BLACK-SCHOLES OPTION PRICING FORMULA

The Black-Scholes model is a special case of the binomial model. In order to understand how to go from the binomial model to the Black-Scholes model we need go over a few technical details.

■ Distributional Assumptions:

■Suppose an option expires in one year and suppose we <u>model</u> the stock price as changing two times prior to expiration. This results in four possible paths. Suppose each "path" is equally likely:

		u^2S	(2 ups which happens with prob. 1/4)
	uS		
S		udS	(1 up and 1 down which happens with prob. ½)
	dS		
		d^2S	(2 downs which happens with prob. 1/4))

As we noted earlier, the continuously compounded return is equal to $\ln(S_T/S)$. For example, if the stock price at expiration is S_T=udS, then the return is ln(udS/S) = ln(ud). The probability distribution of c.c. stock returns is:

```
.5    |                x
      |
.25   |     x                    x
      |_____________________________________
            ln(d^2)    ln(ud)    ln(u^2)
```

■ Suppose we model the stock price as changing four times prior to expiration and suppose each "path" is equally likely:

				u^4S	(1)
			u^3S		
		u^2S		u^3dS	(4)
	uS		u^2dS		
S		udS		u^2d^2S	(6)
	dS		ud^2S		
		d^2S		ud^3S	(4)
			d^3S		
				d^4S	(1)

The probability distribution of stock returns is

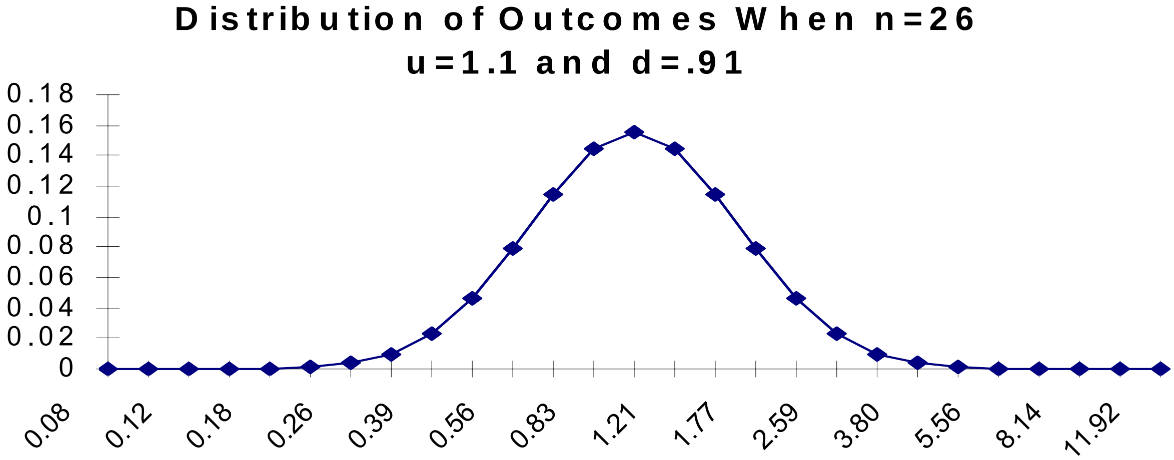
	$\ln(d^4)$	$\ln(ud^3)$	$\ln(u^2d^2)$	$\ln(u^3d)$	$\ln(u^4)$
6/16			x		
4/16		x		x	
1/16	x				x

■ In general, if I model the stock price as changing n times prior to expiration, there are

1. 2^n paths

2. $\frac{n!}{(n-j)!j!}$ ways to have paths with j upward jumps in the price of the underlying asset

■ If I let n get very large (i.e. infinite), the distribution of the continuously compounded returns becomes normally distributed.



■ If the c.c return, $\ln(S_T/S)$, is normally distributed, then S_T ,the stock price at expiration, is lognormally distributed.

Note: A random variable X is lognormally distributed if ln(X) is normally distributed.

Economically this means:

- limited liability since $S_t \geq 0$
- no upper bound on stock price movements but large moves are quite unlikely

■ Modeling Consistency

Suppose the underlying asset has an annual continuously compounded expected rate of return equal to μ and a variance of the annual continuously compounded rate of return equal to σ^2.

Assuming the distribution of returns is stationary, consistency requires that the expected return and variance over a 6 month period should be half the annual expected return, ½μ, and half the annual variance, ½σ^2, respectively.

In general, if T is the time to expiration and I divide the time to expiration up into n periods, each period is T/n in length and should have expected return equal to (T/n)μ and variance equal to (T/n)σ^2. That is, the expected return and variance over the period is proportional to the length of the period.

The expected c.c. return over one "period" is equal to

$$\mu' = \mu\frac{T}{n} = q * \ln(u) + (1 - q) * \ln(d)$$

The variance of the c.c. return over one "period" is equal to

$$\sigma^{2}{}' = \sigma^2 \frac{T}{n} = q * [\ln(u) - \mu']^2 + (1 - q)[\ln(d) - \mu']^2$$

where:

u = the size of the upward jump for that period
d = the size of the downward jump for that period
q = probability of an upward jump

As you can see, the expected return and variance depend on the size of the jumps. Therefore when you change the number of periods, you need to change u and d. For example, jumps over a 6 month period should be smaller than jumps over a 1 year period to insure that the expected return and variance over 6 months is half the expected return and variance over 1 year.

The formulas for u and d which maintain consistency are

$$u = e^{\sigma\sqrt{\frac{T}{n}}} \qquad d = \frac{1}{u} = e^{-\sigma\sqrt{\frac{T}{n}}}$$

where

n = number of periods
T = the time to expiration in fractions of a year
σ = the standard deviation of the c.c. rate of return

The derivation of these formulas is in Cox &Rubenstein if you are interested.

■ Just as the expected rate of return on the underlying asset needed to be adjusted as the number of periods change, the risk-free rate also must be adjusted. In particular, if the time to expiration, T, is divided up into n periods and the annual continuously compounded risk-free rate is r then the per period risk-free rate is $e^{r*T/n}$

For example, if the annual continuously compounded risk-free rate is 10%, then the 6 month interest rate (n=2) is $R=e^{.10*.5}$ = 1.0513 and the monthly interest rate (n = 12) is $R=e^{.10*1/12}$ = 1.0084

Now we are done with the technical details, so we can see how to get the Black-Scholes model.

BLACK-SCHOLES FORMULA:

To get the Black-Scholes formula we start with the binomial formula for a European call on a non-dividend paying stock n periods from expiration

$$C = \frac{\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max(0, u^j d^{n-j} S - K)}{R^n}$$

Let T = time to expiration
n = number of stock price changes prior to expiration
σ = the standard deviation of the annual c.c. return for the stock
R= the per period T-Bill rate

Define u, d and R in the way we discussed earlier:

$$u = e^{\sigma\sqrt{\frac{T}{n}}} \qquad d = \frac{1}{u} = e^{-\sigma\sqrt{\frac{T}{n}}} \qquad R = e^{r\frac{T}{n}}$$

If you substitute u, d and R into the formula given above and let n get very large (i.e. let n go to infinity), you get the Black-Scholes formula:

$$C = SN(d_1) - Ke^{-rt}N(d_2)$$

where

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}$$

N(d) is the probability that a normal random variable with a zero mean and variance equal to 1 is less than or equal to d. (i.e., N(d) is the distribution function for a standard normal random variable).

The Black-Scholes formula has the same interpretation as the binomial formula

The Black-Scholes formula:

$$C = SN(d_1) - Ke^{-rt}N(d_2)$$

has the same form as

$$C = \Delta S + B$$

where $\Delta = N(d_1)$
$B = -Ke^{-rT}N(d_2)$

To get the value of a European put on a non-dividend paying stock, use put-call parity:

$$\begin{aligned} P &= C - S + KR^{-T} \\ &= Ke^{-rT}N(-d_2) - SN(-d_1) \end{aligned}$$

The Black-Scholes put formula has an interpretation which is similar to the binomial formula

$$P = \Delta_{put}S + B$$

where $\Delta_{put} = -N(-d_1)$
$B_{put} = Ke^{-rT}N(-d_2)$

USING THE BLACK-SCHOLES FORMULA

■ To compute the Black-Scholes formula, we need

S = current stock price
K = strike price
T = # of calendar days to expiration/365
r = annualized c.c. rate on a T-Bill maturing as close as possible to the expiration date
σ = annualized standard deviation of the cc rate of return on the underlying asset

Example:

In the example below, the inputs are:

S = $40
K = $50
T = 182/365 = .5
r = 10%
σ = 30%

				Call	Put
Stock Price	40.00			**Call**	**Put**
Exercise Price	50.00		**Price**	1.077207	8.638678
Volatility	30.00%		**Delta**	0.238808	-0.761192
Risk-free interest rate	10.00%		**Gamma**	0.036537	0.036537
Time to Expiration	0.5		**Vega**	0.08769	0.08769
Dividend Yield	0		**Theta**	-0.009529	0.003501
			Rho	0.042376	-0.195432

Black-Scholes Formula with Dividend Yield:

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(Se^{-\delta T}/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

and

δ = continuous dividend yield

Black Formula for Options on Currencies:

$$C = S\,e^{-r_fT}\,N(d_1) - K\,e^{-r_dT}\,N(d_2)$$

where

$$d_1 = \frac{\ln(S\,e^{-r_fT}/K) + (r_d + \sigma^2/2)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

S = the price of the foreign currency in term of the domestic currency
r_f = the foreign risk-free rate
r_d = the domestic risk-free rate

Black Formula for Options on Futures:

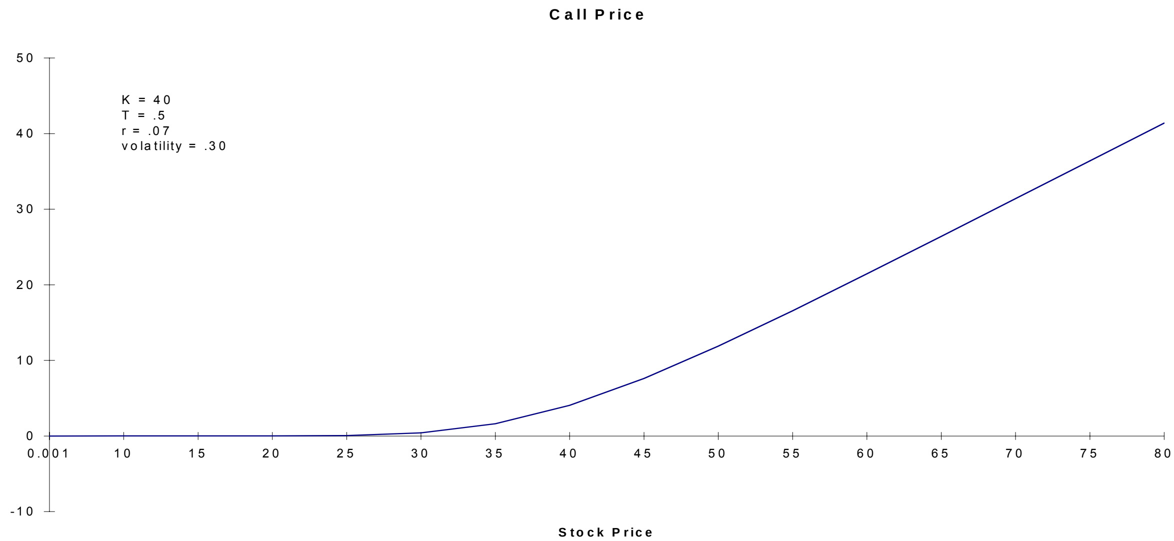
$$C = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2)$$

