Module 9: Dynamic Principal-Agent Problems

Information Economics (Ec 515) · George Georgiadis

- Effects of dynamics:

  1. Consumption smoothing
     - Bad shocks can be smoothed over time.
     - Agent becomes more risk-accepting, increasing surplus.
  2. Statistical inference
     - Repeated observations provide better information, increasing surplus.
  3. Agents' action set increases
     - An agent can offset bad performance in one period by working harder in the next.
  4. Renegotiation
     - Principal and agent may have an incentive to change their contract as time evolves.
  5. Formal contracts become less important
     - Can use relational contracts to provide incentives (moral hazard).
     - Use reputational concerns (implicit incentives) to motivate (adverse selection).

Examples of dynamic models:

1. One action, many outputs.

   - Time $t \in \{1, .., T\}$
   - Agent takes action a at $t = 0$.
   - Output $q_t = a + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$.
     - Then $q_T \sim N(na, n\sigma^2)$

- Optimal contract will punish heavily at the left tail: $w(q) = \begin{cases} w^* & \text{if } q_T \geq \underline{q} \\ -K & \text{otherwise.} \end{cases}$
    - Achieves first best asymptotically (Mirrlees, 1975)

2. Many actions, one output.
    - Agent chooses a_t at time $t \in \{1, .., T\}$.
    - Output q_T is obtained at $t = T$.
    - Just like the static problem where action is $a = \sum_{i=1}^{T} a_t$ at cost $\sum_{i=1}^{T} c(a_t)$.

3. Many actions, many outputs.
    - Agents chooses a_t at time $t \in \{1.., T\}$.
    - Output $q_t = a_t + \epsilon$ in each period t.
    - In period t, agents receives wage w_t that is a function of $\{q_1, .., q_{t-1}\}$.

A Two-period Principal-Agent Model

Rogerson (Econometrica, 1985)

- Time $t \in \{1, 2\}$
- Effort $a_t \in A \subseteq \mathbb{R}$ at cost $c(a_t)$
- Output $q_t \sim f(q \mid a_t)$
- Agent's utility $= \sum_t [u(w_t) - c(a_t)]$
    - *i.e.*, preferences are time-separable.
    - Reservation utility $\bar{u}$ in each period.
- Contract: $w_1(q_1)$, $w_2(q_1, q_2)$
- Principal's profit $= \sum_t (q_t - w_t) = [q_1 - w_1(q_1)] + [q_2 - w_2(q_1, q_2)]$
- Links between periods:
    1. No technological link or changes in preferences of the agent.
    2. Principal can use w_2 to reward a_1.

- Agent has no access to credit markets (*i.e.*, cannot borrow or save).

- Action at $t = 2$ will depend on q_1; *i.e.*, $a_2(q_1)$.
- Principal's maximization problem:

$$\begin{aligned} \max_{w_1, w_2, a_1, a_2} \quad & \mathbb{E}\left[q_1 - w_1(q_1) + q_2 - w_2(q_1, q_2) \mid \mathbf{a}\right] \\ \text{s.t.} \quad & \{a_1, a_2\} \in \arg\max_{\tilde{a}_1, \tilde{a}_2} \mathbb{E}\left[u(w_1(q_1)) - c(\tilde{a}_1) + u(w_2(q_1, q_2)) - c(\tilde{a}_2(q_1)) \mid \tilde{a}_1, \tilde{a}_2\right] \quad \text{(IC)} \\ & \mathbb{E}\left[u(w_1(q_1)) - c(a_1) + u(w_2(q_1, q_2)) - c(a_2(q_1)) \mid a_1, a_2\right] \geq 2\bar{u} \qquad \text{(IR)} \end{aligned}$$

- *Proposition:* Along the optimal path:

$$\frac{1}{u'(w_1(q_1))} = \mathbb{E}\left[\frac{1}{u'(w_2(q_1, q_2))} \mid q_1\right]$$

Proof.

- Suppose $w_1(q_1)$ and $w_2(q_1, q_2)$ is optimal.
- Consider modifying the contract such that $u(\hat{w}_1(q_1)) = u(w_1(q_1)) - \epsilon$ and $u(\hat{w}_2(q_1, q_2)) = u(w_2(q_1, q_2)) + \epsilon$.
- The agent's (IC) and (IR) are unaffected.
- Increases principal's profit by:

$$\Delta = \mathbb{E}\left[w_1(q_1) - \hat{w}_1(q_1) + w_2(q_1, q_2) - \hat{w}_2(q_1, q_2) \mid q_1\right]$$

