# The Absence of Attrition in a War of Attrition under Complete Information*

George Georgiadis, Youngsoo Kim, and H. Dharma Kwon†

October 22, 2021

**Abstract**

We consider a two-player game of war of attrition under complete information. It is well-known that this class of games admits equilibria in pure, as well as mixed strategies, and much of the literature has focused on the latter. We show that if the players' payoffs whilst in "war" vary stochastically and their exit payoffs are heterogeneous, then the game admits Markov Perfect equilibria in pure strategies only. This is true irrespective of the degree of randomness and heterogeneity, thus highlighting the fragility of mixed-strategy equilibria to a natural perturbation of the canonical model. In contrast, when the players' flow payoffs are deterministic or their exit payoffs are homogeneous, the game admits equilibria in pure and mixed strategies.

## 1 Introduction

In the classic war of attrition, the first player to quit concedes a prize to his opponent. Thus, each player trades off the cost associated with fighting against the value of the prize. These features are common in many managerial and economic problems. Oligopolists in a declining industry may bear losses in anticipation of profitability following a competitor's exit (Ghemawat and Nalebuff, 1985). For example, the rise of Amazon in the mid-1990s made the business model of Barnes & Noble and Borders obsolete, turning traditional bookselling into a declining market. As the demand shrank

*We thank the editor, Nicolas Vieille, the advisory editor, and two anonymous referees, Hector Chade, Johannes Hörner, Mike Powell, and Luis Rayo, as well as participants at several seminars and conferences for helpful comments and suggestions. A previous version of this paper was titled "Equilibrium Selection in the War of Attrition under Complete Information".

†G. Georgiadis: Kellogg School of Management, Northwestern University; g-georgiadis@kellogg.northwestern.edu.;
Y. Kim: Culverhouse College of Business, University of Alabama; ykim@culverhouse.ua.edu.;
H.D. Kwon: Gies College of Business, University of Illinois at Urbana-Champaign; dhkwon@illinois.edu.

sharply, these two major players at the time had to cut down slack in their capacities, but each would prefer its competitors to carry the painful burden of closing stores or exiting the market altogether (Newman, 2011). Similarly, the presently low price of crude oil is often attributed to a war of attrition among the OPEC allies and non-OPEC rivals such as Russia and the many shale-oil producers in the United States (Reed, 2016). Other examples of wars of attrition include the provision of public goods (Bliss and Nalebuff, 1984), lobbying (Becker, 1983), labor disputes (Greenhouse, 1999), court of law battles (McAfee, 2009), races to dominate a market (Ghemawat, 1997), technology standard races (Bulow and Klemperer, 1999), price cycles in oligopolistic collusion (Maskin and Tirole, 1988), all-pay auctions (Krishna and Morgan, 1997), and bargaining games (Abreu and Gul, 2000).

A central feature of wars of attrition is the waste of valuable resources (*a.k.a attrition*): there exist times when players would collectively be better off if one of them quits, but each player strategically resists quitting in anticipation that his opponent will be the first to do so. Canonical, complete-information games of war of attrition typically admit equilibria in both pure and mixed strategies; see, for example, Tirole (1988), Fudenberg and Tirole (1991), and Levin (2004). Attrition, however, is featured only in the latter, while the former are (Pareto) efficient. We study such a two-player model, and show that if the players' flow payoffs whilst fighting for the prize follow an irreducible stochastic process and their exit payoffs are heterogeneous, then the game admits only pure-strategy Markov Perfect equilibria (hereafter MPE), and this is true irrespective of the degree of uncertainty or heterogeneity. In other words, our main result shows that an arguably natural perturbation of the model eradicates all MPE that exhibit attrition. This result has implications for the modeling choices in such games, as well as a growing literature that aims to empirically study strategies in real-world wars of attrition; see, for example, Wang (2009) and Takahashi (2015).

In our continuous-time model, two competing oligopolists contemplate exiting a market. While both firms remain in the market, each receives the flow payoff that depends on the *market conditions* (*e.g.*, the price of a relevant commodity), which fluctuate according to a stochastic diffusion process, hereafter the *state*. At every moment, each firm can exit the market and collect its outside option. Its rival then obtains a (higher) *winner's* payoff, which depends on the state at the time of the opponent's exit; *e.g.*, the net present value of monopoly profits. All payoff-relevant parameters are common knowledge. The firms may have heterogeneous outside options but they are otherwise identical. Given that the state follows a Markov process and mixed strategy equilibria characterized in the literature are typically stationary (e.g., Tirole (1988)), we focus on Markov strategies in the main body of the paper, wherein at every moment, each firm conditions its probability of exit on the current state.

We first characterize the best response of a firm that anticipates its rival will never exit, which turns out to be instrumental for the equilibrium analysis. We show that a firm optimally exits at the first moment that the state drifts below a threshold. This *single-player-optimal* threshold is strictly increasing in the firm's outside option; the better is a firm's outside option, the less it is willing to endure poor market conditions before exiting.

We present our main result in Section 3. To set the stage, Proposition 1 shows that there exists a pure-strategy MPE in which the firm with the larger outside option exits at the first moment that the state drifts below its single-player-optimal threshold. Moreover, if the heterogeneity in outside options is not too large, then there exists another pure-strategy MPE in which the firm with the lower outside option exits at the first moment that the state drifts below its own single-player-optimal threshold. Towards our main result, we show that in any mixed-strategy MPE, (i) the firms must randomize between remaining in the market and exiting on a common set of states, and (ii) each firm exits with nonzero probability if (and only if) the state is below its single-player-optimal threshold. However, (i) and (ii) are incompatible with each other if the firms have heterogeneous outside options, because their single-player-optimal thresholds differ in that case. Therefore, it follows that no mixed-strategy MPE exists in that case. We also extend our main result to non-Markovian Subgame Perfect equilibria subject to a restriction on the firms' strategies (see Online Appendix G for details).

First, this paper contributes to the literature on wars of attrition, which has received widespread attention since the seminal work of Maynard Smith (1974). Our model is closest to Hendricks et al. (1988) and Murto (2004). The former characterizes equilibria in both pure and mixed strategies in a war of attrition under complete information with asymmetric players whose payoffs are deterministic. The latter considers stochastic payoffs, but restricts attention to pure-strategy MPE. In contrast, we allow payoffs to vary stochastically, and we show that if players are heterogeneous, then (subject to a set of restrictions on strategies) the game admits MPE in pure strategies *only*.

We also contribute to a literature that contemplates equilibrium selection in games of war of attrition. This literature has two broad themes. The first considers games which are *backward-inductible*. For example, Ghemawat and Nalebuff (1985) studies a game with asymmetric players in which there is a state (that is reached with probability one) at which both firms have a dominant strategy to exit, while Bilodeau and Slivinski (1996) considers a finite-horizon war of attrition game. In both cases, the game is shown to have a unique equilibrium in pure strategies. In the second theme, with a *small* probability, each player *never* exits. In Fudenberg and Tirole (1986), players are uncertain about their rivals' costs of remaining in the market, whereas in Kornhauser et al. (1989), Kambe (1999), and Abreu and Gul (2000), with a small probability, each player is irrational and never exits. It is shown that the respective games admit a unique equilibrium. Myatt (2005) shows that this uniqueness is insensitive to perturbations having a similar economic interpretation as exit failure. We complement this literature by considering a complete-information framework with rational players, and showing that an arguably natural perturbation of the canonical model eliminates all mixed-strategy MPE.

Touzi and Vieille (2002) introduces the concept of mixed strategies in continuous-time Dynkin games, and proves that the game admits minimax solutions in mixed strategies. With this notion of mixed strategies, Seel and Strack (2016) investigates a war of attrition with privately observed Brownian motions, and Steg (2015) characterizes equilibria in both pure and mixed strategies in a family of continuous-time stochastic timing games. Whereas these articles consider games with identical players, we focus on ones with heterogeneous players and show that the set of equilibria differs drastically. Riedel and Steg (2017) examines mixed-strategy equilibria in continuous-time stopping games with heterogeneous players, but focuses on games with pre-emption incentives, whereas ours is purely one of war of attrition.

## 2 Model

We consider a war of attrition with complete information between two oligopolists. Time is continuous, and firms discount time at rate $r > 0$. At every moment, each firm decides whether to exit the market.

While both firms remain in the market, each earns a flow profit $\pi(X_t)$, where $\pi : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and $X_t$ is a scalar that captures the *market conditions* that the firms operate in (*e.g.*, the size of the market or the price of raw materials). The market conditions fluctuate according to

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t\,, \tag{2.1}$$

where $X_t$ is defined on $\mathcal{X} := (\alpha, \beta) \subseteq \mathbb{R}$, $X_0 \in \mathcal{X}$, the functions $\mu : \mathcal{X} \to \mathbb{R}$ and $\sigma : \mathcal{X} \to \mathbb{R}_+$ are Lipschitz continuous, and $B_t$ is a Wiener process.[1,2] Then because we assume $\sigma(\cdot) > 0$ on $\mathcal{X}$, the functions $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the local integrability condition (Arkin, 2015), which implies that the diffusion process $X$ is *regular* in $\mathcal{X} = (\alpha, \beta)$ (Karatzas and Shreve, 1991): For any $x, y \in \mathcal{X}$, the process $X$ reaches from $x$ to $y$ in finite time with positive probability; *i.e.*, $X$ is *irreducible* (Borodin and Salminen, 1996, p.13). Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ denote the probability space with sample space $\Omega$, $\sigma$-algebra $\mathcal{F}$, probability measure $\mathbb{P}$, and filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that satisfies the *usual* conditions (p. 172, Rogers and Williams 2000). We assume that the process $\{B_t\}_{t\geq 0}$ (or equivalently, $\{X_t\}_{t\geq 0}$) is progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Throughout the paper, we let $\mathbb{E}[\cdot]$ denote the expected value with respect to $\mathbb{P}$.

If firm $i$ chooses to exit at $t$, then it receives its outside option $l_i$, and its opponent, denoted by

[1] Special cases in which $\sigma(\cdot) = 0$ have been analyzed extensively (Ghemawat and Nalebuff, 1985, Hendricks et al., 1988, and others). Therefore, we restrict attention to $\sigma(\cdot) > 0$ in the main body of this paper, and for completeness, we revisit the case in which $\sigma(\cdot) = 0$ in Appendix A.

[2] The boundary points $\alpha$ and $\beta$ are assumed to be *natural* (Borodin and Salminen, 1996, p.18-20); *i.e.*, neither $\alpha$, nor $\beta$ can be reached by $X_t$ in finite time. For example, if $X_t$ is a standard diffusion process, then $\mathcal{X} = \mathbb{R}$. If $X_t$ is a geometric Brownian process, then $\mathcal{X} = (0, \infty)$.

$-i$, receives $w(X_t) \in \mathbb{R}$, the expected payoff associated with being the sole remaining firm; *e.g.*, the net present value of monopoly profits. In this case, we say that firm $i$ is the *loser* and firm $-i$ is the *winner*. We adopt the convention that $l_1 \leq l_2$; *i.e.*, firm 2 has a larger outside option than firm 1. We assume that $w(x) > l_2$ for all $x$ so that the winner's reward is always larger than the loser's. The game ends as soon as a firm exits the market. If both firms exit at the same moment, then each obtains the outside option $l_i$ or $w(X_t)$ with probability $1/2$.[3]

Finally, we make the following assumptions on the functions $\pi(\cdot)$ and $w(\cdot)$: First, we assume that $\pi(\cdot)$ satisfies the absolute integrability condition $\mathbb{E}\left[\int_0^\infty |e^{-rt}\pi(X_t)|\,dt\right] < \infty$, which ensures that each firm's expected discounted payoff is well-defined (see Alvarez, 2001). Second, we assume $w(\cdot) \in C^2(\mathcal{X})$ and $w(x) > \mathbb{E}\left[\int_0^t e^{-rs}\pi(X_s)ds + \; e^{-rt}w(X_t)\,|X_0 = x\right]$ for all $x \in \mathcal{X}$ and $t$, so that each firm prefers to become the winner sooner rather than later.[4] Lastly, we assume that for each $i$, there exists some $x_{ci} \in \mathcal{X}$ such that $\pi(x_{ci}) = rl_i$, which guarantees the existence of an optimal exit threshold in the interior of $\mathcal{X}$ (see Lemma B.1 and the proof of Lemma 1 for details).

### 2.1 Markov Strategies

We assume that both firms employ Markov strategies: At every moment $t$, each firm chooses (probabilistically) whether to exit based on the current state $X_t$, conditional on the game not having ended. Formally, each firm $i$ chooses

i. a closed subset $E_i$ of the state space $\mathcal{X}$ (or an *exit region*) such that it exits with probability $p_i(X_t) = 1$ if $X_t \in E_i$,

ii. a non-negative function $\lambda_i : \mathcal{X} \to \mathbb{R}_+$ (or an *exit rate*) such that $\lambda_i(x)$ represents the firm's hazard rate of exit when $X_t = x$.

Note that we stipulate that $E_i$ is a closed set and the exit probability $p_i(x)$ is always 1. These assumptions can be safely made because $X$ is a regular diffusion process: For any $x \in \mathcal{X}$, the hitting times $\tau_x^+ = \inf\{t > 0 : X_t^x > x\}$ and $\tau_x^- = \inf\{t > 0 : X_t^x < x\}$ are both 0 almost surely (e.g., Revuz and Yor (1991, Exercise 3.22, p.312)), which implies $\tau_x = \inf\{t > 0 : X_t^x = x\} = 0$ almost surely, i.e., $X$ comes back to the original value indefinitely many times within any finite time span. This has two implications for Markov strategies. First, exiting when $X_t \in E_i$ is indistinguishable from exiting when $X_t \in \mathrm{cl}(E_i)$. Second, any Markov strategy in which a firm exits with probability $p_i(x) \in (0,1)$ whenever $X_t = x$ is indistinguishable from one in which the firm exits with probability 1 at the

[3]For simplicity, we assume that the firms can differ only in the loser's exit payoff, $l_1$ and $l_2$, which is independent of $X$. In Online Appendix D, we show that under certain conditions, our main result continues to hold if the firms have heterogeneous discount rates, flow profits, and winner payoffs, and $l_i$ is a function of $X$.

[4]This assumption is satisfied if and only if $\sigma^2(x)w''(x)/2 + \mu(x)w'(x) + \pi(x) > rw(x)$ for all $x \in \mathcal{X}$.

moment $X_t = x$.[5] In the proof of Lemma 2, we prove that $E_i$ always has an equivalent closed set representation even if it is not closed.

Throughout this paper, we impose a regularity condition on the function $\lambda_i$. We stipulate that $\lambda_i(X_t)$ is Riemann integrable over any given time interval $[u,v]$ for $0 \leq u < v < \infty$. Note that firm $i$'s probability of exit within a time interval $[u,v]$ is given by $1 - \exp[-\int_u^v \lambda_i(X_t)dt]$; intuitively, the Riemann integrability thus ensures that, if the time interval is discretized, the process of coarse graining of the time interval does not alter the probability of exit. Conversely, if $\lambda_i(X_t)$ were not Riemann integrable, the lower Riemann summation over time does not coincide with the upper Riemann summation, which implies that the continuation time limit is not well-defined. Therefore, it is natural to impose a condition that $\lambda_i(X_t)$ is Riemann integrable over any time interval.

We represent firm $i$'s strategy as the pair $a_i = (E_i, \lambda_i)$, and $\{a_1, a_2\}$ is a *strategy profile*.[6] Intuitively, during any *small* interval $[t, t+dt)$, firm $i$ exits with probability

$$\rho_i(X_t) = \begin{cases} 1 & \text{if } X_t \in E_i\ , \\ \lambda_i(X_t)dt & \text{otherwise.} \end{cases}$$

If firm $i$ does not exit with probability 1 at all, we write $E_i = \emptyset$. If it does not exit with a positive hazard rate (*i.e.,* $\lambda_i(X_t) = 0$ for all $t$ almost surely, hereafter a.s), we write $\lambda_i \equiv \mathbf{0}$. Finally, we say that firm $i$'s strategy is *pure* if $\lambda_i \equiv \mathbf{0}$, and it is *mixed* otherwise.

