Module 1: Decision Making Under Uncertainty

Information Economics (Ec 515) · George Georgiadis

- Today, we will study settings in which decision makers face uncertain outcomes.
  - Natural when dealing with asymmetric information.
  - Need to have a model of how agents make choices / behave when they face uncertainty.
  - Prevalent theory: Expected utility theory.
- States of the world (or states of nature):
  - Relevant pieces of information that are mutually exclusive.
  - The state of the world affects your payoff (or utility, or welfare).

An Example: Will Greece default on its debt or not?

- Two possible "states of nature": default (D), or no default (N).
  - An investor's return may be affected by the state of nature.
  - This may affect whether you invest in stocks or cash.
- Agent has two options: invest in cash or in stocks.
- If the agent invests in stocks:
  - Return equal to 5% under state N.
  - Return equal to -10% under state D.
- If he invests in cash:
  - Return equal to 0% under either state.
- In which assets should the agent invest?
  - Need a model of decision making under uncertainty.

Preferences over lotteries

- Let X be a set of "prizes".
  - For instance, X could be monetary payoffs (returns). In this case $X = \mathbb{R}$.
- A "lottery" is a function $p: X \to [0,1]$ such that $\sum_{x \in X} p(x) = 1$.
  - $p(x)$ is the probability with which lottery p pays $x \in X$.
- *Example:* $X = \{1, 10, 100\}$, $p(1) = 0.4$, $p(10) = 0.2$, $p(100) = 0.4$.
- Let $\mathcal{L}(X)$ denote the set of all lotteries on X.
- A lottery $p \in \mathcal{L}(X)$ is non-trivial if it has at least two distinct prizes with positive probability.

Expected utility

- Let $\succeq$ be a "preference relation" over lotteries in $\mathcal{L}(X)$.
  - The preference relation $\succeq$ tells us how the agent whose decisions we are studying ranks the lotteries in $\mathcal{L}(X)$.
  - For two lotteries p, q, $p \succeq q$ means that p is preferred to q.
  - $p \succ q$ means p is strictly preferred to q ($p \succeq q$ and $q \not\succeq p$).
- We would like there to be a utility function function $u: X \to \mathbb{R}$ such that $p \succeq q$ if and only if

$$\sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x)$$

- In this case, we can evaluate lotteries by computing their expected payoffs.
- Under *certain conditions* such a utility function exists.
  1. $\succeq$ is complete and transitive:
     - For any pair of lotteries p, q, either $p \succeq q$, or $q \succeq p$ (or both).
     - If $p \succeq q$ and $q \succeq r$, then $p \succeq r$.
  2. $\succeq$ satisfies the independence axiom:

- For any lotteries p, q, r, and any $a \in (0,1)$, if $p \succeq q$ then

$$ap + (1-a)r \succeq aq + (1-a)r.$$

3. $\succeq$ satisfies continuity:
   - For any p, q, r, if $p \succ q \succ r$, then there exists $a, b \in (0,1)$ such that

$$ap + (1-a)r \succ q \succ bp + (1-b)r.$$

- From now on, we will work directly with expected utility.

- Suppose that an agent's preferences are represented by $u : X \to \mathbb{R}$.
  - This agent's preferences satisfy the conditions of expected utility.
- Let $v(x) = \alpha u(x) + \beta$ for some $\alpha > 0$ and β.
- Then, for all $p, q \in \mathcal{L}(X)$,

$$\begin{aligned} \sum_{x \in X} p(x)v(x) &\geq \sum_{x \in X} q(x)v(x) \\ \iff \sum_{x \in X} p(x)u(x) &\geq \sum_{x \in X} q(x)u(x). \end{aligned}$$

- The utility function $v(x)$ also represents the preferences of this agent.

Monetary Consequences

- Suppose that $X = \mathbb{R}$; think of elements in X as money.
- In this case, natural to assume that $u(x)$ is increasing in x:
  - If $x_1 \geq x_2$, then $u(x_1) \geq u(x_2)$.

Attitudes towards risk

- Suppose that $X = \mathbb{R}$ (monetary outcomes).
- Let u be the utility function of the agent.

Definition 1. The certainty equivalent of a lottery p is the value x_p^c such that $u\left(x_p^c\right) = \sum_{x \in X} p(x)u(x)$.

- The agent is indifferent between facing a lottery p or obtaining x_p^c for sure.

