Motivation

Imagine you are a manager designing employee performance pay plan. To find optimal contract, need to know all payoff-relevant parameters. 

- i.e., agent’s utility and cost function, output distribution for every effort
- Otherwise, agency theory gives us guiding principles (trade-offs, CS).

This paper: How to optimally improve current performance pay plan?

1. What information do you need?
2. And how should you use that information?
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A Simple Example with Linear Contracts

- Suppose you restrict attention to linear contracts; i.e., \( w(x) = \alpha x \)
  - How to modify your current piece rate, \( \alpha \), to increase profits?

- Denote the principal’s profit by \( \Pi = (m - \alpha) a \)
  - \( m \) is marginal profit, and \( a = \mathbb{E}[x|\text{effort}(\alpha)] \) is expected output.

- Consider the impact of a small increase in \( \alpha \):
  \[
  \frac{d\Pi}{d\alpha} = -a + (m - \alpha) \frac{da}{d\alpha} \geq 0 \iff \frac{\alpha}{a} \frac{da}{d\alpha} \geq \frac{\alpha}{m - \alpha}
  \]

Observation:

\( a \) and \( da/d\alpha \) are sufficient statistics for finding profitable perturbation.

(i.) Need observational data corresponding to \( \alpha \), plus an experiment.
(ii.) If contract is characterized by \( n \) parameters need \( \geq n \) experiments!
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Preview of Results

- **Framework**: Static agency model with a risk-averse agent
  - Principal knows the distribution of $x$ following some $\hat{w}$ and some $\hat{w} + \hat{t}$.
  - **Objective**: Perturb $\hat{w}$ in direction that increases profits at fastest rate.

**Key Lemma:**
If the principal knows the agent’s marginal utility for money, then she can estimate his response to any arbitrary perturbation of $\hat{w}$.

- If principal takes stance on $v'$, she can find the optimal perturbation.
- True even if she restricts attention to a parametric class (e.g., affine).
- Illustrate how to apply using dataset from DellaVigna & Pope (2017).
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Related Literature

- *Agency theory:* Mirrlees (1976), Holmström (1979), etc...

- Agency problems — Applications:
  - Gibbons (1998), Murphy (1999), Prendergast (2002), etc...

- Agency problems — Empirics:
  - Lazear (2000), Shearer (2004), Bandiera et al. (2007, 2009), etc...
  - Chiappori & Salanie (2002)

- Sufficient statistics:
  - Monopoly pricing: Lerner (1934), Tirole (1988), etc...
  - Optimal taxation: Saez (2001), Golosov et al. (2014), etc...
  - General framework: Chetty (2009)
Principal-agent model with the standard timing:

1. Principal offers a contract $w(\cdot)$.
2. Agent observes $w(\cdot)$ and chooses effort $a(w)$.
3. Output $x \sim f(\cdot|a(w))$ and payoffs are realized. (Normalize $\mathbb{E}[x|a] = a$.)

Preferences:

- Agent’s utility: $\mathbb{E}_a[v(w(x))] - c(a)$
- Principal’s profit: $\pi(w) := ma(w) - \mathbb{E}_{a(w)}[w(x)]$.

Information:

- Principal has an estimate for $f(\cdot|a(\hat{w}))$ and $f(\cdot|a(\hat{w} + \hat{t}))$, where $\hat{w}$ is a status quo contract and $\hat{w} + \hat{t}$ is a perturbation thereof.

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Assumptions

A.1. Performance measure $x$ does not suffer from multitasking problem, and the data corresponding to $\widehat{w} + \widehat{t}$ is not “tainted” by ratchet effects.

A.2. The first-order approach is valid, so that the agent’s IC constraint can be replaced by the corresponding first-order condition.