## 2.2 Payoffs

Fix an initial value $X_0 = x \in \mathcal{X}$ and a strategy profile $\{a_i, a_{-i}\}$, and define $\tau_i := \inf\{s \geq 0 : X_s \in E_i\}$. Then firm $i$'s exit probability, that is, the probability that firm $i$ has exited by time $t$, given that $X_0 = x$, can be written as

$$G_i(t) := 1 - (1 - \mathbf{1}_{\{t \geq \tau_i\}}(t))e^{-\int_0^t \lambda_i(X_s)ds}\ .$$

[5] Note that the rival's strategy is not relevant to this property of our Markov strategy. First, in any MPE, only one firm may exit with positive probability at any $x \in \mathcal{X}$. This is because the payoff from simultaneous exit $(l_i + w(x))/2$ is strictly less than the winner's payoff $w(x)$. Second, suppose that one firm exits with positive probability at the hitting time of $x$ while its rival exits with positive probability at the hitting time of $y$ where $|y - x| = \delta > 0$. Then because of the mentioned property of a regular diffusion process $X$, if $X$ starts from $x$, it will return to $x$ indefinitely many times within an arbitrarily small time interval without hitting $y$, no matter how small $\delta > 0$ is.

[6] Our definition of a strategy implies that it can be alternatively expressed as a sum of the absolutely continuous function in time and the discontinuous jumps without the singularly continuous component. In Online Appendix E, we explain how this formulation of a strategy can be justified.

We can define firm $i$'s expected payoff under $\{a_i, a_{-i}\}$ when $X_0 = x$ as follows:

$$
\begin{aligned}
V_i(x; a_i, a_{-i}) = & \mathbb{E}\Bigg[\int_0^\infty \int_0^\infty \Bigg[\int_0^{t\wedge s} e^{-ru}\pi(X_u)du \\
& + e^{-r(t\wedge s)}\Big(l_i \mathbf{1}_{\{t<s\}}(t) + w(X_s)\mathbf{1}_{\{t>s\}}(t) + m_i(X_t)\mathbf{1}_{\{t=s\}}(t)\Big)\Bigg] dG_{-i}(s)dG_i(t) | X_0 = x\Bigg],
\end{aligned}
$$

where $m_i(x) := (l_i + w(x))/2$. The first line represents the firm's discounted flow payoff until either firm exits. The second line captures the lump-sum payoff from becoming the winner, becoming the loser, and exiting simultaneously, respectively.

A strategy profile $\{a_1^*, a_2^*\}$ is a Markov Perfect equilibrium (MPE) if

$$
V_i(x; a_i^*, a_{-i}^*) \geq V_i(x; a_i, a_{-i}^*)
$$

for each firm $i$, every initial value $x$, and every strategy $a_i$.

# 3 Equilibrium Analysis

In Section 3.1, we characterize the best response of a firm that anticipates its rival will never exit, which is instrumental for the equilibrium analysis. We then characterize pure-strategy MPE in Section 3.2, and in Section 3.3, we consider mixed-strategy MPE. In particular, we establish necessary conditions that any mixed-strategy MPE must satisfy, and our first main result follows immediately: The game has no mixed-strategy MPE if the firms have heterogeneous exit payoffs (*i.e.,* $l_1 \neq l_2$).

## 3.1 Best Response to a Firm which Never Exits

We characterize firm $i$'s best response assuming that its rival never exits; *i.e.,* the best response to $a_{-i} = (\emptyset, \mathbf{0})$. In this case, the firm's best response can be determined by solving a single-player optimal stopping problem as in Alvarez (2001). Because $X$ is a time-homogeneous process and the time horizon is infinite, it is without loss of generality to restrict attention to pure strategies such that $\lambda_i = \mathbf{0}$, and the firm's expected payoff at $t$ depends solely on the current value of the state $x = X_t$.[7] Thus, it can be expressed as

$$
\sup_{\tau_i \geq t} \mathbb{E}\left[\int_t^{\tau_i} e^{-r(s-t)}\pi(X_s)ds + e^{-r(\tau_i - t)} l_i | X_t = x\right]. \tag{3.1}
$$

[7] Note that any strategy with $\lambda_i \neq \mathbf{0}$ *mixes* pure strategies. Thus, if a strategy with $\lambda_i \neq \mathbf{0}$ is a best response to $a_{-i} = (\emptyset, \mathbf{0})$, then there must exist more than one stopping times that are solutions of the single-player optimal stopping problem, (3.1). However, this optimal stopping problem admits a unique solution, which is a hitting time, given in Lemma 1 (*e.g.*, see Alvarez (2001); Arkin (2015)).

Using Proposition 2 in Alvarez (2001), we can characterize the firm's optimal exit region as follows.

**Lemma 1** *Suppose firm $-i$ never exits. There exists a unique threshold $\theta_i^*$ such that $E_i^* = \{X_t \leq \theta_i^*\}$ is optimal for firm i, that is, firm i optimally exits whenever $X_t \leq \theta_i^*$. If $l_1 < l_2$, then $\theta_1^* < \theta_2^*$.*

The proof is relegated to Online Appendix F. Intuitively, a firm's value of remaining in the market decreases as the market conditions deteriorate, and once they become sufficiently poor, the firm is better off exiting and collecting its outside option. As the firms earn identical flow payoffs while in the market, the firm with the higher outside option optimally exits at a higher threshold.

### 3.2 Pure-strategy MPE

The following proposition shows that there is a pure-strategy MPE in which firm 2 exits at the first moment that $X_t$ drifts below $\theta_2^*$, and firm 1 never exits. Moreover, if the firms are not too heterogeneous, there is another pure-strategy MPE in which firm 1 exits at the first moment that $X_t \leq \theta_1^*$ and firm 2 never exits.

**Proposition 1**
*(i) The strategy profile $\{a_1, a_2\} = \{(\emptyset, \mathbf{0}), (E_2^*, \mathbf{0})\}$ is a pure-strategy MPE, where $E_i^* = (\alpha, \theta_i^*]$ and $\theta_i^*$ is given in Lemma 1.*
*(ii) There exists a threshold $\kappa > 0$ that is independent of $l_1$ such that $\{a_1, a_2\} = \{(E_1^*, \mathbf{0}), (\emptyset, \mathbf{0})\}$ is also a pure-strategy MPE if $|l_2 - l_1| < \kappa$.*
*(iii) If $X_0 \geq \max\{\theta_1^*, \theta_2^*\}$, every pure-strategy MPE is payoff-/outcome-equivalent to one of the above.*

The proof is provided in Appendix C. If firm $i$ expects its rival to never exit, then by Lemma 1, it will optimally exit at the first time such that $X_t \leq \theta_i^*$. Therefore, it suffices to show that if firm $i$ employs the exit region $E_i^*$, then its opponent's best response is to never exit.

Suppose that firm 1 expects its rival to exit at the first moment that $X_t \leq \theta_2^*$. Recall that firm 2 has a better outside option than firm 1 (*i.e.*, $l_2 \geq l_1$), so by Lemma 1, $\theta_1^* \leq \theta_2^*$, which implies that firm 1 has no incentive to exit until at least $X_t \leq \theta_1^*$. Therefore, firm 1 expects that the game will end before the state hits $\theta_1^*$, and hence the strategy of never exiting is incentive compatible. If instead firm 2 anticipates that its rival chooses $(E_1^*, \mathbf{0})$, then the strategy $a_2 = (\emptyset, \mathbf{0})$ is incentive compatible as long as it does not need to wait too long until $X_t$ hits $\theta_1^*$ and firm 1 exits. As a result, never exiting is a best response for firm 2 as long as $|l_2 - l_1|$, and hence $\theta_2^* - \theta_1^*$ is not too large.

If $X_0 < \max\{\theta_1^*, \theta_2^*\}$ and $\sigma(\cdot)$ is sufficiently large, as shown in Proposition 5 in Murto (2004), there may also exist pure-strategy MPE with multiple exit thresholds. As such equilibria do not affect our analysis of mixed-strategy equilibria, we do not consider them here. Finally, because along the path of any equilibrium characterized in Proposition 1, at most one player resists exiting below his single-player optimal threshold, it follows that both equilibria are Pareto-efficient.

### 3.3 Mixed-strategy MPE

We now consider mixed-strategy MPE. First, we define the support of firm $i$'s mixed strategy as the subset of the state space in which firm $i$ randomizes between remaining in the market and exiting,

$$\Gamma_i = \{x \in \mathcal{X} : \lambda_i(x) > 0\}\,. \tag{3.2}$$

The support $\Gamma_i$ can be represented as a union of open intervals in $\mathcal{X}$ because of the regularity condition imposed in Section 2.1 that $\lambda_i(X_t)$ is Riemann integrable over time intervals (see Lemma B.2 for the proof of this statement). Intuitively, this property of $\Gamma_i$ implies that whenever $X_t \in \Gamma_i$, firm $i$ exits with probability $\lambda_i(X_t)dt > 0$ during the time interval $[t, t+dt)$ and continues to randomize its decision until $\tau_{\Gamma_i} := \inf\{s \geq t : X_s \notin \Gamma_i\}$.

The following lemma shows that the firms' mixed strategies must have common support and neither firm exits with probability 1.

**Lemma 2** *Suppose that $\sigma(\cdot) > 0$, and $\{a_1, a_2\}$ constitutes a mixed-strategy MPE. Then the firms' mixed strategies have common support $\Gamma = (\alpha, \theta_1^*) = (\alpha, \theta_2^*)$, where $\theta_i^*$ is given in Lemma 1, and $E_1 = E_2 = \emptyset$.*

We give a sketch of the proof below, while the formal proof is relegated to Appendix C. At any time $t$ such that $X_t \in \Gamma_i$, firm $i$ must be indifferent between exiting immediately and remaining in the market, which implies that its expected payoff must be equal to its outside option, that is,

$$l_i = \lambda_{-i}(X_t)dt\, w(X_t) + (1 - \lambda_{-i}(X_t)dt)\left[\pi(X_t)dt + (1 - rdt)l_i\right]. \tag{3.3}$$

The left-hand-side of (3.3) represents firm $i$'s payoff if it exits at $t$, while the right-hand side represents its payoff if it remains. To be specific, with probability $\lambda_{-i}(X_t)dt$, it receives the winner's payoff, $w(X_t)$, whereas with the complementary probability, it earns the flow payoff $\pi(X_t)$ during $(t, t+dt)$, and its (discounted) continuation profit, $l_i$, at $t+dt$.[8] Thus, firm $-i$'s exit rate must satisfy

$$\lambda_{-i}(X_t) = \frac{rl_i - \pi(X_t)}{w(X_t) - l_i}\,. \tag{3.4}$$

Note that $\pi(x) < rl_i$ for any $x \in \Gamma_i$.[9] We shall now argue that $\Gamma_1 = \Gamma_2$. Towards a contradiction, suppose that there exists a non-empty interval that is a subset of $\Gamma_i$ but not of $\Gamma_{-i}$. Then for any $x$ in that interval, $\pi(x) < rl_i$ and $\lambda_{-i}(x) = 0$, because by assumption, $x \in \Gamma_i$ and $x \notin \Gamma_{-i}$, respectively.

[8] We ignore the event that both firms exit simultaneously. As the proof shows, this is an innocuous simplification.

[9] If $\pi(X_t) > rl_i$, then the right-hand-side of (3.3) is strictly larger than $l_i$, so firm $i$ strictly prefers to remain in the market regardless of its rival's strategy.

This implies that the right-hand-side of (3.3) is strictly smaller than $l_i$, so firm $i$ strictly prefers to exit, which contradicts that $x \in \Gamma_i$. Hence, we conclude that $\Gamma_i \backslash \Gamma_{-i}$ is empty, and so $\Gamma_1 = \Gamma_2$.

Next, recall that even if firm $i$ anticipates that its rival will never exit, it is unwilling to exit until $X_t$ hits $\theta_i^*$. Hence, if this firm expects its rival to exit with positive probability, then *ceteris paribus*, this *decreases* its incentive to exit. Consequently, firm $i$ *always* strictly prefers to remain in the market whenever $X_t > \theta_i^*$, which, together with the fact that $\Gamma$ is open, implies that $\Gamma \subseteq (\alpha, \theta_i^*)$.

We now argue that in a mixed-strategy MPE, neither firm exits with probability 1, that is, $E_1 = E_2 = \emptyset$. Towards a contradiction, suppose that $E_i \neq \emptyset$ for some $i$. Because exiting at any $X_t > \theta_i^*$ is a strictly dominated strategy for firm $i$, it must be the case that $E_i \subseteq (\alpha, \theta_i^*]$. Moreover, because firm $-i$ strictly prefers to remain in the market when $X_t$ is sufficiently close to $E_i$ (anticipating that firm $i$ will soon exit with probability one), it must also be the case that $E_i$ and $\Gamma$ are disjoint and separated by a non-empty interval $(c,d)$ wherein neither firm exits. As both $E_i$ and $\Gamma$ are subsets of $(\alpha, \theta_i^*]$, so must be $(c,d)$. Then because $\pi(x) < rl_i$ for any $x \in (c,d) \subseteq (\alpha, \theta_i^*]$, firm $i$ strictly prefers to exit instantaneously if $X_t \in (c,d)$, instead of waiting until the state hits $E_i$ or $\Gamma$, contradicting the premise that $a_i$ is a best response to $a_{-i}$. Hence, we conclude that $E_i = \emptyset$.

We have already argued that $\Gamma \subseteq (\alpha, \theta_i^*)$. It remains to argue that this inclusion is an equality. Suppose that $\Gamma = (\alpha, \theta)$ for some $\theta < \theta_i^*$. Because firm $-i$ does not exit at any $X_t > \theta$, firm $i$'s expected payoff at any $x \in (\theta, \theta_i^*)$ from exiting at the first time that $X_t \leq \theta$ is strictly less than $l_i$ by Lemma 1, which implies that this firm strictly prefers to exit instantaneously–a contradiction.

Because $\theta_1^* < \theta_2^*$ whenever $l_1 < l_2$ by Lemma 1, we have the following immediate implication.

**Theorem 1** *Suppose that* $\sigma(\cdot) > 0$ *and* $l_1 < l_2$. *Then the game admits no mixed-strategy MPE.*

This theorem, together with Proposition 1, shows that if there is even a small amount of uncertainty about the payoff from remaining in the market and the firms are even slightly heterogeneous, then none of the MPE feature *attrition*, *i.e.,* both firms resisting exit below their single-player-optimal thresholds.

Both conditions in Theorem 1 are necessary to eliminate mixed-strategy MPE. If the firms are homogeneous (*i.e.,* $l_1 = l_2$) or payoffs are deterministic (*i.e.,* $\sigma(\cdot) \equiv 0$), then as shown in Steg (2015) and Hendricks et al. (1988), respectively, and, for completeness, as we show in Appendix A, the game admits MPE in both pure and mixed strategies.

The key driver behind this result is that the state follows an irreducible stochastic process. If the state is deterministic, as in Hendricks et al. (1988) for example, then in any mixed-strategy equilibrium, the firm with the smaller outside option (firm 1) exits with a positive probability when the state hits its single-player optimal threshold, and from then onward, during every interval of length $dt$, each firm exits with some probability that is proportional to $dt$ and makes the other firm indifferent between exiting and not. If the state is stochastic, any hypothetical candidate mixed-strategy MPE must also be of the form described above. However, when the state is just below that threshold, due to the irreducibility of the state, with some likelihood, it will hit that threshold in short order and firm 1 will exit. As a result, firm 2 strictly prefers to not exit near that threshold, which in turn leads firm 1 to strictly prefer to exit, leading to a pure-strategy MPE.

