Module 5: Generalized Principal Agent Problem

Information Economics (Ec 515) · George Georgiadis

- An agent and a firm / principal.
- Principal offers a wage contract $w(q)$; agent accepts or rejects it.
- Agent takes action $a \in A \subseteq \mathbb{R}$, where A is compact.
- Output $q \sim f(\cdot \mid a)$, which is observable to both parties.
- Parties observe q, agent is paid $w(q)$, and the game ends.

- Agent's expected utility $= \mathbb{E}[u(w)] - c(a)$
  - $u(\cdot)$ is increasing and concave, while $c(\cdot)$ is increasing and convex.
- Agent has outside option $\bar{u}$.

- Principal's expected payoff $\Pi = \mathbb{E}[q - w(q)]$.
- Principal makes a take-it-or-leave-it (TIOLI) offer $w(q)$ to the agent.
- Principal's Problem: Choose $w(\cdot)$ such that there exists an equilibrium action a^* that maximizes the principal's expected payoff.

Nonlinear Optimization (Kuhn-Tucker Theorem)

- Consider the following problem:

$$\begin{array}{rcl} \max\limits_{x_1,x_2,\ldots,x_k} f(x_1,x_2,...,x_k) & & s.t. \\ g_1(x_1,x_2,...,x_k) & \leq & 0 \\ g_2(x_1,x_2,...,x_k) & \leq & 0 \\ & \vdots & \\ g_n(x_1,x_2,...,x_k) & \leq & 0 \end{array}$$

- Suppose f is concave, and g_j is convex for all j.
- Note that all constraints have to be written as inequalities with $\leq$.

∘ Write the Hamiltonian:

$$L(x, \lambda) = f(x_1, x_2, ..., x_k) - \sum_{j=1}^{n} \lambda_j g_j(x_1, x_2, ..., x_k)$$

∘ Suppose that $x^* = (x_1^*, ..., x_k^*)$ and $\lambda^* = (\lambda_1^*, .., \lambda_n^*)$ solves

$$\min_{\lambda_1,..,\lambda_n} \max_{x_1,..,x_k} L(x, \lambda)$$

∘ Then $x^* = (x_1^*, ..., x_k^*)$ also solves the original problem.

∘ Note that:

- If $\lambda_j^* > 0$, then constraint $g_j(x)$ binds (*i.e.*, $g_j(x^*) = 0$).
- If $\lambda_j^* = 0$, then constraint $g_j(x)$ doesn't bind (*i.e.*, $g_j(x^*) < 0$).

Full Information Benchmark (First Best)

∘ Assume that the agent's action is verifiable; *i.e.*, contractible.

∘ Effectively, the action is chosen by the principal, who solves:

$$\max_{a, w(\cdot)} \mathbb{E}[q - w(q) \mid a]$$
$$\text{s.t. } \mathbb{E}[u(w(q)) \mid a] - c(a) \geq \bar{u} \qquad \text{(IR)}$$

∘ If (IR) is not satisfied, then the agent will reject the offer.

∘ *Claim:* (IR) binds.

Proof.

∘ Suppose $\mathbb{E}[u(w(q)) \mid a] - c(a) = \bar{u} + \delta$, where $\delta > 0$. Then reduce wage such that $\mathbb{E}[u(\tilde{w}(q)) \mid a] = c(a) + \bar{u}$.

∘ The agent's IR constraint is still satisfied and the principal's expected payoff increases.

$\square$

- Optimal Contract: 2-step approach.

    1. Fix the action a to implement, and find the optimal wage schedule $w(\cdot)$.
    2. Choose the optimal action a^*.

- **Step 1:** Find optimal wage $w(q)$ for a given action a.

    - Fix a. Then the principal solves

    $$\min_{\lambda \geq 0} \max_{w(\cdot)} \int_{\mathbb{R}} q - w(q) + \lambda \left[u(w(q)) - c(a) - \bar{u} \right] f(q \mid a) \, dq$$

        * λ is Lagrangian multiplier of (IR).