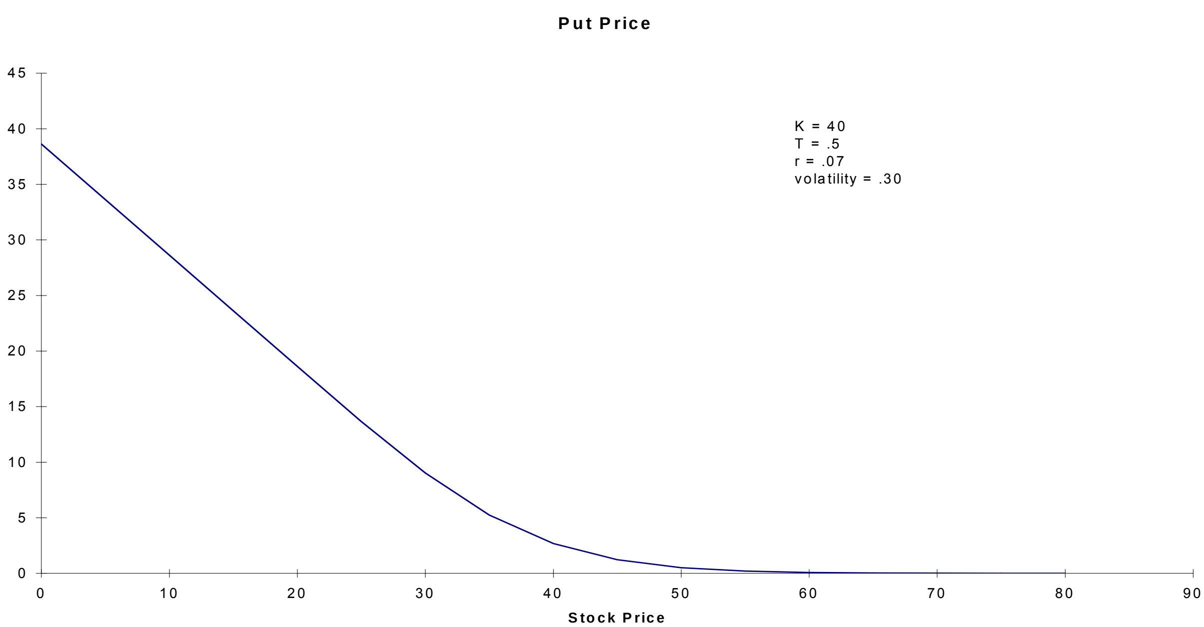
where

$$d_1 = \frac{\ln(Fe^{-rT}/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Now that we have an exact pricing formula we can see how an option behaves prior to expiration.





■ Delta, denoted Δ, gives the sensitivity of the option price to a small change in the stock price. It is the number of shares needed to make a synthetic call and it is a measure of the riskiness of an option position.

■The delta for a call is equal to $\partial C/\partial S = N(d_1)$. Recall that

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

If S, K, R, T or σ change, then the value of delta will change.

■ The delta of a call is always between 0 and 1.

Example: Suppose that the delta is .58. This means that if the value of the underlying asset increases by $1, the value of the call increases by approximately $.58.

■ The delta for a put is equal to $\partial P/\partial S = \partial C/\partial S - 1$. Therefore if the delta of a call is .58, the delta of an identical put (i.e. same K, same underlying asset and same time to expiration) is equal to .58 -1 = -.42.

■The delta of a put is always between 0 and -1.

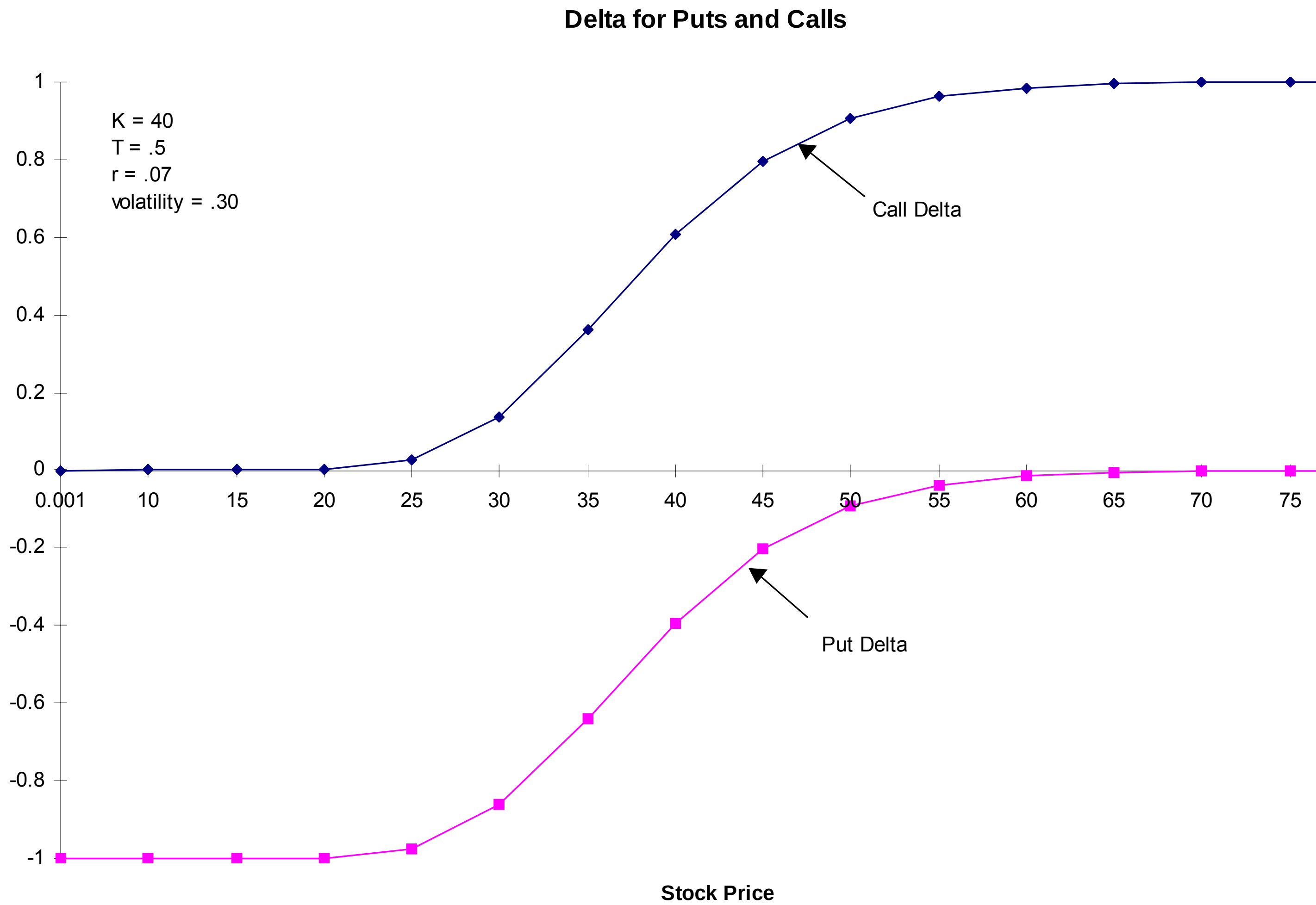
Example: Suppose that the delta of a put is -.42. This means that if the value of the underlying asset increases by $1, the value of the put decreases by $.42.

■ The delta of a stock is 1
The delta of a T-Bill is 0

■ The delta of a portfolio is the sum of the individual deltas. The formula is

$$\Delta_{portfolio} = \sum N_i \Delta_i$$

where N_i is the number of units of asset i and Δ_i is the delta of asset i.


Delta for Puts and Calls
K = 40
T = .5
r = .07
volatility = .30
Call Delta
Put Delta
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0.001
10
15
20
25
30
35
40
45
50
55
60
65
70
75
Stock Price

Example: Consider the following information: T=.5, r = 7% and σ =.30

S=39		Price	Delta	Gamma
Call	K=40	3.46	.56	.0477
	K=50	.75	.18	.0321
	K=70	.016	.0064	.0022
Put	K=40	3.092	-.44	.0477
	K=50	10.04	-.82	.0321
	K=70	28.61	-.9936	.0022

S=40		Price	Delta	Gamma
Call	K=40	4.05	.61	.0454
	K=50	.95	.22	.0347
	K=70	.0234	.0089	.0028
Put	K=40	2.6747	-.39	.0454
	K=50	9.2368	-.78	.0347
	K=70	27.6223	-.99	.0028

S=41		Price	Delta	Gamma
Call	K=40	4.68	.65	.0426
	K=50	1.19	.25	.0368
	K=70	.0338	.0121	.0036
Put	K=40	2.3033	-.35	.0426
	K=50	8.4714	-.75	.0368
	K=70	26.6326	-.989	.0036

Suppose the current stock price is $40 and I have a position which consists of 2 long calls with K=40, 2 short calls with K=50, 1 long share and short $40 in T-Bills.

The portfolio delta is

$$(2)(.61)+(-2)(.22)+(1)(1)+(0)(-40) = 1.78$$

This means that if the value of the stock goes up by $1, the value of the portfolio will increase by approximately $1.78.

■ Traders are often interested in a <u>delta neutral position</u>.
This is a position where

$$\Sigma N_i \Delta_i = 0$$

This means that for a <u>small</u> change in the stock price in either direction, the value of the position remains the same. A hedged position is delta neutral.

<u>Example</u>:
Suppose you are long 1 call with K=40 and 6 months to expiration. The current stock price is $40. The Black-Scholes price is 4.05. The perfect hedge would involve shorting an identical call. I would have no risk and the net value of the position would always to zero.

How does delta hedging compare to the perfect hedge?
The delta of the call is .61. If I want to be delta neutral what should I do?

I need to determine the number of shares to buy or sell such that

$$n_c\Delta_c + n_s\Delta_s = (1)(.61) + n_s(1) = 0$$

$$\text{=====>}\;\; n_S = -\Delta_C = -.61$$

To see how well the hedge worked we need to see what happens if the stock price increases or decreases by $1.

The initial cost of the position is

$$(1)(.4.05) + (-.61)(40) = -20.35$$

Note that a negative cost means that I receive $20.35 up front. If the stock price rises to 41, the value of the option is 4.68. If I closed out my position at that point, I would sell the call and buy the stock to cover the short position in the stock. This would cost me

$$(-1)(4.68) + (.61)(41) = 20.33$$

Therefore my net profit would be

$$20.35 - 20.33 = .02$$

If the stock price drops to 39, the value of the option is 3.46. If I closed out my position at that point, I would sell the call and buy the stock to cover the short position in the stock. This would cost me

$$(-1)(3.46) + (.61)(39) = 20.33$$

Therefore my net profit would be

$$20.35 - 20.33 = .02$$

The net value of the position changed by less than .1% when the stock price changed by $1.

If the stock prices changes by a large amount, being delta neutral will not guarantee that you are hedged.

If the stock price rises to 45, the value of the option is 7.59. If I closed out my position at that point, I would sell the call and buy the stock to cover the short position in the stock. This would cost me

$$(-1)(7.59) + (.61)(45) = 19.86$$

Therefore my net profit would be

$$20.35 - 19.86 = .49$$

If I didn't hedge I would have had a profit equal to

$$7.59 - 4.05 = 3.54$$

Hedging results in an approximately 2.5% gain while an unhedged position results in a 87% gain.

