Equilibrium Selection in the War of Attrition under Complete Information∗

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Abstract

We consider a two-player war of attrition under complete information. It is well-known that this class of games admits equilibria in both pure and mixed strategies; much of the literature has focused on the latter. We show that if the players’ payoffs whilst in “war” follow an irreducible stochastic process and their exit payoffs are heterogeneous, then all equilibria are in pure strategies. This result holds irrespective of the degree of uncertainty or heterogeneity. Moreover, this result holds irrespective of the degree of uncertainty or heterogeneity, thus highlighting the fragility of mixed-strategy equilibria in this class of games. In contrast, if either the players’ flow payoffs are deterministic, or their exit payoffs are homogeneous, then the game admits equilibria in both pure and mixed strategies.

1 Introduction

In the classic war of attrition, the first player to quit concedes a prize to his opponent. Each player trades off the cost associated with fighting against the value of the prize. These features are common in many managerial and economic problems. Oligopolists in a declining industry may bear losses in anticipation of profitability following a competitor’s exit (Ghemawat and Nalebuff, 1985). For example, the rise of Amazon in the mid-1990s made the business model of Barnes & Noble and Borders obsolete, turning traditional bookselling into a declining market. As the demand shrank

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sharply, these two major players at the time had to cut down slack in their capacities, but each would prefer its competitors to carry the painful burden of closing stores or exiting the market altogether (Newman, 2011). Similarly, the presently low price of crude oil is often attributed to a war of attrition among Saudi Arabia, its Persian Gulf OPEC allies, and non-OPEC rivals such as Russia and the many shale-oil producers in the United States (Reed, 2016). Other examples of wars of attrition include the provision of public goods (Bliss and Nalebuff, 1984), lobbying (Becker, 1983), labor disputes (Greenhouse, 1999), court of law battles (McAttee, 2009), races to dominate a market (Ghemawat, 1997), technology standard races (Bulow and Klemperer, 1999), price cycles in oligopolistic collusion (Maskin and Tirole, 1988), all-pay auctions (Krishna and Morgan, 1997), and bargaining games (Abreu and Gul, 2000).

It is well-known that the canonical model of war of attrition admits equilibria in both pure and mixed strategies; see for example, Tirole (1988), Fudenberg and Tirole (1996), and Levin (2004). Moreover, much of the (applied) literature has focused on the mixed-strategy equilibria, owing to the fact that only they feature certain attractive properties such as attrition (i.e., costly waste of resources), and in certain cases, symmetry. We study a simple model of war of attrition under complete information. Our main results show that if the players’ flow payoffs whilst fighting for the prize follow an irreducible stochastic process and their exit payoffs are heterogeneous, then subject to certain restrictions on the players’ strategies, the game admits only pure-strategy Subgame Perfect equilibria, and this is true irrespective of the degree of uncertainty or heterogeneity. As it is unlikely that players are precisely identical or payoffs are deterministic, this result highlights the fragility of mixed-strategy equilibria in this class of games. Moreover, it has implications for a growing literature that aims to empirically characterize strategies in real-world games of war of attrition; see for example Wang (2009) and Takahashi (2015).

In our continuous-time model, two competing oligopolists contemplate exiting a market. While both firms remain in the market, each receives a flow payoff that depends on the market conditions (e.g., the price of a relevant commodity), which fluctuate according to a Brownian motion, hereafter the state. At every moment, each firm can exit the market and collect its outside option. Its rival then obtains a (higher) winner’s payoff, which depends on the state at the time of exit; e.g., the net present value of monopoly profits. All payoff-relevant parameters are common knowledge. The firms may have heterogeneous outside options but are otherwise identical.

We first characterize the best response of a firm that anticipates its rival will never exit the market, which turns out to be instrumental for the equilibrium analysis. We show that a firm optimally exits at the first moment that the state drifts below a threshold. This single-player-optimal threshold is strictly increasing in the firm’s outside option, which is intuitive: the better is a firm’s outside option, the less it is willing to endure poor market conditions before exiting.

In Section 4 we analyze Markov Perfect equilibria (MPE), wherein at every moment, each
firm conditions its probability of exit on the current state. Proposition 1 shows that there exists a
pure-strategy MPE in which the firm with the larger outside option exits at the first moment that
the state drifts below its single-player-optimal threshold. Moreover, if the heterogeneity in outside
options is not too large, then there exists another pure-strategy MPE in which the firm with the
lower outside option exits at the first moment that the state drifts below its own single-player-optimal
threshold. Towards our first main result, we show that in any mixed-strategy MPE, (i) the firms must
be randomizing between remaining in the market and exiting on a common set of states, and (ii)
each firm exits with nonzero probability if (and only if) the state is below its single-player-optimal
threshold. However, (i) and (ii) are incompatible with each other if the firms have heterogeneous
outside options, because their single-player-optimal thresholds differ in that case. Therefore, we
conclude that no mixed-strategy MPE exists in that case. An implication of this result is the non-
existence of MPE that feature attrition when the state is stochastic and the firms have heterogeneous
outside options; i.e., equilibria in which both firms resist exit below their single-player-optimal
thresholds.

The key driving force behind this result is that the state follows an irreducible stochastic process.
If the state is deterministic, as in [Hendricks et al., 1988] for example, then in any mixed-strategy
equilibrium, the firm with the smaller outside option (say firm 1) exits with a positive probability
when the state hits its single-player optimal threshold, and from then onwards, during every interval
\((t, t + dt)\), each firm exits with some probability that is of order \(dt\) and makes the other firm indif-
f erent between exiting and not. If the state is stochastic, any hypothetical candidate mixed-strategy
MPE must be also of the form described above. However, when the state is just below that threshold,
due to the irreducibility of the state, with some likelihood, it will hit that threshold in short order and
firm 1 will exit. As a result, firm 2 strictly prefers to not exit, which in turn leads firm 1 to strictly
prefer to exit, leading to a pure-strategy MPE.

A natural concern is the restrictiveness of Markov strategies. In Section 5, we consider the
possibility that firms condition their exit decision on the entire history. We show that under either one
of two restrictions on the firms’ strategies, with heterogeneous outside options, the game admits no
mixed-strategy Subgame Perfect equilibria (SPE). To explain these restrictions, we first note that a
strategy consists of (i) a collection of stopping times at which the firm exits with positive probability,
and (ii) an exit rate function, specifying the probability that the firm exits during an interval \((t, t + dt)\), which is of order \(dt\) and depends on the history of the state up to \(t\). The first restriction imposes
that the firms exit with positive probability at no more than finitely-many stopping times and they
do not exit with probability one after any history. The second restriction mandates that the (possibly
infinitely-many) stopping times are also hitting times and the firms’ exit rates depend only on the
current state. We remark that both restrictions are satisfied by the strategies in the mixed-strategy
SPE that appear in the literature when the state evolves deterministically [Hendricks et al., 1988], or
the firms are homogeneous (Steg, 2015), or both (Tirole, 1988).

First and foremost, this paper contributes to the literature on wars of attrition, which has received widespread attention since the seminal work of Maynard Smith (1974). Our model is closest to Hendricks et al. (1988) and Murto (2004). The former characterizes equilibria in both pure and mixed strategies in a war of attrition under complete information with symmetric players whose payoffs vary deterministically over time. The latter considers stochastic payoffs, but restricts attention to pure-strategy MPE. In contrast, we allow payoffs to vary stochastically, and we show that if players are heterogeneous, then (subject to a set of restrictions on strategies) the game admits SPE in pure strategies only.

We also contribute to a literature that contemplates equilibrium selection in games of war of attrition. This literature has two broad themes. The first considers games which are backward-inductible. For example, Ghemawat and Nalebuff (1985) studies a game with asymmetric players in which there is a state (that is reached with probability one) at which both firms have a dominant strategy to exit, while Bilodeau and Slivinski (1996) considers a finite-horizon war of attrition game. In both cases, the game is shown to have a unique equilibrium in pure strategies. In the second theme, with a small probability, each player never exits. In Fudenberg and Tirole (1986), players are uncertain about their rivals’ costs of remaining in the market, whereas in Kornhauser et al. (1989), Kambe (1999), and Abreu and Gul (2000), with a small probability, each player is irrational and never exits. It is shown that the respective games admit a unique equilibrium. See also Myatt (2005) and the references therein. We complement this literature by considering a complete-information framework with rational players, and showing that an arguably natural perturbation of the canonical model eliminates all mixed-strategy equilibria.

Touzi and Vieille (2002) introduces the concept of mixed strategies in continuous-time Dynkin games (a class of stopping games), and proves that the game admits minimax solutions in mixed strategies. With this notion of mixed strategies, Seel and Strack (2016) investigates a war of attrition (specifically, an all-pay auction) with privately observed Brownian motions, and Steg (2015) characterizes equilibria in both pure and mixed strategies in a family of continuous-time stochastic timing games. Whereas these articles consider games with identical players, we focus on ones with heterogeneous players and show that the set of equilibria differs drastically. Riedel and Steg (2017) examines mixed-strategy equilibria in continuous-time stopping games with heterogeneous players, but restricts attention to pre-emption games, whereas our model is one of war of attrition.

Finally, our paper is related to the literature in real option games in the context of timing decisions with externalities under uncertainty. Dixit and Pindyck (1994) establishes a framework for analyzing real option games. Grenadier (2002), Lambrecht and Perraudin (2003), and Mason and Weeds (2010) examine the interplay between the option value of waiting and externalities due to competition, learning, and network effects. These papers focus on the role of a preemptive threat in
real option games, whereas our focus is on a free-riding incentive.

2 Model

We consider a war of attrition with complete information between two oligopolists. Time is continuous, and firms discount time at rate $r > 0$. At every moment, each firm decides whether to remain in, or exit the market.

While both firms remain in the market, each earns a flow profit $\pi(X_t)$, where $\pi : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and $X_t$ is a scalar that captures the market conditions that the firms operate in (e.g., the size of the market or the price of raw materials). The market conditions fluctuate according to

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t,$$

where $X_t$ is defined on $X := (\alpha, \beta) \subseteq \mathbb{R}$, $X_0 \in X$, the functions $\mu : X \to \mathbb{R}$ and $\sigma : X \to \mathbb{R}_+$ are Lipschitz continuous, and $B_t$ is a Wiener process.\footnote{Special cases in which $\sigma(\cdot) = 0$ have been analyzed extensively \cite{GhemawatNalebuff1985, Hendricks1988, Alvarez2001} and others). Therefore, we restrict attention to $\sigma(\cdot) > 0$ in the main body of this paper, and for completeness, we revisit the case in which $\sigma(\cdot) = 0$ in Appendix A.} Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ denote the probability space with sample space $\Omega$, $\sigma$-algebra $\mathcal{F}$, probability measure $\mathbb{P}$, and filtration $\{\mathcal{F}_t\}_{t \geq 0}$ that satisfies the usual conditions \cite[p. 172]{RogersWilliams2000}. We assume that the process $\{B_t\}_{t \geq 0}$ (or equivalently, $\{X_t\}_{t \geq 0}$) is progressively measurable with respect to $\{\mathcal{F}_t\}_{t \geq 0}$. Throughout the paper, we let $\mathbb{E}[\cdot]$ denote the expected values with respect to $\mathbb{P}$.

If firm $i$ chooses to exit at time $t$, then it receives its outside option $l_i$, and its opponent, denoted by $-i$, receives $w(X_t) \in \mathbb{R}$, the expected payoff associated with being the sole remaining firm; e.g., the net present value of monopoly profits. In this case, we say that firm $i$ is the loser and firm $-i$ is the winner. We adopt the convention that $l_1 \leq l_2$; i.e., firm 2 has a larger outside option than firm 1. We assume that $w(x) > l_2$ for all $x$ so that the winner’s reward is always larger than the loser’s. The game ends as soon as a firm exits the market. If both firms exit at the same moment, then each obtains the outside option $l_i$ or $w(X_t)$ with probability $1/2$.\footnote{The boundary points $\alpha$ and $\beta$ are assumed to be natural \cite[p.18-20]{BorodinSalminen1996}; i.e., neither $\alpha$, nor $\beta$ can be reached by $X_t$ in finite time. For example, if $X_t$ is a standard diffusion process, then $X = \mathbb{R}$. If $X_t$ is a geometric Brownian process, then $X = (0, \infty)$.}

Finally, we make the following assumptions on the functions $\pi(\cdot)$ and $w(\cdot)$: First, we assume that $\pi(\cdot)$ satisfies the absolute integrability condition $\mathbb{E} \left[ \int_0^\infty |e^{-rt}\pi(X_t)| dt \right] < \infty$, which ensures that each firm’s expected discounted payoff is well-defined (see \cite{Alvarez2001}). Second, we assume $w(\cdot) \in C^2(X)$ and $w(x) > \mathbb{E} \left[ \int_0^t e^{-rs}\pi(X_s)ds + e^{-rt}w(X_t) | X_0 = x \right]$ for all $x \in X$ and $t$, so that each

$$\int_0^\infty e^{-rt}\pi(X_t) dt,$$
firm prefers to become the winner sooner rather than later. Lastly, we assume that for each $i$, there exists some $x_{ci} \in X$ such that $\pi(x_{ci}) = rl_i$, which guarantees the existence of an optimal exit threshold in the interior of $X$.

### 2.1 Strategies

At every moment, given the history $h^t = \{X_s\}_{s \leq t}$ and conditional on the game not having ended, each firm chooses (probabilistically) whether to exit. Formally, each firm $i$ chooses

i. a set of histories $I_i$ (or an exit region) such that it exits with probability 1 if $h^t \in I_i$,

ii. a set of stopping time and exit probability pairs, denoted by $\mathcal{P}_i = \{(\tau_{i,n}, p_{i,n})\}_{n=1}^{\infty}$, such that the firm exits at $t = \tau_{i,n}$ with probability $p_{i,n} \in (0, 1)$, and

iii. a non-negative process $\Lambda_i = \{\lambda_{i,t}\}_{t > 0}$, which represents the firm’s hazard rate of exit at $t$.

We assume that each stopping time - probability pair $(\tau_{i,n}, p_{i,n})$ and the process $\lambda_{i,t}$ is progressively measurable with respect to $\mathcal{F}_{\tau_{i,n}}$ and $\mathcal{F}_t$, respectively, and so the exit probability $p_{i,n}$ at $\tau_{i,n}$, and the exit rate $\lambda_{i,t}$ may depend on the entire history of $X$ up to $\tau_{i,n}$ and $t$, respectively.\footnote{This assumption is satisfied if and only if $\sigma^2(x)w''(x)/2 + \mu(x)w'(x) + \pi(x) > rw(x)$ for all $x \in X$.}

Then we can represent firm $i$’s strategy as the three-tuple $a_i = (I_i, \mathcal{P}_i, \Lambda_i)$, and $\{a_1, a_2\}$ is a strategy profile. As each firm’s decision at $t$ can be conditioned on the entire history up to $t$, it is without loss to assume that each firm chooses its strategy at time 0. Intuitively, during any small interval $[t, t + dt)$, firm $i$ exits with probability

$$
\rho_{i,t} = \begin{cases} 
1 & \text{if } h^t \in I_i, \\
p_{i,n} + (1 - p_{i,n})\lambda_{i,t}dt & \text{if } t = \tau_{i,n}, \text{ and} \\
\lambda_{i,t}dt & \text{otherwise.}
\end{cases}
$$

If firm $i$ does not exit with probability 1 after any history, we write $I_i = \emptyset$. If it does not choose any stopping time - exit probability pairs, we write $\mathcal{P}_i = \emptyset$. If it does not exit with a positive hazard rate (i.e., $\lambda_{i,t} = 0$ for all $t$ almost surely, hereafter a.s), we write $\Lambda_i = \emptyset$. Finally, we say that firm $i$’s strategy is pure if $\mathcal{P}_i = \emptyset$ and $\Lambda_i \equiv 0$, and it is mixed otherwise.

### 3 Preliminaries

In Section 3.1 we write each firm’s payoff as a function of an arbitrary strategy profile. Then in Section 3.2 we characterize the best response of a firm who anticipates that its rival will never exit.\footnote{When we restrict attention to Markov strategies in Section 4, we impose appropriate restrictions so that the probability of exit at $t$ depends only on $X_t$.}
3.1 Payoffs

Fix a strategy profile \( \{a_1, a_2\} \) and history \( h' \), and define \( \tau_{i,0} := \inf\{s \geq 0 : h^x \in I_i\} \) and \( p_{i,0} := 1 \). The survival probability, that is, the probability that the game does not end during \([t, u]\) is given by

\[
S_{t,u} := e^{-\int_t^u (\lambda_{1,s} + \lambda_{2,s})ds} \prod_{\{n,m \geq 0 : t \leq \tau_{1,n} < u, t \leq \tau_{2,m} < u\}} (1 - p_{1,n})(1 - p_{2,m}).
\]

Firm \( i \)'s payoff at time \( t \) (conditional on the game not having ended by \( t \)) can be written as

\[
V_i(h'; a_1, a_2) = \mathbb{E}\left[ \int_t^{\infty} e^{-r(s-t)}S_{t,s}[\pi(X_s) + \lambda_{i,s}l_i + \lambda_{j,s}w(X_s)]ds + \sum_{n \geq 0} S_{t,\tau_{i,n}}e^{-r(\tau_{i,n}-t)}p_{i,n}l_i + \sum_{m \geq 0} S_{t,\tau_{j,m}}e^{-r(\tau_{j,m}-t)}p_{j,m}w(X_{\tau_{j,m}}) - \frac{1}{2} \sum_{n,m \geq 0} \mathbb{I}_{\{\tau_{i,n} = \tau_{j,m}\}} S_{t,\tau_{i,n}}e^{-r(\tau_{i,n}-t)}p_{i,n}p_{j,m}(l_i - w(X_{\tau_{j,m}})) \right].
\]

The first line represents the firm’s discounted flow payoff with survival chances taken into account, plus the reward from the end of the game through the exit rate by either firm. The second line captures the payoff from either firm’s instantaneous exit probability, while the third line accounts for the possibility of simultaneous exit and the double counting from the second line.