A natural concern is the restrictiveness of Markov strategies. Toward investigating this, we consider the possibility that firms condition their exit decision on the history. Under a set of restrictions on the firms' strategies, we show that with heterogeneous outside options, the game admits no mixed-strategy Subgame Perfect equilibria in Online Appendix G. To elaborate on the restrictions, note that a non-Markovian strategy consists of (i) a set of stopping times at which the firm exits with positive probability, and (ii) an *exit rate* function, specifying the probability that the firm exits during an interval $(t, t+dt)$, which depends on the history of the state up to $t$. Then our restriction imposes that the firms exit with positive probability at no more than finitely-many stopping times, they do not exit with probability one, and the exit rate function satisfies a regularity condition analogous to that in Markov strategies. We remark that these restrictions are satisfied by the strategies in the mixed-strategy SPE that appear in the literature when the state evolves deterministically (Hendricks et al., 1988), or the firms are homogeneous (Steg, 2015), or both (Tirole, 1988).[10]

## 4 Discussion

We consider a two-player war of attrition under complete information. Our main result shows that if the players are heterogeneous and their flow payoffs whilst in war follow a diffusion process, then the game admits no mixed-strategy MPE. We also extend this non-existence result to a class of Subgame Perfect equilibria (subject to a set of restrictions), where the players' strategies can depend on the entire history. Because the pure-strategy equilibria are Pareto-efficient, these results indicate that an arguably natural perturbation of the canonical model eradicates equilibria that possess a central feature of wars of attrition—the waste of valuable resources, suggesting that the complete-information model may be unsuitable for studying this class of problems.

Much of the recent theoretical and empirical literature on wars of attrition has focused on asymmetric-information models (*e.g.,* Myatt (2005), Wang (2009) and Takahashi (2015)), which admit equilibria that feature attrition. However, as these equilibria are often obtained by purifying mixed strategies in the complete-information game, it is an open question whether there exist any equilibria that feature attrition in an incomplete-information counterpart of our model. We view this as a promising avenue for future research.

[10] If the state $X$ always drifts downward, then there is a one-to-one correspondence between the state $X$ and time $t$, which implies that every SPE is also an MPE, i.e., the class of SPEs coincides with that of MPEs.

## References


Abreu, D., F. Gul. 2000. Bargaining and reputation. *Econometrica* **68**(1) 85–117.

Alvarez, L. H. R. 2001. Reward functionals, salvage values and optimal stopping. *Mathematical Methods of Operations Research* **54** 315–337.

Arkin, V. I. 2015. Threshold strategies in optimal stopping problem for one-dimensional diffusion processes. *Theory of Probability and Its Applications* **59**(2) 311–319.

Becker, G. S. 1983. A theory of competition among pressure groups for political influence. *Quarterly Journal of Economics* **98**(3) 371–400.

Bilodeau, M., A. Slivinski. 1996. Toilet cleaning and department chairing: Volunteering a public service. *Journal of Public Economics* **59**(2) 299–308.

Bliss, C., B. Nalebuff. 1984. Dragon-slaying and ballroom dancing: The private supply of a public good. *Journal of Public Economics* **25**(1-2) 1–12.

Borodin, A., P. Salminen. 1996. *Handbook of Brownian motion - Facts and Formulae*. Birkhauser, Basel.

Bulow, J., P. Klemperer. 1999. The generalized war of attrition. *American Economic Review* **89**(1) 175–189.

Dutta, P. K., A. Rustichini. 1993. A theory of stopping time games with applications to product innovations and asset sales. *Economic Theory* (3) 743–763.

Folland, G. B. 1999. *Real Analysis*. Wiley, New York.

Fudenberg, D., J. Tirole. 1986. A theory of exit in duopoly. *Econometrica* **54**(4) 943–960.

Fudenberg, D., J. Tirole. 1991. *Game Theory*. The MIT Press.

Ghemawat, P. 1997. *Games businesses play: cases and models*. The MIT Press.

Ghemawat, P., B. Nalebuff. 1985. Exit. *Rand Journal of Economics* **16**(2) 184–194.

Greenhouse, S. 1999. A 15-month labor dispute turns into a war of attrition. *New York Times* Retrieved from http://mobile.nytimes.com/1999/12/26/us/a-15-month-labor-dispute-turns-into-a-war-of-attrition.html.

Hendricks, K., A. Weiss, C. Wilson. 1988. The war of attrition in continuous time with complete information. *International Economic Review* **29**(4) 663–680.

Kambe, S. 1999. Bargaining with imperfect commitment. *Games and Economic Behavior* **28** 2177–2237.

Karatzas, I., S. E. Shreve. 1991. *Brownian Motion and Stochastic Calculus*. Springer-Verlag Berlin.

Kornhauser, L., A. Rubinstein, C. Wilson. 1989. Reputation and patience in the war of attrition. *Economica* **56**(221) 15–24.

Krishna, V., J. Morgan. 1997. An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory* **72** 343–362.

Levin, J. 2004. Wars of attrition. *Lecture Note* .

Lon, P. C., M. Zervos. 2011. A model for optimally advertising and launching a product. *Mathematics of Operations Research* **36**(2) 363–376.

Maskin, E., J. Tirole. 1988. A theory of dynamic oligopoly, ii: Price competition, kinked demand curves, and edgeworth cycles. *Econometrica* **56**(3) 571–599.

Maynard Smith, J. 1974. Theory of games and the evolution of animal conflicts. *Journal of Theoretical Biology* **47** 209–221.

McAfee, R. P. 2009. *Competitive Solutions: The Strategist's Toolkit*. Princeton University Press.

McKean, J., H.P. 1965. Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics. *Industrial Management Review* **2**(6) 32–39.

Merton, R. 1973. Theory of rational option pricing. *Bell Journal of Economics and Management Science* **4** 141–183.

Murto, P. 2004. Exit in duopoly under uncertainty. *RAND Journal of Economics* **35**(1) 111–127.

Myatt, D. P. 2005. Instant exit from the asymmetric war of attrition. *Working Paper* .

Newman, R. 2011. Why barnes & noble is thumping borders. *US News* Retrieved from http://money.usnews.com/money/blogs/flowchart/2011/01/10/why-barnes–noble-is-thumping-borders.

Oksendal, B. 2003. *Stochastic Differential Equations: An Introduction with Applications*. Springer.

Reed, S. 2016. Saudi arabia keeps pumping oil, despite financial and political risks. *New York Times* Retrieved from http://www.nytimes.com/2016/01/28/business/energy-environment/saudi-arabia-keeps-pumping-oil-despite-financial-and-political-risks.html.

Revuz, D., M. Yor. 1991. *Continuous Martingales and Brownian Motion*. Springer-Verlag Berlin Heidelberg.

Riedel, F., J.-H. Steg. 2017. Subgame-perfect equilibria in stochastic timing games. *Journal of Mathematical Economics* **72** 36–50.

Rogers, L. C. G., D. Williams. 2000. *Diffusions, Markov Processes and Martingales*, vol. 1. 2nd ed. Cambridge University Press, Cambridge, UK.

Samuelson, P. 1965. Rational theory of warrant pricing. *Industrial Management Review* **6**(2) 13–31.

Seel, C., P. Strack. 2016. Continuous time contests with private information. *Mathematics of Operations Research* **41**(3) 1093–1107.

Steg, J.-H. 2015. Symmetric equilibria in stochastic timing games. *Center for Mathematical Economics Working Papers* .

Takahashi, Y. 2015. Estimating a war of attrition: The case of the us movie theater industry. *American Economic Review* **105**(7) 2204–2241.

Tirole, J. 1988. *The Theory of Industrial Organization*. The MIT Press.

Touzi, N., N. Vieille. 2002. Continuous-time dynkin games with mixed strategies. *SIAM Journal on Control and Optimization* **41**(4) 1073–1088.

Wang, Z. 2009. (mixed) strategy in oligopoly pricing: Evidence from gasoline price cycles before and under a timing regulation. *Journal of Political Economy* **117**(6) 987–1030.

# A Mixed Strategy MPE in two Special Cases ($l_1 = l_2$ or $\sigma(\cdot) \equiv 0$)

Recall that if $\sigma(\cdot) > 0$ and $l_1 < l_2$, then the game admits no mixed-strategy MPE (Theorem 1). In this section, we show that if either of these conditions is removed, a mixed-strategy MPE does exist.

First, let us consider the case in which the firms are homogeneous (*i.e.*, $l_1 = l_2$) and $\sigma(\cdot) > 0$. It follows from Lemma 1 that $\theta_1^* = \theta_2^*$. Following Steg (2015), it is easy to show that the strategies $a_1 = (\emptyset, \lambda_1(\cdot))$ and $a_2 = (\emptyset, \lambda_2(\cdot))$, where

$$\lambda_i(x) := \mathbb{I}_{\{x \leq \theta_1^*\}} \frac{rl_{-i} - \pi(x)}{w(x) - l_{-i}}, \tag{A.1}$$

constitute a mixed strategy MPE. Note that $\lambda_1 = \lambda_2$ in this case because $l_1 = l_2$.

Next, we consider the case in which $X$ evolves deterministically (*i.e.*, $\sigma(\cdot) \equiv 0$) and $l_1 < l_2$. From Lemma 1, we have that $\theta_1^* < \theta_2^*$. Following Hendricks et al. (1988), we let $\mu(\cdot) \leq 0$, i.e., the market condition always goes down over time. Then unlike the case in which $\sigma(\cdot) > 0$ considered in Section 3.3, when $X$ is deterministic, it is not without loss of generality to assume that $p_i(x) = 1$, i.e., $p_i(x) < 1$ is possible in Markov strategies. Let $E_1^* = (\alpha, \theta_1^*]$ with $p_1(x) = q_1 \in (0,1)$ for all $x \in E_1^*$. Consider the strategies $a_1 = (E_1^*(p_1), \lambda_1(\cdot))$ and $a_2 = (\emptyset, \lambda_2(\cdot))$, where $E_1^*(p_1)$ indicates the exit probability $p_1$ whenever $X_t \in E_1^*$ and $\lambda_i$ is given in (A.1). Because $X$ always moves downwards, the strategies defined above are Markov. Using similar arguments to Hendricks et al. (1988), one can show that if $|l_1 - l_2|$ is not too large, then there exists a $q_1 \in (0,1)$ such that $(a_1, a_2)$ constitutes a mixed strategy MPE.

## B Mathematical Supplement

This section provides supplementary lemmas that are used to prove the results (lemmas and propositions) in the body of the manuscript.

We first define the following functions that will be used later.

$$R(x) := \mathbb{E}^x \left[ \int_0^\infty \pi(X_t) e^{-rt} dt \right], \tag{B.1}$$

$$\beta_i(x) := \frac{l_i - R(x)}{\phi(x)}, \tag{B.2}$$

where $\phi : \mathcal{X} \to \mathbb{R}$ satisfies the differential equation[11] $\frac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) - r\phi(x) = 0$ with the properties of $\phi(\cdot) > 0$ and $\phi'(\cdot) < 0$. The function $R(\cdot)$ is well-defined because we assume that $\pi(\cdot)$ satisfies the absolute integrability condition in Section 2. The following lemma establishes some properties of the function $\beta_i$. This lemma will be used to prove Lemma 1 and Proposition 1.

**Lemma B.1** *The function $\beta_i(x)$ has a unique interior maximum at $\theta_i^* \leq x_{ci}$ where $\pi(x_{ci}) = rl_i$. Furthermore, $\beta_i'(x) > 0$ for $x < \theta_i^*$ and $\beta_i'(x) < 0$ for $x > \theta_i^*$.*

**Proof of Lemma B.1**: To prove this lemma, it is enough to examine the behavior of the first derivative of $\beta_i(x) = [l_i - R(x)]/\phi(x)$.

According to the theory of diffusive processes (Alvarez, 2001, p.319), the function $R(\cdot)$, given

[11]This second-order linear ordinary differential equation (ODE) always has two linearly independent fundamental solutions, one of which is monotonically decreasing (see Alvarez, 2001, p.319). Note that if $f(\cdot)$ solves this equation, then so does $cf(\cdot)$ for any constant $c \in \mathbb{R}$ because it is a homogeneous equation. Hence, we can always find the one which is always positive.

in (B.1), can be expressed as

$$R(x) = \frac{\phi(x)}{B}\int_a^x \psi(y)\pi(y)m^{'}(y)dy + \frac{\psi(x)}{B}\int_x^b \phi(y)\pi(y)m^{'}(y)dy\,. \tag{B.3}$$

Here, $a$ and $b$ are the two boundaries of the state space $\mathcal{X}$, $\psi(\cdot)$ and $\phi(\cdot)$ are the increasing and decreasing fundamental solutions to the differential equation $\frac{1}{2}\sigma^2(x)f^{''}(x)+\mu(x)f^{'}(x)-rf(x)=0$, $B=[\psi^{'}(x)\phi(x)-\psi(x)\phi^{'}(x)]/S^{'}(x)$ is the *constant* Wronskian determinant of $\psi(\cdot)$ and $\phi(\cdot)$, $S^{'}(x)=\exp(-\int 2\mu(x)/\sigma^2(x)dx)$ is the density of the scale function of $X$, and $m^{'}(y)=2/[\sigma^2(y)S^{'}(y)]$ is the density of the speed measure of $X$.

By virtue of (B.3), differentiation of $R(x)$ with respect to $x$ leads to

$$R^{'}(x)\phi(x)-R(x)\phi^{'}(x) = S^{'}(x)\int_x^b \phi(y)\pi(y)m^{'}(y)dy\,. \tag{B.4}$$

Moreover, because $l_i=\mathbb{E}^x[\int_0^\infty rl_ie^{-rt}dt]$, we can write

$$R(x)-l_i = \mathbb{E}^x\left[\int_0^\infty [\pi(X_t)-rl_i]e^{-rt}dt\right], \tag{B.5}$$

which implies that we can treat the functional $R(x)-l_i$ as the expected cumulative present value of a flow payoff $\pi(\cdot)-rl_i$. Combining (B.4) and (B.5), therefore, we obtain

$$\beta_i^{'}(x) = -\frac{R^{'}(x)\phi(x)-[R(x)-l_i]\phi^{'}(x)}{\phi^2(x)} = -\frac{S^{'}(x)}{\phi^2(x)}\int_x^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy\,. \tag{B.6}$$

Now, because $\pi(\cdot)$ is strictly increasing and $\pi(x_{ci})=rl_i$, it must be the case that $\pi(x)<rl_i$ for $x<x_{ci}$ and $\pi(x)>rl_i$ for $x>x_{ci}$. Thus, $\beta_i^{'}(x)<0$ for all $x>x_{ci}$. Note also that if $x<K<x_{ci}$, then

$$\begin{aligned}\int_x^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy &= \int_x^K \phi(y)[\pi(y)-rl_i]m^{'}(y)dy + \int_K^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy \\ &\leq \frac{[\pi(K)-rl_i]}{r}\left(\frac{\phi^{'}(K)}{S^{'}(K)}-\frac{\phi^{'}(x)}{S^{'}(x)}\right)+\int_K^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy \to -\infty\,,\end{aligned}$$

as $x\downarrow a$ because $a$ is a natural boundary, which implies that $\lim_{x\downarrow a}\beta_i^{'}(x)=\infty$. Here we use $\phi^{'}(x)<0$ and $\pi(x)<\pi(K)<rl_i$ for $x<K$. It thus follows that $\beta_i^{'}(\theta_i^*)=0$ for some $\theta_i^*\leq x_{ci}$, which implies that $\int_{\theta_i^*}^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy=0$ because $S^{'}(x)>0$ and $\phi(x)>0$ in (B.6). Moreover, note that $\int_x^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy$ is increasing in $x<x_{ci}$ because $\pi(y)<rl_i$ for $\forall y<x_{ci}$, thus yielding $\int_x^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy<0$ if $x<\theta_i^*\leq x_{ci}$ and $\int_x^b \phi(y)[\pi(y)-rl_i]m^{'}(y)dy>0$ if $\theta_i^*<x\leq x_{ci}$. Combining this with (B.6), we obtain the unique existence of $\theta_i^*$ such that $\beta_i^{'}(x)>0$ for $\forall x<\theta_i^*$ and

$\beta_i'(x) < 0$ for $\forall x > \theta_i^*$, which completes the proof. ■

Recall that we define the support $\Gamma_i = \{x \in \mathcal{X} : \lambda_i(x) > 0\}$ for the hazard rate function $\lambda_i$ where we impose the regularity condition that $\lambda_i(X_t)$ is Riemann integrable over time intervals. In the following lemma, we prove that the regularity condition on $\lambda_i$ implies $\Gamma_i = \{x \in \mathcal{X} : \lambda_i(x) > 0\}$ is a union of open intervals, hence it is an open set. This lemma will be used to prove Lemma 2.