- For small ϵ:
  - $w_1(q_1) - \hat{w}_1(q_1) = \frac{\epsilon}{u'(w_1(q_1))}$
  - $w_2(q_1, q_2) - \hat{w}_2(q_1, q_2) = -\frac{\epsilon}{u'(w_2(q_1, q_2))}$
  - Follows from applying the Taylor expansion $u(\hat{w}) = u(w) + u'(w)(\hat{w} - w) \implies \hat{w} - w = \frac{u(\hat{w}) - u(w)}{u'(w)}$.
- Then:

$$\Delta = \epsilon\mathbb{E}\left[\frac{1}{u'(w_1(q_1))} - \frac{1}{u'(w_2(q_1, q_2))} \mid q_1\right]$$

- Because we can pick $\epsilon \gtrless 0$, the optimal contract must satisfy $\frac{1}{u'(w_1(q_1))} = \mathbb{E}\left[\frac{1}{u'(w_2(q_1,q_2))} \mid q_1\right]$.

$\square$

- Implications:

  1. Contract has memory.
     - Suppose $w_2(q_1,q_2) = w_2(q_2)$. Then $\mathbb{E}\left[\frac{1}{u'(w_2(q_1,q_2))} \mid q_1\right] =$ constant (independent of q_1).
     - Hence $\frac{1}{u'(w_1(q_1))} =$ constant $\Longrightarrow$ no incentives in period 1.
  2. The principal front-loads consumption.
     - Jensen's inequality $\Longrightarrow \mathbb{E}\left[\frac{1}{u'(w_2(q_1,q_2))} \mid q_1\right] \geq \frac{1}{\mathbb{E}[u'(w_2(q_1,q_2)) \mid q_1]} \Longrightarrow u'(w_1(q_1)) \leq \mathbb{E}\left[u'(w_2(q_1,q_2)) \mid q_1\right]$.
     - *Intuition:* The principal forces the agent to consume more in the first period to keep his continuation wealth low, so that the marginal utility for money remains high.
     - The agent would like to save (not borrow).

- What does "front-loading consumption" mean?

  - Suppose the agent has $\$W$ that he can consume over two periods. Then he solves

$$\max_{w_1} \{u(w_1) + u(W - w_1)\}$$

    The first order condition implies that $u'(w_1) = u'(W - w_1)$, so that $w_1 = \frac{W}{2}$.
  - Because $u'(\cdot)$ is decreasing, we say that consumption is front-loaded if $u'(w_1) \leq u'(w_2)$ (because $w_1 \geq w_2$), and back-loaded if $u'(w_1) \geq u'(w_2)$ (because $w_1 \leq w_2$).

Infinitely Repeated Principal-Agent Problem

- We now extend the previous model to an infinitely-repeated relationship between the principal and the agent.

- It turns out that this model is easier to solve than the two-period model.

Spear and Srivastava (REStud, 1987)

- Time $t \in \mathbb{N}$
- Effort $a_t \in \{0, 1\}$ at cost $c(a) = c\,a$.
- Output $q_t \in \{q_L, q_H\}$, where $\Pr\{q_t = q_H | a_t = 1\} = \pi_1$, $\Pr\{q_t = q_H | a_t = 0\} = \pi_0$, and $\Delta = \pi_1 - \pi_0 > 0$.
- Discount rate $\delta \in (0, 1)$.
- Agent's utility:

$$U_t = \underbrace{\mathbb{E}[u(w_t) - c(a_t)]}_{\text{expected payoff in } t} + \delta \underbrace{\mathbb{E}[U_{t+1}]}_{\text{exp. continuation value in } t+1}$$

  - Outside option $\bar{u} = 0$.
  - Define $h = u^{-1}$. Note that $u' > 0$ and $u'' < 0 \implies h' > 0$ and $h'' > 0$.

- Principal's profit: $V_t = \mathbb{E}[S(q_t) - w_t] + \delta \mathbb{E}[V_{t+1}]$
  - $S(q_t)$ is the principal's profit from q_t. Denote $S_H = S(q_H)$ and $S_L = S(q_L)$
  - Assume the principal wants to implement $a_t = 1$ for all t.

- Contract specifies $\{w_t, U_{t+1}\}$ as a function of $\{q_t, U_t\}$; *i.e.*, it exhibits the Markov property.
  - Given the agent's utility U_t and his output q_t in period t, the contract specifies
    1. instantaneous utlity u_H or u_L (or equivalently wages $w = h(u)$); and
    2. continuation utility for the agent U_H or U_L

    if the output is q_H or q_L, respectively.