A strategy profile \( \{a_1^*, a_2^*\} \) is a Subgame Perfect equilibrium (hereafter SPE) if

\[
V_i(h'; a_1^*, a_2^*, a_{-i}) \geq V_i(h'; a_i, a_{-i})
\]

for each firm \( i \), every history \( h' \), and every strategy \( a_i \). An equilibrium is Markov Perfect (hereafter MPE) if the probability that a firm exits at \( t \) depends only on \( X_t \) (as opposed to the entire history \( h' \)).

3.2 Best Response to a Firm that Never Exits

We now characterize firm \( i \)'s best response assuming that its rival never exits; \textit{i.e.}, the best response to \( a_{-i} = (0,0,0) \). In this case, the firm’s best response can be determined by solving a single-player optimal stopping problem as in [Alvarez (2001)]. Because \( X \) is a stationary process and the horizon is infinite, it is without loss to restrict attention to pure strategies such that \( P_i = \emptyset \) and \( \Lambda_i = \emptyset \), and the firm’s expected payoff at \( t \) depends solely on the current value of the state \( x = X_t \). Thus, it can be expressed as

\[
\sup_{\tau_i} \mathbb{E}\left[ \int_t^{\tau_i} \pi(X_s)e^{-r(s-t)}ds + l_i e^{-r(\tau_i-t)} \right].
\]

Using Proposition 2 in [Alvarez (2001)], we can characterize the firm’s optimal exit region as follows.
Lemma 1 Suppose firm $-i$ never exits. There exists a unique threshold $\theta^*_i$ such that $I^*_i = \{X_t \leq \theta^*_i\}$ is optimal for firm $i$, that is, firm $i$ optimally exits whenever $X_t \leq \theta^*_i$. If $l_1 < l_2$, then $\theta^*_1 < \theta^*_2$.

The proof is relegated to Appendix D. Intuitively, a firm’s value of remaining in the market decreases as the market conditions deteriorate, and once they become sufficiently poor, the firm is better off exiting and collecting its outside option. As the firms earn identical flow payoffs while in the market, the firm with the higher outside option optimally exits at a higher threshold.

4 Markov Perfect Equilibria

We characterize pure-strategy MPE in Section 4.1, and in Section 4.2, we consider mixed-strategy MPE. In particular, we establish a necessary condition that any mixed-strategy MPE must satisfy, and our first main result follows immediately: the game has no mixed-strategy MPE if the firms have heterogeneous exit payoffs (i.e., $l_1 \neq l_2$).

Following the definition of MPE by Maskin and Tirole (2001), at every $t$, each firm’s decision to exit may depend only on payoff-relevant state variables, which in this case is $X_t$. Therefore, to study MPE, we must impose appropriate restrictions on the firms’ strategies.

First, because firm $i$’s decision to exit at $t$ must depend only on $X_t$, region $I_i$ must be of the form

$$I_i = \{X_t \in E_i\}$$

for some closed set $E_i \subseteq X$, and each firm’s exit rate at $t$, $\lambda_{i,t}$, must be a function of only $X_t$.

To make this explicit, we write $\lambda_i(x)$ to denote the firm’s exit rate when $X_t = x$. Second, due to the property of Brownian motion of coming back to the original value indefinitely many times within any finite time span, any Markovian strategy in which $p_{i,n} \in (0, 1)$ if $t = \tau_{i,n}$ is indistinguishable from one in which the firm exits with probability 1 at $\tau_{i,n}$. Therefore, it is without loss to assume that $P_i = \emptyset$.

Thus, abusing notation slightly, a Markov strategy for firm $i$ can be expressed as $a_i = (E_i, \lambda_i)$, where $E_i \subseteq X$ is a closed set and $\lambda_i : X \rightarrow \mathbb{R}_+$ is a finite-valued function.

4.1 Pure-strategy MPE

Let $E^*_i = (\alpha, \theta^*_i]$, where $\theta^*_i$ is given in Lemma 1. The following proposition shows that there is a pure-strategy MPE in which firm 2 exits at the first moment that $X_t$ drifts below $\theta^*_2$, and firm 1 never exits. Moreover, if the firms are not too heterogeneous, there is another pure-strategy MPE in which firm 1 exits at the first moment that $X_t \leq \theta^*_1$ and firm 2 never exits.

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6Because $X$ is a diffusive process, exiting when $X_t \in E_i$ is indistinguishable from exiting when $X_t \in cl(E_i)$. Thus, it is without loss to assume that $E_i$ is a closed set.
Proposition 1: The strategy profile \( \{a_1, a_2\} = \{(0, 0), (E^*_2, 0)\} \) is a pure-strategy MPE. Moreover, there exists a threshold \( \kappa > 0 \) that is independent of \( l_1 \) such that \( \{a_1, a_2\} = \{(E^*_1, 0), (0, 0)\} \) is also a pure-strategy MPE if \( |l_2 - l_1| < \kappa \).

The proof is provided in Appendix [D]. If firm \( i \) expects its rival to never exit, then by Lemma [1] it will optimally exit at the first time such that \( X_t \leq \theta^*_i \). Therefore, it suffices to show that if firm \( i \) employs the exit region \( I_i^* \), then its opponent’s best response is to never exit.

Suppose that firm 1 expects its rival to exit at the first moment that \( X_t \leq \theta^*_2 \). Recall that firm 2 has a better outside option than firm 1 (i.e., \( l_2 \geq l_1 \)), so by Lemma [1] \( \theta^*_1 \leq \theta^*_2 \), which implies that firm 1 has no incentive to exit until at least \( X_t \leq \theta^*_1 \). Therefore, firm 1 expects that the game will end before the state hits \( \theta^*_1 \), and hence the strategy of never exiting is incentive compatible. If instead firm 2 anticipates that its rival chooses \( (E^*_1, 0) \), then the strategy \( a_2 = (0, 0) \) is incentive compatible as long as it does not need to wait too long until \( X_t \) hits \( \theta^*_1 \) and firm 1 exits. As a result, never exiting is a best response for firm 2 as long as \( |l_2 - l_1| \) is not too large.

We focus on single-threshold strategies, so that \( (E^*_1, 0) \) and \( (0, E^*_2) \) are the sole candidates for pure-strategy MPE. As shown in [Murto (2004)], there may also exist pure-strategy equilibria with multiple exit thresholds (provided that \( \sigma(\cdot) \) is sufficiently large; see Proposition 5 in [Murto (2004)]). However, such pure-strategy MPE do not affect our characterization of mixed-strategy MPE, and so we do not consider them in this paper.

4.2 Mixed-strategy MPE

We now consider mixed-strategy MPE. First, we define the support of firm \( i \)'s mixed strategy as the subset of the state space in which firm \( i \) randomizes between remaining in the market and exiting,

\[
\Gamma_i = \{x \in \mathcal{X} : \lambda_i(x) > 0\}.
\]

Next, we impose the regularity condition that \( \lambda_i(\cdot) \) is lower semicontinuous. This condition ensures that \( \Gamma_i \) is an open set, and so the Hamilton-Jacobi-Bellman equation (hereafter HJB) corresponding to each firm’s payoff is well-defined.

Recall that in an MPE, it is without loss to assume \( \mathcal{P}_1 = \mathcal{P}_2 = \emptyset \). The following lemma shows that the firms’ mixed strategies must have common support and neither firm exits with probability 1 after any history.

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5In particular, [Murto (2004)] shows that there may exist an equilibrium in which each firm \( i \) exits at the first moment such that \( X_t \in (-\infty, a_i] \cup [b_i, \theta^*_i] \) for some \( a_i < b_i \); i.e., firm \( i \) does not exit within some interval \( (a_i, b_i) \) below the threshold \( \theta^*_i \). Intuitively, for \( x \in (a_i, b_i) \), if \( x - a_i \) or \( b_i - x \) is sufficiently small, then firm 2 can be better off waiting until \( X_t \) hits \( a_i \) or \( b_i \) and becoming the winner rather than exiting immediately. Finally, note that if the initial state \( X_0 \geq \max(\theta^*_1, \theta^*_2) \), then the outcome of this equilibrium coincides with the outcome of the equilibrium characterized in Proposition [1].

6To be specific, \( \Gamma_i \) being an open set implies that whenever \( X_t \in \Gamma_i \), there exists a non-empty interval \( [t, \tau_{\Gamma_i}] \), where \( \tau_{\Gamma_i} = \inf\{s \geq t : X_s \not\in \Gamma_i\} \), such that firm \( i \) randomizes its decision within this interval.
Lemma 2 Suppose that \( \sigma(\cdot) > 0 \), and \( \{a_1, a_2\} \) constitutes a mixed-strategy MPE. Then the firms’ mixed strategies have common support \( \Gamma = (\alpha, \theta^*_i) = (\alpha, \theta^*_j) \), where \( \theta^*_i \) is given in Lemma [7] and \( E_1 = E_2 = \emptyset \).

We give a sketch of the proof below, while the formal proof is relegated to Appendix C. Consider some \( t \) such that \( X_t \in \Gamma_i \). Because firm \( i \) is indifferent between exiting immediately and remaining in the market, its expected payoff must be equal to its outside option, that is,

\[
\ell_i = \lambda_{-i}(X_t) dt \, w(X_t) + (1 - \lambda_{-i}(X_t) dt) \left[ \pi(X_t) dt + (1 - rd_t) \ell_i \right].
\]

The left-hand-side of (5) represents firm \( i \)'s payoff if it exits at \( t \), while the right-hand side represents its payoff if it remains. To be specific, with probability \( \lambda_{-i}(X_t) dt \), it receives the winner’s payoff, \( w(X_t) \), whereas with the complementary probability, it earns the flow payoff \( \pi(X_t) \) during \( (t, t + dt) \), and its (discounted) continuation profit, \( \ell_i \), at \( t + dt \). Thus, firm \(-i\)'s exit rate must satisfy

\[
\lambda_{-i}(X_t) = \frac{rl_i - \pi(X_t)}{w(X_t) - \ell_i}.
\]

Note that \( \pi(x) < rl_i \) for any \( x \in \Gamma_i \). We shall now argue that \( \Gamma_1 = \Gamma_2 \). Towards a contradiction, suppose that there exists a non-empty interval that is a subset of \( \Gamma \) but not of \( \Gamma_{-i} \). Then for any \( x \) in that interval, \( \pi(x) < rl_i \) and \( \lambda_{-i}(x) = 0 \), because by assumption, \( x \in \Gamma_i \) and \( x \notin \Gamma_{-i} \), respectively. This implies that the right-hand-side of (5) is strictly smaller than \( \ell_i \), so firm \( i \) strictly prefers to exit. However, this implies that \( \Gamma_i \setminus \Gamma_{-i} \) is empty, and hence we conclude that \( \Gamma_1 = \Gamma_2 \).

Next, recall that even if firm \( i \) anticipates that its rival will never exit, it is unwilling to exit until \( X_t \) hits \( \theta^*_i \). Hence, if this firm expects its rival to exit with positive probability, then ceteris paribus, this decreases its incentive to exit. Consequently, firm \( i \) always strictly prefers to remain in the market whenever \( X_t > \theta^*_i \), which, together with the fact that \( \Gamma \) is open, implies that \( \Gamma \subseteq (\alpha, \theta^*_i) \).

We now argue that in a mixed-strategy MPE, neither firm exits with probability 1, that is, \( E_1 = E_2 = \emptyset \). Towards a contradiction, suppose that \( E_i \neq \emptyset \) for some \( i \). Because exiting at any \( X_t > \theta^*_i \) is a strictly dominated strategy for firm \( i \), it must be the case that \( E_i \subseteq (\alpha, \theta^*_i) \). Moreover, because firm \(-i\) strictly prefers to remain in the market when \( X_t \) is sufficiently close to \( E_i \) (anticipating that firm \( i \) will soon exit with probability one), it must also be the case that \( E_i \) and \( \Gamma \) are disjoint and separated by a non-empty interval \((c, d)\). In fact, as both \( E_i \) and \( \Gamma \) are subsets of \((\alpha, \theta^*_i)\), so must be \((c, d)\). Then because \( \pi(x) < rl_i \) for any \( x \in (c, d) \subseteq (\alpha, \theta^*_i) \), firm \( i \) strictly prefers to exit instantaneously if \( X_t \in (c, d) \), instead of waiting until the state hits \( E_i \) or \( \Gamma \), contradicting the premise that \( a_t \) is a best

\footnote{We ignore the event that both firms exit simultaneously. As the proof shows, this is an innocuous simplification.}

\footnote{If \( \pi(X_t) > rl_i \), then the right-hand-side of (5) is strictly larger than \( \ell_i \), so firm \( i \) strictly prefers to remain in the market regardless of its rival’s strategy.}
response to $a_{-i}$. Hence, we conclude that $E_i = \emptyset$.

We have already argued that $\Gamma \subseteq (\alpha, \theta^*_i)$. It remains to argue that this inclusion is an equality. Suppose that $\Gamma = (\alpha, \theta)$ for some $\theta < \theta^*_i$. Because firm $-i$ does not exit at any $X_t > \theta$, firm $i$’s expected payoff at any $x \in (\theta, \theta^*_i)$ from exiting at the first time that $X_t \leq \theta$ is strictly less than $l_i$ by Lemma[1] which implies that this firm strictly prefers to exit instantaneously—a contradiction.

Because $\theta^*_1 < \theta^*_2$ whenever $l_1 < l_2$ by Lemma[1] we have the following immediate implication.

**Theorem 1** Suppose that $\sigma(\cdot) > 0$ and $l_1 < l_2$. Then the game admits no mixed-strategy MPE.

While the assumptions that payoffs are deterministic and firms are symmetric may be a good approximation of a particular setting, in reality, payoffs are not set in stone and no firms are exactly alike. This theorem, together with Proposition[1] shows that if there is even a small amount of uncertainty about the payoff from remaining in the market and the firms are even slightly heterogeneous, then the game admits MPE only in pure-strategies. An implication of this result is that if the state is stochastic and the firms have heterogeneous outside options, there there exists no MPE that features attrition; i.e., an equilibrium in which both firms resist exit below their single-player-optimal thresholds.

Both conditions in Theorem[1] are necessary to eliminate mixed-strategy MPE. If the firms are homogeneous (i.e., $l_1 = l_2$) or payoffs are deterministic (i.e., $\sigma(\cdot) \equiv 0$), then as shown in [Steg (2015)] and [Hendricks et al. (1988)], respectively, and, for completeness, as we show in Appendix A, the game admits MPE in both pure and mixed strategies.

The key driver behind this result is that the state follows an irreducible stochastic process. To see why, suppose that $\sigma(\cdot) \equiv 0$ and $l_1 < l_2$ (so that $\theta^*_1 < \theta^*_2$). As shown in [Hendricks et al. (1988)], there exists an essentially unique mixed-strategy equilibrium, in which neither firm exits while $X_t > \theta^*_1$, and at the moment that $X_t$ hits $\theta^*_1$, firm $1$ exits with some probability (say $p$) that makes it in firm $2$’s interest to not exit while $X_t \in (\theta^*_1, \theta^*_2)$.[11] From that moment onwards, each firm exits with a rate that makes the other indifferent. Now suppose that $\sigma(x) > 0$ for all $x$, so that the state follows an irreducible process. In any hypothetical candidate mixed-strategy MPE, firm $1$ must exit with positive probability when the state hits $\theta^*_1$ for the same reason as in the deterministic case. But then, unlike the case of the deterministic model, even if the state is just below $\theta^*_1$, due to irreducibility, it has a chance of hitting $\theta^*_1$ again, in which case firm $1$ has a high chance of exit. Anticipating that firm $1$ is likely to exit soon, firm $2$ strictly prefers to not exit near $\theta^*_1$. In turn, firm $1$ then strictly prefers to exit, leading to a pure-strategy MPE. This argument illustrates why the mixed-strategy MPE disappears when $\sigma(\cdot) > 0$ and $\theta^*_1 \neq \theta^*_2$, which is the case when the firms have heterogeneous outside options.

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[11] This equilibrium is essentially unique in the sense that such a mixed-strategy MPE exists for any $p \in [p_0, 1)$, where $p_0$ is the smallest probability that makes firm $2$ strictly prefer to not exit at any $X_t \in (\theta^*_1, \theta^*_2)$.
5 (Non-Markov) Subgame Perfect Equilibria: Two Non-existence Results

In this section, we extend our analysis to allow firms to condition their decision at \( t \) on the entire history \( h' = \{X_s\}_{s \leq t} \) (as opposed to only the current state, \( X_t \)). We show that if the firms have heterogeneous exit payoffs (i.e., \( l_1 < l_2 \)), then subject to either of two (distinct) sets of restrictions on their strategies, the game admits no mixed-strategy SPE.