**Lemma B.2** *If $\lambda_i(X_t)$ is Riemann integrable over any bounded time interval $[u, v]$, then $\Gamma_i$ can be represented as a union of open intervals in the state space $\mathcal{X}$.*

**Proof of Lemma B.2:** For notational simplicity, we let $\lambda_{i,t} := \lambda_i(X_t)$. For any given bounded time interval $[u, v]$, if $\lambda_{i,t}$ is Riemann integrable for a given sample path of $X_{u \leq t \leq v}$, then $\lambda_{i,t}$ must be bounded within the interval $[u, v]$. Then by virtue of the Riemann-Lebesgue theorem (Folland 1999, p. 57), the set $D_i := \{t \in [u, v] : \lambda_{i,t} \text{ is discontinuous}\}$ has Lebesgue measure zero. Hence, we can construct an equivalent version of $\lambda_{i,t}$ where $\lambda_{i,t} = 0$ whenever $t \in D_i$; because $\int_{D_i} \lambda_{i,t} dt = 0$ even if $\lambda_{i,t} > 0$ for all $t \in D_i$, this transformation does not affect either the firm's payoff or the outcome of the game. By construction, the transformed process $\{\lambda_{i,t}\}$ is continuous for all $t$ at which $\lambda_{i,t} > 0$. Therefore, $C_i := \{t \in [u, v] : \lambda_{i,t} > 0\}$ is an open set of the real line $\mathbb{R}$, and hence, a countable union of disjoint open intervals of time.

Next, because $X_t$ is a continuous process in time $t$, the set $\{X_t : t \in C_i\}$, which is the mapping of the set $C_i$ into the state space via $X$, is a union of disjoint intervals, open or closed; it may contain point sets, but without loss of generality, we may ignore them because these possibilities are zero-probability events. Note that $\{X_t : t \in C_i\} \subseteq \Gamma_i$ by the definition of $C_i$ and $\Gamma_i$. Hence, $\Gamma_i$ is a union of disjoint intervals in the state space $\mathcal{X}$ because otherwise there would be some time interval $[u, v]$ and some sample path $X$ such that $C_i$ cannot be represented as a union of time intervals.

As a final step, we can equivalently represent $\Gamma_i$ as a union of disjoint *open* intervals by taking its interior (i.e., removing all the boundary points of $\Gamma_i$, if any) without affecting the equilibrium payoffs because $X$ is a regular diffusion process: Let $B_i$ be the set of all boundary points of $\Gamma_i$. Then because $\Gamma_i$ is a union of disjoint intervals in the real line $\mathbb{R}$, the cardinality of $B_i$ is at most countably infinite, which implies that $B_i$ has Lebesgue measure zero. Hence, we have $\int_0^t \mathbf{1}_{\{x \in B_i\}}(X_s) \lambda_i(X_s) ds = 0$ because $X$ is a regular diffusion process and neither strategy nor associated payoff is affected by the removal of any of the boundary points of $\Gamma_i$. ■

## C Proofs

**Proof of Proposition 1**:

(i) Define $\tau_i^* := \inf\{t \geq 0 : X_t \in E_i^*\}$, $i \in \{1, 2\}$, where $E_i^* = (\alpha, \theta_i^*]$ is given in Section 3.2. For expositional convenience, we also let $\tau(E_i) := (E_i, \mathbf{0})$ for each $i \in \{1, 2\}$. We first prove that

$\{a_1, a_2\} = \{\tau(\emptyset), \tau(E_2^*)\}$ is an MPE. Because it is shown in Lemma 1 that $a_2 = \tau(E_2^*)$ is firm 2's best response to $a_1 = \tau(\emptyset)$, it only remains to prove that $a_1 = \tau(\emptyset)$ is also firm 1's best response to $a_2 = \tau(E_2^*)$.

Let $\tau(E_1)$ be firm 1's best response to $\tau(E_2^*)$ and $V_{W1}^*(x) := V_1(x; \tau(E_1), \tau(E_2^*))$ be the corresponding payoff to firm 1. We let $C_1 = \mathcal{X} \backslash E_1$ denote the continuation region associated with the strategy $\tau(E_1)$.

First, we show that $E_1 \cap (\theta_2^*, \infty) = \emptyset$. Toward a contradiction, suppose this is not the case. Then pick some $x \in E_1 \cap (\theta_2^*, \infty)$ and observe that $V_{W1}^*(x) = l_1$ due to $x \in E_1$. However,

$$V_{W1}^*(x) \geq V_1(x; \tau(\emptyset), \tau(E_2^*)) = \mathbb{E}^x\left[\int_0^{\tau_2^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_2^*}^x) e^{-r\tau_2^*}\right] = R(x) + \left[\frac{w(\theta_2^*) - R(\theta_2^*)}{\phi(\theta_2^*)}\right]\phi(x)$$
$$> R(x) + \left[\frac{l_1 - R(\theta_2^*)}{\phi(\theta_2^*)}\right]\phi(x) = R(x) + \beta_1(\theta_2^*)\phi(x) > R(x) + \beta_1(x)\phi(x) = l_1\,,$$

where the first inequality follows because $w(X_{\tau_2^*}^x) = w(\theta_2^*) > l_1$ and $\mathbb{E}^x[e^{-r\tau_2^*}] = \phi(x)/\phi(\theta_2^*)$ for $x > \theta_2^*$, and the second inequality holds because $x > \theta_2^* > \theta_1^*$ and $\beta_1'(x) < 0$ for $x > \theta_1^*$ by Lemma B.1. This establishes the contradiction.

Second, we also prove that $E_1 \cap (-\infty, \theta_2^*] = \emptyset$. Towards a contradiction, suppose this is not the case. Then we can pick some $x \in E_1 \cap (-\infty, \theta_2^*]$ such that $V_{W1}^*(x) = m_1(x)$ because $\tau_2^* = \inf\{t \geq 0 : X_t \in E_2^*\}$. However,

$$V_{W1}^*(x) \geq V_1(x; \tau(\emptyset), \tau(E_2^*)) = \mathbb{E}^x\left[\int_0^{\tau_2^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_2^*}^x) e^{-r\tau_2^*}\right] = w(x) > m_1(x)\,,$$

where the second equality uses that $\tau_2^* = 0$ when $X_0 = x \leq \theta_2^*$. This establishes the contradiction. Hence, we can conclude that $E_1 = \emptyset$ and $C_1 = \mathcal{X}$, which implies that $\tau(E_1) = \tau(\emptyset)$.

(ii) Next, we prove the conditions under which $\{a_1, a_2\} = \{\tau(E_1^*), \tau(\emptyset)\}$ is an MPE. Consider the following condition:

$$V_2(x; \tau(E_1^*), \tau(\emptyset)) = \mathbb{E}^x\left[\int_0^{\tau_1^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_1^*}^x) e^{-r\tau_1^*}\right] > l_2 \quad \text{for all } x \in (\theta_1^*, \theta_2^*]\,. \tag{C.1}$$

First, we prove that (C.1) is a sufficient condition for $\{a_1, a_2\} = \{\tau(E_1^*), \tau(\emptyset)\}$ to be an MPE. Let $\tau(E_2)$ be firm 2's best response to $\tau(E_1^*)$, i.e., $V_{W2}^*(x) := V_2(x; \tau(E_1^*), \tau(E_2))$ be the corresponding payoff. We let $C_2 = \mathcal{X} \backslash E_2$ denote the continuation region associated with the strategy $\tau(E_2)$.

We now claim that $E_2 \cap (\theta_2^*, \infty) = \emptyset$. Towards a contradiction, suppose not. Then we can pick some $x \in E_2 \cap (\theta_2^*, \infty)$, which implies that $V_{W2}^*(x) = l_2$. However, because $\tau_1^* > \tau_2^*$ when $X_0 = x$, Lemma 1 implies that firm 2 could obtain a *strictly* higher payoff by exiting at $\tau_2^* > 0$ instead, i.e.,

$$V_{W2}^*(x) \geq V_2(x;\tau(E_1^*),\tau(E_2^*)) = \mathbb{E}^x\left[\int_0^{\tau_2^*}\pi(X_t)e^{-rt}dt + l_2 e^{-r\tau_2^*}\right] > l_2\,,$$

which is a contradiction. We next claim that $E_2 \cap (\theta_1^*, \theta_2^*] = \emptyset$. Towards a contradiction, suppose not. Then we can pick $x \in E_2 \cap (\theta_1^*, \theta_2^*]$, which implies that $V_{W2}^*(x) = l_2$. However, we have

$$V_{W2}^*(x) \geq V_2(x;\tau(E_1^*),\tau(\emptyset)) = \mathbb{E}^x\left[\int_0^{\tau_1^*}\pi(X_t)e^{-rt}dt + w(X_{\tau_1^*}^x)e^{-r\tau_1^*}\right] > l_2\,,$$

where the last inequality follows from (C.1). This establishes the contradiction. We further claim that $E_2 \cap (-\infty, \theta_1^*] = \emptyset$. If not, then there exists $x \in E_2 \cap (-\infty, \theta_1^*]$, which implies that both firms exit simultaneously when $X_t^y = x$, and hence, $V_{W2}^*(x) = m_2(x)$. Because $\tau_1^* = 0$ when $X_0 = x \leq \theta_1^*$, we have

$$V_{W2}^*(x) \geq V_2(x;\tau(E_1^*),\tau(\emptyset)) = \mathbb{E}^x\left[\int_0^{\tau_1^*}\pi(X_t)e^{-rt}dt + w(X_{\tau_1^*}^x)e^{-r\tau_1^*}\right] = w(x) > m_2(x)\,,$$

which is a contradiction. Combining the three claims above, therefore, we conclude that $E_2 = \emptyset$, which implies that $C_2 = \mathcal{X}$, and hence, $\tau(E_2) = \tau(\emptyset)$.

Second, define $\underline{w} := \inf\{w(x) : x \in \mathcal{X}\}$ and $\beta_W(\theta) := [\underline{w} - R(\theta)]/\phi(\theta)$. Note that $\beta_W(\theta) > \beta_2(\theta)$ for $\forall\theta \in \mathcal{X}$ because $\underline{w} > l_2$. Also, observe that for $\forall\theta < \theta_2^*$, we have

$$\begin{aligned}\beta_W'(\theta) &= \Big\{-R'(\theta)\phi(\theta) - \phi'(\theta)[\underline{w} - R(\theta)]\Big\}/\phi^2(\theta)\\ &> \Big\{-R'(\theta)\phi(\theta) - \phi'(\theta)[l_2 - R(\theta)]\Big\}/\phi^2(\theta) = \beta_2'(\theta) > 0\end{aligned}$$

where the first inequality follows because $\phi'(\theta) < 0$, and the last inequality holds because $\beta_2'(\theta) > 0$ for $\theta < \theta_2^*$ from Lemma B.1. Next, pick $\kappa_\theta > 0$ such that

$$\beta_W(\theta_2^* - \kappa_\theta) = \beta_2(\theta_2^*)\,, \tag{C.2}$$

where $\beta_2(\cdot)$ is defined in (B.2). If such $\kappa_\theta$ exists, it must be unique because $\beta_W'(\theta) > 0$ for $\theta < \theta_2^*$. If there does not exist $\kappa_\theta$ which satisfies C.2, then we let $\kappa_\theta = \infty$.

Finally, we show that (C.1) is satisfied *if* $\theta_2^* - \theta_1^* < \kappa_\theta$, which will complete the proof; this is because we can always find the unique $\kappa_l > 0$ for any given $\kappa_\theta > 0$ such that $\theta_2^* - \theta_1^* < \kappa_\theta$ if and only if $l_2 - l_1 < \kappa_l$ from the fact that $\theta_i^*$ given in Lemma 1 strictly increases in $l_i$. Suppose now that $\theta_2^* - \theta_1^* < \kappa_\theta$, i.e., $\theta_1^* > \theta_2^* - \kappa_\theta$. Note that $\beta_W'(\theta) > 0$ for $\forall\theta < \theta_2^*$, and recall that $\theta_1^* < \theta_2^*$. Therefore,

$\beta_W(\theta_1^*) > \beta_W(\theta_2^* - \kappa_\theta) = \beta_2(\theta_2^*)$ by (C.2). Thus, for any $x \in (\theta_1^*, \theta_2^*]$,

$$\mathbb{E}^x\left[\int_0^{\tau_1^*} \pi(X_t^x)e^{-rt}dt + w(\theta_1^*)e^{-r\tau_1^*}\right] \geq \mathbb{E}^x\left[\int_0^{\tau_1^*} \pi(X_t^x)e^{-rt}dt + \underline{w}e^{-r\tau_1^*}\right] = R(x) + \phi(x)\beta_W(\theta_1^*)$$
$$> R(x) + \phi(x)\beta_2(\theta_2^*) \geq R(x) + \phi(x)\beta_2(x) = l_2\,,$$

where the first inequality holds from the definition of $\underline{w}$, the first equality holds because $\mathbb{E}^x[e^{-r\tau_1^*}] = \phi(x)/\phi(\theta_1^*)$ for $x > \theta_1^*$, the second inequality follows because $\beta_W(\theta_1^*) > \beta_2(\theta_2^*)$, the last inequality holds because $\beta_2(\cdot)$ achieves its maximum at $\theta_2^*$ by Lemma B.1, and the last equality follows by the definition of $\beta_2(\cdot)$. Hence, (C.1) is satisfied, which establishes the desired result for $\kappa_\theta > 0$.

(iii) Let $E_i \subset \mathbb{R}$ denote player $i$'s exit region; *i.e.*, it is player $i$'s strategy to exit whenever $X_t$ enters $E_i$. Note that $\overline{E}_1 \cap \overline{E}_2 = \emptyset$ because none of the players have incentive to exit at the same time. Without loss of generality, assume $\sup\{E_1\} = \bar{x}_1 > \sup\{S_2\} = \bar{x}_2$. Recall that we already established that $E_i \subseteq (-\infty, \theta_i^*]$, so $\bar{x}_1 \leq \theta_1^*$.

We now prove that $\bar{x}_1 = \theta_1^*$ so that player 1's equilibrium strategy is to exit at $\tau_1 := \inf\{t > 0 : X_t \leq \theta_1^*\}$. Towards a contradiction, suppose that $\bar{x}_1 < \theta_1^*$ so that player 1's equilibrium strategy is to exit at $\bar{\tau} = \inf\{t \geq 0 : X_t = \bar{x}_1\} > \tau_1$. Then player 1's payoff reduces to one in which player 1 exits at a threshold $\bar{x}_1 < \theta_1^*$ and player 2 never exits. However, this contradicts Lemma 1, which asserts that the best response of player 1 is to exit at threshold $\theta_1^*$. We conclude that $\bar{x}_1 = \theta_1^*$ if $\bar{x}_1 > \bar{x}_2$, which proves the statement. ∎

**Proof of Lemma 2**: Suppose that $\{a_1, a_2\} = \{(E_1, \lambda_1), (E_2, \lambda_2)\}$ is a mixed-strategy MPE, where $E_i$ is firm $i$'s exit region (*i.e.*, it exits with probability one whenever $X_t \in E_i$), and $\lambda_i(x)$ is its hazard rate of exit function. For the sake of generality, we do not assume that $E_i$ is a closed set at the outset in this proof. Instead, we prove that $E_i$ can be replaced by its closure without affecting the payoff functions (see the remark followed by Claim 2 below).

Let $\Gamma_i$ denote the support of firm $i$'s mixed strategy as defined in (3.2). Without loss of generality, we can assume that $E_i \cap \Gamma_i = \emptyset$ because if firm $i$ exits at some $x \in \mathcal{X}$, it does so either with probability 1 (if $x \in E_i$) or at the hazard rate $\lambda_i(x)$ (if $x \in \Gamma_i$) based on our definition of Markov strategy.

We first show that $E_i \cap E_{-i} = \emptyset$ and $\Gamma_i \cap E_{-i} = \emptyset$. For the purpose of contradiction, suppose that there exists some $y \in E_i \cap E_{-i}$. Then firm $i$'s payoff at $y$ is $\frac{1}{2}(l_i + w(y))$ because both firms attempt to exit simultaneously. However, if firm $i$ deviates from its strategy and chooses not to exit, its payoff at $y$ would be $w(y) > \frac{1}{2}(l_i + w(y))$, so its payoff is improved. This contradicts the assumption that $\{a_1, a_2\}$ is an equilibrium, and hence, we conclude $E_i \cap E_{-i} = \emptyset$.