If the stock price drops to 35, the value of the option is 1.62. If I closed out my position at that point, I would sell the call and buy the stock to cover the short position in the stock. This would cost me

$$(-1)(1.62) + (.61)(35) = 19.73$$

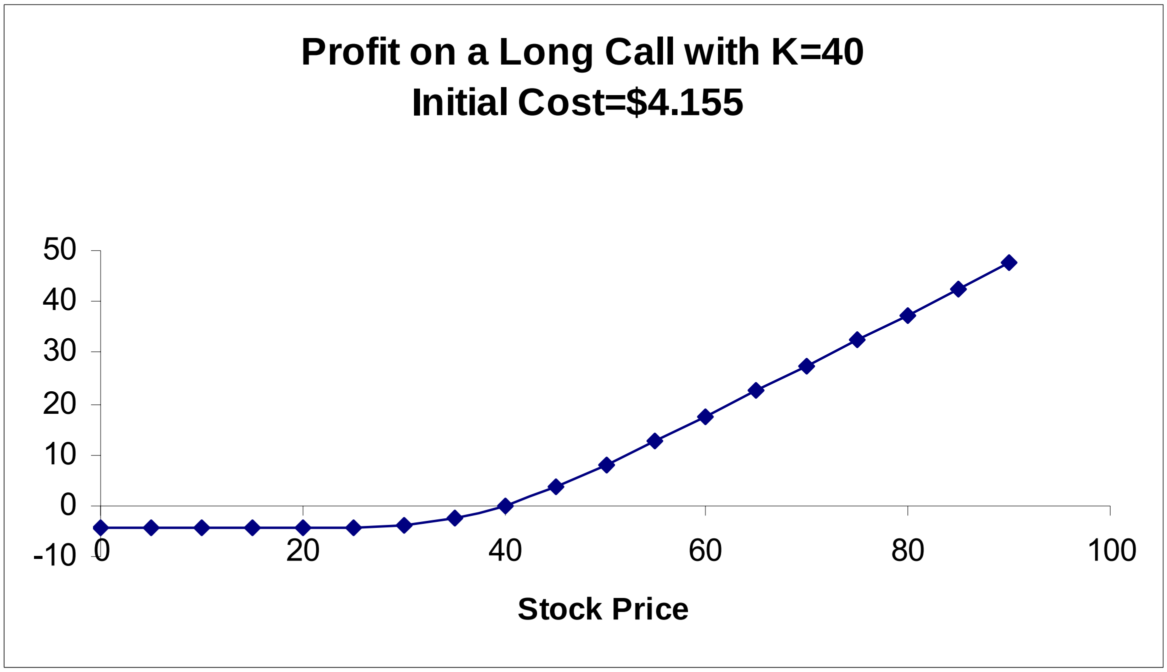
Therefore my net profit would be

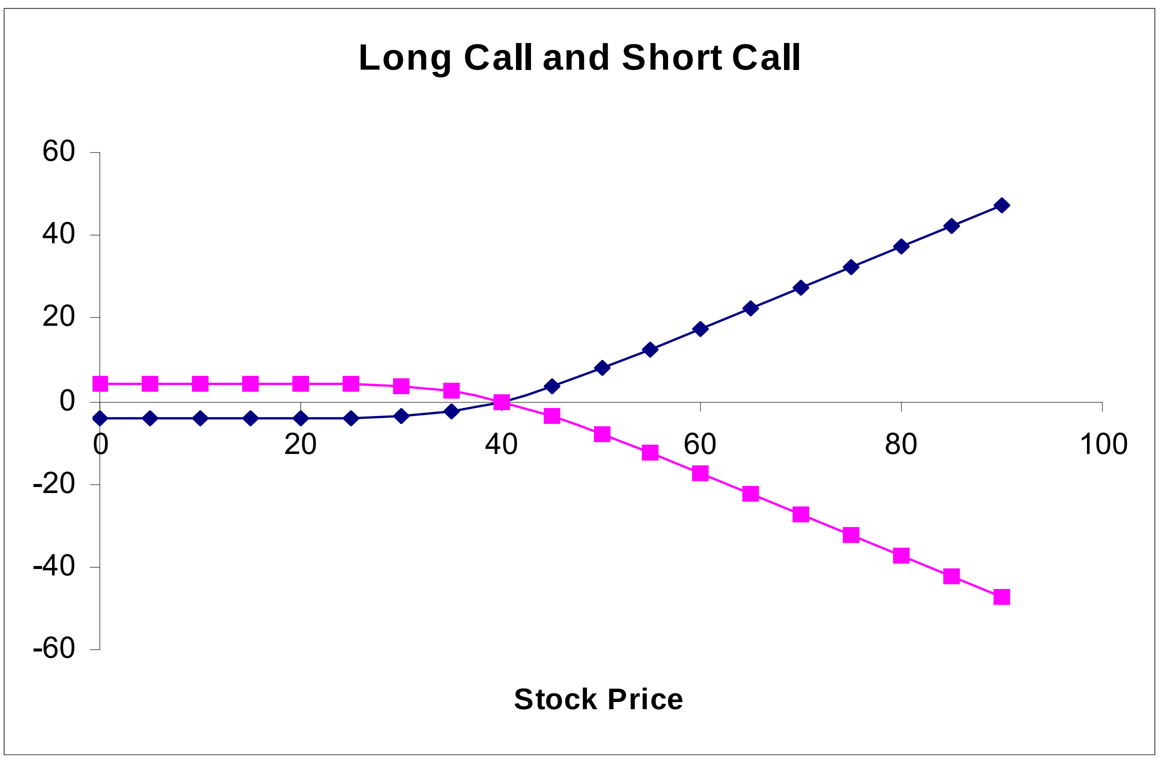
$$20.35 - 19.73 = .62$$

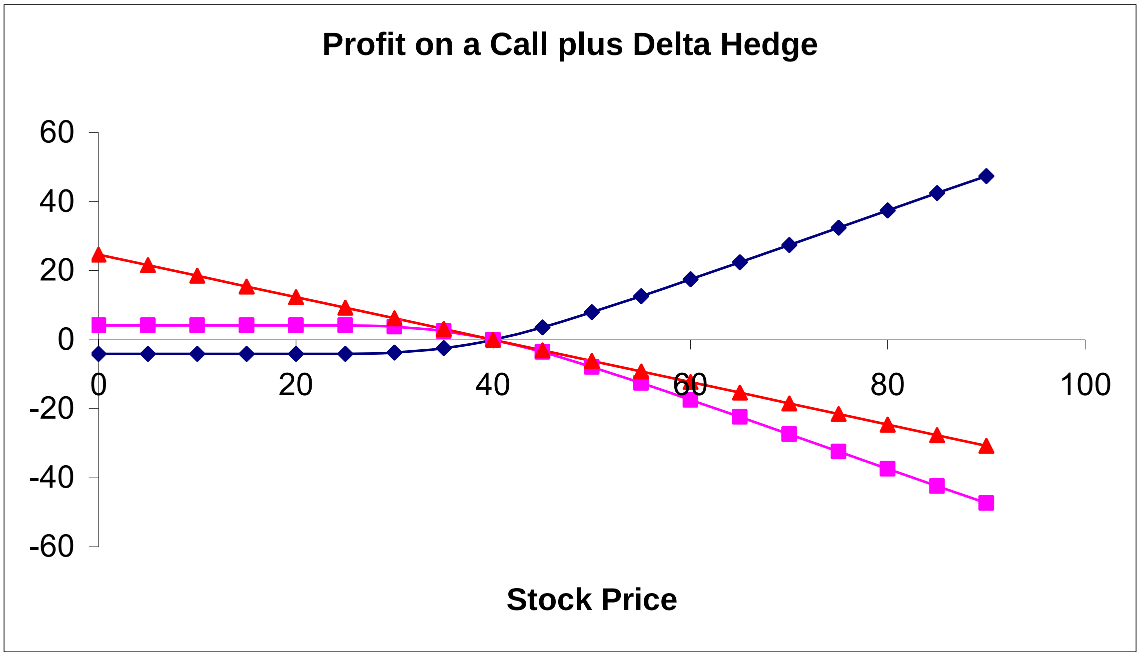
If I didn't hedge I would have had a loss equal to

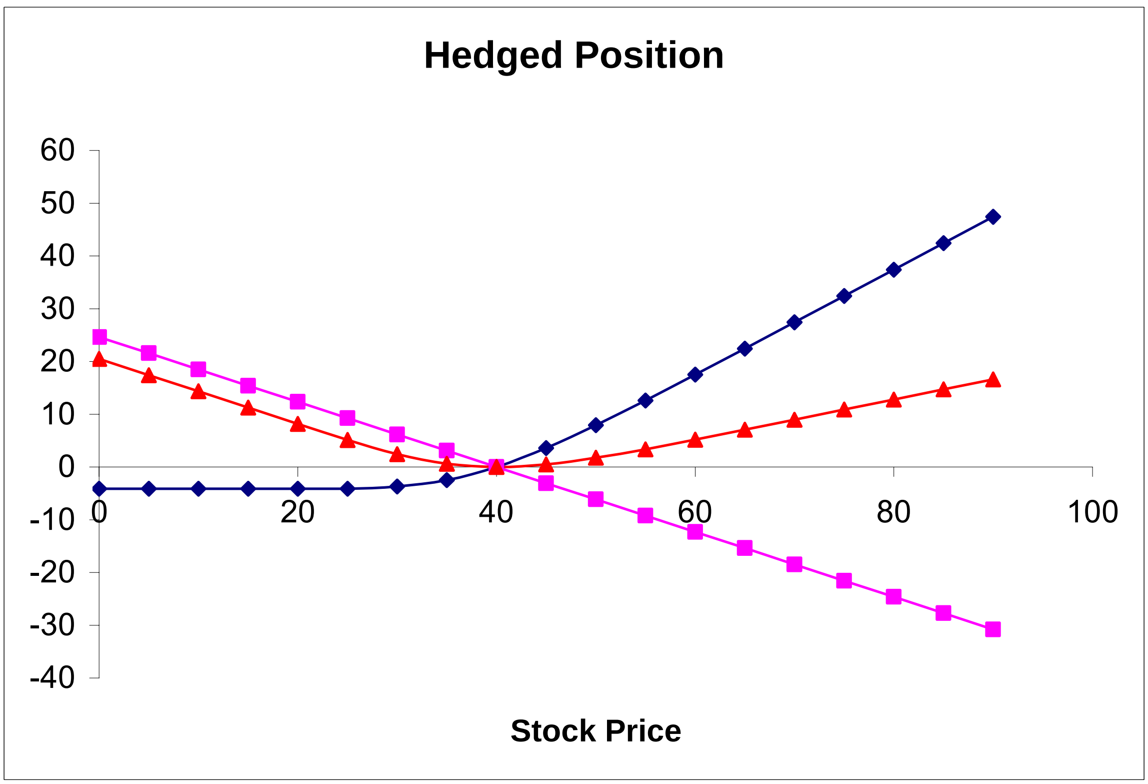
$$1.62 - 4.05 = -2.43$$

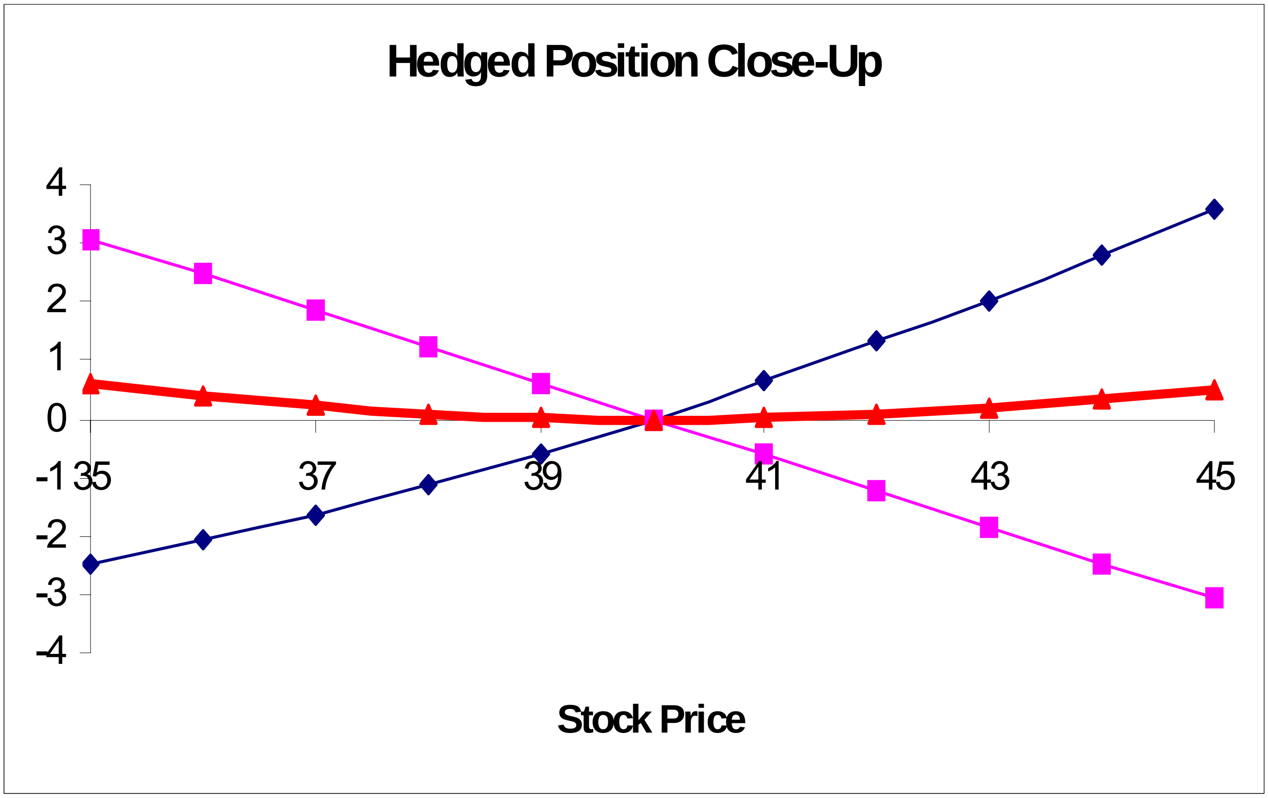
Hedging results in an approximately 3% gain while an unhedged position results in a 60% loss.