Recall that firm \( i \)'s strategy can be summarized by the three-tuple \( (I_i, \mathcal{P}_i, \Lambda_i) \), where \( I_i \) is a set of histories such that firm \( i \) exits instantaneously whenever \( h' \in I_i \), \( \mathcal{P}_i = \{\tau_{i,n}, p_{i,n}\}_{n=1}^{\infty} \) is a collection of stopping times and corresponding exit probabilities, and \( \Lambda_i = \{\lambda_{i,t}\}_{t \geq 0} \) is a non-negative process. To make it explicit that \( \lambda_{i,t} \) can depend on the entire history \( h' \), we will sometimes write \( \lambda_{i}(h') \) to denote firm \( i \)'s exit rate when \( h' = h \).

To help the reader visualize an SPE with history-dependent mixed strategies, we present an example when the firms are homogeneous.

**Example 1** Suppose that \( l_1 = l_2 \) (and so \( \theta_i^* = \theta_j^* \) by Lemma 1). Fix any \( q \in (0, 1) \), and consider the strategies \( a_1 = (\emptyset, \{\tau_1, q\}, \{\lambda_t\}_{t \geq 0}) \) and \( a_2 = (\emptyset, \emptyset, \{\lambda_t\}_{t \geq 0}) \), where \( \tau_1 = \inf \{t \geq 0 : X_t \leq \theta_1^*\} \), and

\[
\lambda_t := \mathbb{I}_{\{X_t \leq \theta_1^*\}} \frac{rl_1 - \pi(X_t)}{w(X_t) - l_1}.
\]

Then \( \{a_1, a_2\} \) constitutes a (non-Markov) mixed-strategy SPE.

In this example, both firms remain in the market until the first time that \( X_t \in (\alpha, \theta_1^*] \). At that moment, firm 1 exits with instantaneous probability \( q \). From that time onwards, whenever \( X_t \leq \theta_1^* \), each firm exits with rate \( \lambda_t \), which is chosen to make its opponent indifferent between remaining in the market and exiting. This strategy profile is non-Markov because firm 1 exits with probability \( q \) only at the first time that \( X_t \) hits \( \theta_1^* \).

We remark that the exit rate in this example is of the same form as (6), and following every history such that a firm exits with a strictly positive rate, this rate at \( t \) depends only on \( X_t \). As shown in the proof of Lemma 3, this is a general property.

5.1 First Non-existence Result

We impose three restrictions on the firms’ strategies, all of which are satisfied by the strategies in the mixed-strategy equilibria that appear in the extant literature; see for instance, [Tirole (1988), Hendricks et al. (1988), Levin (2004), Steg (2015), and Example 1 above. The first is that the exit
regions $I_1 = I_2 = \emptyset$; i.e., neither firm exits with probability one following any history. The second is that $\mathcal{P}_i = \{\tau_{i,n}, p_{i,n}\}_{n=1}^{N_i} \subset H_i$ for some $N_i < \infty$ and $\tau_{i,n} < \infty$ a.s for all $i$ and $n$. That is, the number of events of instantaneous exit is finite and the stopping times are finite a.s. We discuss the role of these assumptions before Lemma 3. Finally, we impose the regularity condition that $\lambda_i(h)$ is lower-semicontinuous in $h$ with respect to a well-defined metric. This condition ensures that

$$H_i = \{h : \lambda_i(h) > 0\}$$

is an open set of histories, and it serves the same purpose as the assumption in Section 4.2 that $\Gamma_i$ is open: It guarantees that whenever $h' \in H_i$, the HJB equation corresponding to firm $i$’s payoff function is well-defined. We summarize these restrictions below.

**Condition 1** Assume that each firm $i$’s strategy $a_i = \left(I_i, \{\tau_{i,n}, p_{i,n}\}_{n=1}^{N_i}, \lambda_i(h)\right)$, where

(i) $I_i = \emptyset$ for each $i$,

(ii) $N_i < \infty$ and $\tau_{i,n} < \infty$ a.s for all $i$ and $n$, and

(iii) $\lambda_i(h)$ is lower-semicontinuous with respect to a well-defined metric so that $H_i$ is an open set.

Conditions (i) and (ii) ensure that there exists a stopping time $\tau := \max_n \{\tau_{1,n}, \tau_{2,n}\}$ such that exit after this time occurs only via the hazard rate. The following lemma shows that in a (mixed-strategy) SPE, the firms must randomize between remaining in the market and exiting over a common set of histories.

**Lemma 3** Suppose Condition 1 is satisfied and $\{a_1, a_2\}$ constitutes a mixed-strategy SPE. Then $H_1 = H_2$.

This is a counterpart of Lemma 2 when strategies are not constrained to be Markov. Towards a contradiction, suppose that there exists a non-empty open set of histories $C \subset H_i \setminus H_{-i}$. For any history $h' \in C$, its payoff, $V_i(h') = l_i$. As $C$ is open and the value of $V_i(h') = l_i$ has no history dependence, for any $h' \in C$, $V_i(\cdot)$ satisfies the HJB equation

$$\mathcal{A}V_i(h') + \pi(X_i) + \lambda_i(h') \left[ l_i - V_i(h') \right] = 0, \tag{8}$$

\[13\] For example, for any two histories, $h_1$ and $h_2$ with $t \leq t'$, we can define the metric distance between the histories as

$$d(h_1, h_2) := \left[ \max_{x \in [0, x]} \left\{ \max_{x \in [t, t']} |X_i^1 - X_i^2|, \sup_{x \in [t, t']} |X_i^1 - X_i^2| \right\} \right] + (t - t')^2. \tag{7}$$

This is a generalization of the uniform norm, which takes into account the different lengths of the histories. It is straightforward to verify that $d(\cdot, \cdot)$ satisfies all the necessary conditions for a metric, and hence, we can define open sets of histories with respect to this metric.

\[14\] Put differently, each firm exits during any interval of length $dt$ with probability $\lambda_i(h') dt$ for all $t > \tau$. The restriction that $\tau_{i,n} < \infty$ a.s simplifies the exposition by ensuring that $\max_n \{\tau_{1,n}, \tau_{2,n}\} < \infty$ a.s. It can be relaxed, and a proof of the results in this section absent this condition is available upon request.
where \( \mathcal{A} = \sigma^2(x)\partial_{xx}/2 + \mu(x)\partial_x - r \) is the \( r \)-excessive infinitesimal generator for the process \( X \) (Alvarez, 2001), and so \( \mathcal{A}v_i(h') = -rl_i \). This implies that \( \pi(X_t) - rl_i = 0 \), contradicting the fact that \( \pi(X_t) < rl_i \) for any \( h' \in H_i \). Therefore, we conclude that \( C \) is empty, and so \( H_1 = H_2 \).

The following theorem shows that if the firms have heterogeneous outside options, then subject to Condition [1] the game admits no mixed-strategy SPE; i.e., there exists no SPE such that \( H_1 \cup H_2 \neq \emptyset \).

**Theorem 2** Suppose that each firm’s strategy must satisfy Condition [1]. If \( l_1 < l_2 \), then no mixed-strategy SPE exists.

We give a sketch of the proof here, and the formal proof is relegated to Appendix C. Condition [1] together with the irreducibility of \( X \), implies that there exists a subgame (starting at \( \bar{\tau} \)) that is reached with positive probability, in which exit occurs only via the hazard rates. Let \( H_i^\tau \) denote the set of histories in that subgame (i.e., the histories \( h' \supseteq h^\tau \)) such that \( \lambda_i(h) > 0 \), and recall that \( H_i^\tau = H_2^\tau \) by Lemma [3]. For any history in \( H_i^\tau \), the indifference condition in (5) must be satisfied, which implies that \( \lambda_{1,t} \) and \( \lambda_{2,t} \) depend only on the current state, \( X_t \). It is shown, using a similar argument as in Lemma [2], that \( H_i^\tau \) consists of the histories such that \( X_t \leq \theta_i^* \). But this implies that \( H_1^\tau \neq H_2^\tau \) whenever \( l_1 < l_2 \), a contradiction. Finally, because the set \( H_i^\tau \) is reached with positive probability (by Condition [1](i)), it follows that no mixed-strategy SPE exists.

### 5.2 Second Non-existence Result

Condition [1] played an important role in proving that the game admits no mixed-strategy SPE. A natural question to ask then, is how restrictive it is. To provide a partial answer to this question, in this section, we establish a non-existence result for mixed-strategy SPE under an alternative set of restrictions on the firms’ strategies. Unlike Condition [1] these restrictions allow firms to exit with probability one after some histories and they do not limit the number of stopping times that each firm can choose. Moreover, similar to Condition [1] they are satisfied by the strategies in the mixed-strategy equilibria that appear in the extant literature, as well as in Example 1 above.

**Condition 2** Assume that each firm \( i \)'s strategy \( a_i = (I_i, \{\tau_{i,n}, p_{i,n}\}_{n=1}^{N_i}, \{\lambda_i(h)\}) \) for \( N_i \leq \infty \), where
(i) \( I_i = \{h' : X_t \in E_i\} \) for some stationary closed set \( E_i \subseteq X \),
(ii) \( \tau_{i,n} = \inf\{t : X_t \in S_{i,n}\} \) for some stationary closed set \( S_{i,n} \subseteq X \) with \( \cup_n S_{i,n} \) being closed, and
(iii) \( \lambda_i(h) \) is a lower-semicontinuous function of \( X_t \) (and does not depend on \( X_s \) for any \( s < t \)).

Part (i) requires that each firm exits with probability one whenever \( X_t \) enters some closed set \( E_i \) that is independent of the history of \( X \) (and \( t \)). Part (ii) mandates that each stopping time is a hitting time; i.e., it is of the form \( \inf\{t : X_t \in S_{i,n}\} \) for some stationary, closed set \( S_{i,n} \) with \( \cup_n S_{i,n} \) being closed. Finally, part (iii) imposes that each firm’s hazard rate of exit at \( t \) depends only on the current state, \( X_t \), and \( \lambda_i \) is a lower-semicontinuous function of \( X_t \).
Essentially, this condition imposes that strategies may depend on the history of $X$ only at a hitting time $\tau_{i,n}$. We remark that even if $\lambda_i$ is allowed to depend on the entire history, whenever it is strictly positive, it must satisfy (5), and so it will be a function of $X_t$ only. Moreover, in the proof of Theorem 2, we showed that in any mixed-strategy SPE, the support of $\lambda_i(h)$, $H_i$, consists of all histories such that $X_t \in (\alpha, \theta_i^*)$. For notational convenience, we let $\Gamma_i = \{x \in X : \lambda_i(h) > 0\}$ denote the region of mixed strategies for firm $i$, where $x$ denotes the current value of $X$ in the given history $h$.

The following theorem states our second non-existence result for mixed-strategy SPE.

**Theorem 3** Suppose that each firm’s strategy must satisfy Condition 2. If $l_1 < l_2$, then no mixed-strategy SPE exists.

We sketch the argument below, and the proof is relegated to Appendix E. Towards a contradiction, assume that the strategies $\{a_1, a_2\}$ satisfy Condition 2 and constitute a mixed-strategy SPE. We first show that the firms exit with positive probability on a disjoint set of states; i.e., $\cup_n S_{1,n}$ and $\cup_m S_{2,m}$ are disjoint.

This is straightforward: If a firm expects its rival to exit with positive probability in the near future, then it is strictly better off not exiting.

Next, we consider the case $\Gamma_1 \cup \Gamma_2 = \emptyset$. Using the same argument as in Section 4.2, it is easy to show that $\Gamma_1 = \Gamma_2 = \Gamma$. If $N_1$ and $N_2$ are finite, then we can use the same argument as in Theorem 2 to prove that a mixed strategy SPE does not exist. (The possibility of $E_1$ and $E_2$ being nonempty follows by the same argument as in Section 4.2.) If $N_1$ or $N_2$ is infinite, then we argue that there must exist a subgame with continuation region $(c, d)$ and $d < \theta_i^*$ that is sandwiched between $\Gamma$ and some $S_{i,n}$. Whenever $X_t \in (c, d)$ in this subgame, the equilibrium strategy stipulates that firm $i$ does not exit (and does not anticipate its rival to exit) until $X_t = c$ or $X_t = d$, at which point its payoff is equal to $l_i$. Because $d < \theta_i^*$ however, at any $X_t \in (c, d)$, its expected payoff is strictly less than $l_i$ by Lemma 1, and so this firm strictly prefers to exit instantaneously, contradicting the premise that $\{a_1, a_2\}$ is an SPE. Next, using the same argument as in Section 4.2, we conclude that $\Gamma = (\alpha, \theta_i^*)$ for each $i$, which is not possible if $\theta_1^* \neq \theta_2^*$.

Finally, consider the case $\Gamma_1 = \Gamma_2 = \emptyset$. The set of states at which (each) firm $i$ exits with positive probability can be expressed as a union of closed and disjoint intervals (per our first argument in the sketch). This implies that there exist continuation intervals $(c_1, c_2)$ sandwiched between $S_{1,m}$ and $S_{2,n}$, and each firm’s payoff functions must satisfy specific boundary and optimality conditions at $c_1$ and $c_2$. However, there exist histories that are reached with positive probability such that some sets $S_{i,n}$ are hit, yet firm $i$ has not exited, in which case the continuation regions expand. Thus, new boundary and optimality conditions must be satisfied for expanded continuation intervals $(c_1', c_2')$, where $c_1' \leq c_1$ and $c_2' \geq c_2$. We show that this is not possible, and for these optimality conditions to hold, it must be the case that $\cup_n S_{1,n} = \cup_n S_{2,n} = \emptyset$, that is, such SPE is in pure strategies.

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15We say that a firm exits with positive probability if it exits with (some) probability $p_{i,n} > 0$ at a hitting time $\tau_{i,n}$.
6 Discussion

This paper shows that in a two-player war of attrition under complete information, if the players are heterogeneous, and their flow payoffs whilst in war follow a Brownian motion, then subject to certain restrictions on the players’ strategies, the game admits Subgame Perfect equilibria in pure strategies only. We argue that the key driver behind the disappearance of mixed-strategy is that the players’ flow payoffs follow an irreducible stochastic process.

An implication of this result is the non-existence of equilibria that feature attrition if the stochastic process that governs the players’ flow payoffs is irreducible and their exit payoffs are heterogeneous.\textsuperscript{16} However, the waste of valuable resources is a central feature of wars of attrition, that is, players delay exit, meanwhile suffering losses in the hope that their opponent will exit first (Ghemawat, 1997). Our non-existence result suggests that the commonly used complete-information model may be unsuitable for studying this class of problems. One strand of models that may be suitable is the asymmetric information models of Nalebuff and Riley (1985) or Fudenberg and Tirole (1986). However, the robustness of the known equilibria in these models, for example, to the type of perturbations that we consider in this paper, is an open course. It is, of course, possible that an appropriate model for studying wars of attrition remains elusive. We think that this is a promising avenue for future research.

References


\textsuperscript{16}We say that an equilibrium features attrition if both firms resist exit below their single-player-optimal thresholds.


A **Mixed Strategy MPE in two Special Cases** ($l_1 = l_2$ or $\sigma(\cdot) \equiv 0$)

Recall from Theorem [1] that if $\sigma(\cdot) > 0$ and $l_1 < l_2$, then the game admits no mixed-strategy MPE. In this section, we show that if either of these conditions is removed, then a mixed-strategy MPE does exist.

First, let us consider the case in which the firms are homogeneous (i.e., $l_1 = l_2$) and $\sigma(\cdot) > 0$. It follows from Lemma [1] that $\theta_1^* = \theta_2^*$. Following Steg (2015), it is easy to show that the strategies $a_1 = (\emptyset, \lambda(\cdot))$ and $a_2 = (\emptyset, \lambda(\cdot))$, where

$$
\lambda(x) := \mathbb{I}_{\{x \leq \theta_1^*\}} \frac{rl_1 - \pi(x)}{w(x) - l_1},
$$

constitute a mixed strategy MPE. (The proof is available upon request.)

Next, we consider the case in which $X$ evolves deterministically (i.e., $\sigma(\cdot) \equiv 0$) and $l_1 < l_2$. From Lemma [1], we have that $\theta_1^* < \theta_2^*$. Unlike the case in which $\sigma(\cdot) > 0$ considered in Section 4.2, when $X$ is deterministic, it is not without loss to assume $P_1 = P_2 = \emptyset$. Let $\tau_1^* = \inf\{t \geq 0 : X_t \leq \theta_1^*\}$ and consider the strategies $a_1 = (\emptyset, \{\tau_1^*, q_1\}, \lambda_1(\cdot))$ and $a_2 = (\emptyset, \emptyset, \lambda_2(\cdot))$, where $q_1 \in (0, 1)$, and for each $i \in \{1, 2\}$,

$$
\lambda_i(x) := \mathbb{I}_{\{x \leq \theta_i^*\}} \frac{rl_i - \pi(x)}{w(x) - l_i}.
$$

It is easy to verify that the strategies defined above are Markov. Using similar arguments to Hendricks et al. (1988), one can show that if $|l_1 - l_2|$ is not too large, then there exists a $q_1 \in (0, 1)$ such that $(a_1, a_2)$ constitutes a mixed strategy MPE.
B Structural Stability

The goal of this section is to investigate the robustness of Theorem 1 to our assumptions regarding the firms’ payoffs. In particular, we argue that Theorem 1 continues to hold even if the firms have heterogeneous discount rates \((r_1 \neq r_2)\), heterogeneous flow profits while they remain in the market \((\pi_1(x) \neq \pi_2(x))\), heterogeneous winner payoffs \((w_1(x) \neq w_2(x))\), and the loser’s payoff is state-dependent \((i.e., \text{if } i \text{ exits at } t, \text{ then it obtains payoff } l_i(X_t))\).