Next, suppose that there exists $y \in \Gamma_i \cap E_{-i}$. Note that by the definition of a mixed strategy equilibrium, one of firm $i$'s best responses to $a_{-i}$ is to attempt to exit immediately whenever the current value of $X$ is within $\Gamma_i$. Since $y$ also belongs to $E_{-i}$, firm $i$'s payoff at $y$ is $\frac{1}{2}(l_i + w(y))$, which is strictly less than the payoff from not exiting at $y$, i.e., $w(y)$. This implies that firm $i$ can improve its payoff by deviating from its candidate equilibrium strategy, which again contradicts the assumption that $\{a_1, a_2\}$ is an equilibrium. Thus, we conclude $\Gamma_i \cap E_{-i} = \emptyset$.

Now, we prove two claims that will be used for the proof of the lemma. Below we let $\mathrm{cl}(S)$ denote the closure of the set $S$ and $\partial S$ denote the boundary of the set $S$.

**Claim 1**: $\Gamma_i = \Gamma_{-i}$ and $\sup \Gamma_i \leq x_{ci}$. Furthermore, $\lambda_i(x) = \frac{rl_{-i} - \pi(x)}{w(x) - l_{-i}}$ for any $x \in \Gamma_i$.

Proof: Recall that $\Gamma_i$ can be expressed as a union of open intervals. Towards a contradiction, suppose that for some $i$, there exists an open interval $G$ such that $G \subseteq \Gamma_i$ and $G \cap \Gamma_{-i} = \emptyset$. We let $D_i := \{x \in \mathcal{X} : \pi(x) > rl_i\} = (x_{ci}, \beta)$ denote the set of states at which firm $i$'s net-present-value of its flow payoff from remaining in the market exceeds its outside option. Without loss of generality, we can always appropriately choose $G$ such that either $G \subset D_i$ or $G \cap D_i = \emptyset$ holds.

We first consider the case $G \subset D_i$. Assume $X_0 \in G$ and define $\tau_G := \inf\{t : X_t \notin G\}$. Then $\tau_G > 0$ a.s because $G$ is an open set. Now, fix some stopping time $\tau \in (0, \tau_G)$. Then because $X_0, X_\tau \in G$ and $G$ is a subset of $\Gamma_i$, firm $i$'s payoff from an immediate exit should be the same as that from an exit at $\tau$ by the definition of a mixed strategy equilibrium. However, because $G \cap \Gamma_{-i} = \emptyset$ by assumption, firm $i$'s expected payoff if it exits at $\tau$ and its rival's strategy is $a_{-i}$ is equal to

$$l_i + \mathbb{E}\left[\int_0^\tau e^{-rt}\left(\pi(X_t) - rl_i\right) dt\right] > l_i\,, \tag{C.3}$$

where the inequality follows from the fact that $\pi(X_t) > rl_i$ for all $t \in [0, \tau)$. Therefore, firm $i$'s payoff is strictly higher if it exits at $\tau > 0$ than if it exits immediately, which contradicts the assumption that $\{a_1, a_2\}$ is a mixed strategy equilibrium.

Next, we consider the case $G \cap D_i = \emptyset$. We repeat the same argument as the case of $G \subset D_i$ with $\tau \in (0, \tau_G)$. Then firm $i$'s expected payoff if it exits at $\tau$ and its rival's strategy is $a_{-i}$ is equal to

$$l_i + \mathbb{E}\left[\int_0^\tau e^{-rt}\left(\pi(X_t) - rl_i\right) dt\right] < l_i\,,$$

where the inequality is due to $\pi(X_t) < rl_i$ for all $t \in [0, \tau)$. Therefore, firm $i$ can obtain a strictly higher payoff from an immediate exit than an exit at $\tau$. This contradicts the assumption that $\{a_1, a_2\}$ is a mixed strategy equilibrium. We conclude that $\Gamma_i = \Gamma_{-i}$ must hold.

Now we derive the form of the equilibrium rate of exit. For notational convenience, we let $V(x)$ denote the payoff function $V_i(x; a_i, a_{-i})$. Recall that $V(x) = l_i$ for $x \in \Gamma := \Gamma_i = \Gamma_{-i}$ because an immediate exit within $\Gamma$ is each firm's best response. Let $G$ be an open neighborhood of $x$ such that $G \subsetneq \Gamma$, and let $\tau_G := \inf\{t \geq 0 : X_t \notin G\}$. By the definition of the mixed strategy region $\Gamma$, firm $i$'s payoff associated with exit at time $\tau_G$ is still $V(x) = l_i$ because an exit at $\tau_G$ is also one of the best responses of firm $i$. Hence, one of the possible expressions of $V(x)$ is given by the following:

$$V(x) = \mathbb{E}^x\Big[\int_0^{\tau_G} e^{-\Lambda_t^{(-i)}}(\pi(X_t) + \lambda_{-i}(X_t)w(X_t))dt + e^{-\Lambda_{\tau_G}^{(-i)}} V(X_{\tau_G})\Big]\,, \tag{C.4}$$

where $\Lambda_t^{(-i)} := \int_0^t(\lambda_{-i}(X_s)+r)ds$. Intuitively, the discount factor $e^{-rt}$ is replaced by $e^{-\Lambda_t^{(-i)}}$ due to the exit rate $\lambda_{-i}(\cdot)$ of firm $-i$; the additional flow profit $\lambda_{-i}(X_t)w(X_t)$ is because of the exit probability of firm $-i$; the reward $l_i$ from exit at $\tau_G$ is replaced by $V(X_{\tau_G})$ because $V(X_{\tau_G}) = l_i$.

To derive the form of $\lambda_{-i}(\cdot)$, we need to derive the Hamilton-Jacobi-Bellman (HJB) equation:

$$\Big[\frac{\sigma(x)^2}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - r - \lambda_{-i}(x)\Big]V(x) + \lambda_{-i}(x)w(x) + \pi(x) = 0\,. \tag{C.5}$$

For this purpose, we define a function $f(\Lambda,x) := e^{-\Lambda}V(x)$. The new variable $\Lambda$ represents the integral $\Lambda_t^{(-i)}$ which satisfies the differential equation $d\Lambda_t^{(-i)} = (r+\lambda_{-i}(X_t))dt$ by its definition. Note that $f(\Lambda,x)$ is continuously differentiable with respect to $\Lambda$ and twice continuously differentiable with respect to $x$ within $\Gamma$ because $V(x) = l_i$ for $x \in \Gamma$. Thus, based on the stochastic differential equation that the bivariate process $(\Lambda_t^{(-i)}, X_t)$ follows, the following characteristic operator is well-defined for the function $f$:

$$\mathcal{L} := (\lambda_{-i}(x)+r)\frac{\partial}{\partial\Lambda} + \frac{\sigma(x)^2}{2}\frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x}\,.$$

From the expression (C.4), we have the following:

$$\mathbb{E}^x[f(\Lambda_{\tau_G}^{(-i)}, X_{\tau_G})] - f(0,x) = -\mathbb{E}^x\Big[\int_0^{\tau_G} e^{-\Lambda_t^{(-i)}}(\pi(X_t)+\lambda_{-i}(X_t)w(X_t))dt\Big]\,. \tag{C.6}$$

Recall that $G$ is any arbitrary open neighborhood of $x$. From equation (7.5.1) of Oksendal (2003),

$$\lim_{G\downarrow x}\frac{\mathbb{E}^x[f(\Lambda_{\tau_G}^{(-i)}, X_{\tau_G})] - f(0,x)}{\mathbb{E}^x[\tau_G]} = \mathcal{L}f(0,x) = \Big[\frac{\sigma(x)^2}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - r - \lambda_{-i}(x)\Big]V(x)\,. \tag{C.7}$$

Here the limit $G \downarrow x$ represents a series of open sets decreasing to the point $x$. Furthermore,

$$\lim_{G\downarrow x}\frac{-\mathbb{E}^x[\int_0^{\tau_G} e^{-\Lambda_t^{(-i)}}(\pi(X_t)+\lambda_{-i}(X_t)w(X_t))dt]}{\mathbb{E}^x[\tau_G]} = -\pi(x) - \lambda_{-i}(x)w(x)\,. \tag{C.8}$$

Thus, from (C.6), (C.7), and (C.8), we obtain (C.5).

Because $V(\cdot) = l_i$, the HJB equation (C.5) turns into $[-r-\lambda_{-i}(x)]l_i + \lambda_{-i}(x)w(x) + \pi(x) = 0$, which leads to

$$\lambda_{-i}(x) = \frac{rl_i - \pi(x)}{w(x) - l_i}\,. \tag{C.9}$$

Finally, we impose the condition that $\lambda_{-i}(\cdot)$ must be positive within $\Gamma_{-i}$, so we obtain $rl_i > \pi(x)$ because $w(\cdot) > l_i$. Therefore, $\sup\Gamma_i \le x_{ci}$ and $\lambda_i(x) = \frac{rl_{-i}-\pi(x)}{w(x)-l_{-i}}$ for $x \in \Gamma_i$. □

**Claim 2**: $\partial\Gamma_i \cap \partial E_{-i} = \emptyset$ and $\partial E_i \cap \partial E_{-i} = \emptyset$.

Proof: Suppose that there exists $y \in \partial\Gamma_i \cap \partial E_{-i}$. Since $\Gamma_i$ is a union of disjoint open intervals, there is a subset of $\Gamma_i$ of the form $[z,y)$ or $(y,z]$. Without loss of generality, we assume that $[z,y) \subset \Gamma_i$ for some $z < y$.

From Claim 1, $[z,y)$ is also a subset of $\Gamma_{-i}$, within which firm $-i$ exits with a rate $\lambda_{-i}(x)$ given by (C.9). Suppose that, within the interval of $[z,y)$, firm $i$ employs an alternative strategy $a_i'$ of exit at the escape time $\tau = \inf\{t \ge 0 : X_t \notin (z,y')\}$ of $(z,y')$ for some $y' > y$. Then we have $V_i(z;a_i',a_{-i}) = l_i$ by the definition of the strategy $a_i'$. Furthermore, $V_i(y;a_i',a_{-i}) = w(y)$ because firm $-i$ exits at the hitting time of $y \in \partial E_{-i}$ due to the property of the regular diffusion process $X$. Along with the two boundary conditions, because $\lambda_{-i}(\cdot)$ is a continuous function within $\Gamma_{-i}$, $V_i(x;a_i',a_{-i})$ for $x \in (z,y)$ can be obtained as the solution to a boundary value problem, satisfying the following differential equation (Section 9.1, Oksendal 2003):

$$[\frac{\sigma(x)^2}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - r - \lambda_{-i}(x)]V_i(x;a_i',a_{-i}) + \lambda_{-i}(x)w(x) + \pi(x) = 0\,.$$

It is well-known that there exists a unique solution $V_i(x;a_i',a_{-i})$ to this differential equation within $(z,y)$ (see, for example, Appendix of Lon and Zervos 2011), and it follows that $V_i(x;a_i',a_{-i})$ is continuous within $[z,y]$. From the boundary condition $V_i(y;a_i',a_{-i}) = w(y) > l_i$ and the continuity of $V_i(\cdot;a_i',a_{-i})$, there must exist an interval $I \subset [z,y]$ such that $V_i(x;a_i',a_{-i}) > l_i$ for all $x \in I$. This implies that firm $i$ can improve its payoff within $I$ by adopting an alternative strategy $a_i'$, which contradicts the assumption that $\{a_i,a_{-i}\}$ is an equilibrium. We conclude that $\partial\Gamma_i \cap \partial E_{-i} = \emptyset$.

Next, suppose that there exists $y \in \partial E_i \cap \partial E_{-i}$. Because $\Gamma_{-i} \cap E_i = \emptyset$ and $\partial\Gamma_{-i} \cap \partial E_i = \emptyset$, there exists an open neighborhood $G$ of $y$ such that $G \cap \Gamma_{-i} = \emptyset$. Within $G$, we consider an alternative strategy $a_i'$ of exit at the escape time $\tau_G := \inf\{t \ge 0 : X_t \notin G\}$ of $G$. Then for any $x \in G\backslash \mathrm{cl}(E_{-i})$, where $G\backslash \mathrm{cl}(E_{-i})$ is an open set, firm $i$'s payoff function $V_i(x;a_i',a_{-i})$ within $G\backslash \mathrm{cl}(E_{-i})$ can be obtained as the solution to a boundary value problem (Section 9.1, Oksendal 2003), satisfying the following differential equation

$$[\frac{\sigma(x)^2}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - r]V_i(x;a_i',a_{-i}) + \pi(x) = 0\,,$$

along with the boundary conditions (i) $V_i(x;a_i',a_{-i}) = l_i$ at $x \in \partial G\backslash \mathrm{cl}(E_{-i})$, (ii) $V_i(x;a_i',a_{-i}) = (l_i +$

$w(x))/2$ at $x \in \partial G \cap E_{-i}$, and (iii) $V_i(x; a_i', a_{-i}) = w(x)$ for all $x \in G \cap \mathrm{cl}(E_{-i})$ because firm $-i$ exits at the hitting time of $\mathrm{cl}(E_{-i})$ by the diffusive property of $X$. By the solution property of the boundary value problem, $V_i(x; a_i', a_{-i})$ is continuous within the set $\mathrm{cl}(G)$. Recall that $y \in G \cap \partial E_{-i}$, which implies that $V_i(y; a_i', a_{-i}) = w(y) > l_i$ from case (iii) above, and that $y \in \partial E_i$; hence, we can choose $x \in E_i$ arbitrarily close to $y$ such that $V_i(x; a_i', a_{-i}) > l_i$ because of the continuity of $V_i(\cdot; a_i', a_{-i})$. Therefore, because $V_i(x; a_i, a_{-i}) = l_i < V_i(x; a_i', a_{-i})$ for some $x \in E_i$, firm $i$ can improve the payoff by deviating from $a_i$ and adopting an alternative strategy $a_i'$, which contradicts the assumption that $\{a_i, a_{-i}\}$ is an equilibrium. We conclude that $\partial E_i \cap \partial E_{-i} = \emptyset$. $\square$

*Remark*: By Claim 2 and the fact that $E_i \cap E_{-i} = \emptyset$ and $\Gamma_i \cap E_{-i} = \emptyset$, we established that both $E_i$ and $\Gamma_i$ must be separated from $E_{-i}$ by closed neighborhoods (*i.e.*, there must exist a non-empty interval between the two sets). Thus, without loss of generality, we can choose $E_i$ and $E_{-i}$ to be closed sets without affecting the payoff functions because the hitting time of $E_i$ is identical to the hitting time of $\mathrm{cl}(E_i)$ due to the diffusive property of $X$.

