adjusted. The Δ and B that will do the job are

$$\Delta = .849 \qquad\qquad B = -41.33$$

Therefore you need to buy (.849-.719) = .13 more shares and short (41.33-25.78) = $15.55 more in T-Bills. (Note that the adjustment in the number of shares you need to remain hedged is referred to as gamma.)

Notice that it cost (.13)(120) = 15.60 to purchase the additional shares. This is approximately equal to the amount you want to borrow. (This would be exact if there wasn't rounding error in the computation of Δ and B.) Therefore readjusting the hedge is self-financing.

■ Now suppose one period from expiration the stock price is 60.

■ You can close out your arbitrage position if the call price is less than or equal to $5.45 since closing out your position results in a payoff which equals

$$\Delta udS - e^{rh} B - C_{ud} = (.849)(60) - (1.1)(-41.33) - C_{ud} = 5.45 - C_{ud}$$

■ If the call price is still too high, keep the arbitrage going. In particular, adjust your synthetic position so that

$$\Delta = .167 \qquad B = -4.54$$

Since your current Δ is .849 and B = (1.1)(-41.33) = -45.46, you need to sell .682 shares of the stock and use the proceeds to reduce your short position in the T-Bill. At expiration the call price will definitely come back into line and the arbitrage position can be closed out.

■PRICING A EUROPEAN CALL ON A NON-DIVIDEND PAYING STOCK n PERIODS FROM EXPIRATION

If you apply the one period formula over and over again, you get the following formula:

$$C = \frac{\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max[0, u^j d^{n-j} S - K]}{R^n}$$

Note that if the stock does not pay dividends this is also the price of an American call n periods from expiration.