To analyze this model, in addition to the assumptions at the end of the model description in Section 1 we make the following assumptions:

**Condition 3** Assume that for each \(i \in \{1,2\}\),
(i) \(w_i(x) > l_i(x)\) for all \(x \in X\),
(ii) \(l_i(\cdot)\) is twice continuously differentiable on \(X\),
(iii) \(\pi_i(x) + A_i l_i(x)\) is increasing in \(x\), and
(iv) \(\lim_{x \downarrow a} \pi_i(x) + A_i l_i(x) < 0\) and \(\lim_{x \uparrow b} \pi_i(x) + A_i l_i(x) > 0\).

Part (i) ensures that the winner’s payoff is always greater than that of the loser, and it is analogous to the assumption \(w(\cdot) > l_2\) we made in Section 2. Part (ii) implies that we can apply the infinitesimal generator \(A_i\), defined in Section 1, to \(l_i(\cdot)\). Parts (iii) and (iv) guarantee that there exists a unique threshold \(\theta_i^*\) such that firm \(i\)’s best response if its rival never exits \((i.e., \text{if } a_{-i} = \{0,0,0\})\) is to exit at the first time \(\tau_i = \inf\{t \geq 0 : X_t \leq \theta_i^*\}\) (see Theorem 6 (B) in Alvarez (2001) for details).\(^{17}\)

It is straightforward to verify that Lemma 2 continues to hold under this more general model, because the only properties of the payoff-relevant parameters we used in the proof of this lemma are that (a) \(w(x) > l_i\) for all \(x\), and (b) \(w(x) > \mathbb{E}\left[ \int_0^t e^{-rs} \pi_i(X_s) ds + e^{-rt} w(X_t) \right | X_0 = x\) for all \(x \in X\) and \(t\). It thus follows that if Condition 3 is satisfied and the parameters \(\{r_i, \pi_i(\cdot), w_i(\cdot), l_i(\cdot)\}_{i \in \{1,2\}}\) are such that the thresholds \(\theta_1^* \neq \theta_2^*\), then the game admits no mixed-strategy MPE.

C Proofs

**Proof of Lemma 2** Suppose that \(\{a_1, a_2\} = \{(E_1, \lambda_1), (E_2, \lambda_2)\}\) is a mixed-strategy MPE, where \(E_i\) is firm \(i\)’s exit region \((i.e., \text{it exits with probability one whenever } X_t \in E_i)\), and \(\lambda_i(x)\) is its hazard rate of exit function. Let \(\Gamma_i\) denote the support of firm \(i\)’s mixed strategy as defined in (4). Without loss of generality, we can assume that \(E_i \cap \Gamma_i = \emptyset\). In addition, by the same argument made in Hendricks et al. (1988), for this strategy profile to constitute an equilibrium, for each \(i\), both \(E_i\) and \(\Gamma_i\) must

\(^{17}\)As an example, if the state \(X\) is a linear diffusion \((i.e., \mu(x) \equiv \mu < 0 \text{ and } \sigma(x) \equiv \sigma > 0 \text{ in (4)})\), \(\pi_i(x) = A_i x + B_i\) and \(l_i(x) = C_i x + D_i\), then it is easy to verify that Condition 3 (ii)-(iv) are satisfied as long as \(A_i > r_i C_i\). If \(X\) is a geometric Brownian motion \((i.e., \mu(x) \equiv \mu x \text{ and } \sigma(x) \equiv \sigma x \text{ for some } \mu < 0 \text{ and } \sigma > 0)\), then Condition 3 (ii)-(iv) are satisfied for the above choice of \(\pi_i(\cdot)\) and \(l_i(\cdot)\) as long as \(A_i > (r_i + \mu) C_i\) and \(B_i < r_i D_i\).
be separated from \( E_{-i} \) by closed neighborhoods \( (i.e., \) there must exist a non-empty interval between the two sets): if firm \( i \) expects its rival to exit with probability one whenever \( X_t \in E_{-i} \), then firm \( i \) strictly prefers to remain in the market whenever \( X_t \) lies in a neighborhood around \( E_{-i} \).

**Step 1.**—For each \( i \), let \( C_i \) denote the set of states at which firm \( i \) does not exit, that is, \( C_i := (\alpha, \beta) \setminus (\Gamma_i \cup E_i) \). Moreover, let \( D_i := \{ x \in X : \pi(x) > rl_i \} = (x_{ci}, \beta) \) denote the set of states at which firm \( i \)'s net-present-value of its flow payoff from remaining in the market exceeds its outside option, and let \( F_i := (\theta_i^*, \beta) \) denote the set of states at which firm \( i \) would prefer to remain in the market if it expects its rival to never exit. Recall from Lemma 1 that \( D_i \subset F_i \). We will show that \( F_i \subset C_i \), or equivalently, \( (\Gamma_i \cup E_i) \cap F_i = \emptyset \), that is, \( F_i \) is a subset of the continuation region for firm \( i \). Towards a contradiction, suppose there exists some \( x \in (\Gamma_i \cup E_i) \cap F_i \). Fix an \( i \), and define the strategies \( \tilde{a}_i := ((\alpha, \theta_i^*), 0) \) and \( \tilde{a}_{-i} := (0, 0) \), that is, a strategy profile in which firm \( i \) exits (with probability 1) whenever \( X_t \in (\alpha, \theta_i^*) \) and its rival never exits. Then

\[
V_i(x; a_i, a_{-i}) = l_i < V_i(x; \tilde{a}_i, \tilde{a}_{-i}) \leq V_i(x; \tilde{a}_i, a_{-i}).
\]

The equality follows from the assumption that \( x \in \Gamma_i \cup E_i \), which also implies \( x \notin E_{-i} \). The first inequality follows because \( x > \theta_i^* \) (since \( x \in F_i \) by assumption), so by Lemma 1, firm \( i \) can obtain a strictly higher payoff by exiting at \( \theta_i^* \) if its rival never exits. The last inequality follows because firm \( i \) is better off if its rival exits in finite time with positive probability compared to the case in which it never exits. To elaborate, the assumption that \( w(x) > \mathbb{E} \left[ \int_0^t e^{-rt} \pi(X_s) ds + e^{-rt} w(X_t) \mid X_0 = x \right] \) for all \( x \in X \) and \( t \) implies that the payoff process for the winner \( W_i(t) := \int_0^t \pi(X_s) e^{-rt} ds + e^{-rt} w(X_t) \) is a supermartingale. Letting \( \tau_{\theta_i^*} := \inf \{ t : X_t \leq \theta_i^* \} \) denote the first hitting time of the set \( (\alpha, \theta_i^*) \), the supermartingale property of \( W_i(t) \) implies that \( W_i(t) \geq \mathbb{E}[W_i(\tau_{\theta_i^*}) \mid F_t] \) for any \( t < \tau_{\theta_i^*} \), that it, firm \( i \) is better off (in expectation) becoming the winner at any \( t < \tau_{\theta_i^*} \) compared to becoming the winner at \( t = \tau_{\theta_i^*} \). Combining this with the fact that \( w(X_{\tau_{\theta_i^*}}) > l_i \), we conclude that firm \( i \)'s expected payoff \( V_i(x; a_i^*, a_{-i}) \leq V_i(x; a_i^*, \tilde{a}_{-i}) \) for any strategy \( a_{-i} \). We have thus established that \( \Gamma_i \cup E_i \) does not intersect with \( D_i \) or \( F_i \).

**Step 2.**—Next, we show that \( \Gamma_1 = \Gamma_2 \). Towards a contradiction, suppose that for some \( i \), there exists an open interval \( G \) such that \( G \subseteq \Gamma_i \) and \( G \cap \Gamma_{-i} = \emptyset \). Assume \( X_0 \in G \), and define \( \tau_G := \inf \{ t : X_t \notin G \} \). Then \( \tau_G > 0 \) a.s because \( G \) is an open set. Fix some \( \tau \in (0, \tau_G) \), and assume that firm \( i \) exits at \( \tau \) while its opponent employs the (conjectured equilibrium) strategy \( a_{-i} \). Then firm \( i \)'s expected payoff if it exits at \( \tau \) is equal to \( l_i \), the same as if it exits immediately, by the property of a mixed-strategy equilibrium and the assumption that \( X_0 \in \Gamma_i \). Recall that \( \Gamma_i \cap D_i = \emptyset \), and so \( G \cap D_i = \emptyset \), which implies that \( \pi(X_s) < rl_i \) for all \( s \in [0, \tau] \) because \( F \) is an open set and \( \pi(\cdot) \) is

\[\text{Notice that when we restrict attention to MPE, each firm’s payoff at } t \text{ is a function of only the current state, } X_t. \text{ Thus, we write } V_i(X_t; a_i, a_{-i}) \text{ to denote firm } i \text{'s payoff at } t \text{ given the strategy profile } \{a_1, a_2\}.\]
strictly increasing. Hence, firm $i$’s expected payoff if it exits at $\tau$ and its rival’s strategy is $a_{-i}$ is equal to

$$l_i + \mathbb{E} \left[ \int_{0}^{\tau} e^{-rt} (\pi(X_t) - rl_i) \, dt \right] < l_i,$$

where the inequality follows from the fact that $\pi(X_s) < rl_i$ for all $s \in [0, \tau)$. Therefore, firm $i$ can obtain a strictly greater payoff by exiting immediately, which contradicts the assumption that $G \subseteq \Gamma$.

Therefore, we conclude that $\Gamma_1 = \Gamma_2 := \Gamma$.

**Step 3**.— Next, we prove that $E_1 = E_2 = \emptyset$. Towards a contradiction, suppose that $E_i \neq \emptyset$, and define $x_{i,M} := \sup \{ E_i \} \in (\alpha, \beta)$ and pick $i$ such that $x_{i,M} > x_{i-1,M}$—this inequality must hold for some $i$ since $E_1$ and $E_2$ are disconnected from each other. Pick a subinterval $(g_1, g_2) \subseteq \Gamma$ such that $g_1 > x_{i,M}$ and $g_2 \leq \theta_i^+ \wedge \theta_2^+$. (Such an subinterval exists because $\Gamma$ and $E_i$ are disconnected from each other.) Then $V_i(x_{i,M} ; a_1, a_2) = V_i(g_1 ; a_1, a_2) = l_i$, and $(x_{i,M}, g_1)$ is a subset of the continuation region for firm $i$. Fix some $X_0 = x \in (x_{i,M}, g_i)$, and assume that firm $i$ exits at the hitting time $\tau_b = \inf \{ t : X_t \in \{ x_{i,M}, g_1 \} \}$; i.e., at the first time that $X_t \in E_b = \{ x_{i,M}, g_1 \}$. Then $V_i(x; a_i, a_{-i}) = V_i(x; (E_b, \emptyset), a_{-i})$, and because $\pi(X_t) < rl_i$ for all $t < \tau_b$, we can apply (9) to conclude that $V_i(x; a_i, a_{-i}) < l_i$, thus contradicting the assumption that $a_i$ is a best response to $a_{-i}$. Thus, we conclude that $E_1 = E_2 = \emptyset$.

**Step 4**.— Finally, we prove that $\Gamma = (\alpha, \theta_i^+)$. Towards doing so, we will first show that $\Gamma = (\alpha, \theta)$ for some $\theta$. Towards a contradiction, let $J := X \setminus \Gamma$, and suppose that there exists an interval $(c, d)$ such that $\alpha < c < d < \beta$ and $(c, d) \subseteq J$. This implies that $(c, d)$ is disconnected from both $D_1$ and $D_2$. Suppose that $X_0 = x \in (c, d)$ and assume that firm $i$ exits at the hitting time $\tau_{(c,d)} := \inf \{ t : X_t \notin (c, d) \}$; i.e., at the first time that $X_t \in E_{(c,d)} = \{ c, d \}$. It follows from the property of a mixed strategy equilibrium that $V_i(x; E_{(c,d)}, a_{-i})$ is firm $i$’s maximal payoff. However, observe that

$$V_i(x; E_{(c,d)}, a_{-i}) = l_i + \mathbb{E} \left[ \int_{0}^{\tau_{(c,d)}} [\pi(X_t) - rl_i] e^{-rt} \, dt \right] < l_i,$$

where the inequality follows because $\pi(X_s) < rl_i$ for all $s \leq \tau_{(c,d)}$, and so firm $i$ is strictly better off exiting instantaneously—a contradiction. Therefore, $\Gamma = (\alpha, \theta)$ for some $\theta$.

Next, we show that it must be the case $\theta = \theta_i^+$ for each $i$. Recall from step 1 that $\Gamma_i \cup E_i$ does not intersect with $F_i = (\theta_i^+, \beta)$ for any $i$, from step 2 that $\Gamma_1 = \Gamma_2 = \Gamma$, and from step 3 that $E_1 = E_2 = \emptyset$. Therefore, $\Gamma \subseteq (\alpha, \theta_i^+]$, and so it must be the case that $\theta \leq \theta_i^+$ for each $i$. Towards a contradiction, suppose that $\theta < \theta_i^+$ and fix some $x \in (\theta, \theta_i^+)$. Letting $\tau_\theta = \inf \{ t : X_t \leq \theta \}$, notice that

$$V_i(x; a_1, a_2) = \mathbb{E} \left[ \int_{0}^{\tau_\theta} e^{-rt} \pi(X_t) \, dt + e^{-r\tau_\theta} l_i \mid X_0 = x \right] = l_i + \mathbb{E} \left[ \int_{0}^{\tau_\theta} e^{-rt} (\pi(X_t) - rl_i) \, dt \mid X_0 = x \right] < l_i.$$ 

The first equality follows from the fact that $V_i(\theta; a_1, a_2) = l_i$ since firm $i$ is indifferent between exiting.
and remaining in the market when \( x = \theta \), the second equality follows by manipulating terms, and the inequality follows from the fact that \( \pi(X_t) < rl_i \) for all \( t < \tau_0 \). Therefore, firm \( i \) is strictly better off exiting immediately, contradicting the premise that \( \{a_1, a_2\} \) is an MPE. Hence we conclude that \( \theta = \theta_i^* \) for each \( i \).

**Proof of Theorem 1** By noting that \( \theta_i^* = \theta_2^* \) if and only if \( l_1 = l_2 \), it follows immediately from Lemma 2 that if \( l_1 < l_2 \), then the game does not admit any mixed strategy MPE.

**Proof of Lemma 3** Tovers a contradiction, suppose that there exists an open set \( C_i \subset H_i \) such that \( C_i \cap H_{-i} = \emptyset \). Pick a history \( h' \in C_i \), in which case the HJB equation must hold for \( V_i(h'; a) \) before \( h' \) exits \( C_i \):

\[
\mathcal{A}V_i(h'; a) + \pi(X_t) + \lambda_i(h')[l_i - V_i(h'; a)] = 0 ,
\]

where \( \mathcal{A} = \sigma^2(x) \partial_{xx}/2 + \mu(x) \partial_x - r \), as defined in Section 5.1. Since \( h' \in H_i \), we have \( V_i(h'; a) = l_i \), so the HJB equation reduces to \( \pi(X_t) - rl_i = 0 \). However, we have already established that \( X_t < \theta_i^* \) must be satisfied as a necessary condition for a mixed strategy region for \( i \), in which case \( \pi(X_t) - rl_i < 0 \)– a contradiction. We therefore conclude that \( H_i = H_{-i} \) must be satisfied.

**Proof of Theorem 2** Tovers a contradiction, suppose that \( l_1 < l_2 \), the strategies \( a_1 \) and \( a_2 \) satisfy Condition 1 and the strategy profile \( \{a_1, a_2\} \) constitutes a mixed-strategy SPE.

Define the stopping time \( \bar{\tau} := \max_n \{\tau_{1,n}, \tau_{2,n}\} \), and note that it is finite a.s by Condition 1 iii). Denote by \( H_t^\tau \) the set of histories such that \( k_{i,t} > 0 \) and \( t > \bar{\tau} \). Recall from Lemma 3 that \( H_1^\tau = H_2^\tau \) and define \( H_t^\tau := H_1^\tau \).

Fix a finite time \( t \) such that \( t > \bar{\tau} \) a.s, and let \( \tau_H := \inf \{s > t : h^s \in H^\tau \} \) denote the first hitting time of the mixed-strategy region. We have already established that \( H \) does not intersect with the region in which \( X_t > \theta_i^* \) for each \( i \). Because \( X_t \geq \max \{\theta_1^*, \theta_2^*\} \), we have \( \tau_H \geq \tau_i^* := \inf \{s > t : X_t \leq \theta_i^* \} \) for each \( i \) a.s, that is, the first hitting time of \( H^\tau \) is at least as long as the hitting time of \( \theta_i^* \), as well as \( (a, \theta_2^*) \).