With the two claims established, we now proceed with the proof of the lemma:

*Step 1.*—For each $i$, let $C_i$ denote the set of states at which firm $i$ does not exit, that is, $C_i := (\alpha, \beta) \setminus (\Gamma_i \cup E_i)$. Moreover, let $F_i := (\theta_i^*, \beta)$ denote the set of states at which firm $i$ would prefer to remain in the market if it expects its rival to never exit. Recall from Lemma 1 that $D_i \subset F_i$. We will show that $F_i \subset C_i$, or equivalently, $(\Gamma_i \cup E_i) \cap F_i = \emptyset$, that is, $F_i$ is a subset of the continuation region for firm $i$. Towards a contradiction, suppose there exists some $x \in (\Gamma_i \cup E_i) \cap F_i$. Fix an $i$, and define the strategies $\widetilde{a}_i := ((\alpha, \theta_i^*], \mathbf{0})$ and $\widetilde{a}_{-i} := (\emptyset, \mathbf{0})$, that is, a strategy profile in which firm $i$ exits (with probability 1) whenever $X_t \in (\alpha, \theta_i^*]$ and its rival never exits. Then

$$V_i(x; a_i, a_{-i}) = l_i < V_i(x; \widetilde{a}_i, \widetilde{a}_{-i}) \leq V_i(x; \widetilde{a}_i, a_{-i}).$$

The equality follows from the assumption that $x \in \Gamma_i \cup E_i$, which also implies $x \notin E_{-i}$. The first inequality follows because $x > \theta_i^*$ (since $x \in F_i$ by assumption), so by Lemma 1, firm $i$ can obtain a strictly higher payoff by exiting at $\theta_i^*$ if its rival never exits. The last inequality follows because firm $i$ is better off if its rival exits in finite time with positive probability compared to the case in which it never exits. To elaborate, the assumption that $w(x) > \mathbb{E}[\int_0^t e^{-rs}\pi(X_s)ds + e^{-rt}w(X_t) | X_0 = x]$ for all $x \in \mathcal{X}$ and $t$ implies that the payoff process for the winner $W_i(t) := \int_0^t \pi(X_s)e^{-rs}ds + e^{-rt}w(X_t)$ is a supermartingale. Letting $\tau_{\theta_i^*} := \inf\{t : X_t \leq \theta_i^*\}$ denote the first hitting time of the set $(\alpha, \theta_i^*]$, the supermartingale property of $W_i(\cdot)$ implies that $W_i(t) \geq \mathbb{E}[W_i(\tau_{\theta_i^*}) | \mathcal{F}_t]$ for any $t < \tau_{\theta_i^*}$, that is, firm $i$ is better off (in expectation) becoming the winner at any $t < \tau_{\theta_i^*}$ compared to becoming the winner at $t = \tau_{\theta_i^*}$, which is in turn strictly better than becoming the loser at $t = \tau_{\theta_i^*}$ because $w(X_{\tau_{\theta_i^*}}) > l_i$. This implies that firm $i$'s expected payoff $V_i(x; \widetilde{a}_i, \widetilde{a}_{-i}) \leq V_i(x; \widetilde{a}_i, a_{-i})$ for any strategy $a_{-i}$. We have thus established that $\Gamma_i \cup E_i$ does not intersect with $D_i$ or $F_i$.

*Step 2.*—Next, we prove that $E_1 = E_2 = \emptyset$. Towards a contradiction, suppose that $E_i \neq \emptyset$. Then because each $E_i$, $i \in \{1,2\}$, is a closed set, so must be their union $E_1 \cup E_2$, which implies that its complement $\mathcal{X} \backslash (E_1 \cup E_2)$ is an open set in the state space $\mathcal{X} = (\alpha, \beta) \subseteq \mathbb{R}$. Hence, $\mathcal{X} \backslash (E_1 \cup E_2)$ must be a union of open intervals because any open set on the real line can be expressed as a union of open intervals. Now, we can always find a subinterval $(f,g) \subseteq \Gamma$ such that $(f,g)$ is a *proper* subset of an open interval contained in $\mathcal{X} \backslash (E_1 \cup E_2)$; in other words, there exists an interval $(c,d) \subseteq \mathcal{X} \backslash (E_1 \cup E_2)$ such that $c$ or $d \in E_1 \cup E_2$ and $c < f < g < d$. This is because (1) $\Gamma_i$ is always a union of intervals by assumption, (2) $\Gamma = \Gamma_1 = \Gamma_2$ by Claim 1, (3) $\Gamma_i$ is separated from $E_{-i}$ by closed neighborhoods by Claim 2.

Suppose first that $d \in E_i$ for some $i \in \{1,2\}$. Without loss of generality, we can always choose $(f,g)$ in a way that $(g,d)$ is a subset of the continuation region for firm $i$; this is because $(f,g) \subseteq (c,d) \subseteq \mathcal{X} \backslash (E_1 \cup E_2)$ by construction and $\Gamma$ is separated from $E_i$ by closed neighborhoods (so any components of $\Gamma$ contained in $(c,d)$ must be strictly below $d$). Also, because we proved $\Gamma_i \cup E_i$, $i \in \{1,2\}$, does not intersect $F_i = (\theta_i^*, \beta)$ in Step 1, it follows that $g \leq \min\{\theta_1^*, \theta_2^*\}$ (since $\Gamma_1 = \Gamma_2$) and $d \leq \theta_i^*$. Moreover, because $(f,g) \subseteq \Gamma$ and $d \in E_i$, we have $V_i(d; a_1, a_2) = V_i(g; a_1, a_2) = l_i$.

*Remark*: The boundary conditions $V_i(d; a_1, a_2) = V_i(g; a_1, a_2) = l_i$ can be derived as follows. First, $V_i(d; a_1, a_2) = l_i$ follows from the fact that firm $i$ exits at the hitting time of $d \in E_i$ under the strategy $a_i$. Second, $V_i(g; a_1, a_2) = l_i$ holds for the following reason. If $(a_1, a_2)$ is a mixed strategy MPE, by the definition of a mixed strategy equilibrium, firm $i$'s payoff function must remain unchanged even if firm $i$ employs an alternative strategy of pure-strategy exit within $\Gamma$. In this case, firm $i$ exits at the hitting time of $g$, which is the boundary point of $\Gamma$. (Because $X$ is a diffusion process, the hitting time of $\Gamma$ is identical to the hitting time of the closure of $\Gamma$.) It thus follows that firm $i$'s payoff at $g$ is $l_i$. Alternatively, one can invoke the continuity of $V_i(x; a_1, a_2)$ at $x = g$ to arrive at the boundary condition $V_i(g; a_1, a_2) = l_i$ because $V_i(x; a_1, a_2) = l_i$ for all $x \in \Gamma$.

Now, fix some $X_0 = x \in (g,d)$, and assume that firm $i$ exits at the hitting time $\tau_b = \inf\{t : X_t \in \{g,d\}\}$; *i.e.*, the first time that $X_t$ hits $\{g,d\}$. Define $a_i' = (\{g,d\}, \mathbf{0})$. Then we have $V_i(x; a_i, a_{-i}) = V_i(x; a_i', a_{-i})$. Because $\pi(X_t) < rl_i$ for all $t < \tau_b$ (by $g \leq \theta_i^*$ and $d \leq \theta_i^*$), we can get the inequality similar to (C.3) to conclude that $V_i(x; a_i, a_{-i}) < l_i$, which contradicts the assumption that $a_i$ is a best response to $a_{-i}$.

Suppose next that $c \in E_i$ for some $i \in \{1,2\}$. Then we can similarly proceed with firm $i$'s continuation region $(c,f)$ and leads to a contradiction to the assumption that $a_i$ is a best response to $a_{-i}$. Therefore, we conclude that $E_1 = E_2 = \emptyset$.

*Step 3.*— Finally, we prove that $\Gamma = (\alpha, \theta_i^*)$. Towards doing so, we will first show that $\Gamma = (\alpha, \theta)$ for some $\theta$; note that, if this is the case, we must have $\theta \leq \min\{\theta_1^*, \theta_2^*\}$ because $\Gamma_i \cap F_i = \emptyset$ and $\Gamma = \Gamma_1 = \Gamma_2$ by Step 1 and Claim 1 respectively. Let $J := \mathcal{X} \backslash \Gamma$. We now consider the following two cases for proof by contradiction.

(i) First, suppose that there exists an interval $(c,d)$ such that $\alpha < c < d < \min\{\theta_1^*, \theta_2^*\}$ and

$(c,d) \subseteq J$ with $c,d \in \text{cl}(\Gamma)$, i.e., $(c,d)$ belongs to both firms' continuation region, and yet, it borders with $\Gamma$ at both ends. Let $X_0 = x \in (c,d)$ and consider the hitting time $\tau_{(c,d)} := \inf\{t : X_t \notin (c,d)\}$, *i.e.,* the first time that $X_t$ hits the set $\{c,d\}$. Similarly as in Step 2, by the definition of a mixed strategy equilibrium, firm $i$'s payoff function remains unchanged even if he employs an alternative pure strategy of exit at $\tau_{(c,d)}$. In other words, if we define $a_i' = (\{c,d\}, \mathbf{0})$ as the said alternative strategy, then we have $V_i(x;a_i,a_{-i}) = V_i(x;a_i',a_{-i})$, which satisfies the following inequality:

$$V_i(x;a_i',a_{-i}) = l_i + \mathbb{E}\left[\int_0^{\tau_{(c,d)}} [\pi(X_t) - rl_i]e^{-rt}dt\right] < l_i\,, \tag{C.10}$$

where the inequality follows because $\pi(X_s) < rl_i$ for all $s \leq \tau_{(c,d)}$. Therefore firm $i$ is strictly better off employing a strategy of an immediate exit–a contradiction.

(ii) Next, suppose that there exists an interval $(\alpha,c)$ such that $\alpha < c < \min\{\theta_1^*, \theta_2^*\}$ and $(\alpha,c) \subseteq J$ with $c \in \text{cl}(\Gamma)$, i.e., $(\alpha,c)$ belongs to both firms' continuation region yet it borders with $\Gamma$ at the point $c$. Let $X_0 = x \in (\alpha,c)$ and consider the hitting time $\tau_c := \inf\{t : X_t \geq c\}$, *i.e.,* the first time that $X_t$ hits the point $c$. Then similarly as in the case of $(c,d) \subseteq J$ above, $V_i(x;(\{c\},\mathbf{0}),a_{-i})$ must be equal to firm $i$'s equilibrium payoff $V_i(x;a_i,a_{-i})$, but we will get the inequality similar to (C.10) for $V_i(x;(\{c\},\mathbf{0}),a_{-i})$, which leads to a contradiction; here, although $\mathbb{P}(\tau_c = \infty) > 0$ is possible, $V_i(x;(\{c\},\mathbf{0}),a_{-i})$ is still well-defined because we assume in Section 2 that $\pi(\cdot)$ satisfies the absolute integrability condition.

(iii) From (i) and (ii), we conclude that $\Gamma = (a,\theta)$ for some $\theta$.

Finally, we show that $\theta = \theta_i^*$ for each $i$. Recall from Step 1 that $\Gamma_i \cup E_i$ does not intersect with $F_i = (\theta_i^*, \beta)$ for any $i$, from Claim 1 that $\Gamma_1 = \Gamma_2 = \Gamma$, and from Step 2 that $E_1 = E_2 = \emptyset$. Therefore, $\Gamma \subseteq (\alpha, \theta_i^*]$, and so $\theta \leq \theta_i^*$ must hold for each $i$. Towards a contradiction, suppose that $\theta < \theta_i^*$ and fix some $x \in (\theta, \theta_i^*)$. Letting $\tau_\theta = \inf\{t : X_t \leq \theta\}$, notice that

$$V_i(x;a_1,a_2) = \mathbb{E}\left[\int_0^{\tau_\theta} e^{-rt}\pi(X_t)dt + e^{-r\tau_\theta}l_i \,|\, X_0 = x\right] = l_i + \mathbb{E}\left[\int_0^{\tau_\theta} e^{-rt}\left(\pi(X_t) - rl_i\right)dt \,|\, X_0 = x\right] < l_i\,.$$

The first equality follows from the fact that $V_i(\theta;a_1,a_2) = l_i$ since firm $i$ is indifferent between exiting and remaining in the market when $x = \theta$, the second equality follows by manipulating terms, and the inequality follows from the fact that $\pi(X_t) < rl_i$ for all $t < \tau_\theta$ (recall from Step 1 that $D_i \subset F_i$). Therefore, firm $i$ is strictly better off exiting immediately, contradicting the premise that $\{a_1,a_2\}$ is an MPE. Hence we conclude that $\theta = \theta_i^*$ for each $i$. ■

**Proof of Theorem 1**: By noting that $\theta_1^* = \theta_2^*$ if and only if $l_1 = l_2$, it follows immediately from Lemma 2 that if $l_1 < l_2$, then the game does not admit any mixed strategy MPE. ■

## D Online Appendix: Structural Stability

The goal of this section is to investigate the robustness of Theorem 1 to our assumptions regarding the firms' payoffs. In particular, we argue that Theorem 1 continues to hold even if the firms have heterogeneous discount rates ($r_1 \neq r_2$), heterogeneous flow profits while they remain in the market ($\pi_1(\cdot) \neq \pi_2(\cdot)$), heterogeneous winner payoffs ($w_1(\cdot) \neq w_2(\cdot)$), and the loser's payoff is state-dependent (*i.e.*, if firm $i$ exits at $t$, then it obtains payoff $l_i(X_t)$).

To analyze this model, in addition to the assumptions at the end of the model description in Section 2, we make the following assumptions:

**Condition 1** *Assume that for each* $i \in \{1,2\}$,
*(i)* $w_i(x) > l_i(x)$ *for all* $x \in \mathcal{X}$,
*(ii)* $l_i(\cdot)$ *is twice continuously differentiable on* $\mathcal{X}$,
*(iii)* $\pi_i(x) + \mathcal{A}_i l_i(x)$ *is increasing in* $x$, *and*
*(iv)* $\lim_{x \downarrow a} \pi_i(x) + \mathcal{A}_i l_i(x) < 0$ *and* $\lim_{x \uparrow b} \pi_i(x) + \mathcal{A}_i l_i(x) > 0$.

Part (i) ensures that the winner's payoff is always greater than that of the loser, and it is analogous to the assumption $w(\cdot) > l_2$ we made in Section 2. Part (ii) implies that we can apply the infinitesimal generator $\mathcal{A}_i$ to $l_i(\cdot)$. Parts (iii) and (iv) guarantee that there exists a unique threshold $\theta_i^*$ such that firm $i$'s best response if its rival never exits (*i.e.*, if $a_{-i} = \{\emptyset, \emptyset, \mathbf{0}\}$) is to exit at the first time $\tau_i = \inf\{t \geq 0 : X_t \leq \theta_i^*\}$ (see Theorem 6 (B) in Alvarez (2001) for details).[12]

It is straightforward to verify that Lemma 2 continues to hold under this more general model, because the only properties of the payoff-relevant parameters we used in the proof of this lemma are that (a) $w(x) > l_i$ for all $x$, and (b) $w(x) > \mathbb{E}\left[\int_0^t e^{-rs}\pi(X_s)ds + \ e^{-rt}w(X_t) \,|X_0 = x\right]$ for all $x \in \mathcal{X}$ and $t$. It thus follows that if Condition 1 is satisfied and the parameters $\{r_i, \pi_i(\cdot), w_i(\cdot), l_i(\cdot)\}_{i\in\{1,2\}}$ are such that the thresholds $\theta_1^* \neq \theta_2^*$, then the game admits no mixed-strategy MPE.

## E Online Appendix: Singular Strategy

In this section, we provide justification for precluding a singularly continuous component when a strategy is expressed as a cumulative distribution function of exit time.

We first note that firm $i$'s strategy can be alternatively expressed as a non-decreasing and right-continuous process $A_i = \{A_{i,t}\}_{t\geq 0}$ that ranges in the interval $[0,\infty]$; it can be transformed into a

[12] As an example, if the state $X$ is a linear diffusion (*i.e.*, $\mu(x) \equiv \mu < 0$ and $\sigma(x) \equiv \sigma > 0$ in (2.1)), $\pi_i(x) = A_i x + B_i$ and $l_i(x) = C_i x + D_i$, then it is easy to verify that Condition 1(ii)-(iv) are satisfied as long as $A_i > r_i C_i$. If $X$ is a geometric Brownian motion (*i.e.*, $\mu(x) \equiv \mu x$ and $\sigma(x) \equiv \sigma x$ for some $\mu < 0$ and $\sigma > 0$), then Condition 1(ii)-(iv) are satisfied for the above choice of $\pi_i(\cdot)$ and $l_i(\cdot)$ as long as $A_i > (r_i + \mu)C_i$ and $B_i < r_i D_i$.

cumulative distribution function $G_i(t)$ of firm $i$'s exit timing by letting $\int_0^t \frac{dG_i(s)}{1-G_i(s)} = A_{i,t}$. Such a process can be decomposed into three components as follows:

$$A_{i,t} = \int_0^t \lambda_{i,s} ds + \int_0^t dL_{i,s} + \sum_{0<u\le t} \Delta A_{i,u} \, ,$$

where the first component $\int_0^t \lambda_{i,s} ds$ is the absolutely continuous (in time) part, the second component $\int_0^t dL_{i,s}$ is the singularly continuous (in time) part, and the third component $\sum_{0<u\le t} \Delta A_{i,u}$ is the discontinuous part.

