Practice Problems

Information Economics (Ec 515) · George Georgiadis

Problem 1. Static Moral Hazard

Consider an agency relationship in which the principal contracts with the agent. The monetary result of the relationship depends on both agent's effort and state of nature as follows:

states:	θ_1	θ_2	θ_3
result when $e = 6$	$x = 60,000$	$x = 60,000$	$x = 30,000$
result when $e = 4$	$x = 30,000$	$x = 60,000$	$x = 30,000$

Both parties believe that the probability of each state is one third. The payoff functions of the principal and the agent are give by

$$B(x, w) = x - w$$
$$U(w, e) = \sqrt{w} - e^2,$$

where x is the monetary result of the relationship and w is the wage received by the agent. Suppose that the agent will only accept the contract if he obtains an expected utility level of at least 114.

1. What would be the effort and the wage if effort is contractible?
2. What is the optimal contract when effort is not contractible? What wage scheme induces $e = 4$ at the minimum cost for the principal? What wage scheme induces $e = 6$ at the minimum cost for the principal?
3. Which effort level does the principal prefer when effort is not observable? Discuss.

Problem 2: Normal-Linear Model

The following normal-linear model is regularly used in applied models. Given action a, output is $q = a + x$, where $x \sim N\left(0, \sigma^2\right)$. The cost of effort is $c(a)$, where $c(\cdot)$ is increasing and convex. The agent's utility equals $u(w(q) - c(a))$, while the principal's profit is $q - w(q)$. Suppose that the agent's reservation utility is $u(0)$.

Assume that the principal uses a linear contract of the form $w(q) = \alpha + \beta q$, and that the agent's utility is CARA: $u(w) = -e^{-w}$.

1. Suppose that $w \sim N\left(\mu, \sigma^2\right)$. Shows that $\mathbb{E}\left[u(w)\right] = u(\bar{w})$, where $\bar{w} = \mu - \frac{\sigma^2}{2}$.[1]
2. Suppose that effort is unobservable. The principal's problem is

$$\begin{aligned}
\max_{w(\cdot), a} \quad & \mathbb{E}\left[q - w(q)\right] \\
\text{s.t.} \quad & \mathbb{E}\left[u\left(w(q) - c(a)\right) \mid a\right] \geq u(0) \\
& a \in \arg\max_{a' \in \mathbb{R}} \mathbb{E}\left[u\left(w(q) - c(a')\right) \mid a'\right]
\end{aligned}$$

[1] We call $\bar{w}$ the certainty equivalent of w.

Using the first order approach, characterize the optimal contract (α, β, a). [*Hint:* write utilities in terms of their certainty equivalent.]

3. How would the solution change if the agent knows x before choosing his action (but after signing the contract)[2] ? What if $u(\cdot)$ is an arbitrary strictly increasing and concave function ?

Problem 3: Insurance

Consider a risk-averse agent, with increasing and concave utility $u(\cdot)$ and initial wealth W_0, who faces the risk of having an accident and losing an amount x of her wealth. The agent has access to a perfectly competitive market of risk-neutral insurers who can offer schedules $R(x)$ of repayments net of any insurance premium.[3] Assume that the distribution of x, which depends on accident-prevention effort a, is as follows:

$$\begin{aligned} f(0,a) &= 1-p(a) \\ f(x,a) &= p(a)g(x) \quad \text{for } x>0 \end{aligned}$$

where $\int g\left(x\right)dx = 1$ and $p(\cdot)$ is strictly decreasing and convex. The individual's (increasing and convex) cost of effort, separable from her utility of money, is $c(\cdot)$. Therefore, the agent's utility is given by

$$u\left(W_0 - x + R\left(x\right)\right) - c\left(a\right)$$

1. Suppose there is no insurance market. What action $\hat{a}$ does the agent take?

2. Suppose that a is contractible. Describe the first best payment schedule $R(x)$ and the effort choice a^*.

3. Suppose a is not contractible. Describe the second best payment schedule $R(x)$.

4. Interpret the second best payment schedule. Would the agent ever have an incentive to hide an accident? (*i.e.*, report $x=0$ when $x>0$).