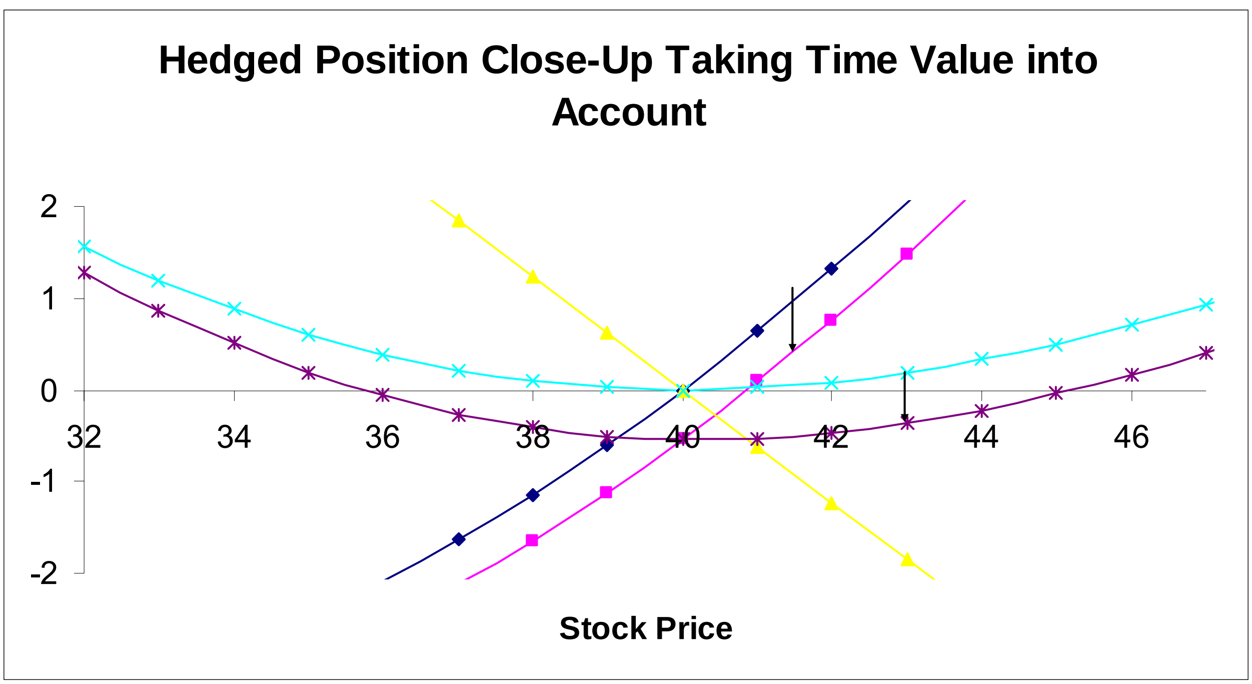

Profit on a Long Call with K=40
Initial Cost=$4.155
50
40
30
20
10
0
-10
0
20
40
60
80
100
Stock Price


Long Call and Short Call
60
40
20
0
-20
-40
-60
0
20
40
60
80
100
Stock Price


Profit on a Call plus Delta Hedge
60
40
20
0
-20
-40
-60
0
20
40
60
80
100
Stock Price


Hedged Position
60
50
40
30
20
10
0
-10
-20
-30
-40
0
20
40
60
80
100
Stock Price


Hedged Position Close-Up
4
3
2
1
0
-1
-2
-3
-4
35
37
39
41
43
45
Stock Price


Hedged Position Close-Up Taking Time Value into Account
2
1
0
-1
-2
32
34
36
38
40
42
44
46
Stock Price

Now lets delta hedge a short straddle which is a slightly more complicated position. A short straddle consists of short call with K=40 and a short put with K=40. When the stock price is 40, the cost of the call is 4.05, the delta of the call is .61, the cost of the put is 2.67 and the delta of the put is -.39. If I want to be delta neutral what should I do?

I need to determine the number of shares to buy or sell such that

$$n_c\Delta_c + n_s\Delta_s + n_p\Delta_p = (-1)(.61) + n_s(1) + (-1)(-.39) = 0$$

$$\text{=====>}\ \ n_S = \Delta_C - \Delta_p = .22$$

The initial cost from the position is

$$(-1)(4.05)+(-1)(2.67) + (.22)(40) = 2.08$$

If the stock price rises to 41, the value of the call is 4.68 and the value of the put is 2.30. If I closed out my position at that point, I would buy back the call and the put and sell the stock. This would cost me

$$(1)(4.68) + (1)(2.30) + (-.22)(41) = -2.04$$

Note that a negative cost means that I receive 2.04. Therefore my net profit would be

$$20.04 - 20.08 = -.04.$$

If the stock price drops to 39, the value of the call is 3.46 and the value of the put is 3.09. If I closed out my position at that point, I would buy back the call and the put and sell the stock. This would cost me

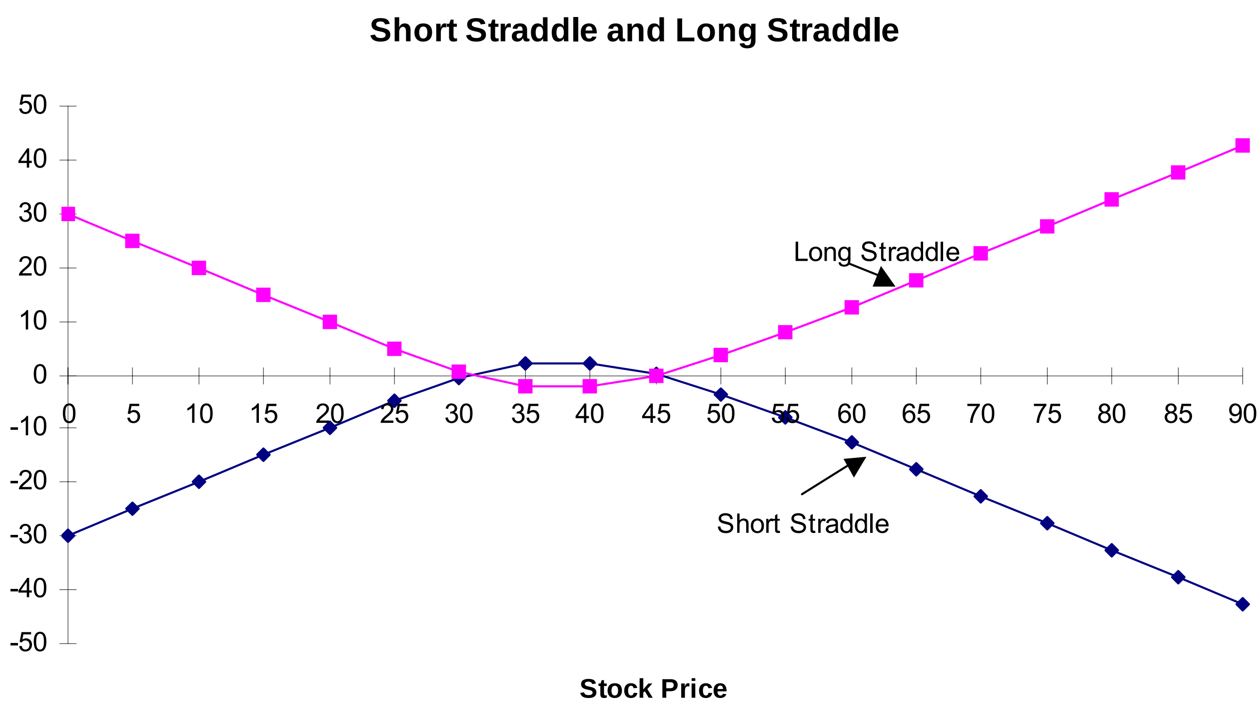
$$(1)(3.46) + (1)(3.09) + (-.22)(39) = -2.03$$

Therefore my net profit would be

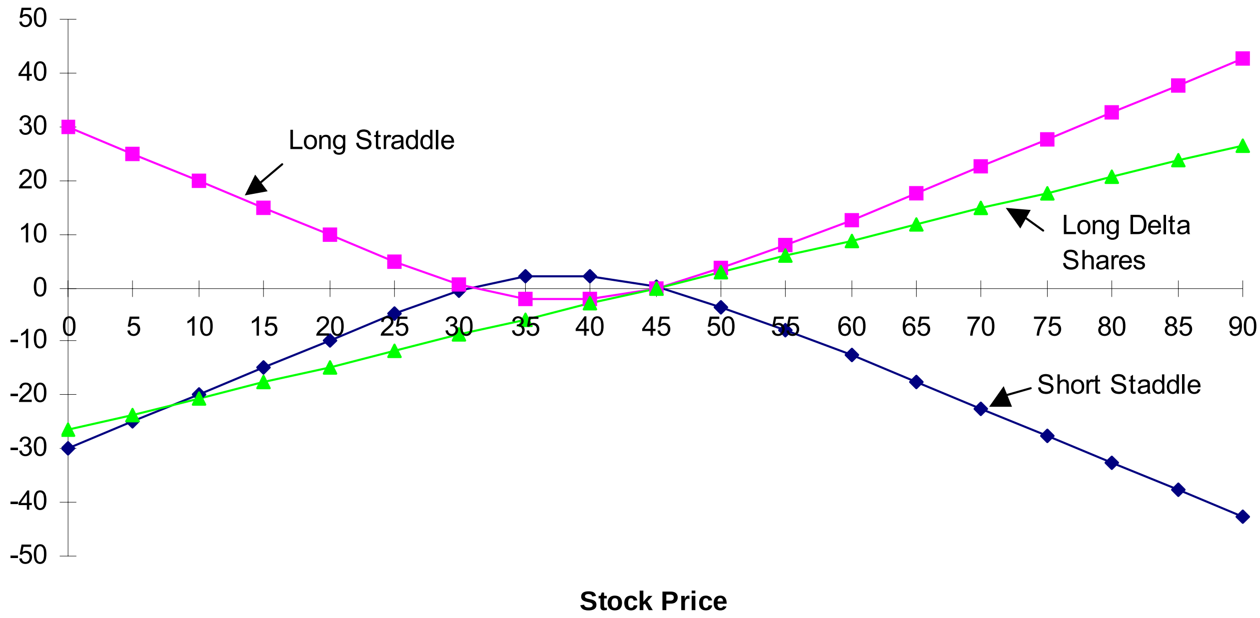
$$2.03 - 2.08 = -.05$$

The net value of the position changed by less than .2% when the stock price changed by $1.