    - Can maximize pointwise:

    $$\max_{w} \{-w + \lambda u(w)\}$$

    - Maximization problem is concave.
    - First order condition: $-1 + \lambda u'(w) = 0 \Longrightarrow \frac{1}{u'(w(q))} = \lambda$ for all q.

    $$\Longrightarrow \text{ Pay a constant wage } w^*.$$

    - Because (IR) binds, the associated Lagrange multiplier must be > 0.
    - (IR): $u(w^*) - c(a) = \bar{u} \Longrightarrow w^* = u^{-1}(\bar{u} + c(a))$.

    - *Intuition:* Because the agent is risk averse, the principal wants to "insure" him. (Draw picture)

- **Step 2:** Find optimal action a^*.

$$\max_{a} \mathbb{E}[q \mid a] - u^{-1}(\bar{u} + c(a))$$

Moral Hazard

- In contrast to the previous case, here, the principal cannot choose the agent's action a.

- Instead, the agent will observe the wage schedule $w(\cdot)$ and choose his action a to maximize his expected utility.

- Principal's Problem:

$$\begin{aligned}
\max_{w(\cdot)} \;& \mathbb{E}\left[q - w\left(q\right) \mid a\right] \\
\text{s.t. } \;& \mathbb{E}\left[u\left(w\left(q\right)\right) \mid a\right] - c\left(a\right) \geq \bar{u} && \text{(IR)} \\
& a \in \arg\max_{\tilde{a}\in A} \mathbb{E}\left[u\left(w\left(q\right)\right) \mid \tilde{a}\right] - c\left(\tilde{a}\right) && \text{(IC)}
\end{aligned}$$

First best is attainable if:

(a) Agent is risk neutral; *i.e.*, $u\left(w\right) = w$.

- As before, (IR) will bind in the optimal contract so that $\mathbb{E}\left[w\left(q\right) \mid a\right] = \bar{u} + c\left(a\right)$.
- Principal's expected payoff is:

$$\mathbb{E}\left[q \mid a\right] - \mathbb{E}\left[w\left(q\right) \mid a\right] = \mathbb{E}\left[q \mid a\right] - c\left(a\right) - \bar{u}$$

- *Claim:* A "sell the firm" contract $w\left(q\right) = q - k$ is first best.

Proof.

- Agent's problem: Choose a^* that solves $\max\left\{\mathbb{E}\left[q \mid a\right] - c\left(a\right) - k\right\}$.
- The agent's optimization problem is identical to that of the principal (up to a constant).
- Pick k such that (IR) binds: $k = \mathbb{E}\left[q \mid a^*\right] - c\left(a^*\right) - \bar{u}$.

□

- *Intuition:* The agent pays a commission to the principal and becomes the residual claimant.
- *Problem:* $w\left(q\right)$ can be negative for some q; *i.e.*, the agent might have to pay the principal.
  - What if the agent cannot pay (*i.e.*, is credit constrained) ?

(b) The "cheapest" action is first best.

- $c\left(\underline{a}\right) = \min_{a\in A} c\left(a\right)$
- Offer a flat wage $w\left(q\right) = w^* = u^{-1}\left(\bar{u} + c\left(\underline{a}\right)\right)$.

- (IC) is trivially satisfied because $u(w^*) - c(\underline{a}) \geq u(w^*) - c(a)$ for all $a \in A$.

(c) Shifting Support.

- Define the set: $S(a) = \{q \, : \, f(q \,|\, a) > 0\}$.
- Shifting support if for any $a \neq a^*$, $S(a) - S(a^*)$ has positive measure.
- *Example:* $q = a + \epsilon$ where $\epsilon \sim U(-1, 1)$.
  - Offer wage $w(q) = \begin{cases} w^* & \text{if } q \in S(a^*) \\ -\infty & \text{otherwise}. \end{cases}$
  - *Problem:* If the agent cannot pay the principal, then this contract is not credible.
- In general, we need: $\frac{f(q \,|\, a)}{f(q \,|\, a^*)} \longrightarrow -\infty$ as $q \to -\infty$.
- Mirrlees: If $q = a + \epsilon$ and $\epsilon \sim N(0, \sigma)$, then the above condition is satisfied $\Longrightarrow$ approximate first best.
  - We will get back to that later.