Problem 4: Private Evaluations with Limited Liability

A principal employs an agent. The agent privately chooses an action $a \in \{L, H\}$ at cost $c(a)$, where $c(H) > c(L)$. The principal privately observes output $q \sim f\left(q \,|\, a\right)$ on $\left[\underline{q}, \bar{q}\right]$. Assume that this distribution function satisfies strict MLRP; *i.e.*, $\frac{f(q\,|\,H)}{f(q\,|\,L)}$ is strictly increasing in q. Suppose the principal reports that output is $\tilde{q}$. The principal then pays out $t(\tilde{q})$, the agent receives $w(\tilde{q})$, where $w(\tilde{q}) \leq t(\tilde{q})$, and the difference is *burned*. The payments $\{t, q\}$ are contractible.

Payoffs are as follows. The principal receives $q - t$. The agent receives $u(w) - c(a)$ where $u(\cdot)$ is strictly increasing and concave. The agent has no (IR) constraint, but does have limited liability; *i.e.*, $w(q) \geq 0$ for all q.

1. First, suppose that the principal wishes to implement $a = L$. Characterize the optimal contract.

2. Second, suppose that the principal wishes to implement $a = H$.

   (a) Write down the principal's problem as maximizing expected profits subject to the agent's (IC) constraint, the principal's (IC) constraint, the limited liability constraint and the constraint that $w(q) \leq t(q)$.

[2] In other words, the agent observes the noise x and chooses his effort a as a function of x.

[3] This assumption implies that the expected profit of each insurer will be equal to 0.

(b) Argue that $t(q)$ is independent of q.

(c) Characterize the optimal contract. How does the wage vary with q?

Problem 5: Moral Hazard with Persistent Effort (25 points)

An agent chooses effort $e \in \{e_L, e_H\}$ at time 0 at cost $c(e) \in \{0, c\}$. At time $t \in \{1, 2\}$, output is binomial; *i.e.*, $q_t \in \{q_L, q_H\}$ is realized according to the i.i.d distribution $\Pr\{q_t = q_H | e_L\} = \pi_L$ and $\Pr\{q_t = q_H | e_H\} = \pi_H$. A contract is a pair of wages $\{w_1(q_1),\, w_2(q_1, q_2)\}$. The agent's utility is then

$$u\left(w_1(q_1)\right) + u\left(w_2(q_1, q_2)\right) - c(e)\,,$$

where $u(\cdot)$ is increasing and concave, while the firm's profits are

$$q_1 + q_2 - w_1(q_1) - w_2(q_1, q_2)\,,$$

where we ignore discounting. The agent has outside option $2u_0$. Also, assume that the principal wishes to implement effort e_H.

1. What is the first best contract, assuming effort is observable?
2. Suppose the firm cannot observe the agent's effort. Set up the firm's problem.
3. Characterize the optimal first-period and second-period wages.
4. How do wages vary over time? In particular, can you provide a full ranking of wages across the different states and time periods?

Problem 6: Screening

Consider the monopoly problem analyzed in section 2.1, but assume that the monopolist has one unit of the good for sale, at zero cost, while the buyer can have the following utility:

$$\theta_L - T$$

or:

$$\log(\theta_H - T).$$

The buyer's risk aversion thus rises with her valuation. Show that the seller can implement the first-best outcome (that is, sell the good for sure, leave no rents to either type of buyer, and avoid any cost of risk in equilibrium) by using a random scheme.

Problem 7: Costly State Verification

Consider a financial contracting problem between a wealth-constrained, risk-neutral entrepreneur and a wealthy risk-neutral investor. The cost of investment at date $t = 0$ is I. The project generates a random return on investment at date $t = 1$ of $\pi\left(\theta, I\right) = 2 \min\left\{\theta, I\right\}$, where θ is the state of nature, uniformly distributed on $[0, 1]$.

