Module 18: VCG Mechanism

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- Society comprising of n agents.
- Set A of alternatives from which to choose.
- Agent i receives valuation $v_i(x)$ if alternative $x \in A$ is chosen.
  - The value function $v_i(\cdot)$ is private information of each agent.
- Monetary transfers $\{t_1, .., t_n\}$.

- *Basic Idea:*
  - Mechanism that makes it each agent's dominant strategy to reveal $v_i(\cdot)$ truthfully.
  - Implements the first-best outcome.

Efficient Outcome

- Suppose (for now) that we know the $v_i(\cdot)$ function for every i.
  1. Compute the socially efficient action: $x^* = \arg\max_x \{\sum_i v_i(x)\}$
  2. Compute the total welfare of the society, not counting i: $\sum_{j \neq i} v_j(x^*)$
- How would this change if i was not a member of the society?
  - Compute $x^*_{-i} = \arg\max_x \left\{\sum_{j \neq i} v_j(x)\right\}$
- The difference

$$\sum_{j \neq i} v_j(x^*) - \sum_{j \neq i} v_j\left(x^*_{-i}\right)$$

  is a measure of how much agent i contributes to the rest of the society (may be negative).

Mechanism

- We will now construct a mechanism in which agent i receives a monetary transfer equal to the amount he contributes to the rest of the society.
  - Each agent i simultaneously announces a valuation $\tilde{v}(\cdot)$; not necessarily equal to $v(\cdot)$.
  - The mechanism must ensure that each agent announces truthfully.
- Outcomes:
  - Suppose the agents announce $\tilde{v} = \{\tilde{v}_1, .., \tilde{v}_n\}$.
  - Let $x^*(\tilde{v}) = \arg\max_x \{\sum_i \tilde{v}_i(x)\}$; and
  - $x^*(\tilde{v}_{-i}) = \arg\max_x \left\{\sum_{j \neq i} \tilde{v}_j(x)\right\}$
- Transfers:
  - Agent i receives
$$t_i(\tilde{v}) = \sum_{j \neq i} \tilde{v}_j(x^*(\tilde{v})) - \sum_{j \neq i} \tilde{v}_j(x^*(\tilde{v}_{-i}))$$
  - Intuitively, each agent receives his "contribution" to the rest of the society.

Efficiency Properties

Proposition. *The VCG mechanism is efficient:*

1. *All agents have a dominant strategy to announce their true valuation (i.e., announcing truthfully $\tilde{v}_i = v_i$ is the best strategy irrespective of the other agents' announcements).*
2. *When they do so, the efficient outcome is enacted by the VCG mechanism.*

Proof.

- Suppose that the others announce $\tilde{v}_{-i}$, and agent i announces $\tilde{v}_i$. Then his utility is

$$\begin{aligned} & v_i(x^*(\tilde{v}_i, \tilde{v}_{-i})) + t_i(\tilde{v}_i, \tilde{v}_{-i}) \\ = \ & v_i(x^*(\tilde{v}_i, \tilde{v}_{-i})) + \sum_{j \neq i} \tilde{v}_j(x^*(\tilde{v}_i, \tilde{v}_{-i})) - \sum_{j \neq i} \tilde{v}_j(x^*(\tilde{v}_{-i})) \end{aligned}$$

- Agent i will choose $\tilde{v}_i$ to maximize the above expression.

- Suppose (for now) that agent i could choose the alternative x directly. He would solve

$$x^* = \arg\max_x \left\{ v_i(x) + \sum_{j \neq i} \tilde{v}_j(x) \right\}$$

- Observe that x^* coincides with $x^*(v_i, \tilde{v}_{-i})$.
  - Agent i cannot choose x directly, but he can choose $\tilde{v}_i = v_i$, which will in turn induce the mechanism to choose $x^*(v_i, \tilde{v}_{-i})$.
- So announcing truthfully is optimal.
- Because $x^*(v)$ is the efficient alternative, property #1 implies property #2.