Two Actions:

- $a \in \{L, \, H\}$ and $c(H) > c(L) = 0$
- Two-step approach:
  1. Find cheapest wage schedule $w(q)$ to implement a.
  2. Find "best" action subject to (IC) and (IR).
- Implement $a = L$:
  - Flat wage: $w^* = u^{-1}(\bar{u})$
  - $\Pi = \mathbb{E}[q \,|\, L] - u^{-1}(\bar{u})$
- Implement $a = H$:

$$\begin{aligned} &\max_{w(\cdot)} \; \mathbb{E}[q - w(q) \mid H] \\ &\text{s.t. } \mathbb{E}[u(w(q)) \mid H] - c(H) \geq \bar{u} \qquad (\lambda) \\ &\qquad \mathbb{E}[u(w(q)) \mid H] - c(H) \geq \mathbb{E}[u(w(q)) \mid L] \qquad (\mu) \end{aligned}$$

- Write Lagrangian:

$$\begin{aligned} L(\lambda,\mu) &= \max_{w(\cdot)} \quad \int_{\mathbb{R}} [q - w(q)] + \lambda [u(w(q)) - c(H) - \bar{u}] + \mu [u(w(q)) - c(H)] \; dF(q \mid H) \\ &\qquad - \underbrace{\int_{\mathbb{R}} \mu [u(w(q))] \; dF(q \mid L)}_{\int_{\mathbb{R}} \mu \left[u(w(q)) \frac{f(q \mid L)}{f(q \mid H)}\right] dF(q \mid H)} \end{aligned}$$

- *Claim:* λ, $\mu > 0$

Proof.

- (IR) binds (same proof as before) $\Rightarrow \lambda > 0$.
- Suppose $\mu = 0$.

  $\Rightarrow$ (IC) is redundant

  $\Rightarrow$ wage $w(q) = w^*$ (i.e., flat)

  $\Rightarrow$ (IC) is not satisfied since $c(H) > 0$, which is a contradiction.

$\square$

- Maximize Lagrangian pointwise (with respect to w):

$$\begin{aligned} -1 + \lambda u'(w(q)) + \mu \left[u'(w(q)) - u'(w(q)) \frac{f(q \mid L)}{f(q \mid H)} \right] &= 0 \\ \Longrightarrow \left[\lambda + \mu - \mu \frac{f(q \mid L)}{f(q \mid H)} \right] u'(w(q)) &= 1 \end{aligned}$$

$$\Longrightarrow \underbrace{\frac{1}{u'(w(q))}}_{\uparrow \text{ in } w} = \underbrace{\lambda}_{>0} + \underbrace{\mu}_{>0} \left[1 - \frac{f(q \mid L)}{f(q \mid H)} \right]$$

- Monotone Likelihood Ratio (MLR): $\frac{f(q \mid L)}{f(q \mid H)}$ decreases in q.
  - *Intuitively:* This implies that if q is larger, then it is more likely that $a = H$ relative to $a = L$.
- MLR $\Longrightarrow w(q)$ increases in q. Why?
  - RHS increases in q, so the LHS must also increase in q.
  - $u(\cdot)$ is concave, so $u'(w)$ decreases in w, so $\frac{1}{u'(w)}$ increases in w.
  - Therefore, $w(q)$ must increase in q.

Continuum of Actions

- Principal's Problem:

$$\begin{aligned} \max_{w(\cdot)}\ & \mathbb{E}\left[q - w(q) \mid a\right] \\ \text{s.t. } & V(a) \geq \bar{u} \qquad\qquad \text{(IR)} \\ & a \in \arg\max_{\tilde{a}\in A} V(\tilde{a}) \qquad \text{(IC)} \end{aligned}$$

  where $V(a) = \mathbb{E}\left[u(w(q)) \mid a\right] - c(a) = \int_{\mathbb{R}} u(w(q))\, f(q \,|\, a)\, dq - c(a)$.