- For any lottery p, let $\overline{x}_p = \sum_{x \in X} p(x)x$ be the expected payment of lottery p.
- An agent is risk-averse if, for all non-trivial lotteries $p \in \mathcal{L}(X)$, $x_p^c < \overline{x}_p$.
- An agent is risk-neutral if, for all non-trivial lotteries $p \in \mathcal{L}(X)$, $\overline{x}_p = x_p^c$.
- An agent is risk-loving if, for all non-trivial lotteries $p \in \mathcal{L}(X)$, $x_p^c > \overline{x}_p$.

- *Jensen's inequality:* if u is strictly concave and p is a non-trivial lottery, then

$$\sum_{x \in X} p(x)u(x) < u(\overline{x}_p)$$

- If $u(\cdot)$ is strictly convex, the opposite inequality holds.
- If u is linear, then $\sum_{x \in X} p(x)u(x) = u(\overline{x}_p)$.
- An agent with utility function $u(\cdot)$ is:
  - risk-averse iff $u(\cdot)$ is strictly concave ($u''(x) < 0$ for all x).
  - risk-neutral iff $u(\cdot)$ is linear ($u''(x) = 0$ for all x).
  - risk-loving iff $u(\cdot)$ is strictly convex ($u''(x) > 0$ for all x).
- The risk premium of lottery p is $\overline{x}_p - x_p^c$.

- The following statements are equivalent:
  - An agent is risk-averse.
  - $\sum_{x \in X} p(x)u(x) < u(\overline{x}_p)$ for all non-trivial lotteries $p \in \mathcal{L}(X)$.
  - $x_p^c < \overline{x}_p$ for all non-trivial lotteries $p \in \mathcal{L}(X)$.
  - The risk premium of lottery p is positive for all non-trivial lotteries $p \in \mathcal{L}(X)$.
  - u is strictly concave (u'').

Measuring risk aversion

Absolute risk aversion

- Suppose an individual has wealth w.
- This individual faces the following choice: a sure gain of z or a lottery p.
  - In first case, he gets $u(w+z)$ for sure.
  - In second case, he gets an expected payoff of $\sum_{x \in X} p(x)u(w+x)$.
- How does this agent's choice depends on his wealth w?
- If the agent's willingness to take the lottery increases with wealth, we say that he has decreasing absolute risk aversion (ARA).
  - If agent has decreasing ARA, then if he is willing to take lottery when his wealth is w_1, he will also be willing to take the lottery when his wealth is $w_2 > w_1$.
- Analogous definitions for increasing ARA and constant ARA.

- Coefficient of absolute risk aversion: $A(x) = -\frac{u''(x)}{u'(x)}$:
  - If $A(x)$ is decreasing (or constant, or increasing), then agent with utility u has decreasing (or constant, or increasing) absolute risk aversion.
- *Examples:*
  - $u(x) = -e^{-\alpha x} \Rightarrow A(x) = \alpha$ (CARA).
  - $u(x) = \sqrt{x} \Rightarrow A(x) = \frac{1}{2x}$ (decreasing ARA).
- Let $u_1(x)$ be a utility function, and let $u_2(x) = g\left(u_1(x)\right)$ (with $g' > 0, g'' < 0$). Then, $A_1(x) < A_2(x)$ for all x.
  - If $A_1(x) < A_2(x)$, then agent 1 is less risk-averse than agent 2.

Relative risk aversion

- Suppose again that agent has wealth w.
- Agent faces two assets: one pays return z for sure, and the other pays a random return r.
- Agent considers investing all his wealth in either of these assets.
  - If he invests all wealth in safe asset, he earns $u\left(w\left(1+z\right)\right)$ for sure.
  - If he invests all wealth in risky asset, he earns expected payoff $\sum p\left(r\right)u\left(w\left(1+r\right)\right)$.
- How does this agent's choice depends on his wealth w?
- If an agent's willingness to invest in risky asset increases with wealth, we say that he has decreasing relative risk aversion (RRA).
  - If agent has decreasing RRA, if he is willing to invest in risky asset when his wealth is w_1, he will also be willing to invest in risky asset when his wealth is $w_2 > w_1$.
- Similar definitions for increasing RRA and constant RRA.

- Coefficient of relative risk aversion: $R(x) = -\frac{xu''(x)}{u'(x)}$.
  - If $R(x)$ is decreasing (or constant, or increasing), then agent with utility u has decreasing (or constant, or increasing) relative risk aversion.
- *Examples:*
  - $u(x) = x^{1-\alpha} \Rightarrow R(x) = \alpha$ (CRRA).
  - $u(x) = -e^{-\alpha x} \Rightarrow R(x) = \alpha x$.

References

Mas-Colell, Whinston and Green, (1995), *Microeconomic Theory*, Oxford University Press.

Ortner J., (2013), Lecture Notes.