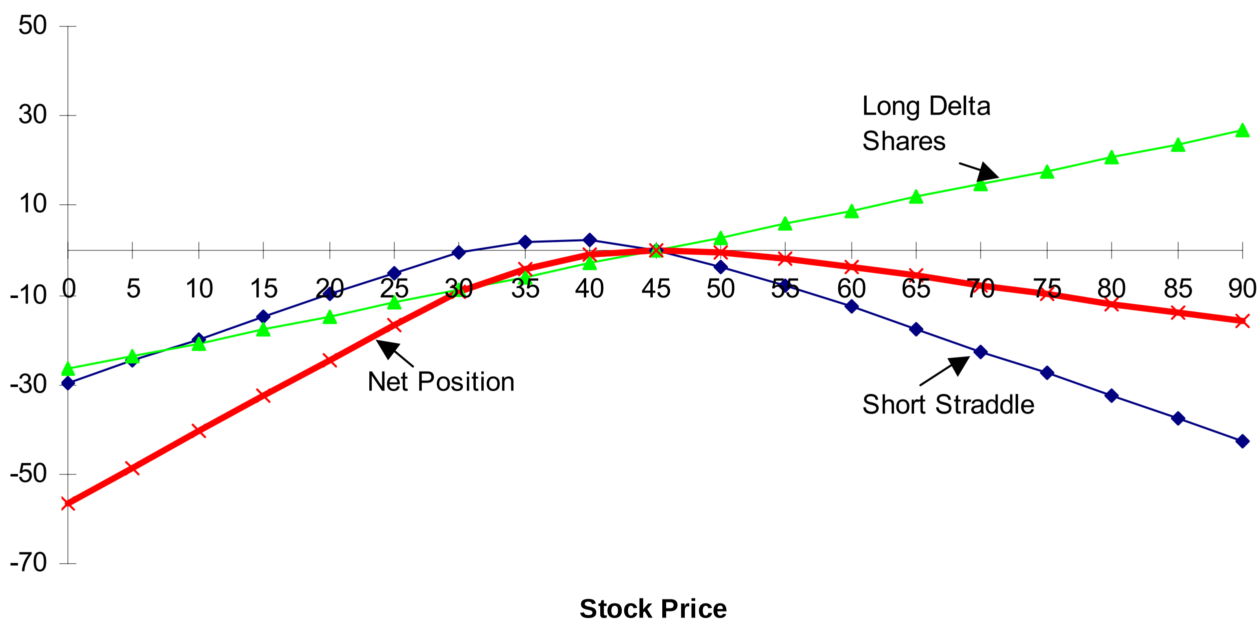

Short Straddle
50
40
30
20
10
0
-10
-20
-30
-40
-50
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
85
90
Stock Price


Short Straddle and Long Straddle
50
40
30
20
10
0
-10
-20
-30
-40
-50
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
85
90
Long Straddle
Short Straddle
Stock Price

Short Straddle Plus Delta Hedge



Hedged Position



Now lets look at how you would stay delta neutral over a period of a couple of days.

Tracking a Delta Neutral Position
Sell 1 Call with K=40, Long Delta Shares. Assume σ=.30 and r=.07

Days to Expiration	91	90	89	88	87
Stock Price	40	39.50	42	42.50	40.50
Call Price	2.7295	2.4339	3.9720	4.3112	2.9568
Delta	.5759	.5422	.6980	.7257	.6072
# of Calls	-100	-100	-100	-100	-100
# of Shares	57.59	54.22	69.80	72.57	60.72
Capital Tied Up in the Position	**57.59*40 - 2.7295*100 = 2030.65**	**54.22*39.50 - 2.4339*100 = 1898.30**	**69.80*42 - 3.9720*100 = 2534.40**	**72.57*42.50 - 4.3112*100 = 2653.105**	**60.72*40.50 - 2.9568*100 = 2163.48**
Capital Gain on Call Position*		29.56	-153.81	-33.92	135.44
Capital Gain on the Stock Position*		-28.79	135.55	34.90	-145.14
Interest Expense		-.39	-.36	-.49	-.51
Profit		.38	-18.62	.49	-10.21
Change in Stock Position		-3.37	15.58	2.77	-11.85

* Computed prior to the adjustment in the stock position

■ Gamma, denoted Γ, gives the sensitivity of the delta to a small change in the stock price (i.e. $\partial\Delta/\partial S = \Gamma$). It can also be interpreted as the change number of shares per dollar change in the stock price needed to maintain a delta neutral position.

■ If S, K, R, t or σ change, then the value of gamma will change.

Example: Suppose that the gamma is .05. This means that if the value of the underlying asset increases by $1, the delta of the call increases by .05. Therefore if the delta was previously .52 and the stock price increases by $1, the delta is now approximately .52 +.05 = .57.

If you were hedging a long position in a call with delta equal to .5, you would have shorted .5 shares. If the stock price increases by $1, you would need to increase your short position by gamma shares (i.e. .05) to maintain a hedged position. Gamma measures how sensitive the hedge is to changes in the stock price.

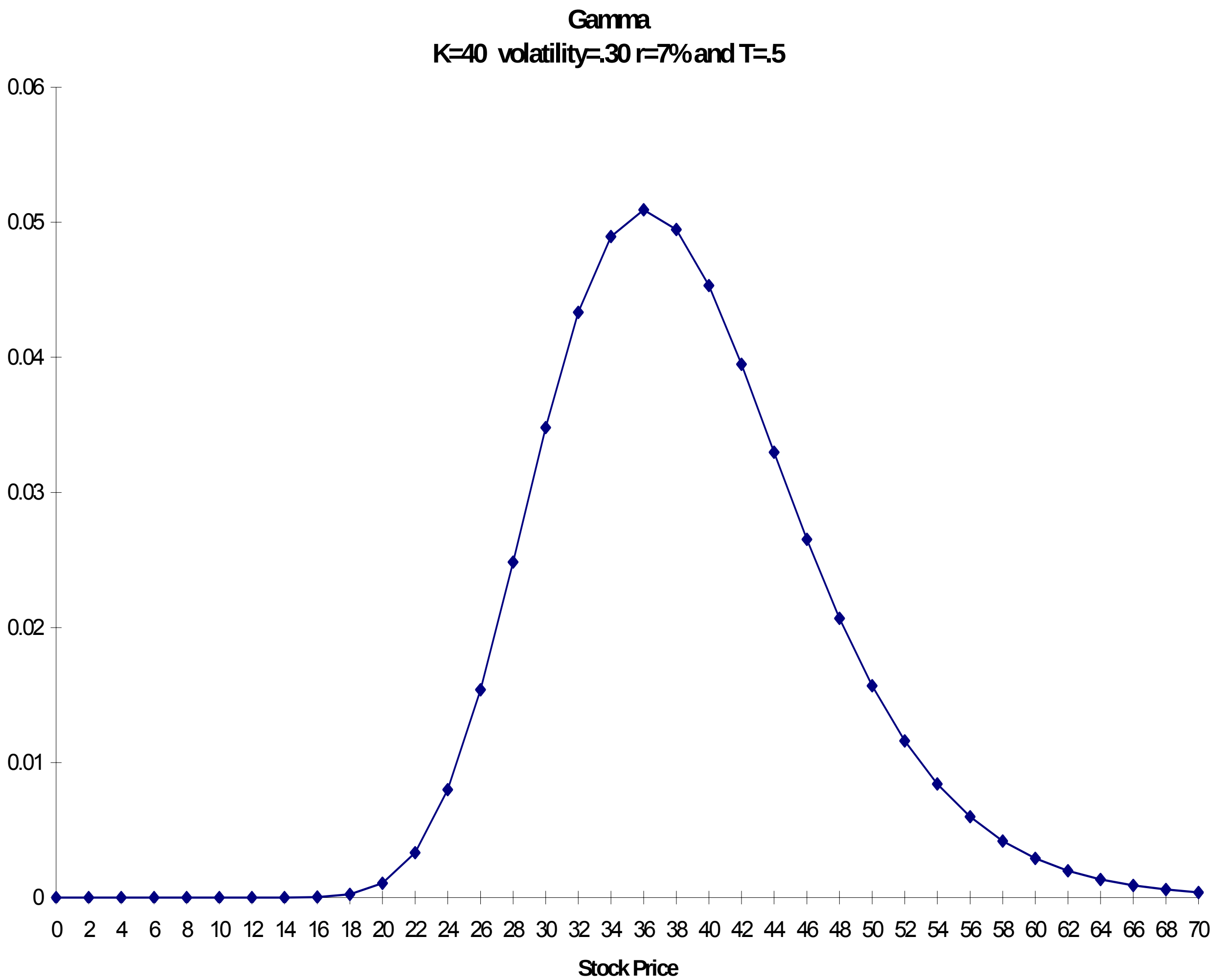
■ The gamma for a put is equal to the gamma for a call. Therefore if the gamma of a call is .05, the gamma of an identical put (i.e. same K, same underlying asset and same time to expiration) is equal to .05.

■ The gamma of a stock is 0
The gamma of a T-Bill is 0

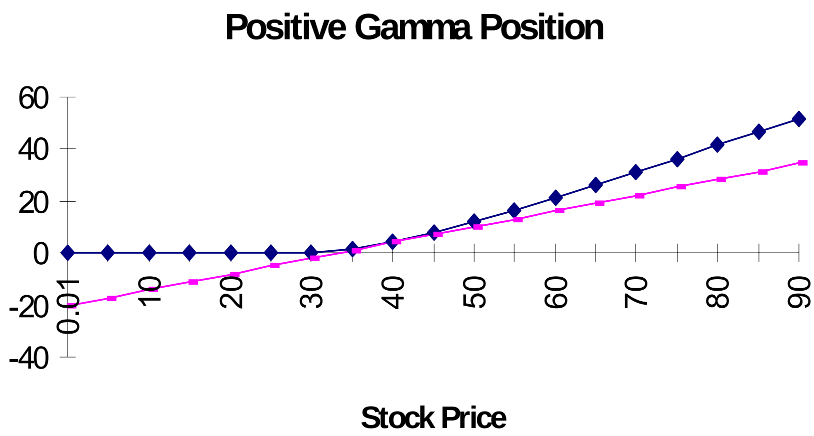
■ The gamma of a portfolio is the sum of the individual gammas. The formula is

$$\Gamma_{portfolio} = \sum N_i \Gamma_i$$

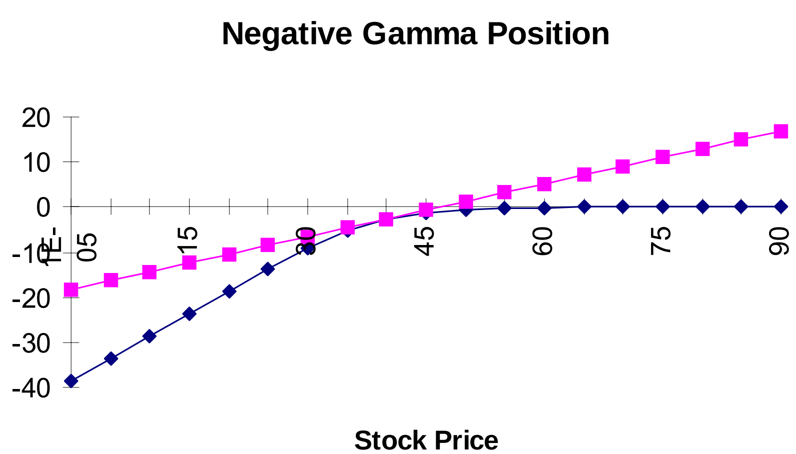
where N_i is the number of units of asset I and Γ_i is the gamma of asset.


Gamma
K=40 volatility=.30 r=7% and T=.5
0.06
0.05
0.04
0.03
0.02
0.01
0
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 64 66 68 70
Stock Price

The gamma of an option position is positive if the tangent line lies below the payoff line. For example long calls and puts have positive gammas.