We now show that \( \tau_H = \tau_i^* \) a.s for both \( i \). Towards a contradiction, suppose that \( \tau_H \neq \tau_i^* \) with positive probability. At \( \tau_H \), firm \( i \)'s payoff

\[
V_i(h^{\tau_H}; a_1, a_2) = l_i = V_i(h^{\tau_H}; \{H^\tau, P_i, 0\}, a_{-i}) = V_i(h^{\tau_H}; \{H^\tau, P_i, 0\}, (\theta, P_{-i}, 0)) .
\]

The first equality follows from the fact that firm \( i \)'s payoff must be equal to \( l_i \) at \( \tau_H \) by definition of a mixed-strategy equilibrium. The third term represents firm \( i \)'s payoff function if it exits with probability one whenever \( h' \in H^\tau \) and its opponent’s strategy is \( a_{-i} \), whereas the last term represents firm \( i \)'s payoff function if it exits with probability one whenever \( h' \in H^\tau \) and its opponent never
exits. The second and third equalities assert that firm $i$'s payoff at $\tau_H$ under the specified strategy profiles equals $l_i$. The third equality follows from the fact that $\tau_H \geq \tau_2^*$ a.s.

Therefore, if $\tau_H \neq \tau_i^*$ with non-zero probability, then there exists a history $h'$ (where $t$ has been fixed in the beginning of the proof) such that

$$V_i(h'; (H^r, P_i, 0), (0, P_{-i}, 0)) < V_i(h'; ((\alpha, \theta_i^*], P_i, 0), (0, P_{-i}, 0)) = V_i(h'; ((\alpha, \theta_i^*], P_i, 0), (H^r, P_{-i}, 0)),$$

The inequality follows because by Lemma 1, exiting whenever $X_t \leq \theta_i^*$ is firm $i$'s unique best response to an opponent who never exits, and $\tau_H \neq \tau_i^*$ with positive probability. The equality follows because $\tau_H \geq \tau_1^*$ a.s. This contradicts the assumption that $(a_1, a_2)$ is an SPE, and hence we conclude that $\tau_H = \tau_i^*$ a.s. However, this is not possible if $\theta_i^* \neq \theta_2^*$, or equivalently, if $l_1 < l_2$, yielding a contradiction.

Finally, note that any such $t$ is reached with positive probability by Condition 1(i) and because $X$ is irreducible. Therefore, it follows that no mixed-strategy SPE exists.

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19Note that the set of stopping times has no impact on the firms' payoffs at $t$ since $t > \max_n \{\tau_{1,n}, \tau_{2,n}\}$ a.s by assumption.
D Online Appendix

We first define the following functions that will be used later.

\[
R(x) := \mathbb{E}^x \left[ \int_0^\infty \pi(X_t) e^{-rt} dt \right], \\
\beta_i(x) := \frac{l_i - R(x)}{\phi(x)},
\]

where \( \phi : X \to \mathbb{R} \) satisfies the differential equation \( \frac{1}{2} \sigma^2(x) \phi''(x) + \mu(x) \phi'(x) - r \phi(x) = 0 \) with the properties of \( \phi(\cdot) > 0 \) and \( \phi'(\cdot) < 0 \). The function \( R(\cdot) \) is well-defined because we assume that \( \pi(\cdot) \) satisfies the absolute integrability condition in Section 2.

**Lemma D.1** The function \( \beta_i(x) \) has a unique interior maximum at \( \theta^*_i \leq x_{ci} \) where \( \pi(x_{ci}) = rl_i \). Furthermore, \( \beta'_{i}(x) > 0 \) for \( x < \theta^*_i \) and \( \beta'_{i}(x) < 0 \) for \( x > \theta^*_i \).

**Proof of Lemma D.1** To prove this lemma, it is enough to examine the behavior of the first derivative of \( \beta_i(x) = [l_i - R(x)]/\phi(x) \).

According to the theory of diffusive processes (Alvarez, 2001, p.319), the function \( R(\cdot) \), given in (10), can be expressed as

\[
R(x) = \frac{\phi(x)}{B} \int_a^x \psi(y) \pi(y) m'(y) dy + \frac{\psi(x)}{B} \int_x^b \phi(y) \pi(y) m'(y) dy.
\]

Here, \( a \) and \( b \) are the two boundaries of the state space \( X \), \( \psi(\cdot) \) and \( \phi(\cdot) \) are the increasing and decreasing fundamental solutions to the differential equation \( \frac{1}{2} \sigma^2(x) \phi''(x) + \mu(x) \phi'(x) - r \phi(x) = 0 \), \( B = [\psi'(x) \phi(x) - \psi(x) \phi'(x)]/\phi'(x) \) is the constant Wronskian determinant of \( \psi(\cdot) \) and \( \phi(\cdot) \), \( S'(x) = \exp(-\int 2\mu(x)/\sigma^2(x) dx) \) is the density of the scale function of \( X \), and \( m'(y) = 2/\sigma^2(y) S'(y) \) is the density of the speed measure of \( X \).

By virtue of (12), differentiation of \( R(x) \) with respect to \( x \) leads to

\[
R'(x) \phi(x) - R(x) \phi'(x) = S'(x) \int_x^b \phi(y) \pi(y) m'(y) dy.
\]

Moreover, because \( l_i = \mathbb{E}^x \left[ \int_0^\infty r_l e^{-rt} dt \right] \), we can write

\[
R(x) - l_i = \mathbb{E}^x \left[ \int_0^\infty \pi(X_t) - rl_i e^{-rt} dt \right],
\]

\footnote{This second-order linear ordinary differential equation (ODE) always has two linearly independent fundamental solutions, one of which is monotonically decreasing (see Alvarez, 2001 p.319). Note that if \( f(\cdot) \) solves this equation, then so does \( cf(\cdot) \) for any constant \( c \in \mathbb{R} \) because it is a homogeneous equation. Hence, we can always find the one which is always positive.}
which implies that we can treat the functional $R(x) - l_i$ as the expected cumulative present value of a flow payoff $\pi(\cdot) - rl_i$. Combining (13) and (14), therefore, we obtain

$$\beta_i'(x) = -\frac{R'(x)\phi(x) - [R(x) - l_i]\phi'(x)}{\phi^2(x)} = -\frac{S'(x)}{\phi^2(x)} \int_{x}^{b} \phi(y)[\pi(y) - rl_i]m'(y)dy. \quad (15)$$

Now, because $\pi(\cdot)$ is strictly increasing and $\pi(x_{ci}) = rl_i$, it must be the case that $\pi(x) < rl_i$ for $x < x_{ci}$ and $\pi(x) > rl_i$ for $x > x_{ci}$. Thus, $\beta_i'(x) < 0$ for all $x > x_{ci}$. Note also that if $x < K < x_{ci}$, then

$$\int_{x}^{b} \phi(y)[\pi(y) - rl_i]m'(y)dy = \int_{x}^{K} \phi(y)[\pi(y) - rl_i]m'(y)dy + \int_{K}^{b} \phi(y)[\pi(y) - rl_i]m'(y)dy$$

$$\leq \frac{[\pi(K) - rl_i](\phi'(K) - \phi'(x))}{r} + \int_{K}^{b} \phi(y)[\pi(y) - rl_i]m'(y)dy \to -\infty,$$

as $x \downarrow a$ because $a$ is a natural boundary, which implies that $\lim_{x \downarrow a} \beta_i'(x) = \infty$. Here we use $\phi'(x) < 0$ and $\pi(x) < \pi(K) < rl_i$ for $x < K$. It thus follows that $\beta_i'(\theta_i^*) = 0$ for some $\theta_i^* \leq x_{ci}$, which implies that $\int_{\theta_i^*}^{b} \phi(y)[\pi(y) - rl_i]m'(y)dy = 0$ because $S'(x) > 0$ and $\phi(x) > 0$ in (15). Moreover, note that $\int_{x}^{b} \phi(y)[\pi(y) - rl_i]m'(y)dy$ is increasing in $x < x_{ci}$ because $\pi(y) < rl_i$ for all $y < x_{ci}$, thus yielding $\int_{x}^{b} \phi(y)[\pi(y) - rl_i]m'(y)dy < 0$ if $x < \theta_i^* \leq x_{ci}$ and $\int_{x}^{b} \phi(y)[\pi(y) - rl_i]m'(y)dy > 0$ if $\theta_i^* < x \leq x_{ci}$. Combining this with (15), we obtain the unique existence of $\theta_i^*$ such that $\beta_i'(x) > 0$ for all $x < \theta_i^*$ and $\beta_i'(x) < 0$ for all $x > \theta_i^*$, which completes the proof.

**Proof of Lemma [1]** The proof of this lemma is available in [Alvarez (2001)], but here, we provide a sketch of the proof based on the verification theorem (Oksendal 2003, Theorem 10.4.1). To that end, we will use the optimality conditions, which are known as “value matching” and “smooth pasting” conditions (Samuelson 1965, McKean 1965, Merton 1973).

First, the state space $X$ must be the union of $C := \{x \in X : V_i^*(x) > l_i\}$ and $\Gamma := \{x \in X : V_i^*(x) = l_i\}$, which are mutually exclusive: This is because (1) $X$ is a stationary process and the time horizon is infinite, and (2) the value function $V_i^*(\cdot)$ from an optimal stopping policy must be always no less than the reward $l_i$ from stopping immediately. Hence, the problem to find an optimal stopping policy can be reduced to identify $C$ or $\Gamma$.

Next, we find the differential equation that $V_i^*(x)$ must satisfy if $x \in C$. Note that the optimal value function $V_i^*(x)$ is the maximum of the reward from waiting an instant and the reward from stopping immediately. For any $x \in C$, therefore, the optimal stopping policy is to wait an instant $dt$, and hence, the optimal value function must satisfy the following equation:

$$V_i^*(x) = \pi(x)dt + (1 - rdt)\mathbb{E}^x[V_i^*(X_t) + dV_i^*(X_t)]. \quad (16)$$
Then applying Ito formula to $V_i^*(X_t)$ and using $\mathbb{E}^x[dB_t] = 0$ yields

$$
\mathbb{E}^x[dV_i^*(X_t)] = [\mu(x)V_i^{**}(x) + \frac{1}{2}\sigma^2(x)V_i^{***}(x)]dt.
$$

(17)

By plugging (17) into (16) and ignoring the term smaller than $dt$, we have

$$
V_i^*(x) = \pi(x)dt + V_i^*(x) + [-rV_i^*(x) + \mu(x)V_i^{**}(x) + \frac{1}{2}\sigma^2(x)V_i^{***}(x)]dt,
$$

from which we obtain the following second-order linear differential equation:

$$
\frac{1}{2}\sigma^2(x)V_i^{***}(x) + \mu(x)V_i^{**}(x) - rV_i^*(x) = -\pi(x).
$$

(18)

Thus, $V_i^*(\cdot)$ can be obtained by solving the differential equation (18). In fact, it can be seen from a series of algebra with the relation (12) that the function $R(\cdot) + A\phi(\cdot)$ with some constant $A \in \mathbb{R}$ is a solution to (18), and hence, we can guess $V_i^*(x) = R(x) + A\phi(x)$ with some constant $A$.

Intuitively, firm $i$ must find it optimal to exit and receive his outside option $l_i$ as soon as the state $X$ hits some lower threshold $\theta_i$. Hence, assume at the moment that the optimal stopping policy is given as $\tau^* := \inf\{t \geq 0 : X_t \leq \theta_i\}$, which implies that $\theta_i$ is the boundary point of the region $C$. Now, we state the value matching condition and the smooth pasting condition, which results in two boundary conditions to the boundary value problem (18) with the free boundary $\theta_i$:

$$
V_i^*(\theta_i) = R(\theta_i) + A\phi(\theta_i) = l_i
$$

(19)

$$
V_i^{**}(\theta_i) = R'(\theta_i) + A\phi'(\theta_i) = 0.
$$

(20)

The value matching condition (19) and the smooth pasting condition (20) are the conditions that $V_i^*(\cdot)$ must satisfy at the boundary $\theta_i$ of $C$. We can first obtain $A = [l_i - R(\theta_i)]/\phi(\theta_i) = \beta_i(\theta_i)$ from (19). Then the condition (20) is equivalent to

$$
0 = R'(\theta_i) + \frac{l_i - R(\theta_i)}{\phi(\theta_i)}\phi'(\theta_i) = \frac{R'(\theta_i)\phi(\theta_i) + [l_i - R(\theta_i)]\phi'(\theta_i)}{\phi(\theta_i)} = -\phi(\theta_i)\beta_i'(\theta_i).
$$

Because $\phi(\cdot) > 0$, it can be seen from Lemma D.1 that this condition is satisfied if and only if $\theta_i = \theta_i^*$, which implies that $A = \beta_i(\theta_i^*)$.

Lastly, it can be easily verified that $R(x) + \beta_i(\theta_i^*)\phi(x) \geq l_i$ for $\forall x \geq \theta_i^*$ and $\pi(x) < rl_i$ for $\forall x \leq \theta_i^* < x_{ci}$. By the verification theorem (Oksendal, 2003, Theorem 10.4.1), therefore, the proposed value function $R(\cdot) + \beta_i(\theta_i^*)\phi(\cdot)$ is, in fact, the optimal value function $V_i^*(\cdot)$, as desired.
Proof of Proposition 1 (i) Define $\tau_i^* := \inf\{t \geq 0 : X_t \in E_i^*\}$, $i \in \{1, 2\}$, where $E_i^* = (\alpha, \theta_i^*]$ is given in Section 4.1. For expositional convenience, we also let $\tau(E_i) := (E_i, 0)$ for each $i \in \{1, 2\}$. We first prove that $\{a_1, a_2\} = \{\tau(0), \tau(E_2^*)\}$ is an MPE. Because it is shown in Lemma 1 that $a_2 = \tau(E_2^*)$ is firm 2’s best response to $a_1 = \tau(0)$, it only remains to prove that $a_1 = \tau(0)$ is also firm 1’s best response to $a_2 = \tau(E_2^*)$.

Let $\tau(E_1)$ be firm 1’s best response to $\tau(E_2^*)$ and $V_{W^1}(x) := V_1(x; \tau(E_1), \tau(E_2^*))$ be the corresponding payoff to firm 1.\footnote{Notice that when we restrict attention to MPE, each firm’s payoff at $t$ is a function of only the current state, $X_t$. Thus, we write $V_i(X_t; a_i, a_{-i})$ to denote firm $i$’s payoff at $t$ given the strategy profile $\{a_1, a_2\}$.} We let $C_1 = X \setminus E_1$ denote the continuation region associated with the strategy $\tau(E_1)$.

First, we show that $E_1 \cap (\theta_2^*, \infty) = \emptyset$. Toward a contradiction, suppose this is not the case. Then pick some $x \in E_1 \cap (\theta_2^*, \infty)$ and observe that $V_{W^1}(x) = l_1$ due to $x \in E_1$. However,

$$V_{W^1}(x) \geq V_1(x; \tau(0), \tau(E_2^*)) = \mathbb{E}^x \left[ \int_0^{\tau_2^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_2^*}^x) e^{-r\tau_2^*} \right]$$

$$= R(x) + \frac{w(\theta_2^*) - R(\theta_2^*)}{\phi(\theta_2^*)} \phi(x)$$

$$> R(x) + \frac{l_1 - R(\theta_2^*)}{\phi(\theta_2^*)} \phi(x)$$

$$= R(x) + \beta_1(\theta_2^*) \phi(x)$$

$$> R(x) + \beta_1(x) \phi(x) = l_1,$$

where the first inequality follows because $w(X_{\tau_2^*}^x) = w(\theta_2^*) > l_1$ and $\mathbb{E}^x[e^{-\tau_2^*}] = \phi(x)/\phi(\theta_2^*)$ for $x > \theta_2^*$, and the second inequality holds because $x > \theta_2^* > \theta_1^*$ and $\beta_1'(x) < 0$ for $x > \theta_1^*$ by Lemma 1.\footnote{Notice that when we restrict attention to MPE, each firm’s payoff at $t$ is a function of only the current state, $X_t$. Thus, we write $V_i(X_t; a_i, a_{-i})$ to denote firm $i$’s payoff at $t$ given the strategy profile $\{a_1, a_2\}$.} This establishes the contradiction.

Second, we also prove that $E_1 \cap (\infty, \theta_2^*) = \emptyset$. Towards a contradiction, suppose this is not the case. Then we can pick some $x \in E_1 \cap (\infty, \theta_2^*)$ such that $V_{W^1}(x) = m_1(x)$ because $\tau_2^* = \inf\{t \geq 0 : X_t \in E_2^*\}$. However,

$$V_{W^1}(x) \geq V_1(x; \tau(0), \tau(E_2^*)) = \mathbb{E}^x \left[ \int_0^{\tau_2^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_2^*}^x) e^{-r\tau_2^*} \right] = w(x) > m_1(x),$$

where the second equality uses that $\tau_2^* = 0$ when $X_0 = x \leq \theta_2^*$. This establishes the contradiction. Hence, we can conclude that $E_1 = \emptyset$ and $C_1 = X$, which implies that $\tau(E_1) = \tau(0)$.

(ii) Next, we prove the conditions under which $\{a_1, a_2\} = \{\tau(E_1^*), \tau(0)\}$ is an MPE. Consider the
which is a contradiction. Combining the three claims above, therefore, we conclude that

\[ V_2(x; \tau(E_1^*), \tau(0)) = \mathbb{E}^x \left[ \int_0^{\tau_1^i} \pi(X_t)e^{-rt}dt + w(X_{\tau_1^i}^x)e^{-r\tau_1^i} \right] > l_2 \text{ for all } x \in (\theta_1^*, \theta_2^*]. \quad (21) \]

First, we prove that (21) is a sufficient condition for \( \{a_1, a_2\} = \{\tau(E_1^*), \tau(0)\} \) to be an MPE. Let \( \tau(E_2) \) be firm 2’s best response to \( \tau(E_1^*) \), i.e., \( V^*_w(x) := V_2(x; \tau(E_1^*), \tau(E_2)) \) be the corresponding payoff. We let \( C_2 = X \setminus E_2 \) denote the continuation region associated with the strategy \( \tau(E_2) \).