A.3. Local Perturbation: Perturbation $\widehat{t}$ leads to a sufficiently small change in the agent’s marginal incentives that the principal can estimate

$$f_a(x|a(\widehat{w})) \approx \frac{f(x|a(\widehat{w} + \widehat{t})) - f(x|a(\widehat{w}))}{a(\widehat{w} + \widehat{t}) - a(\widehat{w})}$$


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Agent’s Problem

- **First-order approach**: optimal effort $a(w)$ satisfies

$$\int v(w(x)) f_a(x|a(w)) \, dx = c'(a(w))$$

(IC)

- Define the Gateaux (directional) derivative

$$\mathcal{D} a(\hat{w}, t) := \frac{d a(\hat{w} + \theta t)}{d \theta} \bigg|_{\theta=0} = \lim_{\theta \to 0} \frac{a(\hat{w} + \theta t) - a(\hat{w})}{\theta},$$

interpreted as marg. change of $a$ when $\hat{w}$ perturbed in direction of $t$.

- Using (IC), $\mathcal{D} a(\hat{w}, t)$ can be written as

$$\mathcal{D} a(\hat{w}, t) = \frac{\int tv'(\hat{w}) \hat{f}_a \, dx}{c''(a(\hat{w})) - \int v(\hat{w}) f_{aa}(x|a(\hat{w})) \, dx},$$

and observe that it is linear in $t$. 

Georgiadis and Powell
Optimal Incentives under Moral Hazard
Northwestern Kellogg
Agent’s Problem

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- The principal’s profit

\[ \pi(w) := ma(w) - \int w(x) f(x | a(w)) \, dx \]

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- Perturbing \( \widehat{w}(\cdot) \) in direction of \( t(\cdot) \) affects profits via two channels:
  1. By inducing a change in effort, as captured by 1\(^{st}\) term.
  2. Mechanically, as captured by 2\(^{nd}\) term.

- Set of Pareto-improving perturbations:

\[ \mathcal{T} = \left\{ t : \left. \frac{d}{d\theta} \int v(\widehat{w} + \theta t) f(x | a(w + \theta t)) \, dx - c(a(w + \theta t)) \right|_{\theta=0} \geq 0 \right\} \]

- (IC) implies constraint \( t \in \mathcal{T} \) can be rewritten as \( \int tv'(\widehat{w}) \widehat{f} \, dx \geq 0 \).
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Interpreting our Assumptions

- Define family of contracts \( \hat{w}_\theta \equiv (1 - \theta) \hat{w} + \theta (\hat{w} + \hat{t}) \equiv \hat{w} + \theta \hat{t} \)

- Using a first-order Taylor expansion, for \( \theta \in (0, 1] \),
  \[
a(\hat{w}_\theta) = a(\hat{w}) + \theta a(\hat{w} + \hat{t}) \text{, and}
\]
  \[
f(x|a(\hat{w}_\theta)) = \hat{f}(x) + \theta \hat{f}_a(x) \left[a(\hat{w} + \hat{t}) - a(\hat{w})\right] \quad \forall x
\]
- Perturbation \( t \) that maximizes \( D_{\pi}(\hat{w}, t) \)? [Steepest ascent]
Principal’s Problem (Cont’d)

- The principal solves

\[
\max_{t \text{ u.s.c}} \left( m - \int \hat{w} \hat{f}_a \, dx \right) D_a(\hat{w}, t) - \int t \hat{f} \, dx \\
\text{s.t} \int t v'(\hat{w}) \hat{f} \, dx \geq 0 \\
\int |t|^p \, dx \leq 1
\]

for some “smoothing parameter” \( p \in \{2, 3, \ldots\} \).

- Given a solution to (P), the principal should replace \( \hat{w} \) with

\[
w(x) = \hat{w}(x) + \theta t(x)
\]

for an appropriately chosen step size \( \theta > 0 \).