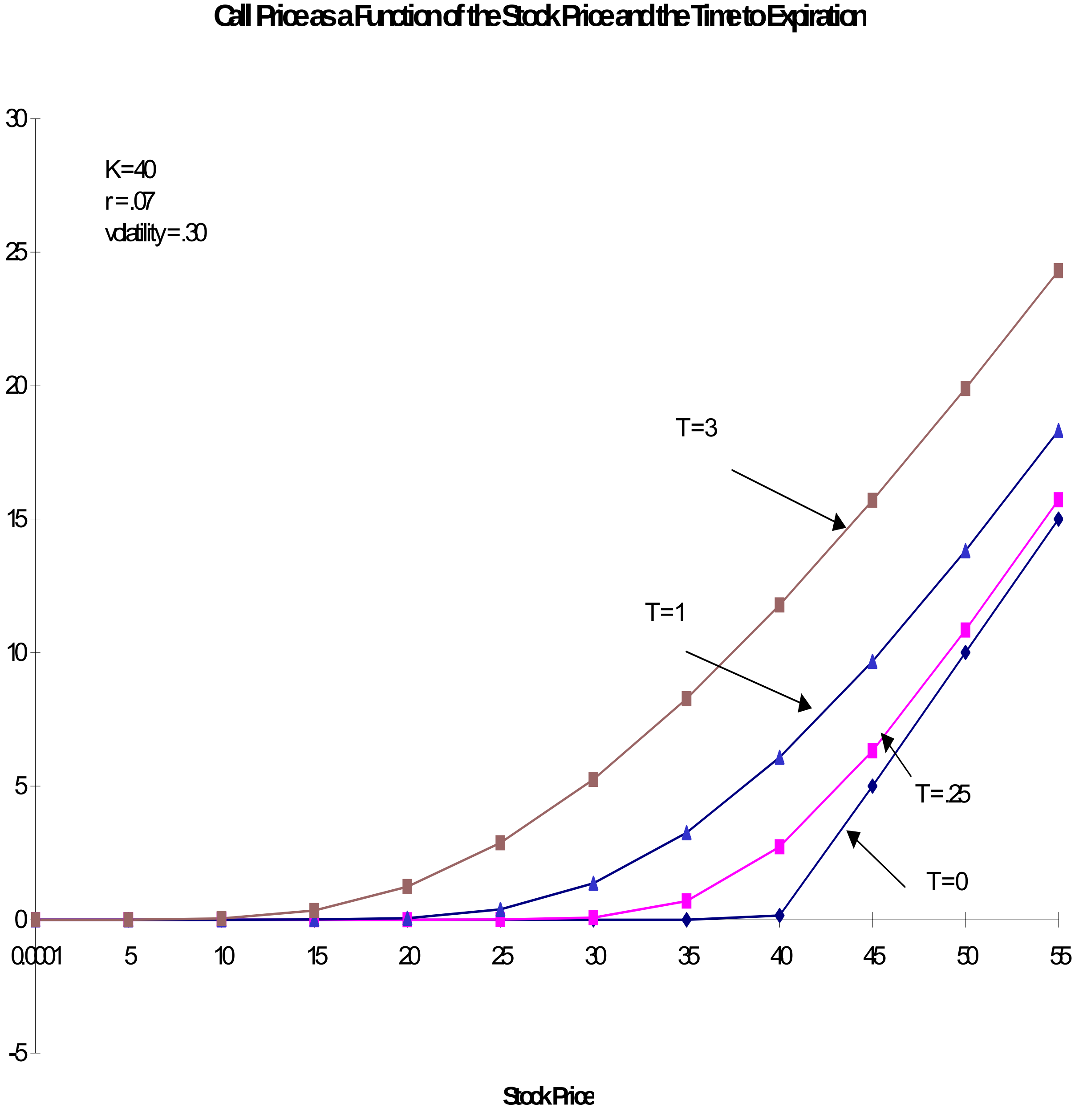


The gamma of an option position is negative if the tangent line lies above the payoff line. For example short calls and puts have negative gammas.

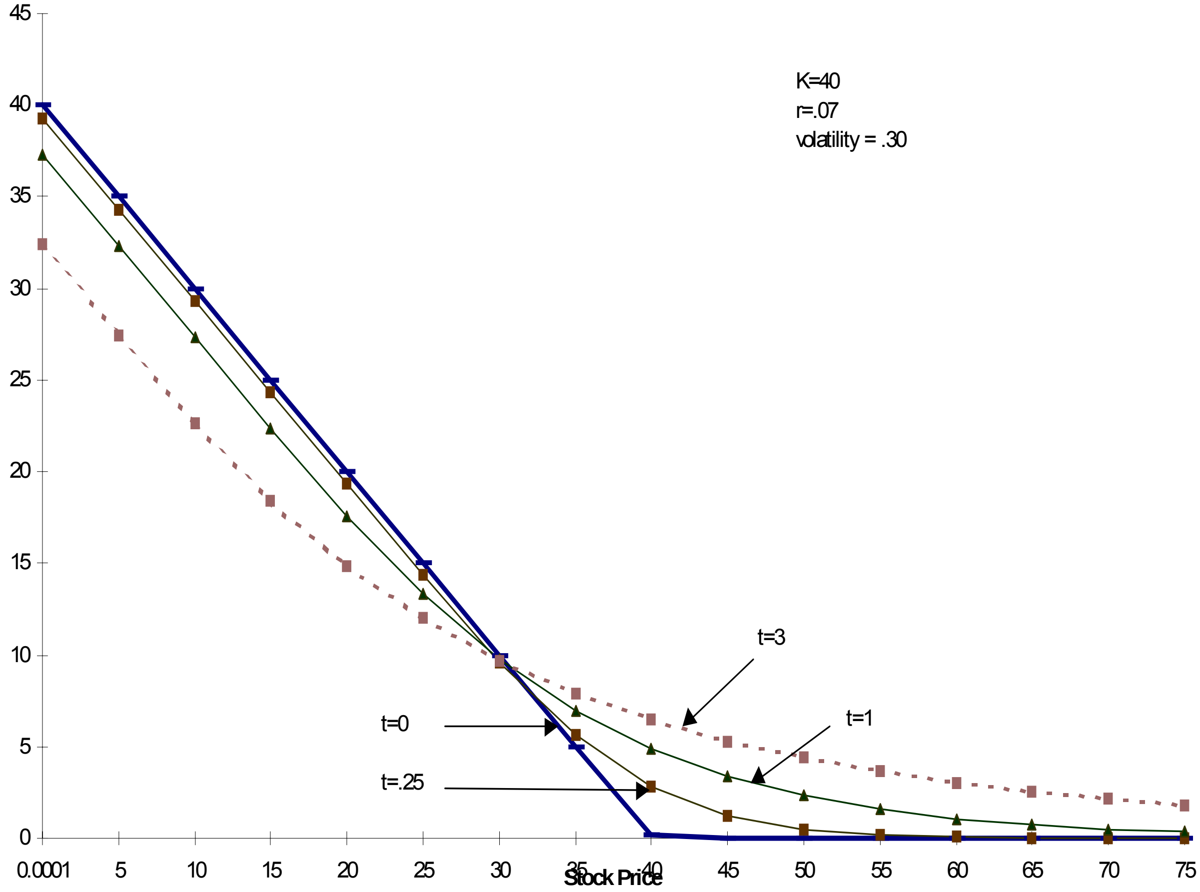


■ Theta

Theta gives the sensitivity of the option price as the time to expiration gets shorter.

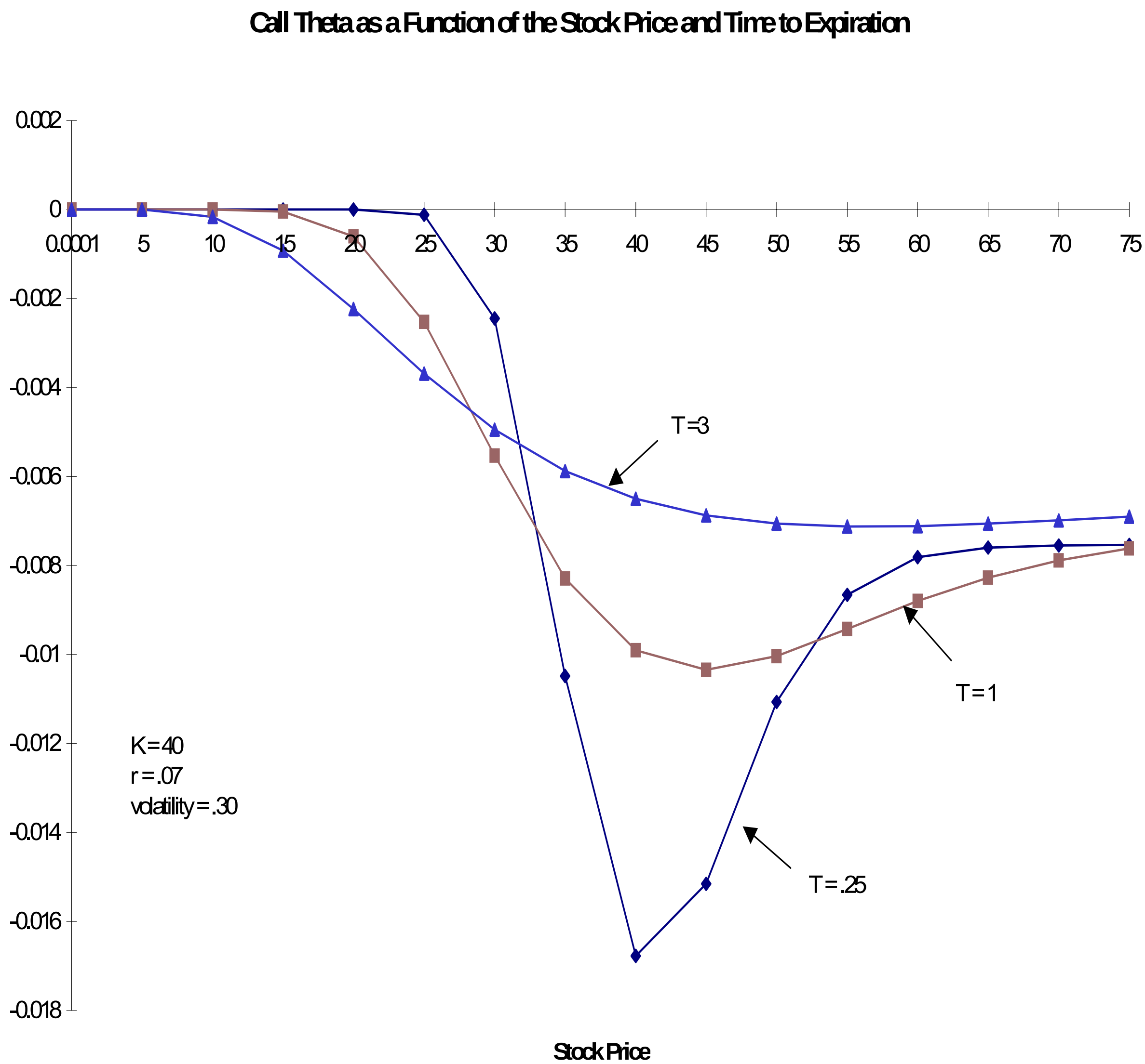


Put Price as a Function of the Stock Price and the Time to Expiration

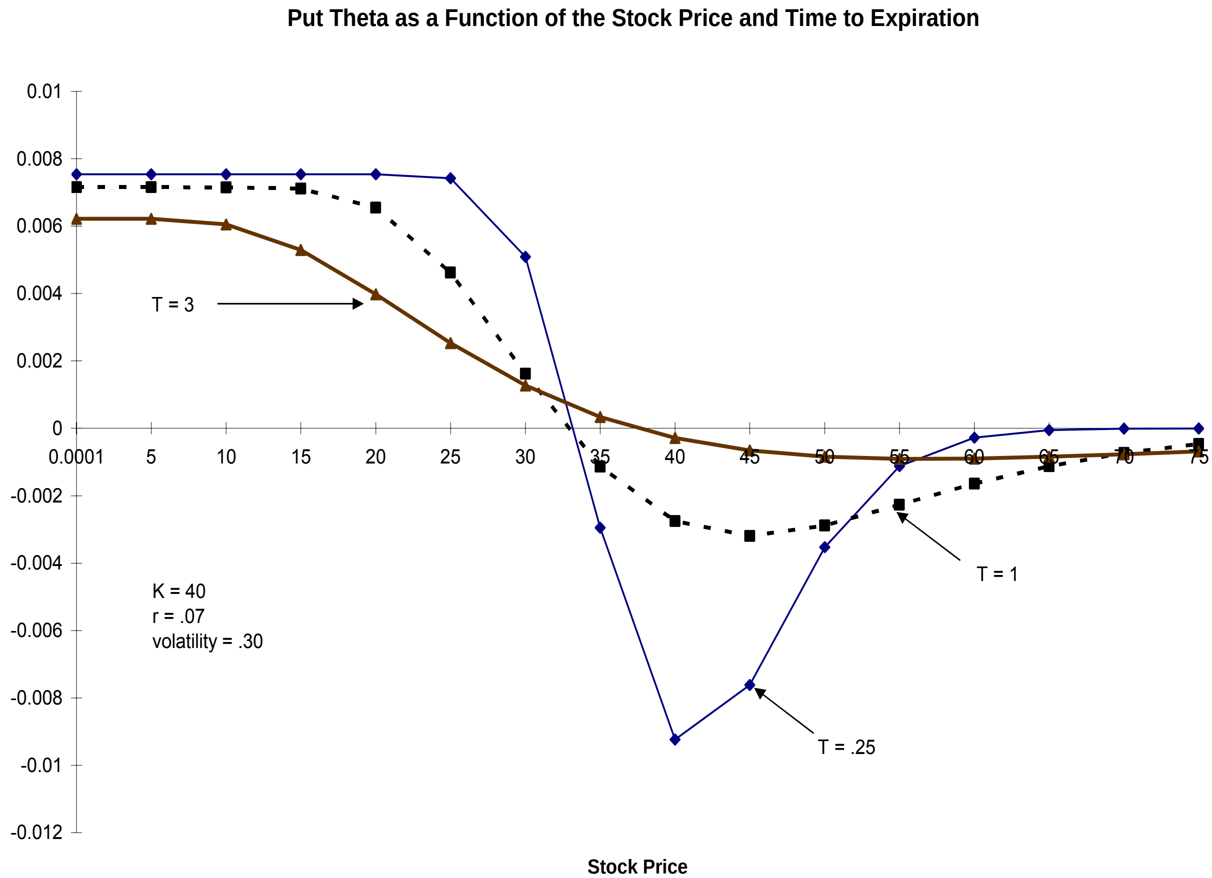


Theta is equal to $\partial C/\partial(T\text{-}t)$ for calls and $\partial P/\partial(T\text{-}t)$ for puts.