1. Characterize the first-best level of investment, I^{FB}.
2. Suppose that the realized return at $t = 1$ is freely observable only to the entrepreneur. A cost $K > 0$ must be paid for the investor to observe $\pi(\theta, I)$. Derive the second-best contract under the assumptions of(a)

deterministic verification and (b) zero expected profit for the investor, taking into account that repayments cannot exceed realized returns (net of inspection costs).

3. Show that the second-best optimal investment level is lower than I^{FB}.

Problem 8. Auctions

Consider a two-person, independent private-value auction with valuations uniformly distributed on $[0,1]$. Compare the following assumptions on utilities: (a) bidder i ($i = 1,2$) has utility $v_i - P$ when she wins the object and has to pay P, while her outside option is normalized to zero; (b) bidder i ($i = 1,2$) has utility $\sqrt{v_i - P}$ when she wins the object and has to pay P, while her outside option is normalized to zero.

1. Compare the seller's expected revenue in cases (a) and (b) for the Vickrey auction.

2. Compare the seller's expected revenue in cases (a) and (b) for the (linear) symmetric bidding equilibrium of the first-price, sealed-bid auction.

3. Discuss.

Problem 9: Public Goods Provision

A firm is considering building a public good (*e.g.*, a swimming pool). There are n agents in the economy, each with i.i.d private value $\theta_i \sim U[0,1]$. The cost of the swimming pool is cn, where $c > 0$.

First suppose the government passes a law that says the firm cannot exclude people from entering the swimming pool. A mechanism thus consists of a build decision $P(\theta_1, .., \theta_n) \in [0,1]$ and a payment by each agent $t_i \in (\theta_1, .., \theta_n) \in \mathbb{R}$. The mechanism must be individually rational and incentive compatible.

1. Consider an agent with type i, whose utility is given by

$$\theta_i P - t_i$$

   Derive her utility in a Bayesian incentive compatible mechanism.

2. Given an build decision $P(\cdot)$, derive the firm's profits.

3. What is the firm's optimal build decision?

4. Show that as $n \to \infty$, the probability of provision $P(\theta_1, .., \theta_n)$ goes to 0 for all $\{\theta_1, .., \theta_n\}$.

Problem 10. VCG Mechanism

Suppose Bill and Linda are buying a new car. They refuse to drive anything other than Italian sports cars, Ferrari ($\mathcal{F}$), Lamborghini ($\mathcal{L}$), or Maserati ($\mathcal{M}$). Bill's type, θ_B, is well known, while Linda is mysterious and can be of two types, θ_L^1 or θ_L^2. Their utility functions are as follows.

$$u_B(x) = \begin{cases} 9 & \text{if } x = \mathcal{F} \\ 7 & \text{if } x = \mathcal{M} \\ -1 & \text{if } x = \mathcal{L} \end{cases}, \quad u_L(x|\theta_L^1) = \begin{cases} 2 & \text{if } x = \mathcal{F} \\ 5 & \text{if } x = \mathcal{M} \\ 7 & \text{if } x = \mathcal{L} \end{cases}, \quad u_L(x|\theta_L^2) = \begin{cases} 6 & \text{if } x = \mathcal{F} \\ 8 & \text{if } x = \mathcal{M} \\ 1 & \text{if } x = \mathcal{L} \end{cases}$$

1. What is the efficient decision rule (*i.e.*, first best?)

2. Is there a mechanism that implements the decision rule? If so, fully write out the mechanism and prove that it implements d, as well as satisfies the agent's IR and IC constraints. Assume payoff is linear, $u_i - t_i$, and each agent's outside option is 0 (i.e., they must both agree, or no car is purchased). Is this mechanism budget balanced?

3. Suppose Linda now has an outside option of 4. Can the above still be implemented? What can be implemented?