- To solve (P), principal must know \( v' \) and \( D_a(\hat{w}, t) \) for all feasible \( t \).
Principal’s Problem (Cont’d)

The principal solves

$$\max_{t \text{ u.s.c}} \left( m - \int \hat{w} \hat{f}_a \, dx \right) Da(\hat{w}, t) - \int t \hat{f} \, dx$$

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- To solve (P), principal must know \( v' \) and \( \mathcal{D}a(\hat{w}, t) \) for all feasible \( t \)!
The principal solves

\[
\max_{t \text{ u.s.c}} \left( m - \int \hat{w} f_a \, dx \right) D_a(\hat{w}, t) - \int t \hat{f} \, dx \\
\text{s.t} \quad \int t v'(\hat{w}) \hat{f} \, dx \geq 0 \\
\int |t|^p \, dx \leq 1
\]

(P)

(Part)

for some “smoothing parameter” \( p \in \{2, 3, \ldots\} \).

Given a solution to (P), the principal should replace \( \hat{w} \) with

\[ w(x) \equiv \hat{w}(x) + \theta t(x) \]

for an appropriately chosen step size \( \theta > 0 \).

To solve (P), principal must know \( v' \) and \( D_a(\hat{w}, t) \) for all feasible \( t \).
Sufficient Statistics

- Recall that

\[ D_a(\hat{w}, t) = \frac{\int t'v'(\hat{w})\hat{f}_a dx}{c''(a(\hat{w})) - \int v(\hat{w})f_{aa}(x|a(\hat{w})) dx} \]

Remark 1.

- For any u.s.c \( t \),

\[ D_a(\hat{w}, t) = \frac{D_a(\hat{w}, t)}{\int t'v'(\hat{w})\hat{f}_a dx} \int t'v'(\hat{w})\hat{f}_a dx \]

- If the principal knows \( v' \), then she can solve (P).
Main Result

Optimal Perturbation

- The principal’s problem can be rewritten as

$$\max_{t \text{ u.s.c}} \mu^* \int tv'(\hat{w}) \hat{f}_a dx - \int t \hat{f} dx$$

s.t. $$\int tv'(\hat{w}) \hat{f} dx \geq 0$$

$$\int |t|^p dx \leq 1$$

where $$\mu^*$$ is a constant that depends on $$\hat{w}, \hat{t}, v',$$ and $$\hat{f}_a$$.

Main Proposition.

i. $$\hat{w}$$ is locally optimal iff $$\lambda + \mu^* \frac{\hat{f}_a}{\hat{f}} = \frac{1}{v'(\hat{w})}$$, where $$\lambda = \int \frac{\hat{f}}{v'(\hat{w})} dx$$.

ii. Otherwise, the optimal perturbation (when $$p = 2$$)

$$t^* = C \times \left[ \lambda^* + \mu^* \frac{\hat{f}_a}{\hat{f}} - \frac{1}{v'(\hat{w})} \right]$$
Optimal Perturbation

- The principal’s problem can be rewritten as

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Main Proposition.

i. \( \hat{w} \) is locally optimal iff \( \lambda + \mu^* \frac{\hat{f}_a}{\hat{f}} \equiv \frac{1}{\nu'(\hat{w})} \), where \( \lambda = \int \hat{f} / \nu'(\hat{w}) dx \).

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Two Remarks & Two Caveats

R.1. While our optimality condition looks the same as Holmström (1979), it contains additional information, as it accounts for perturbations which induce the agent to choose a different effort level.

R.2. Optimal perturbation dictates raising payments if and only if

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(\lambda^* \hat{f} + \mu^* \hat{f_a}) v'(\hat{w}) > \hat{f}
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Proposition 1 is useful in 3 ways

1. Provides an explicit formula for finding the optimal perturbation.

   \[ D a(\hat{w}, \hat{t}) \quad \text{Marg. change of effort if } \hat{w} \text{ is perturbed in direction of } \hat{w} + \hat{t} \]
   \[ f(\cdot | a(\hat{w})) \quad \text{Pdf of } x \text{ given the agent's effort } a(\hat{w}) \]
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   \[ \nu'(\cdot) \quad \text{Agent's marginal utility function} \]

2. Tell us what information principal must estimate / take a stance on.

3. Even if the principal is unwilling to experiment or make assumptions, can infer implicit assumptions (given the premise that \( \hat{w} \) is optimal).
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What about characterizing the *optimal contract*?