The theta of a call is always negative. This is because as the time to expiration grows shorter, both time value and option value decreases.



The theta of a put could be either positive or negative.



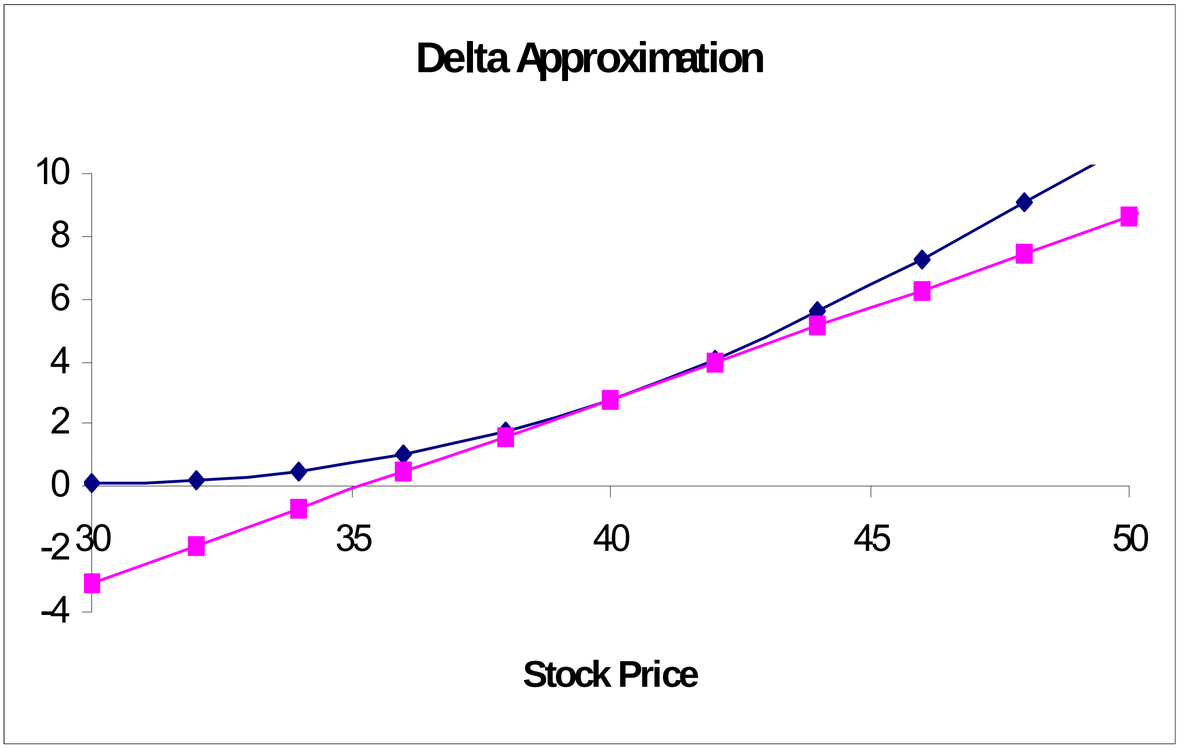
Another Way to Look at Delta Hedging

Suppose the current share price is S and the time to expiration is T-t. Given this, we can write the call price as C(S, T-t).

As time passes, two things will change. The time to expiration gets shorter i.e., time to expiration changes from (T-t) to (T-t-h) and the stock price changes from S to S+ϵ. ϵ is a random variable which can be positive or negative. A simple approximation of the value of the call, referred to as the delta-approximation, is:

$$C(S+\epsilon, T-t-h) = C(S,T-t) + \epsilon\Delta(S, T-t)$$

Recall that delta is the change in the value of the call for a small change in the value of the stock. So $\epsilon\Delta$ is the approximate change in the value of the call when the stock price change is ϵ.



The approximate <u>change in the value of the call</u> is:

$$C(S+\epsilon, T\text{-}t\text{-}h) - C(S,T\text{-}t) = \epsilon\Delta(S, T\text{-}t)$$

When we delta hedge we are hedging the approximate changes in the call price which arise from random changes in the stock. If we go short $\Delta(S, T\text{-}t)$ shares, <u>the change in the value of the share position</u> is:

$$-\Delta(S, T\text{-}t)[(S+\epsilon) - S] = -\epsilon\Delta(S, T\text{-}t).$$

Therefore the change in the value of the total position (call plus delta shares) is approximately zero.

<u>Could we create a better hedge?</u>

One problem with the delta hedge is that the actually delta is changing. Ideally we would like to use an "average" delta rather than the delta at the current stock price, that is, we would like to use:

$$\Delta_{average} = \frac{\Delta(S, T\text{-}t) + \Delta(S+\epsilon, T\text{-}t)}{2}$$

In a similar way to the way we approximated the call value using delta, we can approximate $\Delta(S+, \epsilon\ T\text{-}t)$ using gamma, that is,

$$\Delta(S+\epsilon, T\text{-}t) = \Delta(S, T\text{-}t) + \epsilon\Gamma(S,T\text{-}t)$$

Recall that gamma is the change in the value of delta for a small change in the value of the stock. So $\epsilon\,\Gamma$ s the approximate change in the value of delta when the stock price change is ϵ.

Using this approximation, we get

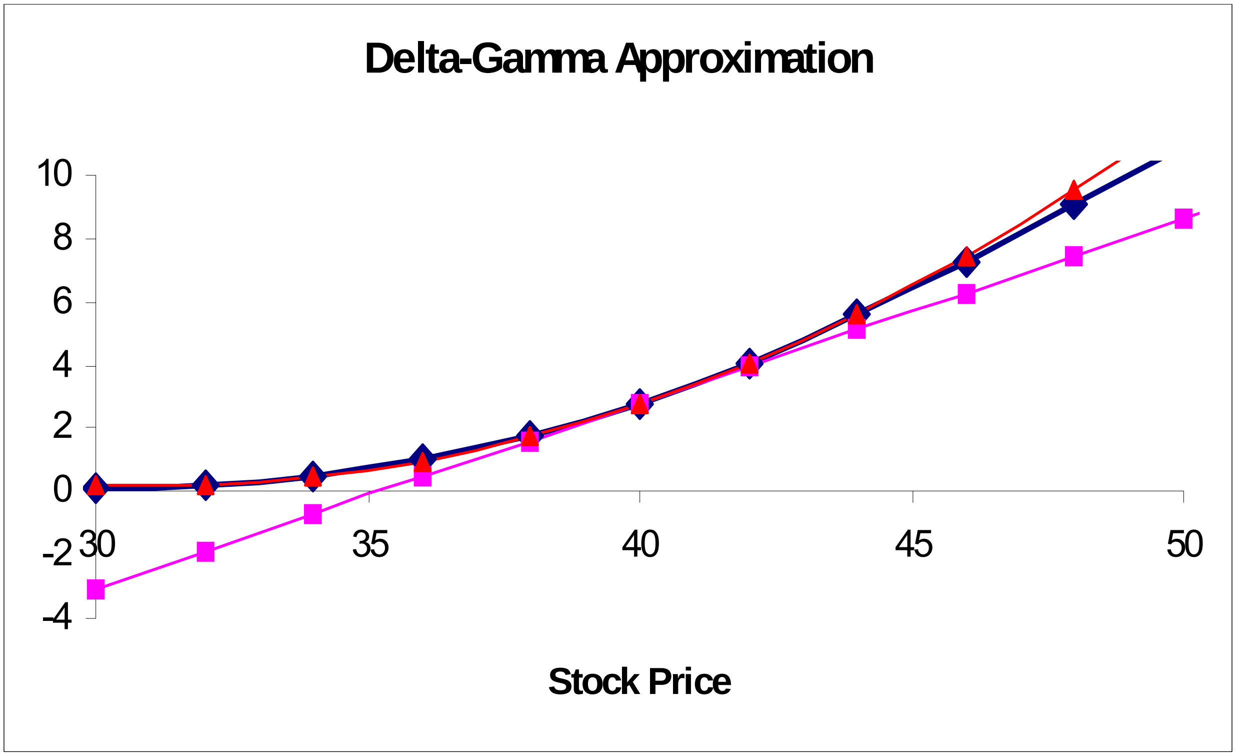
$$\begin{aligned}\Delta_{average} &= [\Delta(S, T\text{-}t) + \Delta(S, T\text{-}t) + \epsilon\Gamma(S,T\text{-}t)]/2 \\ &= \Delta(S, T\text{-}t) + \tfrac{1}{2}\epsilon\Gamma(S,T\text{-}t)\end{aligned}$$

Using this, our approximation for the call becomes:

$$C(S+\epsilon, T-t-h) = C(S,T-t) + \epsilon\Delta_{average}$$

$$= C(S,T-t) + \epsilon\,[\Delta(S, T-t) + \tfrac{1}{2}\epsilon\Gamma(S,T-t)]$$

$$= C(S,T-t) + \epsilon\Delta(S, T-t) + \tfrac{1}{2}\epsilon^2\Gamma(S,T-t)$$



We can improve the approximation still further by taking into account the effect of time passing on the call's value.

$$C(S+\epsilon,T-t-h) = C(S,T-t) + \epsilon\Delta(S, T-t) + \tfrac{1}{2}\epsilon^2\Gamma(S,T-t) + h\theta(S, T-t)$$

Recall that theta is the change in the value of the call for a small change in the time to expiration.

Example:

Stock Price	50
Exercise Price	50
Volatility	40.000%
Risk-free interest rate	5.000%
Time to Expiration (years)	0.249315
Dividend Yield	0.000%

Price	4.270089
Delta	0.564456
Gamma	0.039426
Vega	0.098296
Theta	-0.02488
Rho	0.059718
Elasticity	6.609419

Note :

1. T-t = 91/365 and T-t-h is 90/365
2. The theta in the spreadsheet is defined to be the change in the call price as the time to expiration gets one day closer.
3. A one standard deviation move over one day is

$$S\sigma\sqrt{h} = 50*.40*\sqrt{\tfrac{1}{365}} = 50*.021 = \$1.047$$

S(T-t-h)	ϵ	C(S,T-t)	$\epsilon\Delta$(S, T-t)	$\frac{1}{2}\epsilon^2\Gamma$(S,T-t)	hθ(S,T-t)	Approximation	Actual
52	2	4.27	1.1289	.0788	-.02488	5.453	5.450
51	1	4.27	.5644	.0197	-.02488	4.829	4.829
49	-1	4.27	-.5644	.0197	-.02488	3.700	3.701
48	-2	4.27	–1.1289	.0788	-.02488	3.195	3.198

We can use this approximation to understand how Black and Scholes came up with their formula.

Suppose a market maker hedges a short call by going long delta shares.

The change in the value of his position when the stock price changes from S to S+ϵ and the time to expiration changes from T-t to T-t-h will be the sum of:

1. Change in the value of the stock position $= \Delta^*[S(T-t-h) - S(T-t)]$

$= \Delta^*\epsilon$

minus

2. Change in the value of the call $= [C(S+\epsilon, T-t-h) - C(S, T-t)]$

$\approx \epsilon\Delta + ½\epsilon^2\Gamma + h\theta$

minus

3. Interest expense associated with the cost of the initial position

$= rh[\Delta S(T-t-h) - C(S, T-t)]$

Therefore the market maker's profit is approximately

$$\pi = -\tfrac{1}{2}\epsilon^2\Gamma - \theta h - rh[\Delta S(T\text{-}t\text{-}h) - C(S, T\text{-}t)]$$

Note:

1. Since the profit only depends on ϵ^2 rather than ϵ, gains and losses depend on the magnitude of the stock price change but not the direction of the change.