We now claim that \( E_2 \cap (\theta_2^*, \infty) = \emptyset \). Towards a contradiction, suppose not. Then we can pick some \( x \in E_2 \cap (\theta_2^*, \infty) \), which implies that \( V^*_w(x) = l_2 \). However, because \( \tau_1^* < \tau_2^* \) when \( X_0 = x \), Lemma 1 implies that firm 2 could obtain a strictly higher payoff by exiting at \( \tau_2^* > 0 \) instead, i.e.,

\[ V^*_w(x) > V_2(x; \tau(E_1^*), \tau(E_2)) = \mathbb{E}^x \left[ \int_0^{\tau_2^*} \pi(X_t)e^{-rt}dt + l_2 e^{-r\tau_2^*} \right] > l_2, \]

which is a contradiction. We next claim that \( E_2 \cap (\theta_1^*, \theta_2^*) = \emptyset \). Towards a contradiction, suppose not. Then we can pick \( x \in E_2 \cap (\theta_1^*, \theta_2^*) \), which implies that \( V^*_w(x) = l_2 \). However, we have

\[ V^*_w(x) > V_2(x; \tau(E_1^*), \tau(0)) = \mathbb{E}^x \left[ \int_0^{\tau_1^*} \pi(X_t)e^{-rt}dt + w(X_{\tau_1^*}^x)e^{-r\tau_1^*} \right] > l_2, \]

where the last inequality follows from (21). This establishes the contradiction. We further claim that \( E_2 \cap (-\infty, \theta_1^*) = \emptyset \). If not, then there exists \( x \in E_2 \cap (-\infty, \theta_1^*) \), which implies that both firms exit simultaneously when \( X_1^x = x \), and hence, \( V^*_w(x) = m_2(x) \). Because \( \tau_1^* = 0 \) when \( X_0 = x \leq \theta_1^* \), we have

\[ V^*_w(x) > V_2(x; \tau(E_1^*), \tau(0)) = \mathbb{E}^x \left[ \int_0^{\tau_1^*} \pi(X_t)e^{-rt}dt + w(X_{\tau_1^*}^x)e^{-r\tau_1^*} \right] = w(x) > m_2(x), \]

which is a contradiction. Combining the three claims above, therefore, we conclude that \( E_2 = \emptyset \), which implies that \( C_2 = X \), and hence, \( \tau(E_2) = \tau(0) \).

Second, define \( w := \inf \{ w(x) : x \in X \} \) and \( \beta_w(\theta) := [w - R(\theta)]/\phi(\theta) \). Note that \( \beta_w(\theta) > \beta_2(\theta) \) for all \( \theta \in X \) because \( w > l_2 \). Also, observe that for all \( \theta < \theta_2^* \), we have

\[ \beta'_w(\theta) = \left\{ -R'(\theta)\phi(\theta) - \phi'(\theta)[w - R(\theta)] \right\}/\phi^2(\theta) > \left\{ -R'(\theta)\phi(\theta) - \phi'(\theta)[l_2 - R(\theta)] \right\}/\phi^2(\theta) = \beta'_2(\theta) > 0 \]

where the first inequality follows because \( \phi'(\theta) < 0 \), and the last inequality holds because \( \beta'_2(\theta) > 0 \).
for \( \theta < \theta_2^* \) from Lemma D.1. Next, pick \( \kappa_0 > 0 \) such that

\[
\beta_W(\theta_2^* - \kappa_0) = \beta_2(\theta_2^*), \tag{22}
\]

where \( \beta_2(\cdot) \) is defined in (11). If such \( \kappa_0 \) exists, it must be unique because \( \beta_W'(\theta) > 0 \) for \( \theta < \theta_2^* \). If there does not exist \( \kappa_0 \) which satisfies (22), then we let \( \kappa_0 = \infty \).

Finally, we show that (21) is satisfied if \( \theta_2^* - \theta_1^* < \kappa_0 \), which will complete the proof; this is because we can always find the unique \( \kappa_i > 0 \) for any given \( \kappa_0 > 0 \) such that \( \theta_2^* - \theta_1^* < \kappa_0 \) if and only if \( l_2 - l_1 < \kappa_i \) from the fact that \( \theta_i \) given in Lemma [1] strictly increases in \( l_i \). Suppose now that \( \theta_2^* - \theta_1^* < \kappa_0 \), i.e., \( \theta_1^* > \theta_2^* - \kappa_0 \). Note that \( \beta_W'(\theta) > 0 \) for \( \forall \theta < \theta_2^* \), and recall that \( \theta_1^* < \theta_2^* \). Therefore, \( \beta_W(\theta_1^*) > \beta_W(\theta_2^* - \kappa_0) = \beta_2(\theta_2^*) \) by (22). Thus, for any \( x \in (\theta_1^*, \theta_2^*) \),

\[
\mathbb{E}^x\left[ \int_0^{\tau_1} \pi(X_t^i)e^{-rt} dt + w(\theta_1^*)e^{-r\tau_1} \right] \geq \mathbb{E}^x\left[ \int_0^{\tau_1} \pi(X_t^i)e^{-rt} dt + w\phi_2(x) \right] = R(x) + \phi(x)\beta_W(\theta_1^*) > R(x) + \phi(x)\beta_2(\theta_2^*) \geq R(x) + \phi(x)\beta_2(x) = l_2,
\]

where the first inequality holds from the definition of \( \mathbb{E} \), the first equality holds because \( \mathbb{E}^x[e^{-r\tau}] = \phi(x)/\phi(\theta_1^*) \) for \( x > \theta_1^* \), the second inequality follows because \( \beta_W(\theta_1^*) > \beta_2(\theta_2^*) \), the last inequality holds because \( \beta_2(\cdot) \) achieves its maximum at \( \theta_2^* \) by Lemma D.1, and the last equality follows by the definition of \( \beta_2(\cdot) \). Hence, (21) is satisfied, which establishes the desired result for \( \kappa_0 > 0 \). \( \square \)

E \hspace{1mm} Online Appendix: Proof of Theorem 3

We first recap a strategy that satisfies Condition 2 as follows: Each firm \( i \) chooses (i) a closed set \( E_i \) such that \( I_i = \{ h^i : X_t \in E_i \} \) is the exit region, (ii) a set of hitting times and exit probability pairs, denoted by \( \{(\tau_{i,n}, p_{i,n})\} \), such that firm \( i \) exits at \( \tau_{i,n} := \inf\{ t \geq 0 : X_t \in S_{i,n} \} \) with probability \( p_{i,n} \in (0,1) \) where each \( S_{i,n} \) and \( \bigcup_{n=1}^{\infty} S_{i,n} \) are closed, and (iii) an open set \( \Gamma_i = \{ x \in X : \lambda_i(x) > 0 \} \) where \( \lambda_i(x) \) is firm \( i \)'s exit rate at \( X_t = x \).

As a convention, we stipulate that \( \Gamma_i \) does not intersect with \( E_i \) or \( E_{-i} \), i.e., \( \Gamma_i \cap E_i = \emptyset \) and \( \Gamma_i \cap E_{-i} = \emptyset \). This is because firm \( i \) would have no incentive to plan to exit with the rate of \( \lambda_i(X_t) \) at \( X_t \in E_i \) or \( X_t \in E_{-i} \) where the game ends immediately with probability 1.

\( \footnote{p_{i,n} \text{ can take different values for each point } x \in S_{i,n}; \text{ in other words, } p_{i,n} \text{ is a function of exit probability rather than a single fixed value for } S_{i,n}. \text{ Hence, we sometimes write } p_{i,n}(x) \text{ to denote exit probability at the point } x \in S_{i,n} \text{ if needed. Also, if } S_{i,n} \text{ intersect another } S_{i,n'} \text{ with } n' \neq n, \text{ then the values of } p_{i,n} \text{ and } p_{i,n'} \text{ should coincide for any } x \in S_{i,n} \cap S_{i,n'} \text{ for the consistency of a strategy.}} \)
Given a history \( h' \), we define \( N_i(h') := \{ n \geq 1 : \tau_{i,n} > t \} \), which represents the set of indices of \( S_{i,n} \) that have not been hit by \( X \) in the history \( h' \). We also define \( D_i(h') := (\cup_{n \in N_i(h')} S_{i,n}) \cup \Gamma_i \) and \( C_i(h') := X \setminus (D_i(h') \cup \Gamma_i) \), which are firm \( i \)'s stopping region and continuation region in the history \( h' \) respectively; we will often write them as \( D_i^t \) and \( C_i^t \). We let \( C' := C_1^t \cap C_2^t \) be the common continuation region where neither firm ever exits.

We will use a notation \( U_x \) that denotes an open neighborhood of a point \( x \in X \), i.e., \( x \in U_x \) where \( U_x \) is open. Also, any connected component of a set \( F \) in \( X \) will be called a component of \( F \) for expositional convenience. Note that any component of a given set in \( X \) must be an interval or a point. Lastly, given two sets \( A \) and \( B \) in \( X \), we say that \( A \) is separated from \( B \) if the infimum distance from any point in \( A \) to the set \( B \) is positive. Note that any two closed sets \( A \) and \( B \), which are disjoint \((A \cap B = \emptyset)\), must be separated from each other.

We will provide a proof that a pair of strategies of the above form cannot be a mixed strategy SPE. Note that a player's equilibrium strategy specifies action plans for every subgame, but we only need to focus on each player's best response in a set of subgames that occurs with non-zero probability; the action plans for the set of subgames that occur with zero probability have no bearing on the payoff. Hence, we will focus on characterizing the strategies for the set of subgames the probability measure of which is positive.

We also remark that the strategy profiles must stipulate action plans and payoff functions in subgames which are off the equilibrium path. These are the subgames that can be reached when one firm deviates from his specified strategy. The reason we should consider the strategies and payoffs for the off-the-equilibrium-path subgames is that in order to verify that a strategy profile is an SPE, we need to confirm that the firms have no incentive to deviate from the equilibrium strategies to reach off-the-equilibrium-path subgames.

Before we discuss mathematical arguments in detail, we first provide the roadmap of the proof. Toward a contradiction, we suppose that \( a = \{a_1, a_2\} \) is a mixed strategy SPE in which \( a_i, i \in \{1, 2\}, \) are strategies of the above form. As a preliminary step, we first establish the two necessary conditions that \( a = \{a_1, a_2\} \) must satisfy: (a) \( D_i^0 \) is separated from \( D_{-i}^0 \) (Proposition \[E.1\]) and (b) \( \Gamma_i \cup D_i(h') \) is included in \( (\alpha, \theta_i^* \} \) for any subgame \( h' \) (Proposition \[E.2\]). Then we examine the following two cases separately: (i) \( \Gamma_1 \cup \Gamma_2 \neq \emptyset \) and (ii) \( \Gamma_1 = \Gamma_2 = \emptyset \).

For case (i), it is helpful to recall the following two arguments used for proving the non-existence of mixed strategy MPE: First, \( \Gamma := \Gamma_1 = \Gamma_2 \). Second, \( D_i^0 = \emptyset, i \in \{1, 2\} \). Then the non-existence of mixed strategy MPE follows from showing that \( \Gamma_i = (\alpha, \theta_i^* \} \), \( i \in \{1, 2\} \), i.e., any component of the common continuation region \( X \setminus \Gamma \) cannot be located below \( \theta_i^* \), \( i \in \{1, 2\} \). Indeed, the non-existence of mixed strategy SPE in case (i) can be shown by going through similar steps. More specifically, we first obtain Proposition \[E.3\] which establishes the existence of a subgame \( h' \) in which a small
neighborhood $U_i$ of $x \in \Gamma_i$ does not intersect with $D_i(h')$ or $D_{-i}(h')$. After using this proposition to prove $\Gamma = \Gamma_1 = \Gamma_2$ in Proposition E.4, we extend the results of Proposition E.3 in Lemma E.2. For each component $(f, g)$ of $\Gamma$, we can always find a subgame $h'$ in which $[f, g] \cap D_i(h') = \emptyset$ so that $[f, g]$ is separated from $D_i(h')$, $i \in \{1, 2\}$. Then because $\Gamma_i \subseteq (\alpha, \theta^*_i)$ by Proposition E.2, $\Gamma_i \neq (\alpha, \theta^*_i)$ implies that, between $(f, g)$ and $D_i(h')$ or $D_{-i}(h')$, there must be a component of the common continuation region $C'$ located below $\theta^*_i$ in the subgame $h'$, which is shown to lead to a contradiction in Proposition E.5 similarly as the case of mixed strategy MPE.

For case (ii), it is enough to show that $D^0_i = E_i, i \in \{1, 2\}$: Proving this statement implies that $a = \{a_1, a_2\}$ is a pure strategy equilibrium because $\Gamma_1 = \Gamma_2 = \emptyset$ in this case. Towards this end, we first establish Lemma E.3 that characterizes the structure of $D^0_i \cup D^0_{-i}$ as the disjoint union of closed intervals or points $\{\phi^{(k)}\}_{k=1}^K$ where the upper boundary of $\phi^{(1)}$ is $\theta^*_i$, and the lower boundary of $\phi^{(K)}$ is $\alpha$, and there is a component of $C'$ between $\phi^{(k)}$ and $\phi^{(k+1)}$ for all $k \geq 1$ (See Figure 2). Then in Lemmas E.4 and E.5, we provide the following three characterizations on how $E_1$ and $E_2$ compose $\cup_{k=1}^K \phi^{(k)} = D^0_1 \cup D^0_2$. First, each $\phi^{(k)}$, $1 < k < K$, must contain exactly a single component of $E_1$ or $E_2$ (Lemma E.4(i)). Second, for any $k \geq 1$, if both the upper and lower boundaries of $\phi^{(k)}$ belong to $E_i$, then $\phi^{(k)} \subseteq E_i$ (Lemma E.4(ii)). Third, if one boundary of a component of $C'$ belongs to $E_i$, then the other boundary of this component of $C'$ must belong to $E_{-i}$ (Lemma E.5). Then in Proposition E.6, we go through the following two steps to prove $\phi^{(k)} \subseteq E_1$ or $\phi^{(k)} \subseteq E_2$ for all $k \geq 1$. First, for each $k$ with $1 < k < K$, we use Lemmas E.4(i) and E.5 to show that both the upper boundary of $\phi^{(k-1)}$ and the lower boundary of $\phi^{(k+1)}$ belong to $E_i$ for some $i \in \{1, 2\}$, which implies from Lemma E.4(ii) that $\phi^{(k)} \subseteq E_1$ or $\phi^{(k)} \subseteq E_2$ for all $k > 1$. Second, we show that the upper boundary of $\phi^{(1)}$, which is $\theta^*_i$ for some $i \in \{1, 2\}$, should belong to $E_i$, which means that $\phi^{(1)} \subseteq E_i$ for some $i \in \{1, 2\}$ by Lemma E.4(ii): The argument for this last step is made by constructing a subgame $h'$ in which there is a component of $C'$ located below $\theta^*_i$, $i \in \{1, 2\}$, which leads to a contradiction similarly as in case (i). Lastly, because $D^0_1 \cup D^0_2 = \cup_{k=1}^K \phi^{(k)}$ and $\phi^{(k)} \subseteq E_1$ or $\phi^{(k)} \subseteq E_2$ for all $k \geq 1$, we can conclude that $D^0_i = E_i, i \in \{1, 2\}$, as desired.

**Preliminary**

We first establish that the two sets $D^0_1$ and $D^0_2$ are separated from each other. This is because firm $i$ would have no incentive to exit when firm $-i$ is about to exit.

**Proposition E.1** $D^0_1$ and $D^0_2$ are separated from each other.

**Proof:** It is enough to show that $D^0_1$ and $D^0_2$ are disjoint because they are closed. Towards a contradiction, suppose now that there exists $x \in D^0_1 \cap D^0_2$. Fix an $i \in \{1, 2\}$. Let $X_0 = x$, and $p_i = p_{i,n}$ and $p_{-i} = p_{-i,n}$ for notational convenience.
First, we consider the case \( p_i < 1 \) and \( p_{-i} < 1 \), which implies that \( x \in S_{i,n} \cap S_{-i,m} \) for some \( n,m \geq 1 \). Construct an alternative strategy \( a'_i \) for firm \( i \) in which \( S_{i,n} \) is excluded\(^\text{23}\) from the stopping region \( D_0^i \) associated with \( a_i \); in other words, we have \( x \in D_0^i \) but \( x \notin D_0^i' \) where \( D_0^i' \) is the stopping region associated with \( a'_i \). Then we can express firm \( i \)'s payoff associated with \( a = \{ a_i, a_{-i} \} \) as
\[
V_i(h^0; a_i, a_{-i}) = p_i \left[ p_{-i} \left( \frac{l_i + w(x)}{2} \right) + (1 - p_{-i})l_i \right] + (1 - p_i) \left[ p_{-i}w(x) + (1 - p_{-i})V_i(h^{0+}; a_i, a_{-i}) \right]
\]
where \( h^0 = \{ x \} \) and \( h^{0+} \) is the history immediately after time zero. On the other hand, firm \( i \)'s payoff associated with \( \{ a'_i, a_{-i} \} \) is given by
\[
V_i(h^0; a'_i, a_{-i}) = p_{-i}w(x) + (1 - p_{-i})V_i(h^{0+}; a'_i, a_{-i})
\]
Note that \( V_i(h^{0+}; a'_i, a_{-i}) \neq V_i(h^{0+}; a_i, a_{-i}) \); this is because \( S_{i,n} \) is the only difference between \( a'_i \) and \( a_i \) at time zero, but \( S_{i,n} \) is excluded from both \( a_i \) and \( a'_i \) in the subgame \( h^{0+} \). Note also that \( V_i(h^{0+}; a_i, a_j) \geq l_i \) because of our supposition that \( \{ a_i, a_{-i} \} \) is an SPE; firm \( i \) should earn no less than \( l_i \) by employing a best response \( a_i \) to \( a_{-i} \). Using the inequalities \( V_i(h^{0+}; a'_i, a_{-i}) = V_i(h^{0+}; a_i, a_{-i}) \geq l_i \), \( w(x) > l_i \), and \( p_i > 0 \), therefore, we can conclude that \( V_i(h^0; a'_i, a_{-i}) > V_i(h^0; a_i, a_{-i}) \). This contradicts the premise that \( \{ a_i, a_{-i} \} \) is an equilibrium.