- The characterization itself is standard; see, for example, Grossman and Hart (1983) or Jewitt et al. (2008).

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 Extensions (1/2)

1. *Bounded payments.* Assume that \( \widehat{w}(x) + \theta t(x) \in [w, w] \)
   
   - New constraints are linear, so principal’s problem remains convex.

2. *Heterogeneous abilities.* Assume that the principal offers a common contract to multiple agents who have heterogeneous effort costs.
   
   - Principal must classify the agents into types \((\phi)\), and estimate \( \Pr\{\phi\} \), \( \widehat{r}_\phi \), \( \widehat{r}_a^\phi \), and \( D a^\phi(\widehat{w}, \widehat{t}) \) for each \( \phi \).
   
   - Can induce selection by imposing participation for subset of types.

3. *Multidimensional effort.* Assume agent’s effort \( a \in \mathbb{R}^N \) at cost \( c(a) \)
   
   - *Example:* Agent might be a salesman of \( N \) different products.
   
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- Find optimal perturbation direction $z$. (*New contract:* $w_{\alpha + \theta z}$)
- Same informational requirements as general case.

5. **Other sources of incentives.** (Promotion, firing threat, prestige, etc)

- Results hold verbatim if the agent’s IC constraint can be written as

$$\int v(w(x))f_a(x|a(w))dx + I(a(w)) = c'(a(w))$$

- Key: Additive separability and $I$ not directly dependent on $w$.

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Dataset from DellaVigna and Pope (REStud, 2017).

Real-effort experiment on MTurk: Subjects press a-b keys for 10 min.

7 treatments with different monetary incentives.

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Setting & Objectives

- **Assumptions:**
  - Principal’s profit margin $m = 0.2\$ \text{ (per a-b keystroke, } x)\$
  - Principal has data for
    
    $\hat{w}(x) = 100 + 0.01x$ and $\hat{w}(x) + \hat{t}(x) = 100 + 0.04x$
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Empirical CDFs

\[ a(\hat{w}) = 2029 \]
\[ a(\hat{w} + \hat{t}) = 2132 \]
Step 2: Estimate pdf

- Estimate $f(\cdot | a(\hat{w}))$ and $f(\cdot | a(\hat{w} + \hat{t}))$ using kernels, and compute

$$f_a(x | a(\hat{w})) \approx \frac{f(x | a(\hat{w} + \hat{t})) - f(x | a(\hat{w}))}{a(\hat{w} + \hat{t}) - a(\hat{w})}$$
Step 3: Characterize Optimal Perturbation

$\rho = 0$
$\rho = 0.2$
$\rho = 0.5$
$\rho = 0.8$
A “simple” perturbation

\[ \tilde{t} \text{ is feasible and } D_\pi(\tilde{w}, \tilde{t}) \geq 95\% \times D_\pi(\hat{w}, \hat{t}). \]
Step 4: Perturb the status quo Contract

- Replace $\hat{w}$ with $\tilde{w} = \hat{w} + \theta t^*$ for appropriately chosen step size $\theta$.

Here, $\theta$ is chosen such that $a(\tilde{w}) \simeq a(\hat{w} + \hat{t})$. 
**Effect of “smoothing” parameter \( p \)**

As \( p \) increases, the optimal perturbation focuses more on the sign of \( (\lambda^* \hat{f} + \mu^* \hat{f}_a) v'(\hat{w}) - \hat{f} \) instead of its magnitude.
Discussion

How to use contract theory to improve performance pay plans?

1. To find “optimal” perturbation of status quo contract, principal must:
   a. Estimate how distribution of output responds to a change in incentives.
   b. Take a stance on the agent’s marginal utility for money.

2. To characterize optimal contract, need (much) stronger assumptions.

3. Illustrate how to apply ideas using dataset from DellaVigna & Pope.

Future work?

- Optimal experimentation (ratchet effects, behavioral constraints)
- Extend to other settings or more specific applications
- Other optimization algorithms; e.g., Newton (info. requirements)
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