2. Since the gamma of a long call is positive, market maker who hedges a short call position loses money due to gamma and the loss is increasing in the change in the stock price.

3. The effect of theta is given by - θh. Since the theta of a call is negative, the hedged position is helped by the time decay.

4. The interest expense is a cost since the cost of the shares exceeds the revenue from the calls.

The Black - Scholes prices are designed so that the market maker exactly breaks even when there is a one standard deviation stock price move.

Example:

Stock Price	50
Exercise Price	50
Volatility	40.000%
Risk-free interest rate	5.000%
Time to Expiration (years)	0.249315
Dividend Yield	0.000%

Price	4.270089
Delta	0.564456
Gamma	0.039426
Vega	0.098296
Theta	-0.02488
Rho	0.059718
Elasticity	6.609419

Note :

1. h=1/365
2. A one standard deviation move over one day is

$$S\sigma\sqrt{h} = 50*.40*\sqrt{\tfrac{1}{365}} = 50*.021 = \$1.047$$

π $= -\frac{1}{2}\epsilon^2\Gamma - \theta h - rh[\Delta S(T-t-h) - C(S, T-t)]$

$= -\frac{1}{2}*(1.047)^2(.039426) - (-.02488)-(.05)(1/365)(.564456*50 - 4.2701)$

$= 0$

Since a short call is a negative gamma position, the market maker will make money when the stock price change is less than one standard deviation and lose money when the change is greater than one standard deviation.

Back to Black-Scholes

Suppose that the stock prices moves one standard deviation (either up or down) every minute, that is

$$\epsilon = \pm S\ \sigma\sqrt{h}$$

Suppose a market maker shorts a call and delta hedges the position. If he adjusts his hedge every minute, his hedge will be perfect (i.e., his position will be riskless). Since he has no risk, he should earn the risk free rate on the capital invested, that is, the gain on the hedged position every period should exactly cover the interest expense associate with carrying the position.

$$-\tfrac{1}{2}\sigma^2 S^2 h\Gamma - \theta h = rh[\Delta S - C]$$

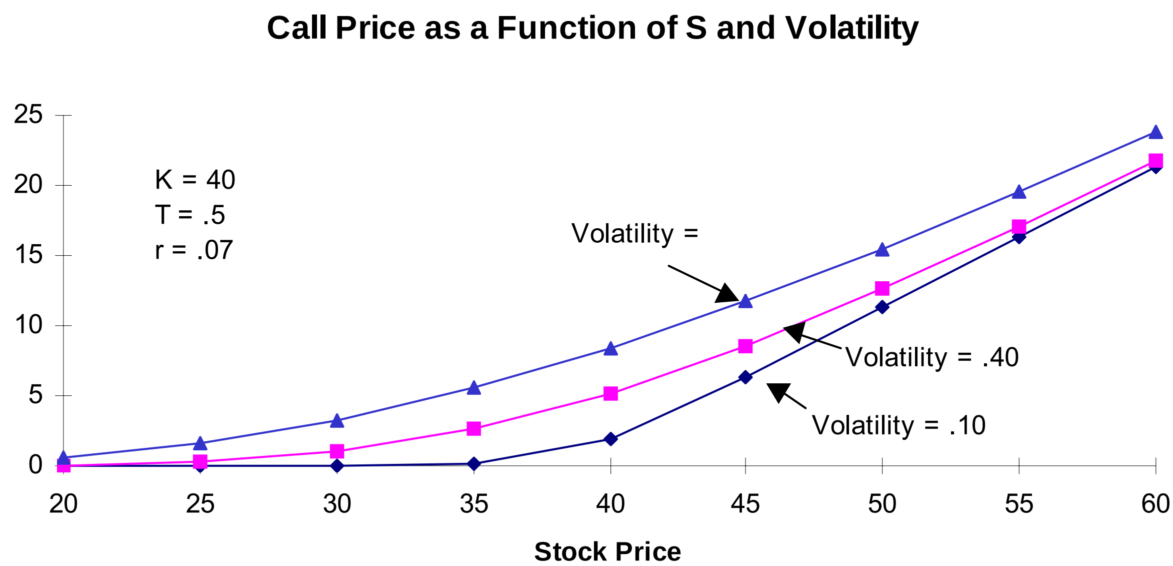
Dividing both sides by h and rearranging terms, we get:

$$rC(S) = \tfrac{1}{2}\sigma^2 S^2 h\Gamma + rS\Delta + \theta = \frac{1}{2}\sigma^2 S^2 h\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial (T-t)}$$

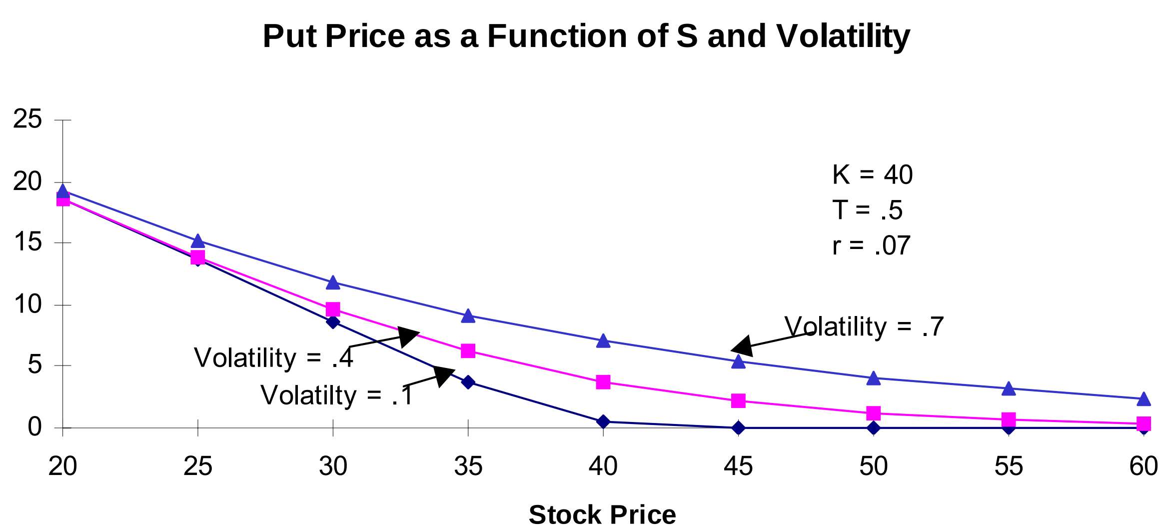
This is the Black-Scholes partial differential equation and the solution to this partial differential equation is the Black-Scholes formula.

■ Vega (this is sometimes referred to as Kappa)

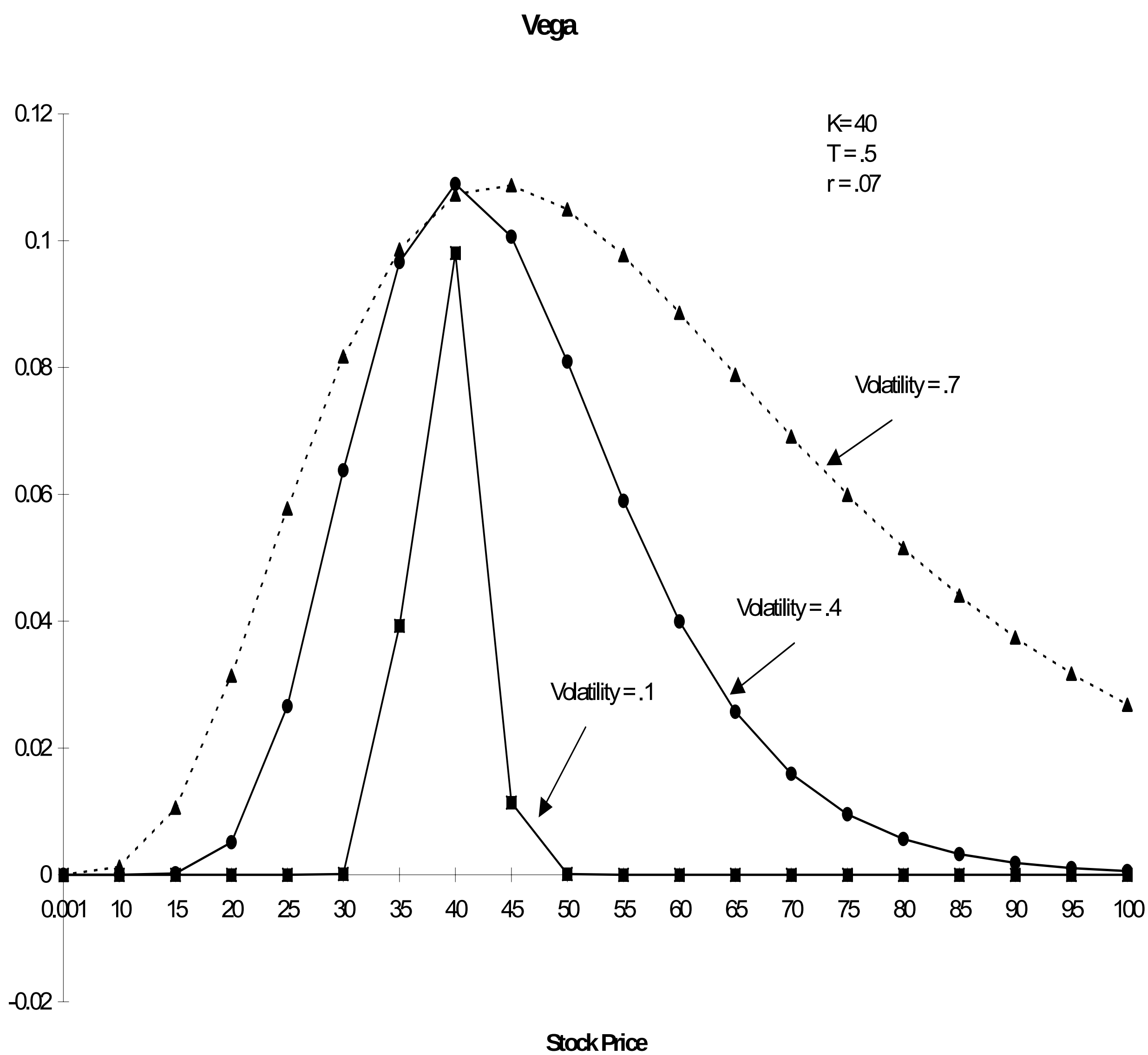
Vega gives the sensitivity of the option price to a small change in the volatility.



Vega is equal to $\partial C/\partial \sigma$ for calls and $\partial P/\partial \sigma$ for puts.



The vega of a put is equal to the vega of an otherwise identical call.


Vega
K=40
T = .5
r = .07
Volatility = .7
Volatility = .4
Volatility = .1
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0.001 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100
Stock Price

■ Implied Volatility

If you know C, S, T, and R, then using the Black-Scholes formula you can back out an estimate of the market's assessment of the volatility. This is known as the implied volatility.

Implied volatilities tend to vary across options with the same underlying asset. As you can see by looking at the graphs, option prices are most sensitive to changes in volatility when they are at-the-money. Therefore at-the-money options probably give the best estimate of the implied volatility.

Inputs			Black-Scholes (European)		
				Call	Put
Stock Price	40		Price	5.770257	3.486964
Exercise Price	40		Delta	0.62422	-0.35598
Volatility	30.000%		Gamma	0.030651	0.030651
Risk-free interest rate	8.000%		Vega	0.147124	0.147124
Time to Expiration (years)	1		Theta	-0.00889	-0.00294
Dividend Yield	2.000%		Rho	0.191986	-0.17726
# Binomial steps	100		Elasticity	4.327159	-4.08353
Type (0=Eur, 1=Amer)	0				
			Binomial European		
Implied Volatility				Call	Put
Observed Call Price	5		Price	5.75901	3.475717
Call Implied Volatility	24.75%		Delta	0.624391	-0.35581
Observed Put Price	3		Gamma	0.030867	0.030867
Put Implied Volatility	26.69%		Theta	-0.00895	-0.003

■ Rho

Rho gives the sensitivity of the option price to changes in the interest rate.

Rho is equal to $\partial C/\partial R$ for calls and $\partial P/\partial R$ for puts.

Option prices are not particularly sensitive to changes in the interest rate.