- Agent's IC constraint:

$$\begin{aligned} \pi_1(u_H + \delta U_H) + (1 - \pi_1)(u_L + \delta U_L) - c &\geq \pi_0(u_H + \delta U_H) + (1 - \pi_0)(u_L + \delta U_L) \\ \implies (u_H + \delta U_H) - (u_L + \delta U_L) &\geq \frac{c}{\Delta} \end{aligned}$$

- Principal's Problem: Given U, she solves

$$\begin{aligned} V(U) = \quad & \max_{u_L, u_H, U_L, U_H} \quad \pi_1[S_H - h(u_H)] + (1 - \pi_1)[S_L - h(u_L)] + \delta[\pi_1 V(U_H) + (1 - \pi_1) V(U_L)] \\ & \text{s.t.} \quad (u_H + \delta U_H) - (u_L + \delta U_L) \geq \frac{c}{\Delta} \\ & \qquad \pi_1(u_H + \delta U_H) + (1 - \pi_1)(u_L + \delta U_L) - c \geq U \end{aligned}$$

- Remarks:
  - First constraint is the agent's IC constraint.
  - Second constraint is the principal's "promise-keeping" (PK) constraint.
  - Note u_L, u_H, U_L, U_H will be functions of U.
  - We will assume that $V(\cdot)$ is concave.
- Write the Lagrangean:

$$\begin{aligned} V(U) = \max \ & \{\pi_1 [S_H - h(u_H)] + (1-\pi_1)[S_L - h(u_L)] + \delta [\pi_1 V(U_H) + (1-\pi_1) V(U_L)] \\ & + \lambda \left[(u_H + \delta U_H) - (u_L + \delta U_L) - \frac{c}{\Delta} \right] \\ & + \mu [\pi_1 (u_H + \delta U_H) + (1-\pi_1)(u_L + \delta U_L) - c - U] \} \end{aligned}$$

- First order conditions w.r.t U_H and U_L:

$$\begin{aligned} \pi_1 V'(U_H(U)) + \lambda + \mu\pi_1 &= 0 \\ (1-\pi_1) V'(U_L(U)) - \lambda + \mu(1-\pi_1) &= 0 \end{aligned}$$

  - Note that we make the dependence on U explicit.
  - Summing these equations gives $\mu = -[\pi_1 V'(U_H(U)) + (1-\pi_1) V'(U_L(U))]$
- First order conditions w.r.t u_H and u_L:

$$\begin{aligned} \pi_1 h'(u_H(U)) &= \lambda + \mu\pi_1 \\ (1-\pi_1) h'(u_L(U)) &= -\lambda + \mu(1-\pi_1) \end{aligned}$$

- *Claim 1:* (PK) binds.

Proof.

- Summing the FOC for u_H and u_L gives $\mu = \pi_1 h'(u_H(U)) + (1-\pi_1) h'(u_L(U)) > 0$, and by complementary slackness, (PK) binds.

$\square$

- *Claim 2:* $U_H(U) \geq U_L(U)$ and $u_H(U) \geq u_L(U)$.

Proof.

- From the FOC for U_H and u_H, and the expression for μ, we get:

$$\begin{aligned}\lambda &= \pi_1 (1-\pi_1) \left[h' (u_H (U)) - h' (u_L (U))\right] \\ &= \pi_1 (1-\pi_1) \left[V' (U_L (U) - V' (U_H (U)))\right]\end{aligned}$$

- Because $h(\cdot)$ is convex and $V(\cdot)$ is concave, this equation implies that $u_H(U) \geq u_L(U)$ if and only if $U_L(U) \leq U_H(U)$.
- To satisfy (IC), we cannot have $U_H(U) \leq U_L(U)$ and $u_H(U) \leq u_L(U)$ simultaneously.
- Therefore, we have $U_H(U) \geq U_L(U)$ and $u_H(U) \geq u_L(U)$.

$\square$

- *Takeaways:*
  - The optimal contract (again) exhibits memory.
  - Good performance today is rewarded by a higher wage today *and* a higher continuation utility (and vice versa); *i.e.*, dynamics provide consumption smoothing.
- Next Problem Set:
  - Assume that $u(w) = \frac{1}{r}\ln(1+rw)$. Then $h(u) = \frac{1}{r}(e^{rw}-1)$.
  - Guess that the principal's profit function has the form $V(U) = \alpha - \frac{1}{\beta}e^{\beta U+\gamma}$, whre α, $\beta \geq 0$, and γ are constants to be determined.
  - Solve for the constants $\{\alpha, \beta, \gamma\}$, and characterize the optimal contract; *i.e.*, $U_H(U)$, $U_L(U)$, $u_H(U)$, and $u_L(U)$.
  - How do $U_H(U) - U_L(U)$ and $u_H(U) - u_L(U)$ depend on δ? Provide some intuition for this result.

References

Board S., (2011), Lecture Notes.

Bolton and Dewatripont, (2005), *Contract Theory*, MIT Press.

Rogerson W.P., (1985), "Repeated Moral Hazard", *Econometrica.*

Spear S.E. and Srivastava S., (1987), "On Repeated Moral Hazard with Discounting", *Review of Economic Studies.*