We now argue that the singularly continuous part of $A_{i,t}$ is absent, i.e., $\int_0^t dL_{i,s} = 0$, in a mixed strategy MPE. Suppose on the contrary that there exists a point $y \in \mathcal{X} \subseteq \mathbb{R}$ at which the increase $dA_{i,t}$ is singularly continuous. Let $L_i(t)$ denote the singularly continuous component of $A_i$. Then, from the definition provided by Karatzas and Shreve (1991), we can express it as

$$L_i(t) = \lim_{\varepsilon \downarrow 0} \int_0^t \frac{f(X_s)}{\varepsilon} \mathbf{1}_{\{y-\varepsilon<X_s<y+\varepsilon\}} ds \, .$$

For example, $L_i(t)$ reduces to the local time if $f(\cdot) = 1/2$ and if $X$ is a Wiener process. Note that we can think of $L_i(\cdot)$ as resulting from a very large exit rate $f(X_t)/\varepsilon$ at and around $y$. Also, for any $\varepsilon > 0$, we can think of $(y-\varepsilon, y+\varepsilon)$ as a mixed strategy exit region with an exit rate of $f(X_t)/\varepsilon$. Then because firm $i$ must be indifferent between exit at time 0 and exit at an infinitesimal time $dt$, firm $-i$ must also have mixed strategy exit region in $(y-\varepsilon, y+\varepsilon)$ with

$$\lambda_{-i}(x) = \frac{rl_i - \pi(x)}{w(x) - l_i} \, .$$

Similarly, by a symmetric argument, firm $i$'s exit rate should be

$$\lambda_i(x) = \frac{rl_{-i} - \pi(x)}{w(x) - l_{-i}} \, ,$$

which cannot be arbitrarily large as $f(X_t)/\varepsilon$ as $\varepsilon \downarrow 0$. Therefore, such a singularly continuous component cannot exist in a mixed strategy MPE.

## F Online Appendix: Proof of Lemma 1

The proof of this lemma is available in Alvarez (2001), but here, we provide a sketch of the proof based on the verification theorem (Oksendal, 2003, Theorem 10.4.1). To that end, we will use the optimality conditions, which are known as "value matching" and "smooth pasting" conditions (Samuelson, 1965; McKean, 1965; Merton, 1973).

First, the state space $\mathcal{X}$ must be the union of $C := \{x \in \mathcal{X} : V_i^*(x) > l_i\}$ and $\Gamma := \{x \in \mathcal{X} : V_i^*(x) = l_i\}$, which are mutually exclusive: This is because (1) $X$ is a time-homogeneous process and the time horizon is infinite, and (2) the value function $V_i^*(\cdot)$ from an optimal stopping policy must be always no less than the reward $l_i$ from stopping immediately. Hence, the problem to find an optimal stopping policy can be reduced to identify $C$ or $\Gamma$.

Next, we find the differential equation that $V_i^*(x)$ must satisfy *if* $x \in C$. Note that the optimal value function $V_i^*(x)$ is the maximum of the reward from waiting an instant and the reward from stopping immediately. For any $x \in C$, therefore, the optimal stopping policy is to wait an instant $dt$, and hence, the optimal value function must satisfy the following equation:

$$V_i^*(x) = \pi(x)dt + (1 - rdt)\mathbb{E}^x[V_i^*(x) + dV_i^*(X_t)] \,. \tag{F.1}$$

Then applying Ito formula to $V_i^*(X_t)$ and using $\mathbb{E}^x[dB_t] = 0$ yields

$$\mathbb{E}^x[dV_i^*(X_t)] = [\mu(x)V_i^{*'}(x) + \frac{1}{2}\sigma^2(x)V_i^{*''}(x)]dt \,. \tag{F.2}$$

By plugging (F.2) into (F.1) and ignoring the term smaller than $dt$, we have

$$V_i^*(x) = \pi(x)dt + V_i^*(x) + [-rV_i^*(x) + \mu(x)V_i^{*'}(x) + \frac{1}{2}\sigma^2(x)V_i^{*''}(x)]dt \,,$$

from which we obtain the following second-order linear differential equation:

$$\frac{1}{2}\sigma^2(x)V_i^{*''}(x) + \mu(x)V_i^{*'}(x) - rV_i^*(x) = -\pi(x) \,. \tag{F.3}$$

Thus, $V_i^*(\cdot)$ can be obtained by solving the differential equation (F.3). In fact, it can be seen from a series of algebra with the relation (B.3) that the function $R(\cdot) + A\phi(\cdot)$ with some constant $A \in \mathbb{R}$ is a solution to (F.3), and hence, we can guess $V_i^*(x) = R(x) + A\phi(x)$ with some constant $A$.

Intuitively, firm $i$ must find it optimal to exit and receive his outside option $l_i$ as soon as the state $X$ hits some lower threshold $\theta_i$. Hence, assume at the moment that the optimal stopping policy is given as $\tau^* := \inf\{t \geq 0 : X_t^x \leq \theta_i\}$, which implies that $\theta_i$ is the boundary point of the region $C$. Now, we state the value matching condition and the smooth pasting condition, which results in two boundary conditions to the boundary value problem (F.3) with the free boundary $\theta_i$:

$$V_i^*(\theta_i) = R(\theta_i) + A\phi(\theta_i) = l_i \tag{F.4}$$

$$V_i^{*'}(\theta_i) = R'(\theta_i) + A\phi'(\theta_i) = 0 \,. \tag{F.5}$$

The value matching condition (F.4) and the smooth pasting condition (F.5) are the conditions that $V_i^*(\cdot)$ must satisfy at the boundary $\theta_i$ of $C$. We can first obtain $A = [l_i - R(\theta_i)]/\phi(\theta_i) = \beta_i(\theta_i)$ from (F.4). Then the condition (F.5) is equivalent to

$$
\begin{aligned}
0 &= R^{'}(\theta_i) + \frac{l_i - R(\theta_i)}{\phi(\theta_i)}\phi^{'}(\theta_i) \\
&= \frac{R^{'}(\theta_i)\phi(\theta_i) + [l_i - R(\theta_i)]\phi^{'}(\theta_i)}{\phi(\theta_i)} = -\phi(\theta_i)\beta_i^{'}(\theta_i)\,.
\end{aligned}
$$

Because $\phi(\cdot) > 0$, it can be seen from Lemma B.1 that this condition is satisfied if and only if $\theta_i = \theta_i^*$, which implies that $A = \beta_i(\theta_i^*)$.

Lastly, it can be easily verified that $R(x) + \beta_i(\theta_i^*)\phi(x) \geq l_i$ for $\forall x \geq \theta_i^*$ and $\pi(x) < rl_i$ for $\forall x \leq \theta_i^* < x_{ci}$. By the verification theorem (Oksendal, 2003, Theorem 10.4.1), therefore, the proposed value function $R(\cdot) + \beta_i(\theta_i^*)\phi(\cdot)$ is, in fact, the optimal value function $V_i^*(\cdot)$, as desired. ■

## G Online Appendix: Extension to (Non-Markov) Subgame Perfect Equilibria

In this appendix, we allow firms to condition their decision at $t$ on the entire history $h^t = \{X_s\}_{s \leq t}$ (as opposed to only the current state, $X_t$). We show that if the firms have heterogeneous exit payoffs (*i.e., $l_1 < l_2$*), then subject to a set of restrictions on their strategies, the game admits no mixed-strategy Subgame Perfect equilibrium.

We first extend the strategy to accommodate the history dependence. At every moment, given the history $h^t = \{X_s\}_{s \leq t}$ and conditional on the game not having ended, each firm chooses (probabilistically) whether to exit. Formally, each firm $i$ chooses

i. a set of histories $I_i$ (or an *exit region*) such that it exits with probability 1 if $h^t \in I_i$,

ii. a set of stopping time and exit probability pairs, denoted by $\mathcal{P}_i = \{(\tau_{i,n}, p_{i,n})\}_{n=1}^{\infty}$, such that it exits at $t = \tau_{i,n}$ with probability $p_{i,n} \in (0,1)$, and

iii. a non-negative process $\Lambda_i = \{\lambda_{i,t}\}_{t>0}$, which represents the firm's hazard rate of exit at $t$.

We assume that $\tau_I := \inf\{t : h^t \in I_i\}$ is a stopping time. We also assume that each $p_{i,n}$ and the process $\lambda_{i,t}$ is progressively measurable with respect to $\mathcal{F}_{\tau_{i,n}}$ and $\mathcal{F}_t$, respectively, and so the exit probability $p_{i,n}$ at $\tau_{i,n}$, and the exit rate $\lambda_{i,t}$ may depend on the entire history of $X$ up to $\tau_{i,n}$ and $t$, respectively. Then we can represent firm $i$'s strategy as the three-tuple $a_i = (I_i, \mathcal{P}_i, \Lambda_i)$, and $\{a_1, a_2\}$ is a *strategy profile*. As each firm's decision at $t$ can be conditioned on the entire history up to $t$, it is without loss of generality to assume that each firm chooses its strategy at time 0. Intuitively, during any *small* interval $[t, t+dt)$, firm $i$ exits with probability

$$\rho_{i,t} = \begin{cases} 1 & \text{if } h^t \in I_i \,, \\ p_{i,n} + (1-p_{i,n})\lambda_{i,t}dt & \text{if } t = \tau_{i,n} \,, \text{and} \\ \lambda_{i,t}dt & \text{otherwise.} \end{cases}$$

If firm $i$ does not exit with probability 1 after any history, we write $I_i = \emptyset$. If it does not choose any stopping time – exit probability pairs, we write $\mathcal{P}_i = \emptyset$. If it does not exit with a positive hazard rate (*i.e.,* $\lambda_{i,t} = 0$ for all $t$ almost surely, hereafter a.s), we write $\Lambda_i = \mathbf{0}$. Finally, we say that firm $i$'s strategy is *pure* if $\mathcal{P}_i = \emptyset$ and $\Lambda_i \equiv \mathbf{0}$, and it is *mixed* otherwise.

Next, we write each firm's payoff as a function of an arbitrary strategy profile. Fix a strategy profile $\{a_1, a_2\}$ and history $h^t$, and define $\tau_{i,0} := \inf\{s \geq 0 : h^s \in I_i\}$ and $p_{i,0} := 1$. The survival probability, that is, the probability that the game does not end during $[t, u)$ is given by

$$S_{t,u} := e^{-\int_t^u (\lambda_{1,s}+\lambda_{2,s})ds} \prod_{\{n,m \geq 0 : t \leq \tau_{1,n} < u, t \leq \tau_{2,m} < u\}} (1-p_{1,n})(1-p_{2,m})\,.$$

Firm $i$'s payoff at time $t$ (conditional on the game not having ended) can be written as

$$\begin{aligned} V_i(h^t; a_1, a_2) = & \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} S_{t,s}[\pi(X_s) + \lambda_{i,s} l_i + \lambda_{j,s} w(X_s)]ds \right. \\ & + \sum_{n \geq 0} S_{t,\tau_{i,n}} e^{-r(\tau_{i,n}-t)} p_{i,n} l_i + \sum_{m \geq 0} S_{t,\tau_{j,m}} e^{-r(\tau_{i,m}-t)} p_{j,m} w(X_{\tau_{j,m}}) \\ & \left. -\frac{1}{2} \sum_{n,m \geq 0} \mathbb{I}_{\{\tau_{i,n} = \tau_{j,m}\}} S_{t,\tau_{i,n}} e^{-r(\tau_{i,n}-t)} p_{i,n} p_{j,m} \left(l_i + w(X_{\tau_{j,m}})\right)\right]. \end{aligned} \tag{G.1}$$

The first line represents the firm's discounted flow payoff with survival chances taken into account, plus the reward from the end of the game through the exit rate by either firm. The second line captures the payoff from either firm's instantaneous exit probability, while the third line accounts for the possibility of simultaneous exit and the double counting from the second line. The dependence of the strategies $a_i$ on history $h^t$ is muted for expositional simplicity.

A strategy profile $\{a_1^*, a_2^*\}$ is a Subgame Perfect equilibrium (hereafter SPE) if

$$V_i(h^t; a_i^*, a_{-i}^*) \geq V_i(h^t; a_i, a_{-i}^*)$$

for each firm $i$, every history $h^t$, and every strategy $a_i$.[13]

Recall that firm $i$'s strategy can be summarized by the three-tuple $(I_i, \mathcal{P}_i, \Lambda_i)$, where $I_i$ is a set of

[13]Dutta and Rustichini (1993) uses a similar formulation in a class of stopping time games.

histories such that firm $i$ exits instantaneously whenever $h^t \in I_i$, $\mathcal{P}_i = \{\tau_{i,n}, p_{i,n}\}_{n=1}^{\infty}$ is a collection of stopping times and corresponding exit probabilities, and $\Lambda_i = \{\lambda_{i,t}\}_{t\geq 0}$ is a non-negative process. To make it explicit that $\lambda_{i,t}$ can depend on the entire history $h^t$, we will sometimes write $\lambda_i(h)$ to denote firm $i$'s exit rate when $h^t = h$.

To help the reader visualize an SPE with history-dependent mixed strategies, we present an example when the firms are homogeneous.

**Example 1** *Suppose that $l_1 = l_2$ (and so $\theta_1^* = \theta_2^*$ by Lemma 1). Fix any $q \in (0,1)$, and consider the strategies $a_1 = \left(\emptyset, \{\tau_1, q\}, \{\lambda_t\}_{t\geq 0}\right)$ and $a_2 = \left(\emptyset, \emptyset, \{\lambda_t\}_{t\geq 0}\right)$, where $\tau_1 = \inf\{t \geq 0 : X_t \leq \theta_1^*\}$, and*

$$\lambda_t := \mathbb{I}_{\{X_t \leq \theta_1^*\}} \frac{rl_1 - \pi(X_t)}{w(X_t) - l_1}.$$

*Then $\{a_1, a_2\}$ constitutes a (non-Markov) mixed strategy SPE.*

In this example, both firms remain in the market until the first time that $X_t \in (\alpha, \theta_1^*]$. At that moment, firm 1 exits with instantaneous probability $q$. From that time onwards, whenever $X_t \leq \theta_1^*$, each firm exits with rate $\lambda_t$, which is chosen to make its opponent indifferent between remaining in the market and exiting. This strategy profile is non-Markov because firm 1 exits with probability $q$ only at the first time that $X_t$ hits $\theta_1^*$. Indeed, this is the stochastic analog of the mixed-strategy equilibrium characterized in Hendricks et al. (1988) when the game is deterministic (*i.e.,* $\sigma(\cdot) \equiv 0$), and in Tirole (1988) when $\mu(\cdot) \equiv \sigma(\cdot) \equiv 0$ and $X_0$ satisfies $\pi(X_0) < rl_i$.[14]

We note that the strategy profile $\{a_1, a_2\}$ in Example 1 cannot constitute a mixed strategy SPE if the firms are heterogeneous, i.e., $l_1 < l_2$. This is because if firm 1 does not exit at the first time $X_t \leq \theta_1^*$, which happens with probability $1 - q > 0$, then both firms' strategies are Markov afterwards. In the subgame firm 1 stays at the first time $X_t \leq \theta_1^*$, therefore, the strategy profile $\{a_1, a_2\}$ constitutes a mixed strategy MPE, which has been proved impossible in Section 3; in the rest of this section, we will formalize and generalize this argument. In addition, we argue that simple variations of the strategy profile $\{a_1, a_2\}$ in Example 1 would not constitute a mixed strategy SPE if $l_1 < l_2$. For instance, one may consider a strategy where firm 1 exits with probability $q$ whenever $X_t$ hits $\theta_1^*$ only from above. This strategy, however, is indistinguishable from a strategy where firm 1 exits whenever $X_t$ hits $\theta_1^*$ from either side because $X$ is a regular diffusion process. As another variation, one may consider the adjustment of exit probability $q$ depending on the history $h^t$ while still keeping the threshold rule: Whenever $X_t$ hits $\theta_1^*$ from above, firm 1 exits with probability $q(h^t)$ depending on the history $h^t$. However, no matter how carefully $q(h^t) > 0$ is chosen, firm 2 would not exit with the rate $\lambda_t$ near but below $\theta_1^*$ because of the chance to become the winner soon.