Second, we consider the case \( p_{-i} = 1 \). In this case, \( p_i \) can be less than or equal to \( 1 \). Let \( a'_i \) denote firm \( i \)'s alternative strategy of never exiting. Then firm \( i \)'s payoff with \( \{ a_i, a_{-i} \} \) and \( \{ a'_i, a_{-i} \} \) are
\[
V_i(h^0; a_i, a_{-i}) = p_i \left( \frac{l_i + w(x)}{2} \right) + (1 - p_i)w(x)
\]
\[
V_i(h^0; a'_i, a_{-i}) = w(x)
\]
Because \( w(x) > l_i \), we have \( V_i(h^0; a'_i, a_{-i}) > V_i(h^0; a_i, a_{-i}) \), which contradicts the assumption that \( \{ a_i, a_{-i} \} \) is an equilibrium.

Therefore, we can conclude that \( D_0^1 \) and \( D_0^2 \) are disjoint, and because they are both closed sets, they must be separated from each other.

We next establish that \( \Gamma_i \cup D_i(h') \) is included in \( (\alpha, \theta_i^+) \) for any subgame \( h' \) (i.e., \( \Gamma_i \cup D_i(h') \cap (\theta_i^+, \beta) = \emptyset \)). This is because firm \( i \) has no incentive to exit at \( X_i > \theta_i^+ \) even in the most pessimistic scenario that firm \( j \) never exits at all.

**Proposition E.2** For each \( i \in \{1, 2\} \), \( \Gamma_i \cup D_i(h') \subseteq (\alpha, \theta_i^+) \) for any subgame \( h' \).

\(\text{If there are multiple } S_{i,n} \text{'s that contain } x \text{ (e.g., } x \in S_{i,a_1} \cap S_{i,a_2}) \text{, we construct } a'_i \text{ by excluding all such } S_{i,n} \text{'s from } D_0^i \text{ to make the rest of the proof work.} \)
Proof: It is enough to show that $\Gamma_i \cup \mathcal{D}^i \subseteq (\alpha, \theta_i^+]$ because $\mathcal{D}_l(h') \subseteq \mathcal{D}^i$ for any subgame $h'$. In fact, the proof of this proposition can be done similarly as in Step 1 of the proof of Lemma 2. To elaborate, suppose that there exists some $x \in (\Gamma_i \cup \mathcal{D}^i) \setminus (\alpha, \theta_i^+]$ and let $h^0 = \{x\}$. Consider the strategies $\tilde{a}_i := ((\alpha, \theta_i^+), \emptyset, 0)$ and $\tilde{a}_{-i} := (\emptyset, 0, 0)$, that is, a strategy profile in which firm $i$ exits (with probability 1) whenever $X_i \in (\alpha, \theta_i^+]$ and its rival never exits. Then we can obtain

$$V_i(h^0; a_i, a_{-i}) = l_i < V_i(h^0; \tilde{a}_i, \tilde{a}_{-i}) \leq V_i(h^0; \tilde{a}_i, a_{-i}).$$

Here the equality and the inequalities follow from the same arguments used in Step 1 of the proof of Lemma 2. Then this leads to a contradiction that $a = \{a_i, a_{-i}\}$ is an equilibrium. 

Case 1: $\Gamma_1 \cup \Gamma_2 \neq \emptyset$

We first show that, for any $x \in \Gamma_i$, we can construct a subgame $h'$ in which a small neighborhood of $x$ does not intersect with $\mathcal{D}^i$ or $\mathcal{D}_{-i}^i$.

Proposition E.3 For any $x \in \Gamma_i$, there is positive probability of a subgame $h'$ in which there exists an open neighborhood $(c, d)$ of $x$ such that $(c, d) \subseteq \Gamma_i$, $(c, d) \cap \mathcal{D}^i = \emptyset$, and $(c, d) \cap \mathcal{D}_{-i}^i = \emptyset$.

In order to prove Proposition E.3, we need to establish that there exists a positive probability of a subgame $h'$ in which the process $X$, starting from some point $x \in \Gamma_i$ has hit every $S_{i,n}$ set (or $S_{-i,m}$ set) that intersects with the interval $(c, d)$. The upshot of this statement is that in that subgame the interval $(c, d)$ does not intersect with $S_{i,n}$ for any $n \in N_i(h')$ or with $S_{-i,m}$ for any $m \in N_i(h')$ so that both firms employ only $\lambda$-strategies (with the exit rate $\lambda(\cdot)$) within the interval $(c, d)$. In this subgame, therefore, each firm’s strategy is Markov before $X$ exits the interval $(c, d)$, which is particularly convenient when we determine the form of $\Gamma_i$, $i \in \{1, 2\}$.

As a first step towards the proof of this proposition, we establish the following lemma regarding a convenient property of a diffusion process, after which the proof of the proposition follows.

Lemma E.1 Given any $x$, some open neighborhood $(c, d)$ of $x$, and some interval $(f, g)$ that contains $(c, d)$, there is a positive probability that $X$ begins at $X_0 = x$, traverses the entire $(c, d)$ without escaping $(f, g)$, and returns to $X_t = x$.

Proof: We first note that there is a non-zero probability that the process $X$, which begins at $X_0 = x$, will reach $c$ within a finite time without first hitting $g$. (This is by the property of a diffusive process $X$.) Similarly, starting from $c$ as the initial point, there is a non-zero probability that the process $X$ will hit $d$ first without ever hitting $f$. Lastly, starting from $d$ as the initial point, there is a non-zero probability that the process $X$ will hit $x$ first without ever hitting $g$. (See also Figure 1).

Proof of Proposition E.3 We first recall that $\Gamma_i \cap E_i = \Gamma_i \cap E_{-i} = \emptyset$ by our stipulation on $\Gamma_i$. It
thus remains to prove the proposition for $S_{i,n}$ and $S_{-i,m}$. Given some $x \in \Gamma_i$, consider the process $X$ with the initial condition $X_0 = x$. Observe that if there is some open neighborhood $U_x = (c, d) \subseteq \Gamma_i$ such that $U_x \cap \mathcal{D}_i^0 = U_x \cap \mathcal{D}_{-i}^0 = \emptyset$, then the claim is trivially true. Now, there remain the following two non-trivial cases: (i) $x \in \mathcal{D}_i^0$ (Any neighborhood $U_x$ intersects with its own stopping set) and (ii) $x \in \mathcal{D}_{-i}^0$ (Any neighborhood $U_x$ intersects with its rival’s stopping set). The cases (i) and (ii) cannot be true at the same time; if they were, then we must have $x \in S_{i,n} \cap S_{-i,m}$ for some $n \in N_i(h^0)$ and $m \in N_{-i}(h^0)$ (because $\mathcal{D}_i^0$ and $\mathcal{D}_{-i}^0$ are closed), which contradicts Proposition E.1. Hence, we need to examine (i) and (ii) separately.

First, we examine case (i), $x \in \mathcal{D}_i^0$, which implies that $x \in \Gamma_i \cap S_{i,n}$ for some $n$. By the separation of $S_{i,n}$ from $\mathcal{D}_i^0$, we can find an open neighborhood $(f, g)$ such that $x \in (f, g)$ and $(f, g) \cap \mathcal{D}_{-i}^0 = \emptyset$. Moreover, because $\Gamma_i$ and $(f, g)$ are open sets that contain $x$, we can find some open neighborhood $U_x = (c, d)$ of $x$ such that $f < c < x < d < g$. Then by Lemma E.1 there is a positive probability of a subgame $h'$ in which $U_x$ does not intersect $\mathcal{D}_i^0$ (i.e., $U_x \cap \mathcal{D}_i^0 = \emptyset$) because the process $X$ traverses the interval $U_x$ over the history $h'$ without escaping $(f, g)$; this subgame can be reached if firm $i$ simply never exits (by possibly deviating from the strategy) until $X$ hits all the sets $S_{i,n}$ that intersect with $U_x$ without escaping $(f, g)$. Moreover, because $U_x \subset (f, g)$ and $(f, g) \cap \mathcal{D}_{-i}^0 = \emptyset$, $U_x$ does not intersect any $\mathcal{D}_{-i}^0 \subset \mathcal{D}_{-i}^0$.

Second, we examine case (ii), $x \in \mathcal{D}_{-i}^0$, which implies that $x \in \Gamma_i \cap S_{-i,m}$ for some $m$. Using the separation of $S_{-i,m}$ from $\mathcal{D}_i^0$, this case can be proved by the arguments used in case (i).

We next show that $\Gamma_1$ and $\Gamma_2$ must coincide as in a mixed strategy MPE.

**Proposition E.4** $\Gamma := \Gamma_1 = \Gamma_2$.

**Proof:** Given any $x \in \Gamma_1$, Proposition E.3 states that there is a positive probability of a subgame in

Figure 1: Sample path of $X$ in which $X$ begins at $x$, hits $c$ and $d$, and then returns to $x$ at some time $\tau$ without escaping the interval $(f, g)$. Lemma E.1 establishes that the probability measure of such sample paths is positive.
which there exists an open neighborhood $U_x \subseteq \Gamma_1$ that does not intersect $\mathcal{D}_1$ and $\mathcal{D}_2$. In this subgame $h'$, we can utilize the same argument as in Step 2 of the proof of Lemma 2 and establish that the neighborhood $U_x$ must be also contained in $\Gamma_2$. Then it follows that $\Gamma_1 \subseteq \Gamma_2$. By a symmetric argument for $\Gamma_2$, we can conclude that $\Gamma_1 = \Gamma_2$. \hfill \blacksquare

Note that because $\Gamma$ is open, $\Gamma$ can be expressed as a disjoint union of open intervals, each of which is a component of $\Gamma$. We state the following lemma regarding each component of $\Gamma$.

**Lemma E.2** For any component $(f, g)$ of $\Gamma$, there is a positive probability of a subgame $h'$ in which the closed interval $[f, g]$ is separated from $\mathcal{D}_1$ and $\mathcal{D}_2$.

**Proof:** Fix any $x \in (f, g)$. Then by Proposition E.3, there is a subgame $h'$ with a neighborhood $U_x = (c, d) \subset (f, g)$ (See Figure 1) such that $U_x$ does not intersect $\mathcal{D}_1$ and $\mathcal{D}_2$. In this subgame $h'$, we first claim that the closed interval $[f, g]$ does not intersect $\mathcal{D}_1$ and $\mathcal{D}_2$. Towards a contradiction, suppose that $[f, g]$ intersects $\mathcal{D}_2$. Consider $U^\delta_x = (x - \delta, x + \delta)$ for some $\delta$ such that $c < x - \delta < x + \delta < d$. Note that $U_x^\delta$ is separated from $\mathcal{D}_2$ because $U_x^\delta \subset U_x = (c, d)$ and $U_x \cap \mathcal{D}_2 = \emptyset$. Choose $y \in [f, g] \cap \mathcal{D}_2$ such that $y$ is the closest point of $\mathcal{D}_2$ to either $x + \delta$ or $x - \delta$.

We first examine the case when $y$ is closest to $x + \delta$, meaning that $y > x + \delta$. In this case, the interval $(x, y)$ does not intersect $\mathcal{D}_2$, but $y \in S_{2, m}$ for some $m \in N_2(h')$. Now, in the subgame $h'$ starting from the point $X_t = x$, we consider the hitting time $\tau_y = \inf\{s \geq t : X_s = y\}$ of $y$. Then because $y$ belongs to the closure of $\Gamma$, firm 1’s equilibrium payoff at $\tau_y^+$ (slightly before $X$ reaches the point $y$) must be $l_1$ because $X_{\tau_y^+} \in \Gamma$. However, because the strategy $a_2$ specifies firm 2’s exit at $\tau_y$ with the probability $p_{2, m} > 0$, firm 1’s payoff from exit at $\tau_y$ is $p_{2, m} M_1(y) + (1 - p_{2, m}) l_1 > l_1$ since $M_1(y) = [w(y) + l_1]/2 > l_1$. This means that firm $i$ can attain a higher payoff at $\tau_y^+$ by simply waiting until $\tau_y$ without exit. This contradicts the supposition that $(a_1, a_2)$ is an equilibrium. In case $y$ is closest to $x - \delta$, using the fact that the interval $(y, x)$ does not intersect $\mathcal{D}_2$, we can similarly arrive at a contradiction. We conclude that $[f, g]$ does not intersect $\mathcal{D}_2$.

Because $[f, g]$ is a component of both $\Gamma_1$ and $\Gamma_2$ (recall $\Gamma = \Gamma_1 = \Gamma_2$), we can show that $[f, g]$ does not intersect $\mathcal{D}_i$ by the symmetric argument. We can thus conclude that $[f, g]$ does not intersect $\mathcal{D}_1$ or $\mathcal{D}_2$ in the subgame $h'$.

Lastly, recall that $X$ does not escape $(f, g)$ in the subgame $h'$ constructed by Proposition E.3. Then because $[f, g]$ and $\mathcal{D}_i^0$, $i \in \{1, 2\}$, are all closed sets, and $\mathcal{D}_i \subseteq \mathcal{D}_i^0$, $i \in \{1, 2\}$, it must be the case that $[f, g]$ is separated from $\mathcal{D}_1$ and $\mathcal{D}_2$ as desired. \hfill \blacksquare

Now, we prove that $\Gamma$ must be the half-line below $\theta_i^+$.

**Proposition E.5** $\Gamma = (\alpha, \theta_i^+)$.  

**Proof:** We first note that $\Gamma \subseteq (\alpha, \theta_i^+)$ by Proposition E.2. Towards a contradiction, suppose that this inclusion is strict, i.e., $\Gamma \subset (\alpha, \theta_i^+)$. Then by Lemma E.2, there is a positive probability of a subgame
can conclude that \( \Gamma \in y \pi X \). This is because at time \( \Gamma \) point of \( \Gamma \) case of \( D \) case of \( C \) equilibrium is not possible. Hence, we only need to consider case (i) below.

Because \( D = \emptyset \) and \( S_{-i,m} \), we can apply the same argument to firm \(-i\)'s subgame payoff function \( V_{-i}(h';a_i,a_{-i}) \) and derive a similar contradiction. Lastly, we can derive a similar contradiction for the case in which \( y \in \Gamma \) and \( z \in \Gamma \).

The only remaining possibility is that \( \Gamma = (\alpha, \theta_0) \) for some \( \theta_0 < \theta^* \) and \( (y,z) = (\theta_0, \theta^*) \). This case can be precluded by the same argument as in Step 4 of the proof of Lemma \[ \] Therefore, we can conclude that \( \Gamma = (\alpha, \theta^*) \).

Because \( \Gamma = (\alpha, \theta^*) \) cannot be reconciled with \( \Gamma = (\alpha, \theta^*_{-i}) \) if \( \theta^*_i \neq \theta^*_{-i} \), we finally conclude from Proposition \[ \] that there is no mixed strategy SPE in case \( \Gamma \neq \emptyset \).

**Case 2:** \( \Gamma_1 = \Gamma_2 = \emptyset \)

In this case, there are two possibilities: (i) \( D_1^0 \neq \emptyset \) and \( D_2^0 \neq \emptyset \), and (ii) \( D_1^0 = \emptyset \) or \( D_2^0 = \emptyset \). (The case of \( D_1^0 = D_2^0 = \emptyset \) cannot be an equilibrium.) In case (ii), one of the firms never exits, in which case the best response of the other firm is to employ a pure strategy of exit, so a mixed strategy equilibrium is not possible. Hence, we only need to consider case (i) below.

Because \( D_1^0 \) and \( D_2^0 \) must be separated (by Proposition \[ \]) and non-empty, there must be continuation regions between \( D_1^0 \) and \( D_2^0 \) at any time \( t \) since \( D_i^0 \subseteq D_i^0 \), \( i \in \{1,2\} \). For any component \( (c,d) \) of a common continuation region \( C' \), it is not possible to have \( \{c,d\} \subseteq D_1^0 \) or \( \{c,d\} \subseteq D_2^0 \) by the same argument as in the proof of Proposition \[ \]. Hence, each component of \( C' \) must be of the
form \((\alpha, c)\) or \((c, \beta)\) where \(c \in D_1^t \cup D_2^t\), or of the form \((c, d)\) where \(c < d, c \in D_1^t \) and \(d \in D_2^t\).