- First Order Approach: Replace (IC) with FOC: $V'(a) = 0$

$$V'(a) = \int_{\mathbb{R}} u(w(q))\, f_a(q \,|\, a)\, dq - c'(a) = 0$$

  - Note: $f_a(q \,|\, a) = \frac{d}{da} f(q \,|\, a)$.
  - We will discuss later when the FOC approach is sufficient.

- Write Lagrangian:

$$L(\lambda, \mu) = \max_{a\,,\, w(\cdot)} \int_{\mathbb{R}} [q - w(q)] + \lambda\left[u(w(q)) - c(a) - \bar{u}\right] + \mu\left[u(w(q))\frac{f_a(q \,|\, a)}{f(q \,|\, a)} - c'(a)\right] dF(q \,|\, a)$$

- Maximize pointwise (with respect to w):

$$-1 + \lambda u'(w(q)) + \mu u'(w(q))\frac{f_a(q \,|\, a)}{f(q \,|\, a)} = 0$$

$$\Longrightarrow \frac{1}{u'(w(q))} = \lambda + \mu\frac{f_a(q \,|\, a)}{f(q \,|\, a)}$$

- Monotone Likelihood Ratio (MLR): $\frac{f(q \,|\, a_L)}{f(q \,|\, a_H)}$ decreases in q for all $a_H > a_L$.
  - Same intuition as previous lecture: If q is larger, then it is more likely that it is the result of a_H relative to a_L.

- *Claim:* MLR $\Longrightarrow \frac{f_a(q \,|\, a)}{f(q \,|\, a)}$ increases in q.

Proof.

- Fix any $a_H > a_L$.

- MLR $\implies \log \frac{f(q \mid a_L)}{f(q \mid a_H)}$ decreases in q

  $\implies \log f(q \mid a_H) - \log f(q \mid a_L)$ increases in q.

- $\frac{f_a(q \mid a)}{f(q \mid a)} = \frac{d}{da} \log f(q \mid a) = \lim_{h \to 0} \frac{\log f(q \mid a+h) - \log f(q \mid a)}{h}$ increases in q.

$\square$

- Sign of μ is unknown.

- *Claim:* MLR $\implies \mu > 0$ (*i.e.*, wages increase in q).

Proof.

- Suppose that $\mu \leq 0$. Then MLR $\implies w(q)$ decreases in q.

- Define $\hat{w}$ such that $\frac{1}{u'(\hat{w})} = \lambda$. Then

$$
\begin{aligned}
V'(a) &= \int_{\mathbb{R}} u(w(q)) f_a(q \mid a)\, dq - c'(a) \\
&= \int_{f_a \geq 0} u(w(q)) \underbrace{f_a(q \mid a)}_{\geq 0} dq + \int_{f_a \leq 0} u(w(q)) \underbrace{f_a(q \mid a)}_{\leq 0} dq - c'(a) \\
&\qquad \because w(q) \leq \hat{w} \qquad\qquad \because w(q) \geq \hat{w} \\
&\leq \int_{f_a \geq 0} u(\hat{w}) f_a(q \mid a)\, dq + \int_{f_a \leq 0} u(\hat{w}) f_a(q \mid a)\, dq - c'(a) \\
&= u(\hat{w}) \int_{\mathbb{R}} f_a(q \mid a)\, dq - c'(a) \\
&= u(\hat{w}) \underbrace{\frac{d}{da} \underbrace{\int_{\mathbb{R}} f(q \mid a)\, dq}_{=1}}_{=0} - c'(a) \\
&= -c'(a) < 0
\end{aligned}
$$

  which is a contradiction. (FOC approach requires that $V'(a) = 0$).

$\square$

- *Corollary:* MLR $\implies \begin{cases} \frac{f_a(q \mid a)}{f(q \mid a)} \uparrow \text{ in } q \\ \mu > 0 \end{cases} \implies w(q) \uparrow$ in q.