[14] The reader is referred to Steg (2015) for a proof that the proposed strategies indeed constitute an SPE.

We impose three restrictions on the firms' strategies, all of which are satisfied by the strategies in the mixed-strategy equilibria that appear in the extant literature; see for instance, Tirole (1988), Hendricks et al. (1988), Levin (2004), Steg (2015), and Example 1 above. The first is that the exit regions $I_1 = I_2 = \emptyset$; *i.e.*, neither firm exits with probability one following any history. The second is that $\mathcal{P}_i = \{\tau_{i,n}, p_{i,n}\}_{n=1}^{N_i}$ for some $N_i < \infty$ and $\tau_{i,n} < \infty$ a.s for all $i$ and $n$. That is, the number of events of instantaneous exit is finite and the stopping times are finite a.s. We discuss the role of these assumptions before Lemma G.1. Finally, we define

$$H_i = \{h : \lambda_i(h) > 0\}$$

and impose the regularity condition that $H_i$ is an open set with respect to the following metric $d(\cdot,\cdot)$ on the space of histories:

$$d(h_1, h_2) := \sqrt{\max\left\{\max_{s\in[0,t]} |X_s^1 - X_s^2|, \sup_{s\in(t,t']} \left|X_t^1 - X_s^2\right|\right\} + (t-t')^2}, \tag{G.2}$$

where $h_1^t$ and $h_2^{t'}$ are two histories with $t \leq t'$. This is a generalization of the uniform metric, which takes into account the different lengths of the histories. It is straightforward to verify that $d(\cdot,\cdot)$ fits the definition of a metric, and hence, we can define open sets of histories with respect to this metric. Note that $H_i$ is the counterpart of $\Gamma_i$ defined in (3.2) when strategies are non-Markov: It comprises the histories in which firm $i$ randomizes between remaining in the market and exiting. With the assumption that $H_i$ is an open set, firm $i$'s strategy is smooth in the sense that if $\lambda(h^t) > 0$, then for any other history $h^{t'}$ close enough to $h^t$, we have $\lambda_i(h^{t'}) > 0$. We summarize these conditions below.

**Condition 2** *Assume that each firm i's strategy* $a_i = \left(I_i, \{\tau_{i,n}, p_{i,n}\}_{n=1}^{N_i}, \lambda_i(h)\right)$, *where*
*(i)* $I_i = \emptyset$ *for each i,*
*(ii)* $N_i < \infty$ *and* $\tau_{i,n} < \infty$ *a.s for all i and n, and*
*(iii)* $H_i = \{h : \lambda_i(h) > 0\}$ *is an open set.*

Conditions 2(i) and (ii) ensure that there exists a stopping time $\overline{\tau} := \max_n\{\tau_{1,n}, \tau_{2,n}\}$ such that exit after this time occurs only via the hazard rate, which implies that there exists a subgame (starting at $\overline{\tau}$) that is reached with positive probability, in which exit occurs only via the hazard rates.[15][16] Let $H_i^{\overline{\tau}}$ denote the set of histories in that subgame (*i.e.,* the histories $h^t \supseteq h^{\overline{\tau}}$) such that $\lambda_i(h) > 0$. The

[15]Put differently, each firm exits during any interval of length $dt$ with probability $\lambda_i(h^t)dt$ for all $t > \overline{\tau}$. The restriction that $\tau_{i,n} < \infty$ a.s simplifies the exposition by ensuring that $\max_n\{\tau_{1,n}, \tau_{2,n}\} < \infty$ a.s. It can be relaxed, and a proof of the results in this section absent this condition is available upon request.

[16]Note that, with Condition 2(i), we can focus on "proper" mixed strategies, in which the probability of exit is always less than one, and this is the case in all the known examples of mixed-strategy equilibria in the literature.

following lemma shows that in a (mixed-strategy) SPE, the firms must randomize between remaining in the market and exiting over a common set of histories in that subgame.

**Lemma G.1** *Suppose Condition 2 is satisfied and $\{a_1, a_2\}$ constitutes a mixed-strategy SPE. Then $H_1^{\bar{\tau}} = H_2^{\bar{\tau}}$.*

**Proof of Lemma G.1**: Define the stopping time $\bar{\tau} := \max_n \{\tau_{1,n}, \tau_{2,n}\}$, and note that it is finite a.s by Condition 2(ii). Let $H_i^{\bar{\tau}} := \{h \supseteq h^{\bar{\tau}} : \lambda_i(h) > 0\}$ denote the set of histories $h$ in the subgame $h^{\bar{\tau}}$ in which firm $i$ exits with hazard rate $\lambda_i(h) > 0$. Next, given a history $h^v$ with $v > \bar{\tau}$, define $H_i^v := \{h^t \supseteq h^v : \lambda_i(h^t) > 0\}$. Then because $H_i^{\bar{\tau}} = \cup_{v > \bar{\tau}} H_i^v$, it is enough to show that $H_1^v = H_2^v$ for any given history $h^v$ with $v > \bar{\tau}$. Now, towards a contradiction, fix some history $h^v$ with $v > \bar{\tau}$ and suppose that there exists an open set $C_1 \subset H_1^v$ such that $C_1 \cap H_2^v = \emptyset$. Pick a history $h^t \in C_1$ and define $\tau(\delta) := \inf\{s \geq t : X_s \notin (X_t - \delta, X_t + \delta)\}$, i.e., $\tau(\delta)$ is the exit time of a $\delta$-neighborhood of $X_t$.

We first note that because $C_1$ is open and $h^t \in C_1$, there exists some $\varepsilon > 0$ such that any history $h$ with $d(h^t, h) < \varepsilon$ must belong to $C_1$ (i.e., a small enough neighborhood of $h^t$ must be contained in $C_1$). Next, based on the definition of the metric $d(\cdot, \cdot)$ on the space of histories of $X$, there must exist some $\delta > 0$ and some time $u > t$ such that $d(h^t, h^{\tau(\delta) \wedge u}) < \varepsilon$, which implies that $h^{\tau(\delta) \wedge u} \in C_1$ by our first note. This is because any history $h^{t'}$ in the subgame $h^t$ (i.e., a history $h^{t'} \supseteq h^t$) must be close enough to $h^t$ with respect to the metric $d(\cdot, \cdot)$ as long as $X_{t'}$ (the value of $X$ in the history $h^{t'}$) is not too much different from $X_t$ and $t' - t$ is not too large. Then for any $s \in [t, \tau(\delta) \wedge u]$, we have $d(h^t, h^s) \leq d(h^t, h^{\tau(\delta) \wedge u}) < \varepsilon$, which implies that $h^s \in C_1$ for any $s \in [t, \tau(\delta) \wedge u]$. Hence, for any $s \in [t, \tau(\delta) \wedge u]$, we have $h^s \notin H_2$ because $C_1 \cap H_2 = \emptyset$.

Next, recall that $h^t \in C_1 \subseteq H_1^v$ and $v > \bar{\tau}$, which implies that $s > \bar{\tau}$ for any $s \in [t, \tau(\delta) \wedge u]$. Let $H^{\tau(\delta) \wedge u} := \{h : h \supseteq h^{\tau(\delta) \wedge u}\}$ and $H^t := \{h : h \supseteq h^t\}$ be the sets of all the histories that contain $h^{\tau(\delta) \wedge u}$ and $h^t$ respectively. Then because $h^s \notin H_2^v$ and $s > \bar{\tau}$ for all $s \in [t, \tau(\delta) \wedge u]$, firm 2 does not exit at all within this time interval, which means that

$$V_1\left(h^t; \left(H^{\tau(\delta) \wedge u}, \mathcal{P}_1, \mathbf{0}\right), a_2\right) - V_1\left(h^t; \left(H^t, \mathcal{P}_1, \mathbf{0}\right), a_2\right) = \mathbb{E}\left[\int_t^{\tau(\delta) \wedge u} \left(\pi(X_s) - r l_1\right) e^{-r(s-t)} ds\right] < 0,$$

where the inequality holds because $X_s < \theta_i^*$ is a necessary condition for a mixed strategy region for firm 1, in which case $\pi(X_s) - r l_1 < 0$. Here $\mathbf{0}$ means that $\lambda_i(h) = 0$ for any history $h$. This, however, contradicts that $h^{\tau(\delta) \wedge u} \in C_1$ because the payoff from exiting at time $\tau(\delta) \wedge u$ is strictly less than that from exiting at time $t$, which implies that $\lambda_1(h^{\tau(\delta) \wedge u}) > 0$ is not a best response to $a_2$. ■

This is a counterpart of Lemma 2 when strategies are not constrained to be Markov. It is helpful to convey the intuition with a heuristic derivation. Towards a contradiction, suppose that there exists a non-empty open set of histories $C \subset H_i^{\bar{\tau}} \backslash H_{-i}^{\bar{\tau}}$. Pick a history $h^t \in C$. Then $V_i(h'; a) = l_i$ for any $h'$ in the vicinity of $h^t$; this is because any history $h'$ close enough to $h^t$ must also belong to $C \subset H_i^{\bar{\tau}}$, and if so, we must have $\lambda_i(h') > 0$, which requires $V_i(h';a) = l_i$ for any mixed-strategy equilibrium $a = (a_i, a_{-i})$. Hence, there is an infinitesimal time $\Delta t$ such that firm $i$ will be indifferent between immediate exit at $t$ and exit at $t + \Delta t$:

$$l_i = \pi(X_t)\Delta t + \exp(-r\Delta t)l_i + o(\Delta t) = l_i + [\pi(X_t) - rl_i]\Delta t + o(\Delta t),$$

where we have used that an exit after an infinitesimal time earns $l_i$ because $h^{t+\Delta t} \in C$. Then the indifference equation leads to $\pi(X_t) = rl_i$, which contradicts the fact that $\pi(X_t) < rl_i$ for any $h^t \in H_i$. Therefore, we can conclude that $C$ is empty, and so $H_1^{\bar{\tau}} = H_2^{\bar{\tau}}$.

The following theorem shows that if the firms have heterogeneous outside options, then subject to Condition 2, the game admits no mixed-strategy SPE; *i.e.,* there exists no SPE such that $H_1 \cup H_2 \neq \emptyset$.

**Theorem 2** *Suppose that each firm's strategy must satisfy Condition 2. If $l_1 < l_2$, then no mixed-strategy SPE exists.*

**Proof of Theorem 2.** Towards a contradiction, suppose that $l_1 < l_2$, the strategies $a_1$ and $a_2$ satisfy Condition 2, and the strategy profile $\{a_1, a_2\}$ constitutes a mixed-strategy SPE. By Lemma G.1, we have $H_1^{\bar{\tau}} = H_2^{\bar{\tau}}$, so define $H^{\bar{\tau}} := H_1^{\bar{\tau}}$.

Fix a finite time $t$ such that $t > \bar{\tau}$ a.s and $X_t \geq \max\{\theta_1^*, \theta_2^*\}$ (reached with positive probability), and let $\tau_H := \inf\{s > t : h^s \in H^{\bar{\tau}}\}$ denote the first hitting time of the mixed-strategy region after time $t$. We first note that $\lambda_i$ is defined as a progressively measurable non-negative process (See Section 2.1), i.e., $\lambda_i = \{\lambda_{i,s}\}_{s \geq 0}$ where $\lambda_{i,s}$ is firm $i$'s hazard rate of exit at time $s$. Then based on the definitions of $H^{\bar{\tau}}$ and $\tau_H$, we have

$$\tau_H = \inf\{s > t : \lambda_{i,s} > 0\},$$

which is a stopping time with respect to $\mathcal{F}_X$.

Next, we have already established that $H$ does not intersect with the region in which $X_t > \theta_i^*$ for either $i$. Because $X_t \geq \max\{\theta_1^*, \theta_2^*\}$, we have $\tau_H \geq \tau_i^* := \inf\{s > t : X_s \leq \theta_i^*\}$ for each $i$ a.s, that is, the first hitting time of $H^{\bar{\tau}}$ is at least as long as the hitting time of $(a, \theta_1^*]$, as well as $(a, \theta_2^*]$.

We now show that $\tau_H = \tau_i^*$ a.s for both $i$. Towards a contradiction, suppose that $\tau_H \neq \tau_i^*$ with positive probability. At $\tau_H$, firm $i$'s payoff

$$V_i(h^{\tau_H}; a_i, a_{-i}) = l_i = V_i(h^{\tau_H}; \left(H^{\bar{\tau}}, \mathcal{P}_i, \mathbf{0}\right), a_{-i}) = V_i(h^{\tau_H}; \left(H^{\bar{\tau}}, \mathcal{P}_i, \mathbf{0}\right), (\emptyset, \mathcal{P}_{-i}, \mathbf{0})).$$

The first equality follows from the fact that firm $i$'s payoff must be equal to $l_i$ at $\tau_H$ by definition of a mixed-strategy equilibrium. The third term represents firm $i$'s payoff function if it exits with probability one whenever $h^t \in H^{\bar{\tau}}$ and its opponent's strategy is $a_{-i}$, whereas the last term represents firm $i$'s payoff function if it exits with probability one whenever $h^t \in H^{\overline{\tau}}$ and its opponent never exits. The third equality follows because $\tau_H > \overline{\tau}$ (so $\mathcal{P}_{-i}$ has no impact on firm $i$'s payoff) and $\lambda_{-i,\tau_H} = 0$ by the definition of $\tau_H$ and $\overline{\tau}$. The second and third equalities assert that firm $i$'s payoff at $\tau_H$ under the specified strategy profiles equals $l_i$.

Therefore, if $\tau_H \neq \tau_i^*$ with non-zero probability, then there exists a history $h^t$ (where $t$ has been fixed in the beginning of the proof) such that

$$\begin{aligned} V_i(h^t; \left(H^{\overline{\tau}}, \mathcal{P}_i, \mathbf{0}\right), a_{-i}) &= V_i(h^t; \left(H^{\overline{\tau}}, \mathcal{P}_i, \mathbf{0}\right), (\emptyset, \mathcal{P}_{-i}, \mathbf{0})) \\ &< V_i(h^t; \left(H^{\tau_i^*}, \mathcal{P}_i, \mathbf{0}\right), (\emptyset, \mathcal{P}_{-i}, \mathbf{0})) = V_i(h^t; \left(H^{\tau_i^*}, \mathcal{P}_i, \mathbf{0}\right), a_{-i}), \end{aligned}$$

where $H^{\tau_i^*} := \{h^t : X_t \leq \theta_i^*\}$ is the set of histories in which $X_t \leq \theta_i^*$. Here the inequality follows because by Lemma 1, exiting whenever $X_t \leq \theta_i^*$ is firm $i$'s unique best response to an opponent who never exits, and $\tau_H \neq \tau_i^*$ with positive probability. The first equality follows because $t > \overline{\tau}$ (so $\mathcal{P}_{-i}$ has no impact on firm $i$'s payoff) and $\lambda_{-i,\tau_H} = 0$. The second equality follows because $t > \overline{\tau}$ and $\tau_H \geq \tau_i^*$ a.s. This contradicts the assumption that $(a_1, a_2)$ is an SPE, and hence we conclude that $\tau_H = \tau_i^*$ a.s. However, this is not possible if $\theta_1^* \neq \theta_2^*$, or equivalently, if $l_1 < l_2$, yielding a contradiction.

Finally, note that any such $t$ is reached with positive probability by Condition 2(i) and because $X$ is irreducible. Therefore, it follows that no mixed-strategy SPE exists. ■

We briefly give a summary of the proof here. First recall that $H_1^{\overline{\tau}} = H_2^{\overline{\tau}}$ by Lemma G.1. For any history in $H_i^{\overline{\tau}}$, an indifference condition similar to (3.3) must be satisfied, which implies that $\lambda_{1,t}$ and $\lambda_{2,t}$ depend only on the current state, $X_t$. It is shown, using a similar argument as in Lemma 2, that $H_i^{\overline{\tau}}$ must consist of histories such that $X_t \leq \theta_i^*$. But this implies that $H_1^{\overline{\tau}} \neq H_2^{\overline{\tau}}$ whenever $l_1 < l_2$, a contradiction. Finally, because $H_i^{\overline{\tau}}$ is reached with positive probability (by Conditions 2(i) and (ii)), it follows that no mixed-strategy SPE exists.