Moreover, because \(D_1^0\) and \(D_2^0\) are closed sets, there is a (finite or infinite) series of closed intervals or points \(\{\phi(k)\}_{k=1}^K\) such that \(D_1^0 \cup D_2^0 = \bigcup_{k=1}^K \phi(k)\), \(\phi(k) = [\phi_L^k, \phi_U^k] \subseteq (\alpha, \beta)\), and \(\phi_L^k > \phi_U^{k+1} \geq \phi_L^{k+1}\) (See Figure 2). Here we always have \(K \geq 2\) because \(D_1^0\) and \(D_2^0\) are separated and non-empty. Indeed, we can further characterize the structure of \(\bigcup_{k=1}^K \phi(k)\) as follows:

**Lemma E.3** The series \(\{\phi(k)\}_{k=1}^K\) with \(\bigcup_{k=1}^K \phi(k) = D_1^0 \cup D_2^0\) satisfies the followings:

(i) If \(\phi(k) \subset D_1^0\), then \(\phi(k+1) \subset D_2^0\) for any \(k \geq 1\).

(ii) \(\phi(1) = [\phi_L^1, \theta_i^1]\) for some \(i \in \{1, 2\}\).

(iii) If \(K < \infty\), then \(\phi(K) = (\alpha, \phi_U^K)\). If \(K = \infty\), then \(\lim_{k \to \infty} \phi_U^k = \alpha\).

**Proof:** (i) If \(\phi(k) \subset D_1^0\) and \(\phi(k+1) \subset D_1^0\) for some \(k \geq 1\), then there must be a component \((c, d)\) of the common continuation region \(C_0\) such that \(\{c, d\} \subset D_1^0\), which was shown to be impossible in the above of the statement of this lemma.

(ii) Towards a contradiction, suppose that \(\phi(1) = [\phi_L^1, \theta_i^1] \subset D_1^0\) for some \(\phi_U^1 \neq \theta_i^1\). Note that \(\phi_U^1 \leq \theta_i^*\) by Proposition E.2. Then, for some \(X_0 = x > \theta_i^*, V_i(h^1; a) = l_i\) if \(\tau = \inf\{t \geq 0 : X_t = \phi_U^1\}\) is the hitting time of the boundary of \(\phi(1)\). It follows that firm \(i\)'s payoff at \(t = 0\) is identical to the value function from the following stopping problem:

\[
\sup_{\tau} V_\tau(x) = \sup_{\tau} \mathbb{E}\left[\int_0^\tau e^{-r\tau} \pi(X_t) + e^{-r\tau} l_i|X_0 = x\right].
\]

However, we recall that \(V_\tau(\cdot)\) attains its optimum if and only if \(\tau = \inf\{t \geq 0 : X_t = \theta_i^*\}\), which contradicts the assumption that \(a\) is an equilibrium. Thus, \(\phi(1) = [\phi_L^1, \theta_i^1]\).

(iii) We first consider the case \(K < \infty\). Towards a contradiction, suppose that \(\phi(k) = [\phi_L^k, \phi_U^k]\) for some \(\phi_L^k > \alpha\). It implies that \((\alpha, \phi_L^k)\) is a common continuation region at time zero. Without loss of generality, we suppose \(\phi(k) \subset D_1^0\), then the payoff to firm \(i\) for \(X_0 = x < \phi_L^k\) is given by

\[
V_i(h^0; a) = \mathbb{E}\left[\int_0^\tau e^{-r\tau} \pi(X_t) + e^{-r\tau} l_i|X_0 = x\right]
\]

where \(\tau = \inf\{t \geq 0 : X_t = \phi_L^k\}\) is the hitting time of \(\phi(k)\). Because \(\pi(X_t) < rl_i\) for \(X_t < \theta_i^*\), we find that \(V_i(h^0; a) < l_i\); it implies that firm \(i\) could improve its payoff by immediate exit, which contradicts the assumption that \(a\) is an equilibrium. We conclude that \(\phi(k) = (\alpha, \phi_U^k)\).

We next consider the case \(K = \infty\). Towards a contradiction, suppose \(\lim_{k \to \infty} \phi_U^k = \phi_L \geq \alpha\). It implies that \(\phi_L\) is a limit point for both \(D_1^0\) and \(D_2^0\), which contradicts the two closed sets \(D_1^0\) and \(D_2^0\) do not intersect with each other.

Recall that \(E_i\) is a subset of \(\bigcup_{k=1}^K \phi(k) = D_1^0 \cup D_2^0\). Below we characterize how the components of \(E_i\) should compose \(\bigcup_{k=1}^K \phi(k)\).
Lemma E.4 (i) Each $\phi^{(k)}_i = [\phi^{(k)}_L, \phi^{(k)}_U]$ for $1 < k < K$ must contain exactly one component of $E_1$ or $E_2$.

(ii) For each $k \geq 1$, if all the boundary point(s) (i.e., $\phi^{(k)}_L$ or $\phi^{(k)}_U$) of $\phi^{(k)}_i \subset D^0_i$ belong to $E_i$, then it must be the case that $\phi^{(k)}_i \subset E_i$.

Proof: (i) Consider $x \in (\phi^{(2)}_U, \phi^{(1)}_L)$ and assume $\phi^{(2)}_i \subset D^0_i$ for some $i \in \{1, 2\}$. By Lemma E.1, there is a positive probability of a subgame $h'$ such that the trajectory $\{X_t\}_{t \in I}$ has traversed the entire interval $\phi^{(2)}$ without ever hitting $\phi^{(1)}$ or $\phi^{(3)}$. In this subgame, $\phi^{(2)}$ does not intersect with $S_{i,n}$ for any $n \in N_{h'}$, but $D^0_{-i} = D^0_i$ because $X$ has not hit $D^0_i$. If $\phi^{(2)}$ does not contain a component of $E_i$, then $(\phi^{(3)}_U, \phi^{(1)}_L)$ is a subset of the common continuation region $\mathcal{C}'$, which leads to a suboptimal payoff function for firm $-i$ because $\phi^{(3)}_U$ and $\phi^{(1)}_L$ belong to $D^0_{-i}$. Thus, $\phi^{(2)}$ must contain a component of $E_i$.

Towards a contraction, suppose $\phi^{(2)}$ contains more than one subinterval of $E_i$. Then in the subgame $h'$ mentioned above, there exists a common continuation region contained within $\phi^{(2)}$ sandwiched between two components of $E_i$: this leads to a suboptimal payoff function for firm $i$, which contradicts the supposition that $a$ is an equilibrium. Therefore, $\phi^{(2)}$ must contain exactly one component of $E_i$.

The same argument above also applies to all $k > 2$ except for $k = K$.

(ii) For $1 < k < K$, the statement follows from (i) because there can be only one component of $E_i$ within $\phi^{(k)}$. We first consider $\phi^{(1)}_i$ when $\phi^{(1)}_L \subset E_i$ and $\theta^*_i \subset E_i$. Towards a contradiction, suppose that there is more than one component of $E_i$ contained within $\phi^{(1)}$. Then there exists a subgame $h'$ in which there is a component $(c,d) \subset \phi^{(1)}$ of the common continuation region $\mathcal{C}'$ where $(c,d) \subset E_i$; this leads to a suboptimal payoff function for firm $i$, which contradicts the assumption that $a$ is an equilibrium. Thus, $\phi^{(1)}_i \subset E_i$ must be satisfied.

Now consider $\phi^{(K)} = (\alpha, \phi^{(K)}_U)$ where $\phi^{(K)}_U \subset E_i$. If there exists more than one component of $E_i$ contained in $\phi^{(K)}$, we encounter the same contradiction as in the case of $\phi^{(1)}$, so this possibility is
precluded. The second possibility is that \( \phi^{(K)} \) a single component \([s, \phi^{(K)}_U]\) of \( E_i \) for some \( s > \alpha \). In this case, by virtue of Lemma [E.1] there is a subgame \( h' \) in which \((c, s) \in C'\) for some \( c \geq \alpha \). Because the boundary point(s) of \((c, s)\) belong to \( D'_i \), the payoff to firm \( i \) is suboptimal in the region \((c, s)\), which contradicts the assumption that \( a \) is an equilibrium. Thus, \( \phi^{(K)} \subset E_i \) if \( \phi^{(K)}_U \subset E_i \).

Next, we examine the structure of the common continuation region \( C' \) in a subgame \( h' \). In particular, consider some subgame \( h' \) with \( C' \) that has a component interval \((c, d)\) which shares boundaries with \( D'_i \) and \( D'_{i-1} \) for some \( i \in \{1, 2\} \). Then we can show that if either of \( c \) or \( d \) belongs to \( E_i \), then the other must belong to \( E_{i-1} \).

**Lemma E.5** Given a component \((c, d)\) of \( C' \) for some subgame \( h' \), suppose \( c \in D'_{i-1} \) and \( d \in D'_i \). If \( c \in E_{i-1} \), then \( d \in E_i \) must be satisfied.

**Proof:** Suppose that \( d \in S_{i,n} \) for some \( n \in N_i(h') \) when \( c \in E_{i-1} \). Define \( \tau_{cd} := \inf\{s \geq t : X_s \notin (c, d)\} \).

If \( X_t = x \in (c, d) \), then \( V_i(h^{x,cd} ; a) = l_i \) if \( X_{\tau_{cd}} = d \) and \( V_i(h^{x,cd} ; a) = w(c) \) if \( X_{\tau_{cd}} = c \). Hence, we have

\[
V_i(h' ; a) = \mathbb{E} \left[ \int_0^{\tau_c \wedge \tau_d} e^{-r(s-t)} \pi(X_s)ds + l_i e^{-r \tau_d} 1_{\{\tau_d < \tau_c\}} + w(c) e^{-r \tau_c} 1_{\{\tau_c < \tau_d\}} \mid X_t = x \right],
\]

where \( \tau_c = \inf\{t \geq 0 : X_t = c\} \) and \( \tau_d = \inf\{t \geq 0 : X_t = d\} \) so that \( \tau_{cd} = \tau_c \wedge \tau_d \). Note that the event \( \tau_c = \tau_d \) does not happen, we excluded it for brevity without influencing the payoff function.

Now consider the following stopping problem for a single player:

\[
\sup_{\tau} V_{\tau}(X_t) = \sup_{\tau} \mathbb{E} \left[ \int_0^{\tau \wedge \tau_c} e^{-r(s-t)} \pi(X_s)ds + l_i e^{-r \tau} 1_{\{\tau < \tau_c\}} + w(c) e^{-r \tau_c} 1_{\{\tau \geq \tau_c\}} \mid X_t = x \right],
\]

where \( \tau \) is the stopping time decision variable and \( \tau_c \) is again the hitting time of \( c \). Note that \( V_i(h' ; a) = V_{\tau_c}(x) \). We now claim that the optimal stopping time for \( \tau \) to solve for the maximization problem of (24) is \( \tau_d \), i.e., the hitting time of \([d, \beta]\). We denote an optimal threshold of exit as \( \theta \in [c, \beta] \) for firm \( i \). (Without loss of generality, there exists such a threshold; if it is never optimal to stop, then we can set \( \theta = \beta \).) Towards a contradiction, we suppose that \( \theta \neq d \). In that case, firm \( i \) can improve its payoff by choosing a different strategy \( a'_i \) of exiting at \( \theta \) to earn \( V_i(h^0 ; a'_i, a_{-i}) > V_i(h^0 ; a) \). This contradicts the assumption that \( a \) is an equilibrium. We conclude that \( d \) must be the optimal stopping threshold for the single player problem.

Next, we consider the subgame after \( \tau_{cd} \) when \( \tau_{cd} = \tau_d \) is the hitting time of \( d \). In that case, we have \( S_{i,n} \notin D'_i \) (i.e., \( n \notin N_i(h') \)) for \( t > \tau_{cd} \). Then there is a positive probability of a subgame at \( t > \tau_{cd} \) in which \( X_t \in (c, f) \subset C' \) where \( f \in E_i \) or \( f \in S_{i,n'} \) for some \( n' \neq n \). In other words, \( f \) is now a new boundary between the continuation region and \( D'_i \). Now we apply the same argument as above to infer that \( f \) is the optimal stopping threshold for the problem (24). This contradicts the
Then we can use the same argument as in (i) to conclude that like any other components. From Lemma E.3 that \( K \) in (24). Since the optimal threshold cannot take two different values, we conclude that \( d \) cannot belong to \( S_{i,n} \) for any \( n \geq 1 \) if \( c \in E_{-i} \).

Lastly, we prove that each \( \phi(k) \), \( k \geq 1 \), must be a component of either \( E_1 \) or \( E_2 \).

**Proposition E.6** Each \( \phi(k) \), \( k \geq 1 \), is a component of \( E_i \) for some \( i \in \{1, 2\} \).

**Proof:** We prove this statement for two separate cases: (i) \( K > 2 \) (i.e., \( K \geq 3 \)), (ii) \( K = 2 \). (Recall from Lemma E.3 that \( K \geq 2 \) if \( D_1^0 \neq \emptyset \) and \( D_2^0 \neq \emptyset \).)

(i) Given \( \phi(k) \subset D_i^0 \) for \( 1 < k < K \), there exists exactly one component \([s, u]\) of \( E_i \) contained in \( \phi(k) \) by Lemma E.4(i). Also, because \( \phi(k) \) is bounded for all \( 1 < k < K \), we can use Lemma E.1 to construct a subgame \( h' \) in which no \( S_{i,n} \) with \( n \in N_1(h') \) (i.e., any \( S_{i,n} \subset D_i^0 \)) intersects with \( \phi(k) \) while \( D_i' = D_i^0 \). In this subgame, there are two components of the common continuation region \( C' \) adjacent to \([s, u]\): \( (\phi(k-1), s) \) and \( (u, \phi(k+1)) \). Then by Lemma E.5, \([s, u] \subset E_i \) implies that \( \phi(k-1) \in E_{-i} \) and \( \phi(k+1) \in E_{-i} \). Because this holds true for all \( 1 < k < K \), it implies that all boundary points of \( \phi(k) \) for all \( k > 1 \) belong to \( E_i \) for some \( i \in \{1, 2\} \). By Lemma E.4(ii), we then conclude that \( \phi(k) \subset E_i \) for some \( i \in \{1, 2\} \) for all \( k > 1 \).

A special case of the conclusion in the above paragraph is \( \phi(2) \subset E_i \) for some \( i \in \{1, 2\} \), which implies that \( \phi(1) \in E_i \) for some \( i \in \{1, 2\} \) by Lemma E.5. It thus remains to prove that \( \phi(1) = \theta_i^* \in E_i \), in which case \( \phi(1) \subset E_i \) by Lemma E.4(ii). Towards a contradiction, suppose that \( \theta_i^* \in S_{i,n} \) for some \( n \geq 1 \). At time zero, \((\theta_i^*, \beta)\) is a component of the common continuation region. Then, by Lemma E.1 there is a subgame \( h' \) with \( X_0 > \theta_i^* \) in which \( X \) has already hit \( \theta_i^* \). In this subgame, the common continuation region has evolved to \((c, \beta)\) for some \( c \in (\phi_L(1), \theta_i^*) \) where \( c \in E_i \) or \( c \in S_{i,n'} \) for some \( n' \neq n \). Then firm \( i \)'s payoff \( V_i(h'; a) \) is suboptimal because \( \theta_i^* \) is the unique optimal stopping threshold while the strategy at \( h' \) for firm \( i \) involves stopping at \( c < \theta_i^* \). This contradicts the fact that \( a \) is an equilibrium. Therefore, \( \theta_i^* \in E_i \), and thus \( \phi(1) \subset E_i \).

(ii) In this case, \( \phi(2) = (\alpha, \phi_U(2)) \). Suppose that \( \phi(1) \subset D_i^0 \) contains no component of \( E_i \). Then because \( \phi(1) \) is bounded, we can invoke Lemma E.1 to construct a subgame \( h' \) in which \( (\phi_U(2), \beta) \) is the common continuation region \( C' \). In this subgame, \((\alpha, \phi_U(2)) \subset D_i' \). In order for firm \( -i \)'s strategy to be a best response in this subgame, \((\alpha, \phi_U(2)) = (\alpha, \theta_i^*) \in E_{-i} \) needs to be satisfied because firm \( i \) never exits in this subgame. By Lemma E.5, it implies that \( \phi_L(1) \in E_i \), which contradicts the supposition that \( \phi(1) \) contains no component of \( E_i \). Thus, \( \phi(1) \) must contain \( E_i \). Furthermore, just like any other components \( \phi(k) \) for \( k > 1 \), \( \phi(1) \) must have a single component of \( E_i \) by Lemma E.4(i). Then we can use the same argument as in (i) to conclude that \( \phi(1) \subset E_i \).

Because \( \bigcup_{k=1}^K \phi(k) = D_1^0 \cup D_2^0 \), Proposition E.6 implies that \( D_i^0 = E_i, i \in \{1, 2\} \). Therefore, \( a_1 \) and \( a_2 \) must be pure strategies in equilibrium if \( \Gamma_1 = \Gamma_2 = \emptyset \). Here note that we do not need to invoke the assumption that \( \theta_i^* \neq \theta_i^* \) in order to prove that there is no mixed strategy SPE.
Combining the results in both cases (\(\Gamma_1 \cup \Gamma_2 \neq \emptyset\) and \(\Gamma_1 = \Gamma_2 = \emptyset\)), we can finally conclude that there is no mixed strategy SPE if \(\theta^*_i \neq \theta^*_{-i}\).