Limited Liability

- The optimal incentive contract may involve negative wages; *i.e.*, $w(q) < 0$ for some q.
  - What if the agent cannot (be forced to) pay ?
  - Desirable to impose the constraint $w(q) \geq 0$ for all q.
- The optimal contract now satisfies

$$1 = \left[\lambda + \mu \frac{f_a(q \mid a)}{f(q \mid a)}\right] u'(w(q)) + \nu(q)$$

  where $\nu(q)$ is the multiplier for the constraint $w(q) \geq 0$ for all q.
- If $w(q)$ was non-negative for all q in the original problem, then $\nu(q) = 0$ for all q, and this problem has the same solution as before.
  - But if $w(q) < 0$ for some q, then the structure of the optimal contract has to change.
- Without limited liability, the agent's IR constraint binds.
  - Generally not true with limited liability.

An Example:

- Effort $a \in \{H, L\}$ and output $q \in \{0, 1\}$.
  - If $a = H$, then $q = 1$ w.p 1.
  - If $a = L$, then $q = 1$ w.p p and otherwise $q = 0$.
- Assume
  - $c_L = 0$, $c_H < 1 - p$;
  - the agent has outside option 0 ; and
  - all parties are risk neutral.

- W/o limited liability, the principal seeks the cheapest way to implement $a = H$:

$$\begin{aligned} \min_{w(0)\,,\,w(1)} \quad & w(1) \\ \text{s.t.} \quad & w(1) - c_H \geq pw(1) + (1-p)\,w(0) \quad (IC) \\ & w(1) - c_H \geq 0 \qquad\qquad\qquad (IR) \end{aligned}$$

- *Solution:*
  - (IR) $\Longrightarrow w(1) = c_H$
  - (IC) $\Longrightarrow pc_H + (1-p)\,w(0) \leq 0 \Longrightarrow w(0) \leq -\frac{p}{1-p}c_H$
    - Observe that $w(0) < 0$!

- Now impose limited liability; *i.e.*, $\bar{w}(q) \geq 0$ for all p.
  - Clearly, $\bar{w}(0) = 0$, so for the IC constraint to be satisfied, we need $\bar{w}(1) \geq \frac{c_H}{1-p}$.
  - Solution: $\bar{w}(1) = \frac{c_H}{1-p} > w(1)$ and $\bar{w}(0) = 0$.

Justifying First Order Approach

- Is effort choice a global maximum?
  - Not always. Counterexample by Mirrlees (see Bolton and Dewatripont).
  - *Problem*: For a given $w(q)$, $V(a)$ need not be concave in a.

- Convex Distribution Function
  - Suppose $F(q\,|\,a)$ is convex in a.
  - Then:

$$\begin{aligned} V(a) &= \int_{\underline{q}}^{\bar{q}} u(w(q))\,f(q\,|\,a)\,dq - c(a) \\ &= u(w(\bar{q}))\underbrace{F(\bar{q}\,|\,a)}_{=1} - u(w(\underline{q}))\underbrace{F(\underline{q}\,|\,a)}_{=0} - \int_{\underline{q}}^{\bar{q}} u'(w(q))\,w'(q)\,F(q\,|\,a)\,dq - c(a) \\ &= u(w(\bar{q})) - \int_{\underline{q}}^{\bar{q}} \underbrace{u'(w(q))}_{\geq 0}\ \underbrace{w'(q)}_{\geq 0\,(MLR)}\ F(q\,|\,a)\,dq - c(a)\ , \end{aligned}$$

which is concave in a.

- *Special case:*
  * Suppose $A = [0,1]$ and $F\left(q\,|\,a\right) = aF_H\left(q\right) + \left(1-a\right)F_L\left(q\right)$ for some CDFs $F_H\left(q\right)$ and $F_L\left(q\right)$.
  * Then $F\left(q\,|\,a\right)$ is linear (and hence concave) in a.

References

Board S., (2011), Lecture Notes.

Bolton and Dewatripont, (2005), *Contract Theory*, MIT Press.

Laffont J-J. and Martimont D., (2002), *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press.

Segal and Tadelis, (2002), *Lectures on Contract Theory*, Stanford University (online link).