Midterm Exam: Sketch of Solutions

Information Economics (Ec 515) · George Georgiadis

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Problem 1. Application of a Principal-Agent Problem to Credit Markets (9-13-13 points)

A risk-neutral entrepreneur has a project that requires capital K to initiate. The project will either succeed, in which case it generates profit R, or it will fail, in which case it generates profit 0. The probability of success is equal to the entrepreneur's effort level $e \in [0,1]$, and his effort cost is $\frac{e^2}{2}$. He seeks outside funding from an investor. The investor receives a repayment r if the project is successful and 0 otherwise. Therefore, the entrepreneur's and the investor's expected profit is $\Pi_E = e(R - r) - \frac{e^2}{2}$ and $\Pi_I = er - K$, respectively. Assume that the entrepreneur makes a take-it-or-leave-it offer to an investor, whose outside option is 0. (This is equivalent to assuming that the investor offers the contract and he operates in a perfectly competitive market.) Furthermore, assume that $R^2 > 4K$.

1. Assume that effort level is contractible. Derive the first-best outcome e^{FB} and r^{FB}.

2. Now suppose that effort is not contractible. Find the optimal solution e^* and r^*.

3. Assume that the entrepreneur has initial wealth $w \in [0, K]$. Therefore, he will invest his own wealth w in the project, and borrow $K - w$ from an investor. Compute the entrepreneur's expected utility $V(w)$. Show that it is increasing and concave in w. Provide some intuition and interpretation.

Solution of Problem 1:

Part 1 Since effort level is contractible, the entrepreneur's problem is

$$\max_{e,r} \quad e(R - r) - \frac{e^2}{2}$$
$$\text{s.t.} \quad er \geq K$$

The entrepreneur's IR constraint binds, and so the entrepreneur's problem can be re-written as

$$\max_e \left\{ eR - K - \frac{e^2}{2} \right\},$$

where $r = \frac{K}{e}$. The FOC gives $e^{fb} = R$ and so $r^{fb} = \frac{K}{R}$.

Part 2 Since the effort level is not contractible, the entrepreneur's optimal effort level must maximize her payoff given payment r:

$$e = \arg\max_e \left\{ e(R - r) - \frac{e^2}{2} \right\}$$

The FOC gives $e = R - r$. Therefore, the entrepreneur's problem is

$$\max_{e,r} \quad e(R-r) - \frac{e^2}{2}$$
$$\text{s.t.} \quad er \geq K$$
$$e = R - r$$

or equivalently

$$\max_{r} \quad \frac{(R-r)^2}{2}$$
$$\text{s.t.} \quad (R-r)r \geq K$$

The entrepreneur's IR constraint can be re-written as

$$(R-r)r \geq K \Longleftrightarrow r^2 - Rr + K \leq 0 \Longleftrightarrow r \in \left[\frac{R - \sqrt{R^2 - 4K}}{2}, \frac{R + \sqrt{R^2 - 4K}}{2} \right]$$

The principal's objective function decreases in r (for $r \leq R$), and hence the optimal solution has

$$r^* = \frac{R - \sqrt{R^2 - 4K}}{2} \quad \text{and} \quad e^* = \frac{R + \sqrt{R^2 - 4K}}{2}$$

Part 3 The only difference here relative to part 2 is the IR constraint:

$$er \geq K - w$$

Using the same approach as in part 2, we get $r = \frac{R - \sqrt{R^2 - 4(K-w)}}{2}$ and so the entrepreneur's profit function becomes

$$V(w) = \frac{(R-r)^2}{2} = \frac{[R + \sqrt{R^2 - 4(K-w)}]^2}{8}$$

It's straightforward to show that $V'(w) \geq 0$ and $V''(w) \leq 0$ for all $w \in [0, K]$; *i.e.*, it is an increasing, concave function in the entrepreneur's wealth. This implies that the poorest population would benefit the most from access to credit markets.

Problem 2. Moral Hazard in Teams and Different Types of Implementation (15-15 points)

A risk-neutral firm employs two identical, risk-neutral workers. Each worker's utility is $w - e$ where w is his wage and e is his effort level. Each worker i can either work or shirk; *i.e.*, $e_i \in \{0, 1\}$, and efforts are not contractible. Assume that the firm wants to incentivize both workers to work at the lowest possible cost.

1. The firm decides to monitor its workers by group performance. In particular, it observes $\hat{e} = \min\{e_1, e_2\}$. If $\hat{e} = 1$, then both workers receive wage w^g. Otherwise, they both receive 0. Find the optimal wage w^g. What is a potential problem with this incentive scheme?

2. The firm now turns to a monitoring technology which detects a worker shirking with probability $q \in [0, 1]$. In this case, the worker receives 0. Otherwise, he receives wage w^M. The cost of this technology is $c(q) = \frac{q^2}{2}$. Assume that the firm first commits to a monitoring level q, and each worker observes the q before choosing his effort level. Find the optimal wage w^M and monitoring level q^M.

Hint: For part 1, you may assume either partial or full implementation.