□

Example

- Two roommates with individual valuations $v_1 = v_2 = 300$ for a Playstation 4.
- The cost of the PS4 is \$400.
  - We must also include the mechanism designer (who sells the PS4; *i.e.*, $v_3 = 400$).
  - Note that $v_1 + v_2 \geq 400$, but $v_1 < 400$ and $v_2 < 400$.
- We denote

$$x = \begin{cases} 1 & \text{if the PS4 is purchased} \\ 0 & \text{if it is not} \end{cases}$$

- Therefore: $v_i(1) = 300$ and $v_i(0) = 0$ for each $i \in \{1, 2\}$, while $v_3(1) = 0$ and $v_3(0) = 400$.
- VCG mechanism specifies that the PS4 should be purchased: $x^*(v) = 1$, but $x^*(v_{-i}) = 0$ for all i.
  - Each individual reports $\tilde{v}_i$ and the mechanism specifies $x = 1$ iff $\tilde{v}_1 + \tilde{v}_2 \geq 400$.
- Individual $i \in \{1, 2\}$ receives $t_i(\tilde{v}) = \sum_{j \neq i} \tilde{v}_j(x^*(\tilde{v})) - \sum_{j \neq i} \tilde{v}_j(x^*(\tilde{v}_{-i})) = \tilde{v}_{-i} - 400$.
  - He finds it optimal to report $\tilde{v}_i = 300$, so each individual receives $t_i(300) = 300 - 400 = -100$.

Special Case

- Assume
  - $A = \{p_1, .., p_n\}$, where $p_i \in \{0, 1\}$ and $\sum_i p_i = 1$.
  - $v_i(x) = \begin{cases} \theta_i & \text{if } x = p_i \\ 0 & \text{otherwise} \end{cases}$
- Interpretation: $x = p_i$ if agent i is allocated the good.
- VCG mechanism is equivalent to a second-price auction.

Problems

1. Pushes complexity onto bidders.
   - With non-linear utility function and many outcomes, revelation mechanism requires that each agent announces his entire utility "curve".
2. Not budget balanced.
   - In the previous example, the two individuals pay $200 \leq 400$, which is the cost of the PS4. This is problematic!
3. Possible to have very low-revenue outcomes.
   - Two items A and B.
   - Bidder $i = 1$ and $i = 2$ values item A and B at 9, respectively.
   - Bidder $i = 3$ values values A and B together at 10.
   - So $x = \{I, P\}$, where $x = I$ if each individual bidder receives an item, while $x = P$ if the package bidder receives both items. Then:
     - $v_1(I) = v_2(I) = 9$ and $v_1(P) = v_2(P) = 0$.
     - $v_3(I) = 0$ and $v_3(P) = 10$.
   - Efficient to award items to the individual bidders (*i.e.*, $x^*(v) = I$).
     - Note: $x^*(v_{-1}) = x^*(v_{-2}) = P$ and $x^*(v_{-3}) = I$

- We know that the VCG mechanism induces each agent to reveal his valuation truthfully, so

$$\begin{aligned}
t_1 &= v_2(x^*(v)) + v_3(x^*(v)) - v_2(x^*(v_{-1})) - v_3(x^*(v_{-1})) = 9 + 0 - 0 - 10 = -1 \\
t_2 &= v_1(x^*(v)) + v_3(x^*(v)) - v_1(x^*(v_{-2})) - v_3(x^*(v_{-2})) = 9 + 0 - 0 - 10 = -1 \\
t_3 &= v_1(x^*(v)) + v_2(x^*(v)) - v_1(x^*(v_{-3})) - v_2(x^*(v_{-3})) = 9 + 9 - 9 - 9 = 0
\end{aligned}$$

- So auction revenue is 2, although bidder 3 would pay 10.

4. Highly susceptible to collusion.

   - Two items A and B.
   - Package bidder values A,B together at 10.
   - One individual bidder for each item, with value 2.
   - With honest bidding, package bidder wins.
   - Suppose individual bidders both report 9.
   - Items are awarded to the individual bidder and each pays 1, so profitable collusion leads to very inefficient outcome.

   - *Note:* Collusion is always a concern in auctions, but in a second-price auction, collusion by even a small number of parties can have a big effect.

5. Perverse incentives for de-mergers.

   - Two items A and B.
   - Bidder 1 is willing to pay 10 for the pair.
   - Bidder 2 is willing to pay 9 for the pair.
   - If honest, bidder 1 wins and pays 9.
   - If bidder 2 enters as 2A and 2B, each of which bids 9 for a single item, it wins both and pays 2.

References

Board S., (2011), Lecture Notes.

Bolton and Dewatripont, (2005), *Contract Theory*, MIT Press.