Solution of Problem 2:

Part 1 The optimal wage w^g is the smallest wage such that there exists an equilibrium in which both workers work; i.e., $e_1 = e_2 = 1$. Given that agent i works, the IC constraint of workers $j \neq i$ is

$$w^g - 1 \geq 0$$

Therefore, the optimal wage is $w^g = 1$. The problem with this incentive scheme is that there exists another Nash equilibrium in which both workers shirk. To see why, suppose that worker i shirks. Then worker j's best response is to also shirk

Part 2 The firm's problem is

$$\min_{w^M, q} \quad w^M + \frac{q^2}{2}$$
$$\text{s.t.} \quad w^M - 1 \geq (1 - q)w^M$$

Note that the worker's IC constraint asserts that the worker prefers to work and incur the cost of effort rather than shirk and receive wage w^M if he isn't caught (which occurs with probability $1 - q$). The IC constraint can be re-written as $w^M q \geq 1$, and by noting that the firm's objective increases in both w^M and q, it must be that the IC constraint binds in the optimal solution. By substituting $w^M = \frac{1}{q}$ into the objective function, we have

$$\min_q \left\{ \frac{1}{q} + \frac{q^2}{2} \right\}$$

The FOC gives $-\frac{1}{q^2} + q = 0 \implies q^M = 1$, and hence $w^M = 1$.

Problem 3. Free-riding in Teams (13-13-9 points)

Two risk-neutral agents collaborate in production. Output $y = 1 - (1 - e_1 - e_2)^2$, where e_i denotes agent i's effort level. Agent i's cost of effort is given by e_i^2. The total profit is equal to yp, where $p > \frac{1}{2}$ is the per-unit price of the product.

1. Assume that the agents can coordinate perfectly so that they choose their efforts to maximize their joint surplus $S = yp - e_1^2 - e_2^2$. Find the optimal effort levels $\left\{ e_1^{fb}, e_2^{fb} \right\}$, and the corresponding joint surplus S.

2. Now suppose that the workers cannot coordinate, so that each agent will choose his effort level to maximize his profit given his expectations about the other agent's effort level. Assume that each agent receives $\frac{1}{2}yp$ (i.e., $\frac{1}{2}$ of the profit). Find the equilibrium effort levels $\{e_1, e_2\}$, and the corresponding joint surplus. How does it compare to your answer from part 1? Provide some intuition.

3. In the pursuit of coordination, the workers can hire a manager. The manager will assign an effort level to each agent to maximize profits (*i.e.*, yp, not surplus $S = yp - e_1^2 - e_2^2$), and the agents must follow her instruction. Assume that each agent receives $\frac{1}{2}yp$, and the manager receives her outside option $\bar{u} = 0$. Under what conditions do the agents prefer to hire a manager (relative to the case from part 2). Provide some intuition.

Hint: You may restrict attention to symmetric strategies; *i.e.*, strategies that have $e_1 = e_2$.

Solution of Problem 3:

Part 1 The efforts that maximize the joint surplus of the two agents solve

$$\max_{e_1,e_2} \left\{ p\left[1 - (1 - e_1 - e_2)^2\right] - e_1^2 - e_2^2 \right\}$$

The FOCs are

$$
\begin{aligned}
2p\left(1 - e_1 - e_2\right) &= 2e_1 \\
2p\left(1 - e_1 - e_2\right) &= 2e_2
\end{aligned}
$$

which implies that $e_1^{fb} = e_2^{fb} = \frac{p}{1+2p}$, and by substituting this into the joint surplus function, we get $S^{fb} = \frac{2p^2}{2p+1}$.

Part 2 Now each agent chooses his effort to maximize his own profit while anticipating the effort of the other agent. Therefore, agent i solves

$$\max_{e_i} \left\{ \frac{p}{2}\left[1 - (1 - e_i - e_j)^2\right] - e_i^2 \right\}$$

where e_j is the effort that agent i anticipates from agent $j \neq i$. The FOC of agent i gives

$$\frac{p}{2}\left(1 - e_i - e_j\right) = 2e_i$$

Solving for a symmetric equilibrium (i.e., for an equilibrium where $e_1 = e_2$), we obtain $e_1 = e_2 = \frac{p}{2+2p}$, and the corresponding joint surplus is $S = \frac{p^2(2p+3)}{2(p+1)^2}$.

It is straightforward to show that $S < S^{fb}$. Intuitively, when the agents choose their efforts non-cooperatively, they exert less effort due to the free-rider problem.

Part 3 Denote e_1^M, e_2^M, S^M as the effort levels and joint surplus when a manager is hired. In this case, the manager assigns effort levels to maximize the profit $yp = p\left[1 - (1 - e_1 - e_2)^2\right]$. Note that the manager does not take the agents' effort costs into account. Observe that the profit is maximized when $e_1 + e_2 = 1$, in which case $yp = p$. Assuming symmetric strategy, we have $e_1^M = e_2^M = \frac{1}{2}$, and the corresponding joint surplus is $S^M = p - \frac{1}{2}$. Since the manager receives her outside option which is 0, each agent's payoff is $\frac{1}{2}S^M$. Therefore, the agents prefer to hire a manager if and only if

$$
\begin{aligned}
S^M &\geq S \\
\iff p - \frac{1}{2} &\geq \frac{p^2(2p+3)}{2(p+1)^2}
\end{aligned}
$$

This inequality can be re-written as $\left(\frac{2p-1}{2p+3}\right)\left(\frac{p+1}{p}\right)^2 \geq 1$. First observe that the LHS is equal to 0 when $p = \frac{1}{2}$, and so the desired inequality is not satisfied. Moreover, one can show that the LHS increases in p, and the LHS approaches 1 as $p \to \infty$. Therefore, this inequality is never satisfied and so the agents never find it optimal to hire a manager in this scenario.